Complex Tauberian theorems for Laplace transforms

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Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory.
- Spectral theory for (pseudo-)differential operators.
- Last three decades: operator theory and semigroups.

We will discuss some developments in complex Tauberians for Laplace transforms. We will be concerned with two groups of statements:

- Wiener-Ikehara theorems.
- Ingham-Karamata theorems.

Main questions:

1. Relax boundary requirements to a minimum.
2. Mild Tauberian hypotheses (one-sided conditions).
3. Optimal Tauberian constants: sharp versions.
4. Best possible error terms

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The classical Wiener-Ikehara theorem

Theorem (Wiener-Ikehara, Laplace-Stieltjes transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{\mathrm{d}S;z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > 1$. If

$$\mathcal{L}\{\mathrm{d}S;z\} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $S(x) \sim Ae^x$.

Theorem (Wiener-Ikehara, version for Dirichlet series)

Let $a_n \geq 0$. Suppose $\sum_{n=1}^{\infty} a_n n^{-z}$ converges for $\Re e\, z > 1$. If

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From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \le x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^\alpha \le x} \log p = \sum_{n \le x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\mathbb{C}$ except for a simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

is analytic in a region containing $\Re e\, z \ge 1$. The rest follows from the Wiener-Ikehara theorem.

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Remarks on the Wiener-Ikehara theorem

- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to "good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
  3. Local pseudofunction boundary behavior (Korevaar, 2005).
  4. "if and only if version" (Debruyne and V., 2016).

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Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g : \widehat{g} \in L^\infty(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{|x|\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x+iy) = g(y) \text{ in } \mathcal{D}'(U)$$

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Extension of the Korevaar-Wiener-Ikehara theorem

We call a function S log-linearly slowly decreasing if for each $\varepsilon > 0$ there exists $\delta > 0$

$$\liminf_{x\to\infty} \inf_{0\le h\le\delta} \frac{S(x+h)-S(x)}{e^x} \ge -\varepsilon.$$

Theorem (Debruyne and V., 2016)

Suppose that $\mathcal{L}\{S;z\} = \int_0^\infty S(t)e^{-zt}\mathrm{d}t$ converges for $\Re e\, z > 1$. Then,

$$S(x) \sim Ae^x$$

if and only if

1. $\mathcal{L}\{S;z\} - \dfrac{A}{z-1}$ *has local pseudofunction boundary behavior on $\Re e\, z = 1$, and*
2. *S is log-linearly slowly decreasing.*

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Séries trigonométriques et séries de Taylor,

Fatou proved the following theorem on analytic continuation of power series.

Theorem (Fatou-Riesz theorem)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < 1$ and $c_n = o(1)$ (this is the Tauberian condition). If $F(z)$ has analytic continuation to a neigborhood of $z = 1$, then $\sum_{n=0}^{\infty} c_n$ converges and

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Marcel Riesz gave three proofs of this theorem (1909, 1911, 1916), so his name is usually associated to this result.

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The Ingham-Karamata theorem for Laplace transforms

In 1935 Ingham and Karamata obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

that is, $\tau(x+h) - \tau(x) > -\varepsilon$ for $x > X_\varepsilon$ and $0 \le h < \delta_\varepsilon$.

Theorem (Ingham and Karamata, independently)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing (Tauberian hypothesis). Suppose its Laplace transform

$$\mathcal{L}\{\tau; z\} = \int_0^\infty \tau(t)e^{-zt}\mathrm{d}t$$

converges on $\Re e\, z > 0$ and has L^1_{loc}-boundary behavior on $\Re e\, z = 0$, then $\lim_{x\to\infty} \tau(x) = 0$.

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converges on $\Re e\, z > 0$ and has L^1_{loc}-boundary behavior on $\Re e\, z = 0$, then $\lim_{x\to\infty} \tau(x) = 0$.

The Ingham-Karamata theorem for Laplace transforms

In 1935 Ingham and Karamata obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

that is, $\tau(x+h) - \tau(x) > -\varepsilon$ for $x > X_\varepsilon$ and $0 \leq h < \delta_\varepsilon$.

Theorem (Ingham and Karamata, independently)

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Developments in the 1980s: Newman's contour integration method

In 1980 Newman gave a simple contour integration proof of the next Tauberian theorem.

Theorem

Let $a_n = O(1)$ *(Tauberian hypothesis). If* $F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ *has analytic continuation beyond* $\Re e\, z = 1$*, then*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = F(1).$$

- It is contained in the Ingham-Karamata theorem; however, Newman's proof method is simple and very attractive.
- In the recent book Twelve landmarks in twentieth century analysis, Choimet and Queffélet chose this theorem as one of such landmarks.

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Newman's short way to the PNT

Newman's Tauberian theorem from above provides a relatively simple way to prove the PNT.

- One works here with the Möbius

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors}, \\ 0 & \text{otherwise.} \end{cases}$$

- Property: μ is the Dirichlet convolution inverse of 1. So,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \frac{1}{\zeta(z)} \quad (\zeta \text{ is the Riemann zeta function})$$

- Applying the previous theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(1)} = 0.$
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Tauberians motivated by applications in semigroups

Newman's contour integration method was adapted to a variety of Tauberian problems in numerous articles.

Its importance was recognized by the semigroup community. Here is a sample (extending a result of Korevaar and Zagier):

Theorem (Arendt and Batty, 1988)

Let $\rho \in L^{\infty}(\mathbb{R})$ (Tauberian hypothesis) vanish on $(-\infty, 0)$. Suppose that $\mathcal{L}\{\rho; z\}$ has analytic continuation at every point of the complement of iE where $E \subset \mathbb{R}$ is a closed null set. If $0 \notin iE$ and

$$\sup_{t\in E}\sup_{x>0}\left|\int_0^x e^{-itu}\rho(u)\mathrm{d}u\right| < \infty,$$

then the (improper) integral of ρ converges to $b = \mathcal{L}\{\rho; 0\}$, that is,

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If $E = \emptyset$, the result is due to Korevaar and Zagier (independently), who also obtained it via Newman's contour integration technique.

In this case, the result is contained in the Ingham-Karamata theorem:

- Set $\tau(x) = \int_0^x \rho(u)\mathrm{d}u - b \Rightarrow \mathcal{L}\{\tau; z\} = \dfrac{\mathcal{L}\{\rho; z\} - b}{z}$ with $b = \mathcal{L}\{\rho; 0\}$.
- $\mathcal{L}\{\rho; z\}$ has analytic continuation beyond $\Re e\, z = 0$ if and only if

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The Arendt-Batty Tauberian theorem readily extends to functions with values on a Banach space. Here is a sample application of the vector-valued version:

Theorem (Arendt and Batty)

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space X. Denote the spectrum of its infinitesimal generator A as $\sigma(A)$. If $\sigma(A)\cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then

$$\lim_{t\to\infty} T(t)x = 0, \quad \forall x \in X.$$

In recent times, Tauberian methods have been revisited to study rates of converge that can be a applied to PDE, e.g. decay estimates for damped wave equations.

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Input from operator theory

In the same spirit, Katznelson and Tzafriri established earlier the following important result in asymptotic operator theory:

Theorem (Katznelson and Tzafriri, 1986)

Let T be a power-bounded operator on a Banach space ($\sup_{n\in\mathbb{N}} \|T^n\| < \infty$). Then,

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$$

if and only if $\sigma(T) \cap \partial\mathbb{D} \subseteq \{1\}$.

- This can be deduced from a power series Tauberian theorem that preceded the Arendt-Batty theorem.
- In their work Katznelson and Tzafriri employed local pseudofunctions on the torus, initiating so the distributional approach in complex Tauberian theory.

Extension of the Ingham-Karamata theorem

Theorem (Debruyne and V.)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

(II) *for each $t \in E$ there is $M_t > 0$ such that*

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} \mathrm{d}u \right| < M_t,$$

(III) $0 \notin E$.

Then

$$\tau(x) = o(1). \qquad (1)$$

Conversely, (??) implies that $\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on the whole imaginary axis.

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Some tools involved in the proof

Our proof is based on:

- Boundedness theorems for Laplace transforms with local pseudo-measure behavior.
- Characterizations of local pseudofunctions through behavior outside exceptional sets.

Other results

- General version of the Katznelson-Tzafriri theorem (for power series), improving the Allan-O'Farell-Ransford theorem (1987).

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Quantified finite forms: Ingham's theorem

Lip(I; M) denotes the class of Lipschitz continuous functions on I with Lipschitz constant M.

Known result: Suppose that

1. $\tau \in \mathrm{Lip}([0,\infty); M)$.
2. $\mathcal{L}\{\tau; z\}$ has "good" boundary behavior on $(-i\lambda, i\lambda)$.

There is an absolute contant $\mathfrak{C} > 0$ such that

$$\limsup_{x\to\infty} |\tau(x)| \leq \frac{\mathfrak{C}M}{\lambda}.$$

Some values of $\mathfrak{C}$:

$\mathfrak{C} = 6$, Ingham (1935)

$\mathfrak{C} = 2$, Korevaar, Zagier, and other people ...

Problem: Find the optimal value of $\mathfrak{C}$.

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Theorem (Debruyne and V., 2018)

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and the value of $\pi/2$ *in this inequality cannot be improved.*

Combining this with the Graham-Vaaler sharp Wiener-Ikehara theorem, one can consider 'Lipschitz continuous functions only from below'. We obtained the sharp inequality

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Inequalities for functions with regular Fourier transform

Define the 'oscillation' and 'decrease' moduli at ∞ as:

$$\Psi(\delta) = \limsup_{x\to\infty} \sup_{h\in[0,\delta]} |\tau(x+h) - \tau(x)|.$$

and

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Theorem (Debruyne and V., 2018)

Let $\tau \in L^1_{loc}(\mathbb{R})$ have at most polynomial growth. Suppose that $\hat{\tau} \in PF_{loc}(-\lambda, \lambda)$ (in particular if continuous there). Then,

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- Optimal Tauberian constant in Ingham's theorem for Laplace transforms, Israel J. Math. 228 (2018), 557–586.
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For some applications of these results in analytic number theory, see:

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Important book references on complex Tauberians

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