

Topological properties of convolutor spaces

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Our results are quantified versions of classical results, mainly motivated by:

- 1 Schwartz' convolution description of \mathcal{D}'_{L^1} .
- 2 Grothendieck's results on the completeness of \mathcal{O}_C and the (ultra-)bornologicity of \mathcal{O}'_C .

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- The space \mathcal{B} consists of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\partial^\alpha \varphi\|_{L^\infty} < \infty, \quad \forall \alpha \in \mathbb{N}^d.$$

- The space \mathcal{B} is a Fréchet space.
- The space $\dot{\mathcal{B}}$ is given by the closure of $\mathcal{D}(\mathbb{R}^d)$ in \mathcal{B} , i.e., all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\lim_{|x| \rightarrow \infty} \partial^\alpha \varphi(x) = 0, \quad \forall \alpha \in \mathbb{N}^d.$$

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Let $f \in \mathcal{D}'(\mathbb{R}^d)$. Then, $f \in \mathcal{D}'_{L^1}$ if and only if $f * \varphi \in L^1$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$.

Two natural topologies on \mathcal{D}'_{L^1} :

- 1 The strong topology $b(\mathcal{D}'_{L^1}, \mathcal{B})$.
- 2 The initial topology op w.r.t. the mapping

$$\mathcal{D}'_{L^1} \rightarrow L_b(\mathcal{D}(\mathbb{R}^d), L^1) : f \rightarrow (\varphi \rightarrow f * \varphi).$$

Theorem (Schwartz, 1950)

The spaces $\mathcal{D}'_{L^1, b}$ and $\mathcal{D}'_{L^1, op}$ have the same bounded sets and null sequences.

Question

Do the topologies b and op coincide on \mathcal{D}'_{L^1} ?

Is $\mathcal{D}'_{L^1, op}$ (ultra-)bornological? If yes, this answers positively the above question.

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The space of rapidly decreasing distributions \mathcal{O}'_C

Schwartz introduced the space of rapidly decreasing distributions as follows:

- \mathcal{B}' stands for the space of bounded distributions, dual of

$$\mathcal{D}_{L^1} = \{\varphi : \partial^\alpha \varphi \in L^1, \forall \alpha \in \mathbb{N}^d\}.$$

- A distribution f belongs to \mathcal{O}'_C if $(1 + |x|^2)^k f \in \mathcal{B}'$, for all $k \in \mathbb{N}$.

Theorem (Schwartz: \mathcal{O}'_C is the space of convolutors of $\mathcal{S}(\mathbb{R}^d)$)

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The space of very slowly increasing smooth functions

A predual of \mathcal{O}'_C

- The space \mathcal{O}_C consists of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that there is $N \in \mathbb{N}$ for which

$$\sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{(1 + |x|)^N} < \infty, \quad \forall \alpha \in \mathbb{N}^d.$$

- \mathcal{O}_C is an (LF)-space (countable inductive limit of Fréchet spaces).
- The space \mathcal{O}'_C of rapidly decreasing distributions is given by the topological dual of \mathcal{O}_C .
- Schwartz wrote in his book: “the space \mathcal{O}_C seems not to play any important role”.
- Grothendieck however made a complete and non-trivial analysis of \mathcal{O}_C and \mathcal{O}'_C , showing that the topological properties of these spaces are very interesting.

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The space of rapidly decreasing distributions

Define the topologies b and op on \mathcal{O}'_C as before.

Theorem (Grothendieck, 1955)

The space $\mathcal{O}'_{C,op}$ is complete, semi-reflexive, and (ultra-)bornological (hence reflexive).

Consequently, $\mathcal{O}'_{C,b} = \mathcal{O}'_{C,op}$ and the (LF)-space \mathcal{O}_C is complete.

Grothendieck method:

- He showed that $\mathcal{O}'_{C,op}$ is isomorphic to a complemented subspace of $s\hat{\otimes}s'$.
- Then proved that that $s\hat{\otimes}s'$ is bornological.
- Moreover, he showed that $(\mathcal{O}'_{C,op})'_b = \mathcal{O}_C$.

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Convolutors in Gelfand-Shilov spaces

of smooth functions

There have been attempts to generalize Grothendieck's result. Let $\mathcal{W} = (w_N)_N$ be an increasing sequence of positive continuous functions such that

$$\forall N \exists M : \lim_{|x| \rightarrow \infty} w_N(x)/w_M(x) = 0.$$

- Define the Fréchet space $\mathcal{K}_{\mathcal{W}} (= \mathcal{K}\{w_N\}) = \{\varphi \in C^\infty(\mathbb{R}^d) : w_N \partial^\alpha \varphi \in L^\infty, \forall N, \alpha\}$
- E.g. $w_N(x) = (1 + |x|)^N$ leads to \mathcal{S} , while if $w_N(x) = e^{N|x|}$ one obtains the space of exponentially decreasing smooth functions \mathcal{K}_1 .
- Associated convolutor space:
 $\mathcal{O}'_C(\mathcal{K}_{\mathcal{W}}) = \{f \in \mathcal{K}'_{\mathcal{W}} : f * \varphi \in \mathcal{K}_{\mathcal{W}} \text{ for all } \varphi \in \mathcal{K}_{\mathcal{W}}\}.$

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Is $\mathcal{O}'_{C,op}(\mathcal{K}_{\mathcal{W}})$ (ultra-)bornological?

- Zielezny claims to have shown this for $\mathcal{O}'_{C,op}(\mathcal{K}_1)$.



Studia Math. 31 (1968), 111–124.

His proofs seem to contain major gaps.

- Abdullah even claims in



Proc. Amer. Math. Soc. 110 (1990), 177–185.

that $\mathcal{O}'_{C,op}(\mathcal{K}_{\mathcal{W}})$ is **always** ultrabornological when the family of weights is of the form

$$w_N(x) = e^{\omega(N|x|)}$$

with ω a positive increasing convex function tending to ∞ .

- It follows from our recent results that Abdullah's claim is false. (E.g. when ω is not polynomially bounded.)
- On the other hand, we showed Zielezny's claim was true.

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- Show the full topological identity $\mathcal{D}'_{L^1,b} = \mathcal{D}'_{L^1,op}$ and extend it to **weighted** \mathcal{D}'_{L^1} spaces.
- Unified approach for \mathcal{D}'_{L^1} and \mathcal{O}'_C , or more generally, convolutor spaces for Gelfand-Shilov spaces \mathcal{K}_W .
- Analyze completeness of weighted inductive limits of spaces of smooth functions (**In particular first direct proof** of completeness of \mathcal{O}_C).
- To this end, we study structural and topological properties of a general class of weighted L^1 convolutor spaces.
- Our arguments are based on the mapping properties of the **short-time Fourier transform**. Inspired by:
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- Show the full topological identity $\mathcal{D}'_{L^1,b} = \mathcal{D}'_{L^1,op}$ and extend it to **weighted** \mathcal{D}'_{L^1} spaces.
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The short-time Fourier transform (STFT)

- $T_x f := f(\cdot - x)$ and $M_\xi f := e^{2\pi i \xi t} f(t)$ for $x, \xi \in \mathbb{R}^d$.
- The STFT of $f \in L^2(\mathbb{R}^d)$ w.r.t. a window function $\psi \in L^2(\mathbb{R}^d) \setminus \{0\}$ is defined as

$$V_\psi f(x, \xi) := (f, M_\xi T_x \psi)_{L^2} = \int_{\mathbb{R}^d} f(t) \overline{\psi(t-x)} e^{-2\pi i \xi t} dt.$$

- The mapping $V_\psi : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^{2d})$ is continuous.
- The adjoint of V_ψ is given by the weak integral

$$V_\psi^* F = \int \int_{\mathbb{R}^{2d}} F(x, \xi) M_\xi T_x \psi dx d\xi, \quad F \in L^2(\mathbb{R}^{2d}).$$

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are well-defined **continuous** mappings for $i = 1, 2$.

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Example: The equality $\mathcal{D}'_{L^1,b} = \mathcal{D}'_{L^1,op}$

- Define $C_{\text{pol}}(\mathbb{R}^d)$ as the space consisting of all $\varphi \in C(\mathbb{R}^d)$ such that there is $N \in \mathbb{N}$ for which

$$\sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{(1 + |x|)^N} < \infty.$$

- $C_{\text{pol}}(\mathbb{R}^d)$ is an (LB)-space.

Theorem

Let $\psi \in \mathcal{D}(\mathbb{R}^d) \setminus \{0\}$ and let $\tau = b$ or op . Then,

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Weighted inductive limits of smooth functions

- Let $\mathcal{W} = (w_N)_N$ be an increasing sequence of continuous functions.
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Completeness of $\mathcal{B}_{\mathcal{W}}$ and $\dot{\mathcal{B}}_{\mathcal{W}}$

- Assume that $\mathcal{W} = (w_N)_N$ satisfies

$$\forall N \exists M \geq N \exists C > 0 \forall x \in \mathbb{R}^d : \sup_{y \in [-1,1]^d} w_N(x+y) \leq C w_M(x).$$

Theorem (Debrouwere and V., 2018)

TFAE:

- $\dot{\mathcal{B}}_{\mathcal{W}}$ is complete.
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Weighted L^1 convolutor spaces

- Define $L^1_{\mathcal{W}}$ as the space consisting of all measurable functions f on \mathbb{R}^d such that

$$\int_{\mathbb{R}^d} f(x)w_N(x)dx < \infty, \quad \forall N \in \mathbb{N}.$$

- $L^1_{\mathcal{W}}$ is a Fréchet space.
- Define

$$\mathcal{O}'_C(\mathcal{D}, L^1_{\mathcal{W}}) := \{f \in \mathcal{D}'(\mathbb{R}^d) \mid f * \varphi \in L^1_{\mathcal{W}} \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^d)\}$$

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Theorem (Debrouwere and V., 2018)

The equality $(\dot{\mathcal{B}}_{\mathcal{W}})' = \mathcal{O}'_{\mathcal{C}}(\mathcal{D}, L^1_{\mathcal{W}})$ always holds algebraically.

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In such a case, the bidual of $\dot{\mathcal{B}}_{\mathcal{W}}$ is (topologically) $\mathcal{B}_{\mathcal{W}}$.

- Suppose $\forall N \exists M : \lim_{|x| \rightarrow \infty} \omega_N(x)/\omega_M(x) = 0$. Then $\mathcal{O}'_{\mathcal{C},op}(\mathcal{K}_{\mathcal{W}}) = \mathcal{O}'_{\mathcal{C}}(\mathcal{D}, L^1_{\mathcal{W}})$ and we have then settled when the spaces of convolutors of Gelfand-Shilov spaces are bornological.
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Let w be a positive measurable function and set

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Assume that

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^d} \frac{w(x + \cdot)}{w(x)} \in L^\infty_{\text{loc}}. \quad (1)$$

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It is worth noticing: the hypothesis (1) is equivalent to L^1_w being translation-invariant.

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