Introduction to Fundamental Solutions and Examples

Note Title 7/6/2008

0 The aim of these notes is to give a brief introduction to the idea of Fundamental Solutions of PDE's. The particular case of Laplace operator is analyze. In these notes Partial Differential Operators are assumed to be linear with constant coefficient.

Content

$\boxed{1}$ The idea. Consider a partial differential operator (PDO) with constant coefficients

That is

$$(1.1)\quad P(D)=\sum_{|\alpha|\leq n} a_\alpha D^\alpha ,$$

here we are using multi-index notation

that is α runs over $\mathbb{N}^n$ and for $\alpha=(\alpha_1,\dots,\alpha_n)$

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Likewise for $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, we set $x^\alpha = x_1^{\alpha_1}\cdots\cdots x_n^{\alpha_n}$. The <u>symbol</u> of the PDO is the polynomial $P(iy)$.

We proceed formally to define fundamental solutions of the PDO (1.1).

Denote by δ the mass measure concentrated at the origin, by abuse of notation

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0).$$

So, δ is a unit for the convolution

$$(f*\delta)(x) = \int_{-\infty}^{\infty} f(x)\delta(y-x)dx = f(x).$$

A fundamental solution E of (1.1) is a solution of the equation

$$P(D)E = \delta \quad (1.2)$$

Suppose E is a fundamental solution and consider

$$P(D)u = f \quad (1.3)$$

Then (formally), $u = E * f$ is a solution of (1.3), whenever the convolution is defined.

Indeed

$$P(D)u = P(D)(E * f) = (P(D)E) * f = \delta * f = f.$$

$\boxed{1.1}$ Connection with the Fourier transform.

We apply Fourier transform to (1.3).

The Fourier transform we use is

$$\mathcal{F}\{u(x);y\} := \hat{u}(y) = \int_{\mathbb{R}^n} u(x) e^{-ix\cdot y}\,dx,$$

so that

$$\mathcal{F}\{D^\alpha u; y\} = (iy)^\alpha \hat{u}(y);$$

hence (1.2) $P(D)u = f$ transforms to

$$P(iy)\hat{u}(y) = \hat{f}(y),$$

Therefore, formally

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(y)\left(\frac{1}{P(iy)}\right) e^{iy\cdot x}\,dy. \quad (1.4)$$

is a solution. Set E such that $\hat{E}(y) = \frac{1}{P(iy)}$.

Using the very well known property of the Fourier transform, (1.4) is equivalent to

$$U = \mathcal{F}^{-1}\left(\frac{1}{P(i\cdot)}\right) * f = E * f.$$

Indeed E is a fundamental solution of the PDO. For this, note that $\hat{\delta} = 1$, so

$P(D)E = \delta$, transforms to

$P(iy)\hat{E}(y) = 1$, so

$$\hat{E}(y) = \frac{1}{P(iy)} \tag{1.5}$$

Of course, in this section we have not justified any step. Moreover, it is not always possible to give sense to some of these steps, but they serve as guide to the basic ideas of the subject.

$\boxed{2}$ Some basic examples.

Let Δ the Laplace operator, that is

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \dots + \frac{\partial^2 u}{\partial x_n^2},$$

where u is a function or distribution defined on an open subset of $\mathbb{R}^n$.

The four basic examples are given

$\boxed{2.1}$ The Laplace equation

below

(2.1) $\quad \Delta u = 0.$

The solutions of Laplace equation are called harmonic functions. In relation to (2.1), we have the Poisson equation

(2.2) $\quad \Delta u = f$

We will show below that the Laplace operator has a nice property implying hight regularity of its solutions, namely, it is analytic-hypoelliptic. This property implies that any solution of (2.2) is real analytic.

2.2 The Cauchy-Riemann Equation

Let x, y denote variables in the plane $\mathbb{R}^2$. The homogeneous Cauchy Riemann equation reads

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0, \tag{2.3}$$

where f is complex valued. If we write $f = u + iv$, then (2.3) is a system of equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

Let $z = x + iy$, $\bar{z} = x - iy$, then (2.3) reads in this cordinates

$$(2.4) \quad \frac{\partial f}{\partial \bar{z}} = 0. \qquad \left(\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \right)$$

Recall the solutions of (2.4) are the analytic functions, which in particular are real analytic functions. So, equations (2.1) and (2.4) share an important property: their solutions are real analytic.

$\boxed{2.3}$ The Heat Equation

The heat operator in $\mathbb{R}^{n+1}$ is defined as

$$\frac{\partial}{\partial t} - \Delta_x .$$

The homogeneous heat equation is

(2.5) $$\frac{\partial u}{\partial t} - \Delta_x u = 0$$

Although (2.5) may seem very different from (2.1) and (2.4), they share a property: their solutions are always C^∞-functions.

However, (2.5) differs from (2.1) and (2.4): it has solutions which are not real analytic.

Let us consider an example. Consider the heat equation on $\Omega = \{(x,t) \in \mathbb{R}^2 / x \neq 0\}$; define

$$u(x,t) = \begin{cases} t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} , & t > 0 \\ 0 , & t \leq 0 \end{cases}$$

Then

i) $U \in C^\infty(\Omega)$

ii) $\dfrac{\partial U}{\partial t} = \dfrac{\partial^2 U}{\partial x^2}$

iii) U is not (real) analytic. Indeed a (real) analytic function on a connected open set which vanish on an open subset must be 0.

$\boxed{2.4}$ The Wave Equation

The wave operator, sometimes also called d'Alembertian, is

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta_x = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \dots - \frac{\partial^2}{\partial x_n^2}.$$

The wave equation is

(2.6) $\Box U = 0.$

Note that the wave operator is invariant under the Lorentz group.

The solutions of the wave equation differ radically from those of the previous three equations: they are not C^∞. Moreover, if we consider distributional solutions, they do not even have to be C^2- functions

to see this, consider the space variable, x, just in $\mathbb{R}$. Then if f and g functions of one variable,

$$u(x,t) = f(t-x) + g(x+t) \tag{2.7}$$

is a solution of the wave equation.

Moreover (2.7) is the most general solution of the wave equation.

To see this claim, consider the change of variables $\eta = t - x$, $\xi = x + t$ then

$$\frac{\partial u}{\partial t^2} - \frac{\partial u}{\partial x^2} = 0,$$

transforms into

$$4 \frac{\partial^2 u}{\partial \eta \partial \xi} = 0;$$

so

$$u(\eta, \xi) = f(\eta) + g(\xi).$$

$\boxed{3}$ Distributions

Here we give some definitions and examples from distribution theory.

$\boxed{3.1}$ Spaces $\mathcal{D}$ and $\mathcal{D}'$

Let $\Omega \subseteq \mathbb{R}^n$ be an open subset.

We denote by $\mathcal{D}(\Omega)$ the space of

C^∞-functions with compact support in Ω. It comes with the usual Schwartz topology. In particular, a sequence $\{\phi_m\}_{m=0}^{\infty}$ of elements of $D(\Omega)$ is said to be convergent to 0 if

i) There is a compact $K \subset \Omega$ such that $\operatorname{supp}\phi_m \subseteq K \ \forall m$

ii) For each $\alpha \in \mathbb{N}^n$ $D^\alpha \phi_m(x) \longrightarrow 0$ as $m \to 0$ uniformly on Ω.

The space $D'(\Omega)$ is the dual space of $D(\Omega)$.

Given $f \in D'(\Omega)$ the evaluation of f at ϕ

is denoted by

$$\langle f,\phi\rangle \quad \text{or} \quad \langle f(x),\phi(x)\rangle$$

The use of the "dummy" variable "x" in the above pairing relation is just for notational aspects, it does not make reference to pointwise existence of $f(x)$.

So $f\in \mathcal{D}'(\Omega)$ if

i) For each $c\in\mathbb{R}$, $\phi_1,\phi_2\in \mathcal{D}(\Omega)$,

$$\langle f(x), c\phi_1(x)+\phi_2(x)\rangle = c\langle f(x),\phi_1(x)\rangle + \langle f(x),\phi_2(x)\rangle$$

ii) $\langle f(x),\phi_m(x)\rangle \longrightarrow 0$ as $m\to\infty$,

for each sequence $\{\phi_m\}_{m=0}^{\infty}$ such that $\phi_m\to 0$ in $\mathcal{D}(\Omega)$.

$\boxed{3.1.1}$ Easy examples:

$\boxed{3.1.1.1}$ The delta Dirac distribution

The delta Dirac distribution $\delta(x)$ is defined as

$$\langle \delta(x), \phi(x) \rangle = \phi(0),$$

$\boxed{3.1.1.2}$ If $f \in L^1_{loc}(\Omega)$; $\Omega \subseteq \mathbb{R}^n$; that is for any compact $K \subset \Omega$

$$\int_K |f(x)|\,dx < \infty$$

then $f \in D'(\Omega)$

$\boxed{3.1.1.3}$ Let Σ be a hypersurface in $\mathbb{R}^n$

The we denote by $\delta(\Sigma)$, the distribution

$$\langle \delta(\Sigma)(x), \phi(x) \rangle = \int_\Sigma \phi(\omega)\, d\Sigma(\omega),$$

here $d\Sigma$ denotes the surface area measure (the natural volume form) of Σ.

Suppose that μ is a function defined on Σ, then $\mu\,\delta(\Sigma)$ is the distribution

$$\langle g\,\delta(\Sigma), \phi \rangle = \int_\Sigma g(\omega)\phi(\omega)\,d\Sigma(\omega)$$

$\boxed{3.1.2}$ Operation with distributions

Change of Variables:

If $\psi: W \to \Omega$ is a (C^∞) diffeomorphism, and $f \in \mathcal{D}'(\Omega)$, then $f \circ \psi \in \mathcal{D}'(W)$ is defined by

$$\langle f \circ \psi(y), \phi(y) \rangle = \langle f(y), \frac{\phi(y)}{|\psi'(y)|} \rangle,$$

where $|\psi'|$ denotes here the Jacobian of the map ψ.

Differentiation

Suppose $g \in C^k(\Omega)$ and $|\alpha| \leq k$, then integration by parts gives

$$\langle D^{\alpha} g(x), \phi(x) \rangle = \int_{\Omega} D^{\alpha} g(x) \phi(x) dx$$

$$= (-1)^{|\alpha|} \int_{\Omega} g(x) D^{\alpha} \phi(x) dx$$

$$= \langle g(x), (-1)^{|\alpha|} D^{\alpha} \phi(x) \rangle;$$

we use the last relation as the definition of the distributional derivative.

If $f \in D'(\Omega)$, then $D^{\alpha} f$ is the distribution

$$\langle D^{\alpha} f(x), \phi(x) \rangle = (-1)^{|\alpha|} \langle f(x), D^{\alpha} \phi(x) \rangle$$

Examples:

1 Let us start with the simplest piecewise differentiable function, the Heaviside function defined in the following form

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

So, as a distribution, we have that

$$\langle H(x), \phi(x) \rangle = \int_0^\infty \phi(x)\,dx.$$

Let us calculate $H'(x)$,

$$\langle H'(x), \phi(x) \rangle = -\langle H(x), \phi'(x) \rangle = -\int_0^\infty \phi'(x)\,dx$$

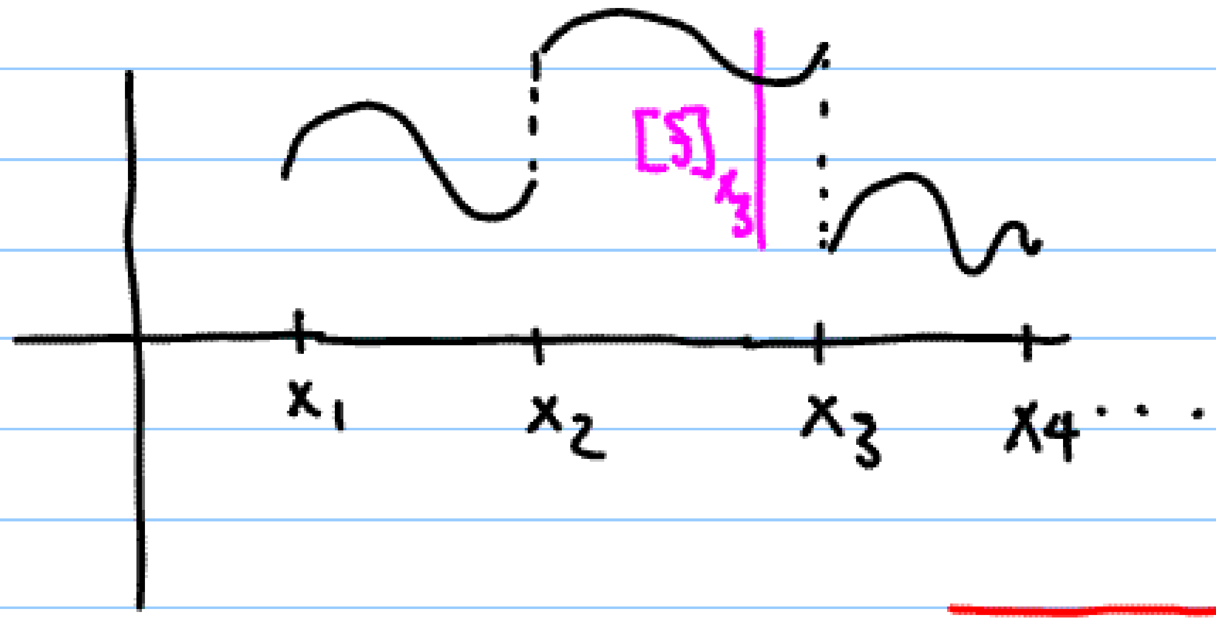
$$= -\left(\phi(x)\right)\Big|_{x=0}^{x=\infty} = \phi(0) = \langle \delta(x), \phi(x) \rangle$$

So $H'(x) = \delta(x)$.

Remark: In general

Let f be a piecewise differentiable function with simple discontinuities at a discreate set $J = \{x_1, x_2, \ldots\} \subseteq \mathbb{R}$

Graph of f



Denote by f'_{ord} its ordinary derivative, which is defined on $\mathbb{R} \setminus J$, so it is a piecewise continuous function. Let $[f]_{x_n} = f(x_n^+) - f(x_n^-)$ the jump then, one easily verifies that

$$f' = f'_{ord} + \sum [f]_{x_n} \delta(x - x_n)$$

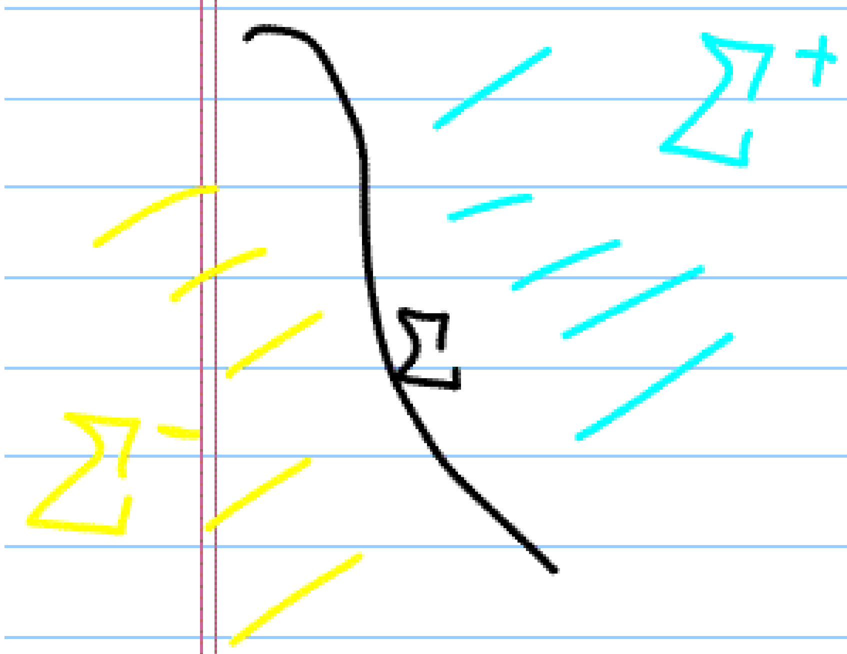
2) Let Σ be a smooth hypersurface in $\mathbb{R}^n$ and μ be a function on Σ. Let n be the

unit normal vector. We denote by $\frac{\partial}{\partial n}(\kappa\delta(\Sigma))$ the distribution

$$\left\langle \frac{\partial}{\partial n}(\kappa\delta(\Sigma)), \phi \right\rangle = -\left\langle \kappa\delta(\Sigma), \frac{\partial}{\partial n}\phi \right\rangle$$

$$= -\int_{\Sigma} \kappa(\omega) \frac{\partial}{\partial n}\phi(\omega)\, d\Sigma(\omega)$$

2 Let Σ be a smooth hypersurface in $\mathbb{R}^n$. Suppose Σ divides $\mathbb{R}^n$ into two connected components, that is



$\mathbb{R}^n \setminus \Sigma = \Sigma^+ \cup \Sigma^-$

In addition, if $x \in \Sigma$ let $n(x)$ be the outer normal unit vector (pointing toward Σ^+ by convention)

Suppose F is a function with a simple discontinuity across Σ that is

$$F(x) = \begin{cases} F^+(x), & x \in \Sigma^+ \\ F^-(x), & x \in \Sigma^- \end{cases},$$

where F^+ and F^- are continuous functions on $\Sigma^+ \cup \Sigma$ and $\Sigma^- \cup \Sigma$ respectively.

We calculate $\frac{\partial F}{\partial x_i}$. Notice that by setting $F^+(x) = 0$ for $x \in \Sigma^-$ and $F^-(x) = 0$ for $x \in \Sigma^+$; we have that $F = F^+ + F^-$.

Now

$$\left\langle \frac{\partial F^\pm}{\partial x_i}, \phi(x) \right\rangle = -\int_{\Sigma^\pm} F^\pm(x) \frac{\partial \phi}{\partial x_i}\, dx$$

$$= \pm \int_{\Sigma} F^\pm(\omega) \phi(\omega) n_i(\omega)\, d\Sigma^\pm(\omega) + \int_{\Sigma^\pm} \frac{\partial F^\pm}{\partial x_i} \phi(x)\, dx;$$

adding, we obtain

$$\left\langle \frac{\partial F}{\partial x_i}, \phi(x) \right\rangle = \int_{\mathbb{R}^n} \frac{\partial F}{\partial x_i} \phi(x)\, dx + \int_{\Sigma} [F](\omega)\phi(\omega) n_i(\omega)\, d\Sigma(\omega)$$

where $[F](\omega) = F^+(\omega) - F^-(\omega)$, is the jump of F across Σ.

We may write this equation as

$$\frac{\partial F}{\partial x_i} = \left(\frac{\partial F}{\partial x_i}\right)_{ord} + [F] n_i \delta(\Sigma)$$

3.2 Tempered Distributions

$S(\mathbb{R})$, the space of rapidly decreasing functions is defined as the vector space of those functions such that

i) $\phi \in C^\infty(\mathbb{R})$

ii) For each $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$

$$(3.1) \quad \|\phi\|_{k,\alpha} := \sup_{x \in \mathbb{R}^n} (1+|x|^2)^{\frac{k}{2}} |D^\alpha \phi(x)| < \infty$$

The topology on $S(\mathbb{R})$ is given by the seminorms (3.1), in particular $S(\mathbb{R}^n)$ is a Frechet space. So $\phi_m \to 0$ in $S(\mathbb{R}^n)$ if for each $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ $\|\phi_m\|_{k,\alpha} \to 0$ as $m \to \infty$.

The space of tempered distributions is the dual space, and it is denoted by $S'(\mathbb{R}^n)$. Note that $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is an isomorphism. On $S'(\mathbb{R}^n)$ the Fourier transform is defined by

$$\langle \hat{f}(x), \phi(x) \rangle = \langle f(x), \hat{\phi}(x) \rangle ,$$

that is the distributional Fourier transform is the transpose of the Fourier transform on $S(\mathbb{R}^n)$.

4 Hypoelliptic Partial Differential Operators with constant coefficients.

Let $L = P(D)$ be a PDO, with

(4.1) $$P(x) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha, \text{ so } L = \sum a_\alpha D^\alpha$$

We can apply L to any distribution $f \in D'(\Omega)$ indeed

$$\langle Lf, \phi \rangle = \langle f, L^* \phi \rangle,$$

where

$$L^* = \sum (-1)^{|\alpha|} a_\alpha D^\alpha$$

is the formal adjoint PDO.

We can now define properly fundamental solutions of PDO

Definition: Let L be a PDO, a distribution $E \in D'(\mathbb{R}^n)$ is called a fundamental solution of L if $LE = \delta$.

Fundamental solutions are never unique. In fact, unlike ODE's, the solution set of a PDE is infinite dimensional

In fact, let L be given by (4.1),

let $y \in \mathbb{C}^n$ then if $u_y(x) = e^{x \cdot y}$,

$(Lu_y)(x) = e^{x \cdot y} P(y)$,

so if the dimension of $n > 1$ (n dimension of $\mathbb{R}^n$), then u_y is a solution of $Lu = 0$ as long as $P(y) = 0$, and the fundamental theorem of algebra shows that there are infinitely many solutions.

In conclusion, if E is a fundamental solution, then $E + u_y$ is a fundamental solution if

$P(y)=0$.

We can read important information from the fundamental solution of a PDO

For example, we commented in $\boxed{2}$ that the heat, Laplace and Cauchy-Riemann equations share a common property: the solutions of the homogeneous equation are always C^∞. This can be related to their fundamental solution.

Let us start with the definition of hypoellipticity.

<u>Definition</u> Let L be a PDO with constant coefficients. L is called hypoelliptic if

$Lu=\psi$ with $u\in D'(\Omega)$ and $\psi\in C^\infty(\Omega) \not\Rightarrow u\in C^\infty(\Omega)$

The first thing we should notice is that if L is hypoelliptic and E is a fundamental solution then E is C^∞ outside the origin. The last condition is sufficient for hypoellipticity.

Theorem (Schwartz)

Suppose that L admits a fundamental solution which is C^∞ off the origin, then L is hypoelliptic

Proof See [Barros-Neto, p.196] or [Treves, p.19] //

So the reader must realize the important of fundamental solutions in the theory of Linear PDE. The conjecture that every PDO with constant coefficients has a fundamental solution was proved in 1954 by Malgrange and independently by Ehrenpreis. Of great importance is the article of [Hörmander] in which by solving the division problem (in polynomial case) conjectured by Schwartz, he discusses the existence of a tempered fundamental solution.

Let us state Malgrange-Ehrenpreis theorem

Theorem: Every Linear PDO with constant coefficients has a Fundamental solution

Proof: See [Barros-Neto, p. 205]

The article by Ortner and Wagner

discusses explicit representation formulas ///

So we get combining the last two results:

Theorem

L is hypoelliptic if and only if it admits a fundamental solution which is C^∞ outside the origin. ///

An important criterion for hypoellipticity is ellipticity.

Definition. Let L be a PDO (constant coefficients) given by $L=\sum_{|\alpha|\leq m} a_\alpha D^\alpha$. The characteristic polynomial is $P_m(y)=\sum_{|\alpha|=m} a_\alpha y^\alpha$.

L is called elliptic if $P_m(y)\neq 0$ for all $y\neq 0$. ///

Theorem: If L is elliptic, then it is hypoelliptic.

Proof: The standard proof uses Sobolev space theory. See [Rudin, p. 201].

Corollary: Laplace and Cauchy-Riemann operators are hypoelliptic. ///

Another useful concept is analytic-hypoellipticity. It also can be described by fundamental solutions

Definition Let L be a PDO with constant coefficients. L is called analytic-hypoelliptic if for any distribution $u \in D(\Omega)$

$Lu = \psi$, with ψ is real analytic $\Rightarrow$ u real analytic.

The corresponding characterization is as follows.

Theorem:

L is analytic-hypoelliptic if and only if there is a fundamental solution of L which is real analytic away from the origin.

5 Fundamental Solution of Laplacian.

In this section we calculate the fundamental solution of the Laplacian. We use two methods.

5.1 Fundamental solution via Fourier transform. $(n>2)$

We have to solve $\Delta E = \delta$, taking Fourier transform

$$-|x|^2 \hat{E} = 1,$$

So, a good choice is the Fourier inverse transform of "$-|x|^{-2}$". When $n>2$, $|x|^{-2}$ is locally integrable, so it makes sense to talk about the distribution $|x|^{-2}$. Indeed using polar coordinates

$$\langle |x|^{-2}, \phi(x)\rangle = \int_{\mathbb{R}^n} \frac{\phi(x)}{|x|^2}\,dx = \int_0^\infty r^{n-3}\left(\int_{S^{n-1}} \phi(r\omega)\,d\omega\right)dr,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$ and I am using $d\omega$ for the surface measure on S^{n-1}.

But notice that when $n=2$, polar coordinates show that

$$\langle |x|^{-2}, \phi(x)\rangle = \text{``}\int_0^\infty \frac{1}{r}\int_0^{2\pi} \phi(re^{i\theta})\,d\theta\text{''}$$

do not define a distribution directly, indeed the last integral is "singular" at $r=0$.

We should restrict ourselves to $k>2$.

We want to calculate the Fourier inverse transform of $r^{-2} := |x|^{-2}$, say $E = \mathcal{F}^{-1}(r^{-2})$.

At this point, we can proceed to make a direct calculation (or use a table!) but we rather appeal to some properties of the Fourier transform.

Note that E is rotation invariant, indeed r^{-2} is rotation invariant and the Fourier transform preserves this property. So E depends only on the radius. Next we note that r is homogeneous of degree -2, since the Fourier transform sends homogeneous distributions of degree α to homogeneous distributions of degree $-n-\alpha$, we see that E is homogeneous of degree $-n+2$, so E must have the form

$$E(x) = C\, r^{-n+2}$$

It remains to calculate C,

we have that $\hat{E} = -r^{-n+2}$. Consider the function $\phi(x) = e^{-|x|^2}$; note that $\hat{\phi}(x) = \pi^{\frac{n}{2}} e^{-\frac{|x|^2}{4}}$

Now

$$\langle \hat{E}(x), e^{-|x|^2} \rangle$$

$$= \langle E(x), \pi^{\frac{n}{2}} e^{-\frac{|x|^2}{4}} \rangle = \pi^{\frac{n}{2}} C \int_{\mathbb{R}^n} |x|^{-n+2} e^{-\frac{|x|^2}{4}} dx$$

$$= \pi^{\frac{n}{2}} C\, |S^{n-1}| \int_0^\infty r \quad e^{-\frac{r^2}{4}} dr$$

$\boxed{|S^{n-1}| = \int_{S^{n-1}} d\omega}$ $\boxed{t = \frac{r^2}{4}}$

$$= 2\pi^{\frac{n}{2}} c|S^{n-1}|$$

On the other hand

$$\langle \hat{E}(x), e^{-|x|^2} \rangle = \langle -r^{-2}, e^{-|x|^2} \rangle$$

$$= -|S^{n-1}| \int_0^\infty r^{n-3} e^{-r^2} dr$$

$$= -\frac{1}{2} |S^{n-1}| \int_0^\infty t^{\frac{n}{2}-2} e^{-t} dt$$

$$= -\frac{1}{2} |S^{n-1}| \Gamma\left(\frac{n}{2} - 1\right).$$

Therefore,

$$c = -\frac{\Gamma\left(\frac{n}{2}-1\right)}{4\pi^{\frac{n}{2}}} = -\frac{\Gamma\left(\frac{n}{2}\right)}{4\left(\frac{n}{2}-1\right)\pi^{\frac{n}{2}}}$$

So

$$\boxed{E = \frac{r^{2-n}}{(2-n)|S^{n-1}|}}$$

$\boxed{5.2}$ Calculation for the general case.

We give a different procedure to calculate the fundamental solution, this includes $n=2$.

Consider

$$\Delta E = \delta$$

Note that δ is rotation invariant, and the Laplacian as well. So it is natural to look for solutions which are rotation invariant. So assume E depends only on r. If a distribution depends only on r, then

$$\Delta E = \frac{\partial^2 E}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}.$$

So we solve the equation

$$f''(r) + \frac{n-1}{r} f'(r) = 0 \qquad r > 0;$$

$$\parallel$$

$$\left(r^{n-1} f'\right)' = 0 \qquad , r > 0.$$

Then $f'(r) = c\, r^{-n}$ so

$$f(r) = \begin{cases} c_1 r^{-n+2} + c_2 \;; & n > 2 \\ c_1 \ln r + c_2 \;; & n = 2 \end{cases}$$

are the candidates; we take $c_2 = 0$;

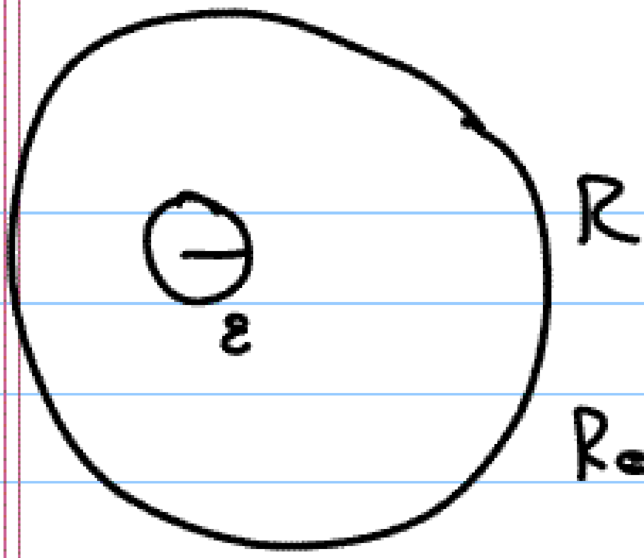
when $n > 2$ $c_1 = -\frac{1}{(n-2)|S^{n-1}|}$ works.

We make the computation for $n=2$

$$\langle \Delta \ln r, \phi(x) \rangle = \int_{\mathbb{R}^2} \ln|x| \, \Delta\phi(x)\, dx$$

$B(\varepsilon) = \{ |x| < \varepsilon \}$

$$= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2 - B(\varepsilon)} \ln|x| \, \Delta\phi(x)\, dx$$



with supp $\phi \subseteq B(R)$

Recall green formula

$$\int_{\Omega} (\Delta u\, \phi - u\, \Delta \phi) = \int_{\partial\Omega} \left(\phi \frac{du}{dn} - u \frac{d\phi}{dn} \right)$$

So

$$\langle \Delta \ln|r|, \phi \rangle$$

$$= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^2 - B(\varepsilon)} \phi \overset{=0}{\Delta \ln|r|} - \varepsilon \int_0^{2\pi} \ln|\varepsilon| \phi(\varepsilon e^{i\theta})\, d\theta$$

$$+ \varepsilon \int_0^{2\pi} \frac{d \ln}{dr}\Big|_{r=\varepsilon} \phi(\varepsilon e^{i\theta})\, d\theta$$

$$= \lim_{\varepsilon \to 0^+} \int_0^{2\pi} \phi(\varepsilon e^{i\theta})\, d\theta = 2\pi \phi(0) = \langle 2\pi \delta(x), \phi(x) \rangle$$

So $E(x) = \frac{1}{2\pi} \ln|x|$ is the fundamental solution

$\boxed{6}$ Fundamental solution of Cauchy-Riemann operator

$\boxed{6.1}$ <u>An ODE</u>: Since we will use Fourier transform to reduce to an ODE, we first discuss the case of the operator

$$\frac{d}{dt} - A \qquad (6.1)$$, where A is a constant

So we want a solution of

$$\frac{dE}{dt} - AE = \delta$$, multiplying by e^{-At}

$$\frac{d(e^{-At}E)}{dt} = \delta$$, so

it is enough to assume $A=0$;

For $\frac{d}{dt}f = \delta$, the general solution is

$f(x) = H(x) + B$, B a constant. So

the general fundamental solution of (6.1) is

$$E(t) = (H(t) + B)e^{At}, \; B \text{ constant}$$

$\boxed{6.2}$ The calculation

Start with

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \delta(x)\delta(y) \tag{6.2}$$

We take Fourier transform with respect to

y, so let $F(x,t) = \mathcal{F}_y(f(x,\cdot))(t)$, then

(6.2) becomes

$$\frac{\partial F(x,t)}{\partial x} - t\,F(x,t) = \delta(x).$$

So

$$F(x,t) = (H(x) + B(t))e^{xt} \tag{6.3}$$

We come to a problem: in general (6.3) is not tempered with respect to t.

Indeed e^{xt} is not tempered when $x>0, t\to\infty$;

e^{xt} is not tempered when $x<0, t\to-\infty$.

To solve this we choose

$$B(t)=\begin{cases} -1 & , t>0 \\ 0 & , t<0. \end{cases}$$

So

$$F(x,t)=\begin{cases} -H(-x)e^{xt} & t>0 \\ H(x)e^{xt} & t<0 \end{cases}$$

Therefore

$$2\pi E(x,y)= H(x)\int_{-\infty}^{0} e^{(x+iy)t}\,dt - H(-x)\int_{0}^{\infty} e^{(x+iy)t}\,dt$$

$$=\frac{H(x)+H(-x)}{x+iy}=\frac{1}{x+iy}=\frac{1}{z}$$

Since $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$,

Theorem: $\frac{1}{\pi z}$ is a fundamental solution of the Cauchy-Riemann operator.

$\boxed{7}$ Exercises

7.1 Find a fundamental solution for the heat operator

7.1 Find a fundamental solution for the wave operator

$\boxed{8}$ References

[Barros-Neto] An introduction to the theory of distributions, Marcel DeKKer Inc., N.Y., 1973.

[Hörmander] On the division problem

by polynomials, Ark. Mat., 3 (1958), 555-568.

[Rudin] Functional Analysis, McGraw-Hill Inc., 1973.

[Ortner-Wagner] A survey on explicit representation formulae for fundamental solutions of Linear Partial Differential Operators, Acta Applicandae Mathematicae 47 (1997), 101-124.

[Treves] Basic Linear Partial Differential Equations, Academic Press Inc., N.Y., 1975.