On some classical Tauberian theorems of Hardy and Littlewood

(by Jasson Vindas, Ghent, 27-10-20)

$\boxed{1}$ Introduction: About a century ago, Hardy and Littlewood proved a series of Tauberian theorems for power series. In this talk we will discuss some of these results and their variants. In particular, we will pose an open question related to the work of Halász on the high-indices theorem. Some of arguments below are based on the distributional method of R. Estrada and the speaker from:

[1] A Tauberian theorem for distributional point values, Arch. Math. (Basel) 91 (2008), 247-253.

[2] Distributional point values and convergence of Fourier series and integrals, J. Fourier Anal. Appl. 13 (2007), 551-576.

$\boxed{2}$ Tauber's theorem: Everyone who has taken a basic analysis course should be familiar with the next theorem due to Abel:

Theorem 1 (Abel): Let $\{a_n\}_{n\in\mathbb{N}}$ be a numerical series. If $\sum_{n=0}^{\infty} a_n = \ell \Rightarrow \lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^n = \ell$.

The proof of this theorem is easy if we express series as Stieltjes integrals. It is easier to work with $r = e^{-y}$, so that $y \to 0^+$ if $r \to 1^-$. We write

$$S(x) = \sum_{n \le x} a_n ,$$

so that the assumption of Theorem 1 is $S(x) \to \mathscr{A}$, $x \to \infty$. Therefore,

$$\sum_{n=0}^{\infty} a_n e^{-yn} = \int_{0^-}^{\infty} e^{-yx} \, dS(x) = y \int_0^{\infty} S(x) e^{-yx} \, dx$$

$$= \int_0^{\infty} S\left(\frac{u}{y}\right) e^{-u} \, du \xrightarrow[y \to 0]{} \mathscr{A} \int_0^{\infty} e^{-u} \, du = \mathscr{A}.$$

Note that the converse of Abel's theorem is not true in general. Example: $\sum_{n=0}^{\infty} (-1)^n$ diverges but

$$\sum_{n=0}^{\infty} (-1)^n e^{-yn} = \frac{1}{1+e^{-y}} \longrightarrow \frac{1}{2} \text{ as } y \to 0^+.$$

The beginning of Tauberian theory goes back to the following partial converse, whose proof is easy as well:

Theorem 2 (Tauber, 1897). Suppose that

$$F(y) := \sum_{n=0}^{\infty} a_n e^{-yn} \text{ converges for all } y > 0$$ and that

$F(y) \to \mathscr{A}$ as $y \to 0^+$. If the ensuing Tauberian condition is satisfied:

$$a_n = o\left(\frac{1}{n}\right) \qquad (1)$$

then $\displaystyle\sum_{n=0}^{\infty} a_n = \gamma$. ///

Proof. We have $|a_n| \leq \frac{M}{n}$ globally for some M and $|a_n| \leq \varepsilon/n$ for, say, $n \geq x_0$. Hence,

$$\left|\sum_{n\leq x} a_n - F\left(\tfrac{1}{x}\right)\right| \leq \left|\sum_{n\leq x_0} a_n\left(1-e^{-\frac{n}{x}}\right)\right|$$

$$+ \varepsilon \sum_{\frac{n}{x}\leq 1} \left(1-e^{-\frac{n}{x}}\right)\cdot\frac{1}{x} + \varepsilon \sum_{1<\frac{n}{x}} e^{-\frac{n}{x}}\cdot\frac{1}{x}$$

$$\leq \left|\sum_{n\leq x_0} a_n\left(1-e^{-\frac{n}{x}}\right)\right| + \varepsilon\cdot\left[\int_0^1 (1-e^{-u})\,du + \int_1^\infty e^{-u}\,du\right]$$

$$= \left|\sum_{n\leq x_0} a_n\left(1-e^{-\frac{n}{x}}\right)\right| + \varepsilon\cdot\left(1-\frac{1}{e}\right).$$

Therefore, the result follows by taking $\displaystyle\limsup_{x\to\infty}$ and then $\varepsilon \to 0^+$. ///

$\boxed{3}$ Littlewood's theorem:

In 1910 Littlewood proved that Tauber's theorem still holds if (1) is replaced by the weaker condition

$$a_n = O\left(\frac{1}{n}\right) \qquad (2).$$

This change makes the theorem deeper and more difficult to show. There are several variants. For example, Hardy and Littlewood conjectured that the condition (2) could still be weakened a the one-sided condition

$$a_n \geq -\frac{M}{n} \qquad (\text{for some } M>0),$$

and in fact provided a proof of their conjectured in 1914. We state here a further generalization proposed by Littlewood[1] in his influential paper from 1911, but first shown by Ananda Rau in 1928. We give here a short distributional proof.

<u>Theorem 3 (Ananda Rau, 1928)</u>. Let $\{a_n\}_{n\in\mathbb{N}}$ satisfy the Tauberian condition:

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right) \qquad (3),$$

where $\{\lambda_n\}_{n\in\mathbb{N}}$ is an unbounded and increasing sequence. Then,

$$F(y) := \sum_{n=0}^{\infty} a_n e^{-y\lambda_n} \longrightarrow \gamma \qquad (4)$$

implies

$$\sum_{n=0}^{\infty} a_n = \gamma \qquad (5)$$

<u>Proof.</u> Reasoning as in the proof of Theorem 2, one readily shows that (all that is need is (3) and $F(y) = O(1)$)

$$S(x) := \sum_{\lambda_n \leq x} a_n = O(1). \qquad (6)$$

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[1] Littlewood actually proved a weaker version additionally assuming that $\lambda_{n+1} \sim \lambda_n$.

We consider the family of tempered distributions

$$f_y(t) = \sum_{n=0}^{\infty} a_n \delta(t - y\cdot\lambda_n) \quad , \quad y>0,$$

namely, $\langle f_y, \varphi \rangle = \sum_{n=0}^{\infty} a_n \varphi(y\cdot\lambda_n)$.

Our first goal is to show that this family converges to the Dirac delta δ times $\mathcal{L}$ as $y \to 0^+$ i.e., for each Schwartz rapidly decreasing smooth function φ on $[0,\infty)$, we claim that

$$\lim_{y\to 0^+} \langle f_y, \varphi \rangle = \lim_{y\to 0^+} \sum_{n=0}^{\infty} a_n \varphi(y\cdot\lambda_n) = \mathcal{L}\cdot\varphi(0) \quad (7) .$$

Our hypothes (4) says that (7) holds for any linear combination of exponentials in the Schwartz class $S[0,\infty)$, which forms a dense subspace. We recall from basic functional analysis that a family of linear continuous linear functionals is weakly* convergent if it is convergent on a dense set and it is equicontinuous. It is enough then to show the latter and we do it via (6):

$$\langle f_y, \varphi \rangle = \int_{0^-}^{\infty} \varphi(y\cdot t)\, dS(t) = \int_0^{\infty} S\left(\frac{t}{y}\right) \varphi'(t)\, dt$$

$$= O(1) \cdot \sup_{t\geq 0} t^2 \cdot |\varphi'(t)| ,$$

So, we have established (7). To show (5), we make a choice for φ in (7). Let $\varepsilon>0$ and choose M such that

$$|a_n| \leq M \cdot \frac{\lambda_n - \lambda_{n-1}}{\lambda_n},$$

Pick $\varphi \in C^{\infty}[0,\infty)$ such that $0 \leq \varphi \leq 1$ and

(i) $\varphi(t)=1$ for $t \in [0,1]$ and (ii) $\varphi(t)=0$ for $t > 1+\varepsilon$

Now,

$$\limsup_{x\to\infty} \Big| \sum_{\lambda_n \leq x} a_n - \gamma \Big|$$

$$\limsup_{x\to\infty} \Big| \sum_{n=0}^{\infty} a_n \varphi\Big(\frac{\lambda_n}{x}\Big) - \gamma\varphi(0) - \sum_{1<\frac{\lambda_n}{x}\leq 1+\varepsilon} a_n \varphi\Big(\frac{\lambda_n}{x}\Big) \Big|$$

$$\leq M \cdot \limsup_{x\to\infty} \sum_{1<\frac{\lambda_n}{x}\leq 1+\varepsilon} \frac{\lambda_n - \lambda_{n-1}}{x} \cdot \varphi\Big(\frac{\lambda_n}{x}\Big)$$

$$= M \cdot \int_1^{1+\varepsilon} \varphi(t)\,dt \leq \varepsilon \cdot M. \quad ///$$

$\boxed{4}$ The high-indices theorem:

Another interesting Tauberian theorem of Hardy and Littlewood is the so-called high-indices theorem, conjectured in 1910 by Littlewood but shown until 1926 by H-L.

Theorem 4 (Hardy-Littlewood high-indices theorem)

Suppose that $\{\lambda_n\}_{n\in\mathbb{N}}$ is lacunary in the sense of Hadamard, i.e., for some $\alpha > 1$

$$\frac{\lambda_{n+1}}{\lambda_n} \geq \alpha, \quad \forall n. \quad (8)$$

Then, (4) always implies (5). ///

In this case

$$1 \leq \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \leq 1 - \frac{1}{\alpha},$$

so that Theorem 4 reduces to Theorem 5 if one is able to show that the coefficients a_n are bounded. Ingham proved the stronger result.

Theorem 5 (Ingham, 1937): Suppose $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfies (8). There is a constant C_1, only depending on α, such that for all sequence $\{a_n\}_{n\in\mathbb{N}}$

$$\sup_{n\in\mathbb{N}} |a_n| \leq C_1 \cdot \sup_{y>0} \left| \sum_{n=0}^{\infty} a_n e^{-y\lambda_n} \right|$$

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Halász showed in 1967 a related inequality by using complex analysis methods that also lead to remainder versions of Theorem 4 (see his article for details on such remainder versions).

<u>Theorem 6 (Halász, 1967).</u> With the same hypotheses as in Theorem 5 and for some $c_2 > 0$,

$$\sup \left| \sum_{n=0}^{N} a_n \right| \leq c_2 \cdot \sup_{y>0} \left| \sum_{n=0}^{\infty} a_n e^{-y\lambda_n} \right|.$$

We point out that, except for the numerical values of the constants, Theorems 5 and 6 are equivalent. We end with an open problem:

<u>Open problem:</u> Find the optimal values of c_1 and c_2 in Theorems 5 and 6.