Asymptotic distribution of Beurling's generalized prime numbers

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1 Introduction:

Shortly after Hadamard and de la Vallée Poussin established the prime number theorem (PNT) in 1896, Landau found in 1903 another proof which used relatively weak information about integers. He moved further and applied his idea to show that the counting function of norms of prime ideals of the ring of integers of an algebraic number field follows exactly the same asymptotic law as that of the ordinary prime numbers. Consequently, the PNT is not an exclusive property of the integers, as it is also satisfied by other rather general multiplicative systems. This suggested Beurling that the PNT could be established for a system of "generalized integers" with a counting function having significantly less regularity than that of the integers. In his seminal work (1937), Beurling settled a general setting for the PNT. In this talk we will discuss classical and recent results concerning the asymptotic distribution of Beurling's generalized primes.

2 Beurling problem:

- Let $1 < p_1 \leq p_2 \leq \cdots \leq p_k \to \infty$ be non-decreasing sequence of real numbers tending to ∞. Following Beurling, we call $P = \{p_k\}_{k=1}^{\infty}$ a set of generalized primes. Arrange all possible products of the p_j in a non-decreasing sequence $1 = n_1 < n_2 \leq n_3 \leq \cdots$, where every n_k is repeated as many times as represented by $p_{\nu_1}^{\alpha_1} p_{\nu_2}^{\alpha_2} \cdots p_{\nu_m}^{\alpha_m}$ $\nu_j < \nu_{j+1}$. The sequence $\{n_k\}_{k=1}^{\infty}$ is called the set of generalized integers.

Set $N(x) = \sum_{n_k \leq x} 1$ and $\pi(x) = \sum_{p_k \leq x} 1$.

$\boxed{\text{Beurling's problem:}}$ Find conditions on $N(x)$ to ensure the

the validity of the PNT in the form $\pi(x) \sim \frac{x}{\log x}$, $x \to \infty$.

Landau's theorem quoted above was:

<u>Theorem (Landau's PNT, 1903)</u>: If $N(x) = ax + O(x^{\theta})$ where $a > 0$ and $\theta < 1$, then $\pi(x) \sim \frac{x}{\log x}$. ///

The condition of Landau's theorem is referred to as the axiom A by Knopfmacher; see his book (1975) on abstract analytic number theory for numerous applications.

Implicit in Beurling's problem is the search for "minimal conditions" on N. In this talk we will focus on generalized number systems having a positive density i.e.,

$$N(x) \sim ax \quad , \quad \text{where } a > 0. \tag{2.1}$$

Most arguments in this theory are based on the analysis of the zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n_k^s} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_k^s}\right)^{-1}, \quad \operatorname{Re} s > 1. \tag{2.2}$$

<u>Remark 1</u>: By introducing $\Pi(x) = \sum_{m=1}^{\infty} \frac{1}{m} \pi(x^{\frac{1}{m}})$. One can rewrite (2.2) as

$$\zeta(s) := \int_1^{\infty} x^{-s} dN(x) = \exp\left(\int_1^{\infty} x^{-s} d\Pi(x)\right), \operatorname{Re} s > 1. \tag{2.3}$$

Most results in this theory are still valid if we simply assume that N and Π are non-decreasing functions linked via (2.3). The function Π is then referred as a system of generalized primes, which may be "continuous". For simplicity, we will only considered the case of "discrete" generalized number systems as defined at the begining of this section. ///

<u>Remark 2</u>: The weighted prime counting functions make sense here. For instance, one can speak about $\psi(x) = \sum_{p_k^m \leq x} \log p^m = \sum_{n_k \leq x} \Lambda(n_k)$,

where Δ is defined in the obvious way. The PNT is easily seen equivalent to $\psi(x) \sim x$. ///

3 The PNT for Beurling's primes.

Beurling was able to considerably relax the hypothesis of Landau's PNT. He showed:

<u>Theorem</u> (Beurling's PNT, 1937). If

$$(3.1) \quad N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right),$$

where $a>0$ and $\boxed{\gamma > \frac{3}{2}}$, then

$$(\text{PNT}) \qquad \pi(x) \sim \frac{x}{\log x}, \quad x \longrightarrow \infty. \;///$$

Beurling actually showed that the condition $\gamma > \frac{3}{2}$ is obtimal in the sense that the PNT may fail if $\gamma = \frac{3}{2}$. He did this by constructing a "continuous" prime system counterexample (see Remark 1). His argument was refined by Diamond

<u>Theorem</u> (Diamond, 1970). Beurling's condition $\gamma > \frac{3}{2}$ is sharp, namely, there is a set of generalized integers whose counting function satisfies (3.1) but for which (PNT) fails. ///

Diamond couterexample was announced in Bateman and Diamond's paper from 1969. This paper has been very influential since it posed a number of conjectures in the theory of Beurling's generalized numbers. The most central of this conjectures was the following one. Bateman and Diamond conjectured that the L^2-condition

$$(3.2) \quad \int_1^{\infty} \left| \frac{(N(x) - ax)\log x}{x} \right|^2 \frac{dx}{x} < \infty$$

would suffice for the PNT. The truth of the conjecture was established almost 30 years latter by Kahane:

Theorem (Kahane's PNT, 1997). The L^2-hypothesis ensures the validity of the PNT. ///

Schlage-Puchta and I have recently discovered another condition yet for the PNT.

Theorem (Schlage-Puchta and Vindas, 2012). The average asymptotic behavior

$$(3.3) \qquad N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right) \quad (C),$$

with $\gamma > \frac{3}{2}$, implies the PNT.

The condition (3.3) means in the Cesàro sense, i.e., it means that there is m (which may be arbitrary large!) such that

$$(3.4) \qquad \int_1^x \frac{N(t) - at}{t}\left(1 - \frac{t}{x}\right)^m dt = O\left(\frac{x}{\log^{\gamma} x}\right).$$

We have also shown that our theorem is a proper extension of Beurling's one.

Proposition (Schlage-Puchta and Vindas, 2012). There is a set of generalized numbers such that $N(x) = x + \Omega\left(\frac{x}{\log^{4/3} x}\right)$ but $N(x) = x + O\left(\frac{x}{\log^{5/3} x}\right)$ in the Cesàro sense. ///

More recently, I have been able to produce a PNT that contains all previous cases.

Theorem (Vindas, unpublished). Let $M : [0, \infty) \longrightarrow [0, \infty)$ be a function that satisfies:

i) $M(e^t)$ is convex, $\frac{M(t)}{t}$ is concave and non-decreasing.

ii) The "non-quasianalyticity" condition:

$$(3.5) \qquad \int_1^\infty \frac{M(t)}{t^3} dt < \infty.$$

Set $G(s) = \zeta(s) - \frac{a}{s-1}$, $\operatorname{Re} s > 1$. If $\exists \lambda > 0$ such that

$$(3.6) \qquad \sup_{1<\sigma<2} \int_{-\infty}^{\infty} |G'(\sigma+it)| e^{-\lambda M(|t|)} dt < \infty,$$

then the PNT holds true. ///

We compare the four conditions occuring in all these PNT.

In order to do so, we write

$$(3.7) \qquad N(x) = ax + \frac{x}{\log x} E(\log x).$$

The function E is the central object for the PNT, it tells us information about the boundary behavior of the derivative of $\zeta'(s)$.

Beurling's condition reads:

$$(B) \qquad N(x) = ax + O\left(\frac{x}{\log x}\right) \Leftrightarrow E(u) = O\left(\frac{1}{u^{\gamma-1}}\right)$$

Kahane's condition (3.2) takes the form:

$$(K) \qquad E \in L^2(\mathbb{R}).$$

Using a little bit of the theory of asymptotic behavior of Schwartz distributions, one shows that (3.3) is equivalent to

$$(SP\text{-}V) \qquad (\varphi * E)(u) = O\left(\frac{1}{u^{\gamma-1}}\right), \quad \forall \varphi \in S(\mathbb{R}).$$

Here $S(\mathbb{R})$ stands for the Schwartz class of rapidly decreasing smooth functions.

Finally, one can show that (3.6) is equivalent to the following condition:

(V) There is $f \in L^1(\mathbb{R})$ such that its Fourier transform satisfies $\hat{f}(t) >> e^{-\lambda M(|t|)}$ for some $\lambda > 0$ and $f * E \in L^2(\mathbb{R})$. //

That (B) and (K) are contained in (V) is obvious. Using the non-quasianalyticity condition (3.5), one can show that there is $\varphi \in S(\mathbb{R})$ that satisfies $e^{-M(|t|)} \ll \hat{\varphi}(t)$. So (SP-V) is also contained in (V).

Remark 1: The condition (3.5) implies that $M(t) = O\left(\frac{t^2}{\log t}\right)$. Is this condition a border for the PNT? I conjecture so ...

Remark 2: The idea behind the proof of all these PNT is as follows. All conditions ensure that $G(s) = \zeta(s) - \frac{a}{s-1}$ has continuous extension to $\operatorname{Re} s = 1$ and that $G'(\sigma + it)$ tends locally as $\sigma \to 1^+$ to a function $g(t) \in H^1_{loc}(\mathbb{R})$, the local Sobolev space. So, provided that $\zeta(1+it) \neq 0$, all conditions are good enough to ensure that $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$ tends in the boundary line $\operatorname{Re} s = 1$ to a good function. A Tauberian argument the leads to $\psi(x) \sim x$, which is equivalent to the PNT. The problem then reduces to show that $\zeta(s)$ has no zeros on $\operatorname{Re} s = 1$. In the case of the conditions (B) and (SP-V) one gets that $\zeta(1+it)$, except for $t=0$, is (locally) Hölder continuous with exponent $\gamma' - 1$, where $\gamma' = \min\{\gamma, 2\}$. If $\gamma > \frac{3}{2}$, then $\gamma' - 1 > \frac{1}{2}$. One then uses a variant of Hadamard's classical argument, by replacing $3 + 4\cos t + \cos 2t \geq 0$ by the Fejér kernel $F_n(t) = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=-k}^{k} e^{ijt} = \frac{1}{n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2 \geq 0$ to show that if $\zeta(1+it) = 0$, then one would have

$$1 \ll_n \frac{|\zeta(\sigma+t)|^{2-\frac{1}{n}}}{(\sigma-1)} = O_n\left((\sigma-1)^{2(\gamma'-1)-1-\frac{(\gamma'-1)}{n}}\right),$$

leading to contradiction for n large enough. The case of (K) and (V) is more complicated. One one gets $|\zeta(1+it) - \zeta(1+it')| = O(|t-t'|^{\frac{1}{2}})$, which only implies that there are at most 2 zeros of ζ. A further elaborated analysis is needed in order to show that such zeros cannot occur. The condition (V) is involved here.