Some recent developments on complex Tauberian theorems for Laplace transforms

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Complex Tauberian theorems have been strikingly useful tools in diverse areas such as:

- Analytic number theory.
- Spectral theory for (pseudo-)differential operators.
- Last three decades: operator theory and semigroups.

We will discuss some recent developments on complex Tauberians for Laplace transforms and power series. We will be concerned with two groups of statements:

- Wiener-Ikehara theorems.
- Ingham-Karamata-Fatou-Riesz theorems.

Main questions:

1. Relax boundary requirements to a minimum.
2. Mild Tauberian hypotheses (one-sided conditions).
3. Optimal Tauberian constants: Sharp versions.

This talk is based on collaborative works with G. Debruyne.

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The classical Wiener-Ikehara theorem

Theorem (Wiener-Ikehara, Laplace-Stieltjes transforms)

Let S be a non-decreasing function (Tauberian hypothesis) such that $\mathcal{L}\{\mathrm{d}S; z\} = \int_{0^-}^{\infty} e^{-zt}\mathrm{d}S(t)$ converges for $\Re e\, z > 1$. If

$$\mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

has analytic continuation through $\Re e\, z = 1$, then $S(x) \sim Ae^x$.

Theorem (Wiener-Ikehara, version for Dirichlet series)

Let $a_n \geq 0$. Suppose $\sum_{n=1}^{\infty} a_n n^{-z}$ converges for $\Re e\, z > 1$. If

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From the Wiener-Ikehara theorem to the PNT:

The Prime Number Theorem (PNT) asserts that

$$\pi(x) = \sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

- PNT is equivalent to $\psi(x) = \sum_{p^{\alpha} \leq x} \log p = \sum_{n \leq x} \Lambda(n) \sim x$.
- $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ has analytic continuation to $\mathbb{C}$ except for a simple pole with residue 1 at $z = 1$.
- Logarithmic differentiation of $\zeta(z) = \prod_p (1 - p^{-z})^{-1}$ leads to

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} = -\frac{\zeta'(z)}{\zeta(z)}, \quad \Re e\, z > 1.$$

- $(z-1)\zeta(z)$ has no zeros on $\Re e\, z = 1$, so

$$-\frac{d}{dz}(\log((z-1)\zeta(z))) = -\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$$

is analytic in a region containing $\Re e\, z \geq 1$. The rest follows from the Wiener-Ikehara theorem.

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Remarks on the Wiener-Ikehara theorem

- Historically, the Wiener-Ikehara theorem improved a Tauberian theorem of Landau (1908) by eliminating the unnecessary hypothesis $G(z) = O(|z|^N)$ on

$$G(z) = \mathcal{L}\{\mathrm{d}S; z\} - \frac{A}{z-1}$$

- The hypothesis $G(z)$ has analytic continuation to $\Re e\, z = 1$ can be significantly relaxed to “good boundary behavior":
  1. G(z) has continuous extension to $\Re e\, z = 1$.
  2. L^1_{loc}-boundary behavior: $\lim_{x \to 1^+} G(x + iy) \in L^1(I)$ for every finite interval I.
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Pseudofunctions and pseudomeasures

Pseudofunctions and pseudomeasures are notions that naturally arise in harmonic analysis.

- Pseudomeasures: $PM(\mathbb{R}) = \{g \in \mathcal{S}'(\mathbb{R}) : \widehat{g} \in L^{\infty}(\mathbb{R})\}$
- Pseudofunctions: $PF(\mathbb{R}) = \{g \in PM(\mathbb{R}) : \lim_{x\to\infty} \widehat{g}(x) = 0\}$

Given an open set $U \subseteq \mathbb{R}$, we define the local spaces:

- $PM_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PM(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $PF_{loc}(U) = \{g \in \mathcal{D}'(U) : \varphi g \in PF(\mathbb{R}), \forall \varphi \in \mathcal{D}(U)\}$.
- $L^1_{loc}(U) \subset PF_{loc}(U)$.
- Every Radon measure on U is a local pseudomeasure.

Let G be analytic on $\Re e\, z > \alpha$ and $U \subset \mathbb{R}$ be open.

We say that G has local pseudofunction boundary behavior on $\alpha + iU$ if it has distributional boundary values there, i.e.

$$\lim_{x\to\alpha^+} G(x + iy) = g(y) \text{ in } \mathcal{D}'(U)$$

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Extension of the Korevaar-Wiener-Ikehara theorem

We call a function S log-linearly slowly decreasing if for each $\varepsilon > 0$ there exist $\delta > 0$

$$\liminf_{x\to\infty} \inf_{0\leq h\leq\delta} \frac{S(x+h)-S(x)}{e^x} \geq -\varepsilon.$$

Theorem (Debruyne and V., 2016)

Suppose that $\mathcal{L}\{S;z\} = \int_0^\infty S(t)e^{-zt}\mathrm{d}t$ *converges for* $\Re e\, z > 1$. *Then,*

$$S(x) \sim Ae^x$$

if and only if

1. $\mathcal{L}\{S;z\} - \dfrac{A}{z-1}$ *has local pseudofunction boundary behavior on* $\Re e\, z = 1$, *and*
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The Fatou-Riesz theorem

In his very influential 1906 paper

Séries trigonométriques et séries de Taylor,

Fatou proved the following theorem on analytic continuation of power series.

Theorem (Fatou-Riesz theorem)

Suppose that $F(z)=\sum_{n=0}^{\infty} c_n z^n$ converges for $|z|<1$ and $c_n = o(1)$ (this is the Tauberian condition). If $F(z)$ has analytic continuation to a neigborhood of $z=1$, then $\sum_{n=0}^{\infty} c_n$ converges and

$$\sum_{n=0}^{\infty} c_n = F(1).$$

Marcel Riesz gave three proofs of this theorem (1909, 1911, 1916), so his name is usually associated to this result.

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Theorem (Fatou-Riesz theorem)

Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ *converges for* $|z| < 1$ *and* $c_n = o(1)$ *(this is the Tauberian condition). If* $F(z)$ *has analytic continuation to a neigborhood of* $z = 1$*, then* $\sum_{n=0}^{\infty} c_n$ *converges and*

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The Ingham-Karamata theorem for Laplace transforms

In 1935 Ingham and Karamata obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h)-\tau(x)) > -\varepsilon.$$

that is, $\tau(x+h)-\tau(x) > -\varepsilon$ for $x > X_{\varepsilon}$ and $0 \leq h < \delta_{\varepsilon}$.

Theorem (Ingham and Karamata, independently)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing (Tauberian hypothesis). Suppose its Laplace transform

$$\mathcal{L}\{\tau; z\} = \int_0^\infty \tau(t)e^{-zt}\mathrm{d}t$$

converges on $\Re e\, z > 0$ and has L^1_{loc}-boundary behavior on $\Re e\, z = 0$, then $\lim_{x\to\infty} \tau(x) = 0$.

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converges on $\Re e\, z > 0$ and has L^1_{loc}-boundary behavior on $\Re e\, z = 0$, then $\lim_{x\to\infty} \tau(x) = 0$.

The Ingham-Karamata theorem for Laplace transforms

In 1935 Ingham and Karamata obtained a Fatou-Riesz type Tauberian theorem for Laplace transforms. The result makes use of *slow decrease*.

A function τ is called slowly decreasing if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\liminf_{x\to\infty} \inf_{h\in[0,\delta]} (\tau(x+h) - \tau(x)) > -\varepsilon.$$

that is, $\tau(x+h) - \tau(x) > -\varepsilon$ for $x > X_\varepsilon$ and $0 \leq h < \delta_\varepsilon$.

Theorem (Ingham and Karamata, independently)

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In 1980 Newman gave a simple contour integration proof of the next Tauberian theorem.

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Let $a_n = O(1)$ (Tauberian hypothesis). If $F(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$ has analytic continuation beyond $\Re e\, z = 1$, then

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Newman's short way to the PNT

Newman's Tauberian theorem from above provides a relatively simple way to prove the PNT.

- One works here with the Möbius

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors,} \\ 0 & \text{otherwise.} \end{cases}$$

- Property: μ is the Dirichlet convolution inverse of 1. So,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} = \frac{1}{\zeta(z)} \quad (\zeta \text{ is the Riemann zeta function})$$

- Applying the previous theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(1)} = 0.$
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Tauberians motivated by applications in semigroups

Newman's contour integration method was adapted to a variety of Tauberian problems in numerous articles.

Its importance was recognized by the semigroup community. Here is a sample (extending a result of Korevaar and Zagier):

Theorem (Arendt and Batty, 1988)

Let $\rho \in L^{\infty}(\mathbb{R})$ (Tauberian hypothesis) vanish on $(-\infty, 0)$. Suppose that $\mathcal{L}\{\rho; z\}$ has analytic continuation at every point of the complement of iE where $E \subset \mathbb{R}$ is a closed null set. If $0 \notin iE$ and

$$\sup_{t \in E} \sup_{x>0} \left| \int_0^x e^{-itu} \rho(u) \mathrm{d}u \right| < \infty,$$

then the (improper) integral of ρ converges to $b = \mathcal{L}\{\rho; 0\}$, that is,

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If $E = \emptyset$, the result is due to Korevaar and Zagier (independently), who also obtained it via Newman's contour integration technique.

In this case, the result is contained in Ingham's Fatou-Riesz type theorem:

- Set $\tau(x) = \int_0^x \rho(u)\mathrm{d}u - b \Rightarrow \mathcal{L}\{\tau; z\} = \dfrac{\mathcal{L}\{\rho; z\} - b}{z}$ with $b = \mathcal{L}\{\rho; 0\}$.
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The Arendt-Batty Tauberian theorem readily extends to functions with values on a Banach space. Here is a sample application of the vector-valued version:

Theorem (Arendt and Batty)

Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a reflexive Banach space X. Denote the spectrum of its infinitesimal generator A as $\sigma(A)$. If $\sigma(A) \cap i\mathbb{R}$ is countable and no eigenvalue of A lies on the imaginary axis, then

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In recent times, Tauberian methods have been revisited to study rates of converge that can be a applied to PDE, e.g. decay estimates for damped wave equations.

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In 1986 Katznelson and Tzafriri proved the next interesting theorem for power series. Denote as $\mathbb{D}$ the unit disc in the complex plane.

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Suppose that $F(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < 1$ and $S_n = \sum_{k=0}^{n} c_k = O(1)$ (Tauberian condition). If $F(z)$ has analytic continuation to every point $\partial\mathbb{D} \setminus \{1\}$, then $c_n = o(1)$.

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The above theorem has also a Banach space valued counterpart, which Katznelson and Tzafriri used to prove:

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Let T be a power-bounded operator on a Banach space ($\sup_{n\in\mathbb{N}} \|T^n\| < \infty$). Then,

$$\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$$

if and only if $\sigma(T) \cap \partial\mathbb{D} \subseteq \{1\}$.

Proof: The contraposition of the direct implication follows by standard functional calculus. For the converse, if $\lambda I - T$ is invertible for all $|\lambda| \geq 1$, $\lambda \neq 1$, then $g(z) = \sum_{n=0}^{\infty} T^n z^n$ is analytic on $\partial\mathbb{D} \setminus \{1\}$, the same is true for

$$F(z) = (I - z)g(z) = \sum_{n=0}^{\infty} (T^n - T^{n+1}) z^n \Rightarrow \|T^{n+1} - T^n\| \to 0.$$

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Extension of the Ingham-Karamata theorem

Theorem (Debruyne and V., 2016)

Let $\tau \in L^1_{loc}(\mathbb{R})$ be slowly decreasing, vanish on $(-\infty, 0)$, and have convergent Laplace transform on $\Re e\, z > 0$. Suppose there is a closed null set $E \subset \mathbb{R}$ such that:

(I) *$\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on $i(\mathbb{R} \setminus E)$,*

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(III) *$0 \notin E$.*

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$$\tau(x) = o(1). \tag{1}$$

Conversely, (1) implies that $\mathcal{L}\{\tau; z\}$ has local pseudofunction boundary behavior on the whole imaginary axis.

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Then, $c_n = o(1)$. Moreover, the series $\sum_{n=0}^{\infty} c_n e^{in\theta_0}$ converges at every point where there is a constant $F(e^{i\theta_0})$ such that

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Some tools

Our proof is based on:

1. Boundedness theorems for Laplace transforms with local pseudo-measure behavior.
2. Characterizations of local pseudofunctions.
3. The following simplified version of the theorem:

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$\tau \in L^1_{loc}(\mathbb{R})$ slowly decreasing with convergent $\mathcal{L}\{\tau; z\}$ on $\Re e\, z > 0$. Then,

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Quantified finite forms: Ingham's theorem

Lip(I; M) denotes the class of Lipschitz continuous functions on I with Lipschitz constant M.

Known result: Suppose that

1. $\tau \in L^1_{loc}[0,\infty)$,
2. $\mathcal{L}\{\tau; z\}$ has "good" boundary behavior on $(-i\lambda, i\lambda)$,
3. $\tau \in \mathrm{Lip}([X,\infty); M)$ for some X.

There is an absolute contant $\mathfrak{C} > 0$ such that

$$\limsup_{x\to\infty} |\tau(x)| \leq \frac{\mathfrak{C} M}{\lambda}.$$

Some values of $\mathfrak{C}$:

$\mathfrak{C} = 6$, Ingham (1935)

$\mathfrak{C} = 2$, Korevaar, Zagier, and other people ...

Problem: Find the optimal value of $\mathfrak{C}$.

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and the value of $\pi/2$ *in this inequality cannot be improved.*

Combining this with the Graham-Vaaler sharp Wiener-Ikehara theorem, one can consider functions that are 'Lipschitz continuous only from below'. We obtained the sharp inequality

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Inequalities for functions with regular Fourier transform

Define the 'oscillation' and 'decrease' moduli at ∞ as:

$$\Psi(\delta) = \limsup_{x\to\infty} \sup_{h\in[0,\delta]} |\tau(x+h) - \tau(x)|.$$

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Theorem (Debruyne and V., 2017)

Let $\tau \in L^1_{loc}(\mathbb{R})$ have at most polynomial growth. Suppose that $\hat{\tau} \in PF_{loc}(-\lambda, \lambda)$ (in particular if continuous there). Then,

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and

$$\limsup_{x\to\infty} |\tau(x)| \leq \inf_{\delta>0} \left(1 + \frac{\pi}{\delta\lambda}\right) \Psi_{-}(\delta).$$

The contants $\pi/2$ and π being sharp in these inequalities.

Some references

The talk is based on recent collaborative works with G. Debruyne:

- Optimal Tauberian constant in Ingham's theorem for Laplace transforms, Israel J. Math., to appear.
- Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior, J. Anal. Math., to appear.
- Generalization of the Wiener-Ikehara theorem, Illinois J. Math. 60 (2016), 613-624.

For applications of these results in analytic number theory, see:

- On PNT equivalences for Beurling numbers, Monatsh. Math. 184 (2017), 401-424.
- On Diamond's L^1 criterion for asymptotic density of Beurling generalized integers, Michigan Math. J., to appear.

Important book references on complex Tauberians

- W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, Vector-valued Laplace transforms and Cauchy problems, Birkhäuser/Springer Basel, 2011.
- J. Korevaar, Tauberian theory. A century of developments, Springer-Verlag, 2004.

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