The pointwise behavior of Riemann's function

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CIRM, Marseille, September 27, 2023

Analysis seminar
University of Reading, May 26, 2023

Functional analysis seminar
University of Lille, January 27, 2023

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$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2} \qquad (1)$$

as an example of a nowhere differentiable function.

- Weierstrass could not show that claim, but gave his own example

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \quad 0 < a < 1, \qquad (2)$$

  which he showed to be nowhere differentiable under extra assumptions: $b \in \mathbb{N}$ odd and $ab > 1 + 3\pi/2$.
- In 1916 Hardy completed the analysis of (2): $ab > 1$, $b \in \mathbb{R}$.
- In the same paper, Hardy was able to show that (1) is not differentiable at the following points:
  - irrationals;
  - rationals of the forms $\frac{2r+1}{2s}$ and rationals $\frac{2r}{4s+1}$.

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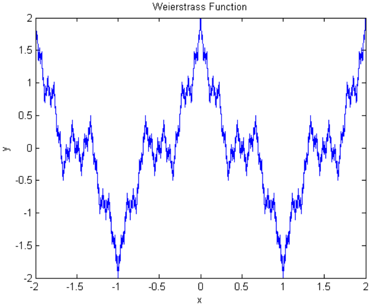

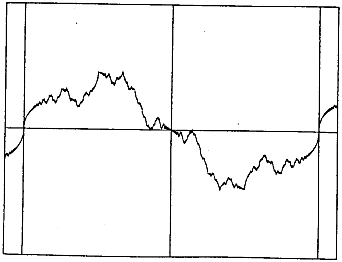

FIGURE 1.1. $y=\sum_{n=1}^{\infty}\frac{1}{n^2\pi}\sin(n^2\pi x); -0.127<x<2.127; |y|<0.845.$

More history

- Hardy's results seemed to confirm the non-differentiability belief.
- It was then a surprise when Gerver showed in 1970-1971 that Riemann's function is in turn differentiable at every rational that is the quotient of two odd integers, and not differentiable elsewhere.
- Gerver proof is elementary, but long and difficult to grasp.
- Smith (1972) and Itatsu (1981) gave simpler treatments of all rational points.
- They both use the Poisson summation formula, i.e.,

$$\sum_{n\in\mathbb{Z}} g(n) = \sum_{n\in\mathbb{Z}} \widehat{g}(n),$$

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$$\widehat{g}(u) = \int_{-\infty}^{\infty} g(x) e(-xu) \mathrm{d}\, x \qquad (e(t) = e^{2\pi \mathrm{i} t})$$

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Pointwise Hölder exponent

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- Their finer estimates (essentially) yield pointwise Hölder exponent.
- This left open the determination of the pointwise Hölder exponents at irrational points.
- Duistermaat (1991) exhibited upper bounds for Hölder exponents at irrationals in terms of diophantine approximation properties of the point.
- Jaffard finally settled the problem in 1996, when he showed that Duistermaat's upper bound was actually the Hölder exponent at every irrational.

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We will sketch a new and simple method to compute the pointwise Hölder exponent of Riemann's function at every point.

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Some words about the idea of our method

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- We compute $\alpha(x) = \sup\{\alpha > 0 \mid \phi(x+h) = P_x(h) + O_x(|h|^{\alpha})\}$.
- Restricting the complex variable z to the upper half-plane, one has

$$\phi'(z) = \frac{1}{2}(\theta(z) - 1), \quad \text{where } \theta(z) = \sum_{n\in\mathbb{Z}} e(n^2 z).$$

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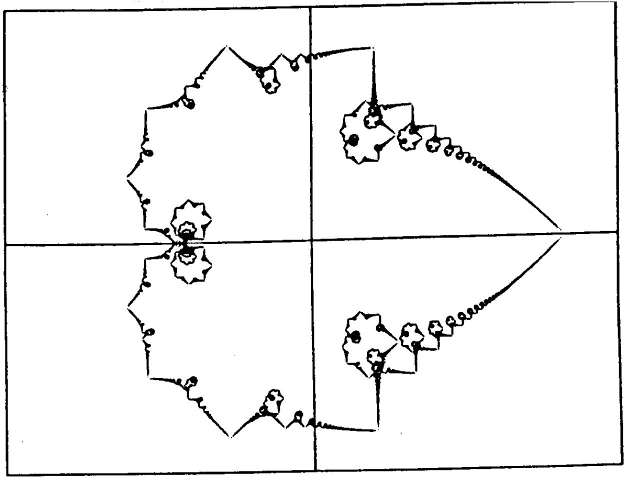
Graph of Riemann's complex function



$$z(t) = 2i\phi\left(t/2\right) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{e^{i\pi n^2 t}}{n^2}$$

Number theoretic preliminaries: Gauss sums

Quadratic Gauss sums and generalized quadratic Gauss sums

Let q, p, m be integers with $(p, q) = 1$. These sums are defined as

$$S(q,p) = \sum_{j=1}^{q} e\left(\frac{pj^2}{q}\right) \quad \text{and} \quad S(q,p,m) = \sum_{j=1}^{q} e\left(\frac{pj^2 + mj}{q}\right).$$

The quadratic Gauss sums were evaluated by Gauss himself:

$$S(q,p) = \begin{cases} \varepsilon_q \left(\frac{p}{q}\right)\sqrt{q} & \text{if } q \text{ is odd,} \\ 0 & \text{if } q \equiv 2 \mod 4, \\ (1+\mathrm{i})\overline{\varepsilon_p}\left(\frac{q}{p}\right)\sqrt{q} & \text{if } q \equiv 0 \mod 4. \end{cases}$$

$\left(\frac{p}{q}\right)$ is the Jacobi symbol and (n odd) $\varepsilon_n = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4, \\ \mathrm{i} & \text{if } n \equiv 3 \bmod 4. \end{cases}$

Elementary manipulations lead to the bounds:

$$S(q,p,m) \ll \sqrt{q}.$$

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The behavior at the rationals: behavior of θ

Lemma

Suppose $(p,q)=1$ and $y=\operatorname{Im} z>0$. Then

$$\theta\left(\frac{p}{q}+z\right)=\frac{e^{\pi i/4}}{q\sqrt{2}}z^{-1/2}\left(S(q,p)+2\sum_{m=1}^{\infty}S(q,p,m)\exp\left(-\frac{i\pi m^2}{2q^2 z}\right)\right).$$

Proof. Note that $e(pn^2/q)$ is q-periodic in n, writing $n=j+mq$,

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$$\phi_{q,p}(z)=\sum_{m=1}^{\infty}\frac{S(q,p,m)}{2\pi i m^2}e(m^2z)\ll\sqrt{q}$$

For $y>0$,

$$\phi\left(\frac{p}{q}+h+iy\right)=\phi\left(\frac{p}{q}+iy\right)-\frac{1}{2}h+\frac{1}{2}\int_{iy}^{h+iy}\theta\left(\frac{p}{q}+\zeta\right)d\zeta,$$ and the very last term is

$$=\frac{e^{\pi i/4}}{q\sqrt{2}}\left(S(q,p)\left[2\zeta^{1/2}\right]_{iy}^{h+iy}+2\int_{iy}^{h+iy}\zeta^{-1/2}(4q^2\zeta^2)\left(\phi_{q,p}\left(-\frac{1}{4q^2\zeta}\right)\right)'d\zeta\right)$$

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$$\theta\left(\frac{p}{q}+z\right) = \frac{\mathrm{e}^{\pi \mathrm{i}/4}}{q\sqrt{2}} z^{-1/2}\left(S(q,p) + 2\sum_{m=1}^{\infty} S(q,p,m)\exp\left(-\frac{\mathrm{i}\pi m^2}{2q^2 z}\right)\right).$$

$$\phi_{q,p}(z) = \sum_{m=1}^{\infty} \frac{S(q,p,m)}{2\pi \mathrm{i} m^2} e(m^2 z) \ll \sqrt{q}$$

For $y > 0$,

$$\phi\left(\frac{p}{q}+h+\mathrm{i}y\right) = \phi\left(\frac{p}{q}+\mathrm{i}y\right) - \frac{1}{2}h + \frac{1}{2}\int_{\mathrm{i}y}^{h+\mathrm{i}y} \theta\left(\frac{p}{q}+\zeta\right) \mathrm{d}\zeta,$$ and the very last term is

$$= \frac{\mathrm{e}^{\pi \mathrm{i}/4}}{q\sqrt{2}}\left(S(q,p)\left[2\zeta^{1/2}\right]_{\mathrm{i}y}^{h+\mathrm{i}y} + 2\int_{\mathrm{i}y}^{h+\mathrm{i}y} \zeta^{-1/2}(4q^2\zeta^2)\left(\phi_{q,p}\left(-\frac{1}{4q^2\zeta}\right)\right)' \mathrm{d}\zeta\right)$$

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The behavior at the rationals: behavior of ϕ

We have thus obtained:

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Let p and q be integers, $q \geq 1$, $(p,q)=1$. Then

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ϕ is differentiable at p/q iff $q\equiv 2 \mod 4$; otherwise $\alpha(p/q)=1/2$.

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Further integration by parts leads to a full trigonometric chirp expansion at the rationals. In particular, $\alpha(p/q)=3/2$ if $q \equiv 2 \mod 4$.

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The behavior at the rationals of Re ϕ

TABLE 1. Behavior of $\mathrm{Re}\big(\phi(p/q+h)-\phi(p/q)\big)$

q mod 4	p mod 4	$h<0$	$h>0$
1	any	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{h}+O_q(h)$
3	any	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$-\left(\frac{p}{q}\right)\frac{1}{2\sqrt{q}}\sqrt{h}+O_q(h)$
2	any	$-\frac{1}{2}h+O\big(q^{3/2}\lvert h\rvert^{3/2}\big)$	$-\frac{1}{2}h+O\big(q^{3/2}h^{3/2}\big)$
0	1	$-\left(\frac{q}{p}\right)\frac{1}{\sqrt{q}}\sqrt{\lvert h\rvert}+O_q(\lvert h\rvert)$	$-\frac{1}{2}h+O\big(q^{3/2}h^{3/2}\big)$
0	3	$-\frac{1}{2}h+O\big(q^{3/2}\lvert h\rvert^{3/2}\big)$	$\left(\frac{q}{p}\right)\frac{1}{\sqrt{q}}\sqrt{h}+O_q(h)$

Corollary

If p and q are both odd, then $f(x)=\sum_{n=1}^{\infty} n^{-2}\sin(\pi n^2 x)$ is differentiable at $x=p/q$; otherwise the Hölder exponent of f at r equals $1/2$.

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- Irrational has continued fraction $\rho = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}$
- Its nth convergent is $r_n = \dfrac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$
- We define τ_n via

$$|\rho - r_n| = \left(\frac{1}{q_n}\right)^{\tau_n}.$$

- Finally, let n_k be the indices for which $q_{n_k} \not\equiv 2 \bmod 4$, and set

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Our proof uses the following bound on the θ function.

Proposition

Suppose $z = x + iy$ with $y > 0$. For each $\varepsilon > 0$,

$$\theta(\rho + z) \ll |z|^{\frac{1}{2\tau(\rho)} - \varepsilon - \frac{1}{2}} + y^{-1/2}|z|^{\frac{1}{2\tau(\rho)} - \varepsilon} \qquad (|z| \ll 1) \qquad (4)$$

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$$\int_0^{\mathrm{i}|h|}\theta(\rho+z)\,\mathrm{d}z + \int_{\mathrm{i}|h|}^{h+\mathrm{i}|h|}\theta(\rho+z)\,\mathrm{d}z - \int_h^{h+\mathrm{i}|h|}\theta(\rho+z)\,\mathrm{d}z =: I_1 + I_2 + I_3.$$

Using the bounds $\theta(\rho+z) \ll |z|^{\frac{1}{2\tau(\rho)}-\varepsilon-\frac{1}{2}} + y^{-1/2}|z|^{\frac{1}{2\tau(\rho)}-\varepsilon}$, we get

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Our proof of the lower bound: $\alpha(\rho) \geq \frac{1}{2} + \frac{1}{2\tau(\rho)}$

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Conclusive remarks: modular forms

- Recently (2019), Pastor has calculated the pointwise Hölder exponent (at every point!) of fractional integrals of modular forms.
- In the case of irrational points, his main result has been obtaining the pointwise Hölder exponent for a modular form that is not a cusp form. (Cusp forms had been already treated by Chamizo et al. in 2017.)
- His result is in terms of diophantine approximations by noncuspidal rationals.
- Pastor analysis is inspired by that of Jaffard; in particular, based on Tauberian theorems for the wavelet transform.

Our ideas can also be adapted to substantially simplify Pastor's proof, at least without having to resort on Tauberians for the wavelet transform.

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Some other references:

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