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Riemann's Example of a continuous, "non-differentiable" function

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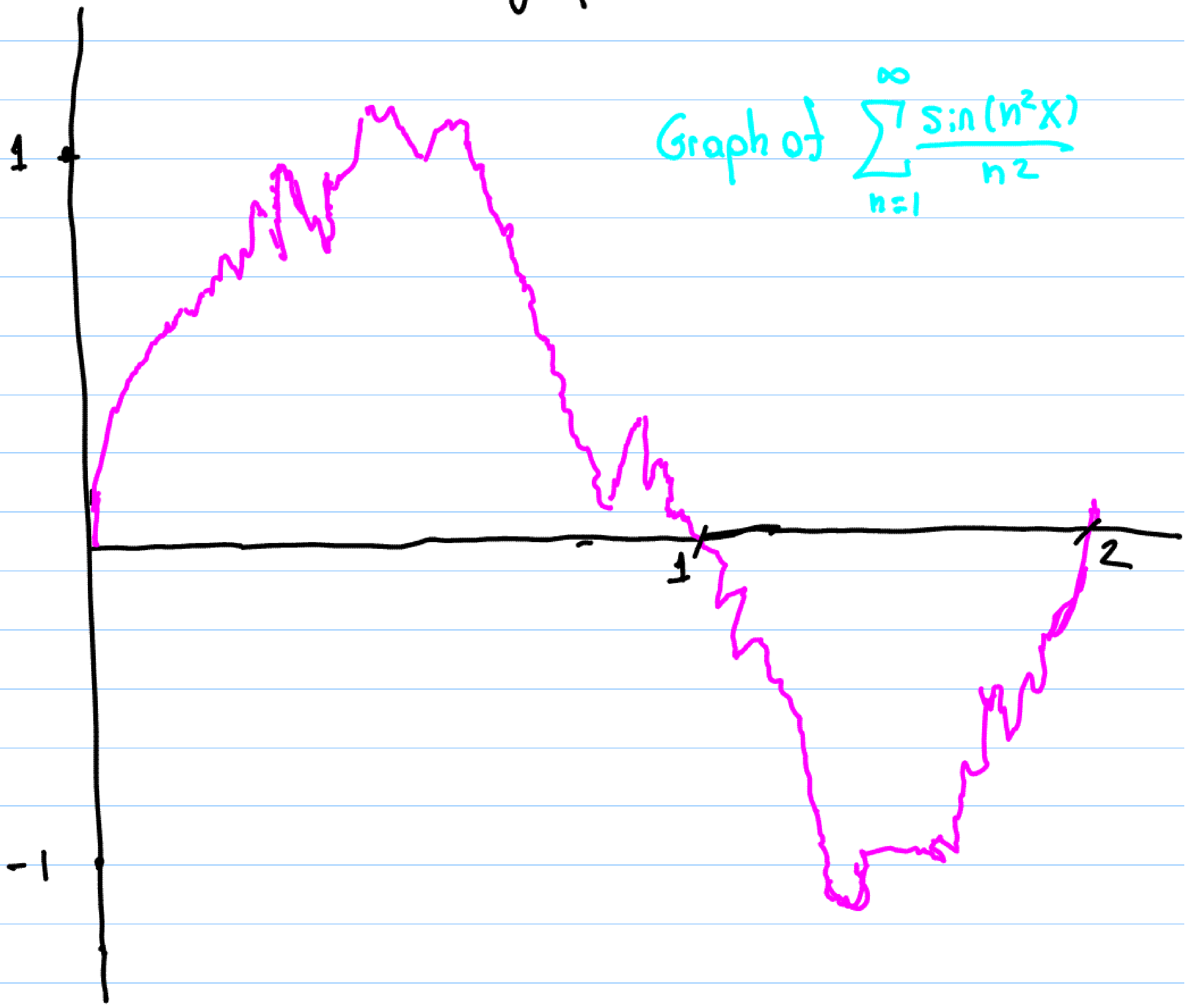
I Introduction

The trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2}, \quad x \in \mathbb{R}, \tag{1.1}$$

has attracted much attention over the past 150 years. It was originally introduced by Riemann. According to the tradition, Riemann would have proposed this function as a continuous function nowhere differentiable function. This was stated by Hardy [12], based on a comment of Bois-Reymond. However, contrary to this claim, it seems that Riemann anticipated much of the work on this function, as discovered by Neuenschwander in a reference to Riemann's work in the diary of Casorati [13].

The interest in this function lies in the fact that it display a very irregular pointwise behavior. Its behavior radically and suddenty changes from point to point, depending essentially in Diophantine properties of the considered point. The function (1.1) is perhaps the chief example of a multifractal function [16]. Its graph looks more or less like



Hardy was the first who published results about (1.1). He showed in 1916 that (1.1) is non-differentiable at any irrational point [12]. Thus, he confirmed the general belief that (1.1) was nowhere differentiable. Later on, the result of Gerver came as a big surprise: (1.1) has actually derivatives at certain rational points [10]. The proof of Gerver was later simplified in [15,20]. The history of the study of (1.1) continued with the development of new tools from wavelet analysis in the 90's [5,14,17] (see particularly the article of Duistermaat).
The complete understanding of the pointwise behavior of Riemann's function was only settled down until 1996 by Jaffard [16].
In this note we briefly discuss the properties of this remarkable series. We will rather work with the complex trigonometric series

$$(1.2) \qquad R(x) = \sum_{n=1}^{\infty} \frac{e^{i\pi n^2 x}}{n^2},$$

From now on, we refer to R as Riemann's function. In section $\boxed{2}$ we discuss its differentiability properties. The remaining sections are devoted to more refined measuments of pointwise behavior for (1.2).

A related challenging problem is the study of the series

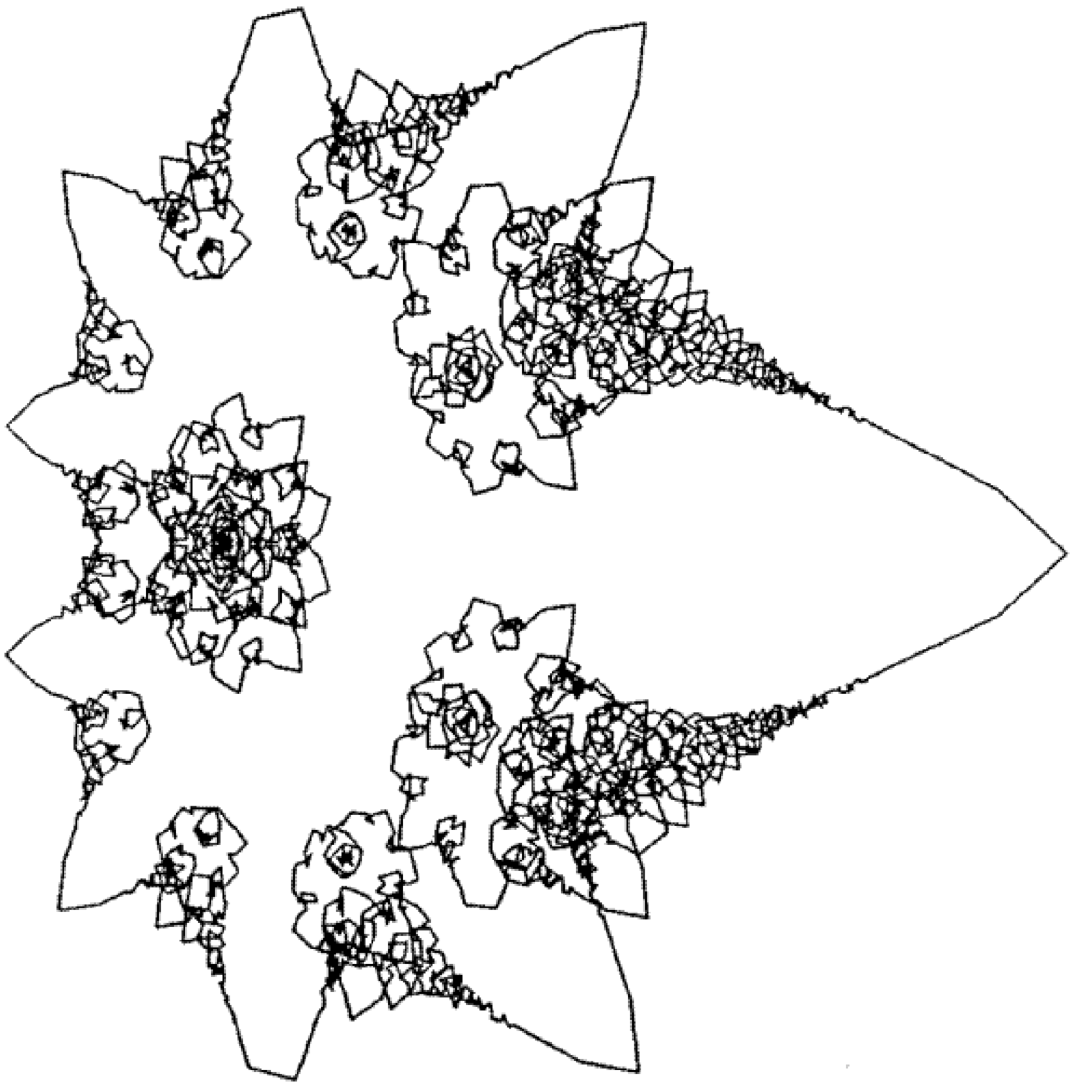
$$\sum_{n=1}^{\infty} \frac{e^{in^{\alpha}x}}{n^{\beta}}. \tag{1.3}$$

The determination of the point behavior of (1.3) at every point is a still a fascinating open problem (out of the scope of these notes). We refer to [1, 3, 4, 9, 19, 23] for some advances in the problem.

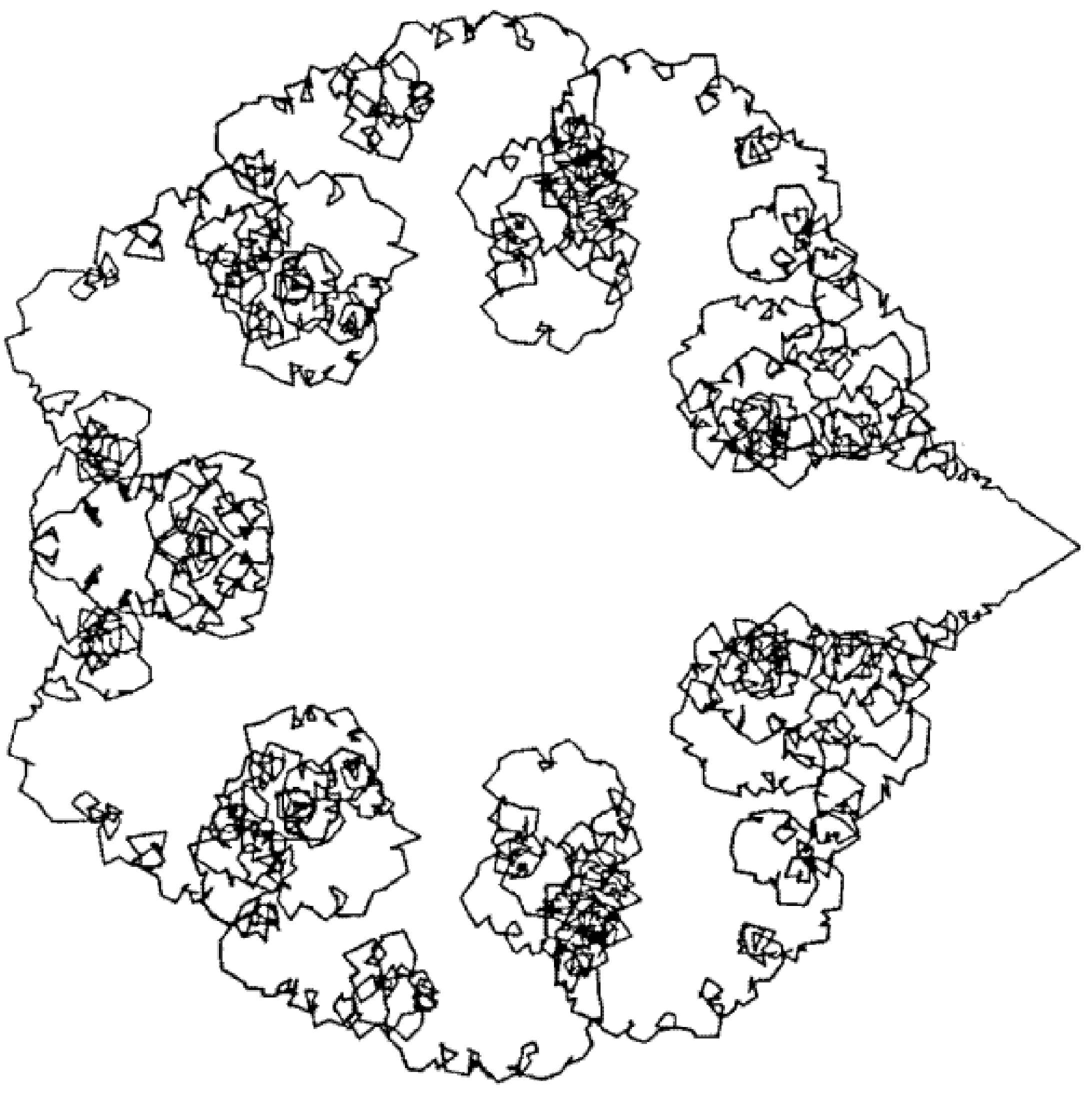
In the next two pages we show some plottings of (1.3) corresponding to some choices of α and β (the graphs are taken from [3]).

Interestingly, when $0<\alpha<1$, (1.3) is real analytic everywhere except at $x=0$. This result has been recently obtained by the author of this note and Jaffard (unpublished material). This was suggested by a particular case studied by Boersma [1], who, when solving a problem proposed by Glasser [9], gave the beautiful formula

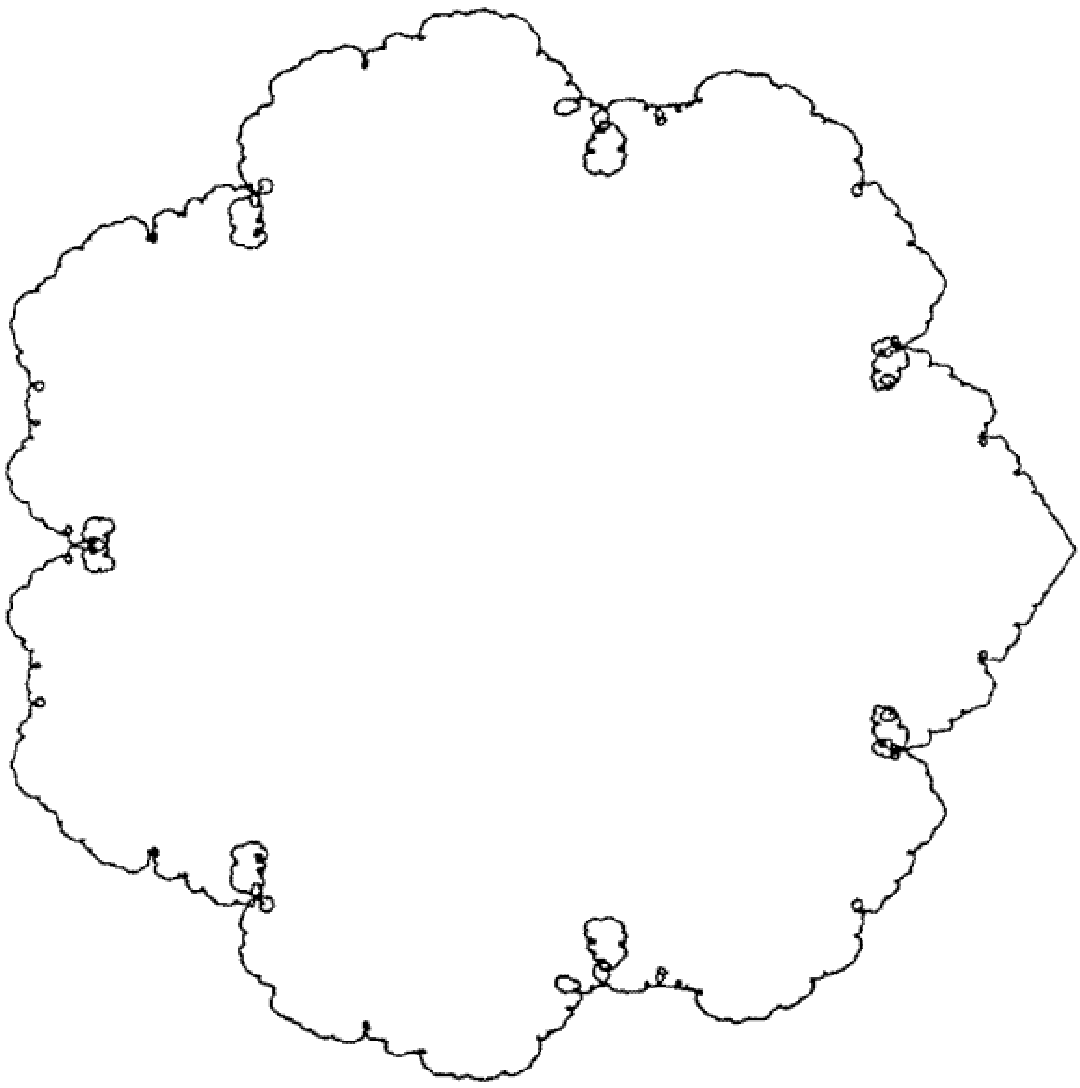
$$\sum_{n=1}^{\infty} \frac{\sin(xn^{\alpha})}{n} = \frac{\pi}{2\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\,\zeta(1-(2k+1)\alpha)\ ; \quad x>0.$$



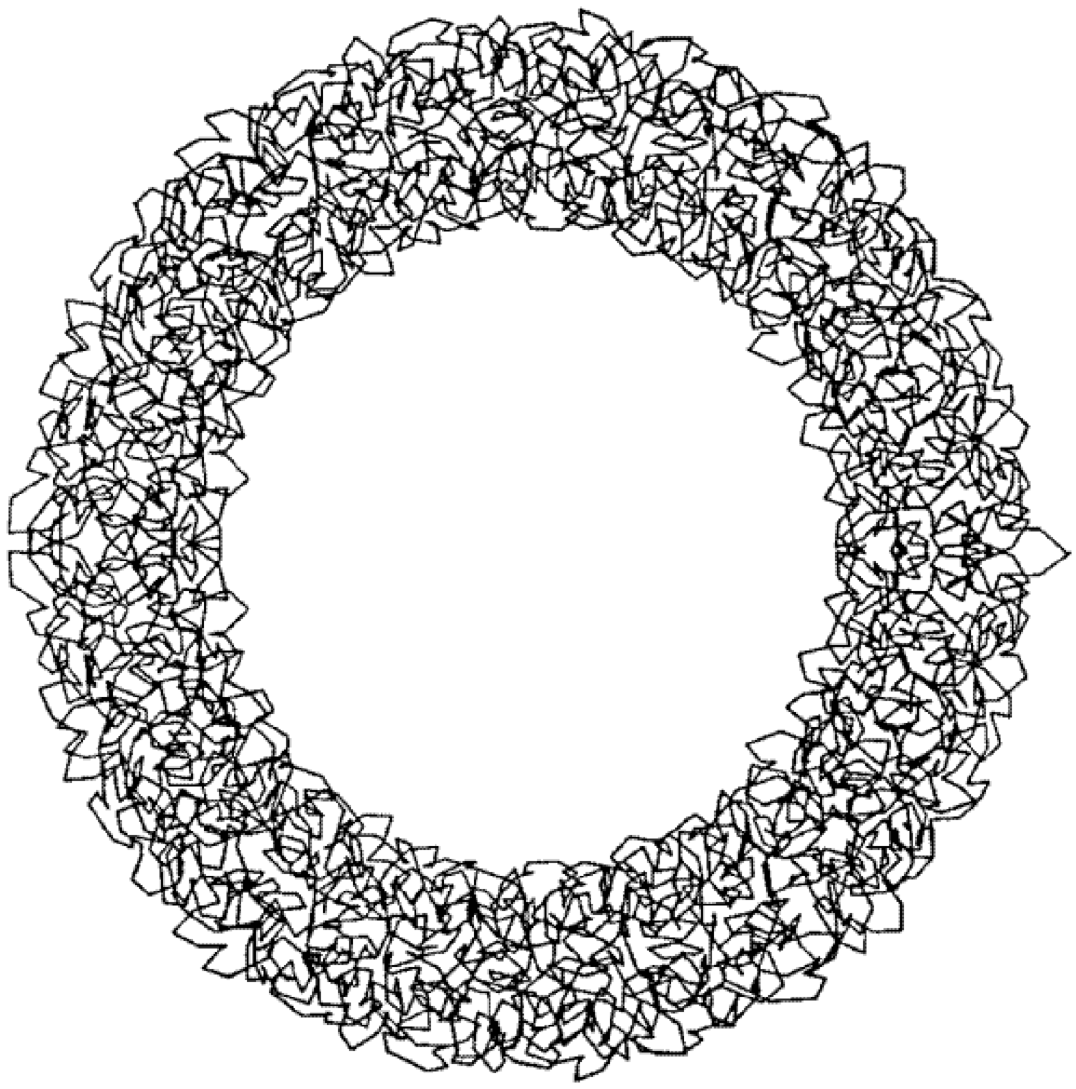
Scheme 1. $\sigma_1(t) = \left(\sum \frac{\sin(n^2 t)}{n^{3/2}}, \sum \frac{\cos(n^2 t)}{n^{3/2}} \right).$



Scheme 2. $\sigma_2(t) = \left(\sum \frac{\sin(n^3 t)}{n^2}, \sum \frac{\cos(n^3 t)}{n^2} \right).$



Scheme 3. $\sigma_3(t)=\left(\sum \frac{\sin(n^3 t)}{n^3}, \sum \frac{\cos(n^3 t)}{n^3}\right).$



Scheme 4. $\sigma_4(t)=\left(\sum \frac{\sin(n^5 t)}{n^{5/2}}, \sum \frac{\cos(n^5 t)}{n^{5/2}}\right).$

2 Differentiability Properties

We begin with Hardy's result [12]:

Theorem 1 The function $R(x)=\sum_{n=1}^{\infty}\frac{e^{i\pi n^2 x}}{n^2}$,

as well as its real and imaginary parts, are non-differentiable at $x=x_0$ if x_0 is irrational or x_0 is a rational that belong to

$$S_0=\left\{\frac{2j}{2m+1}:\ j,m\in\mathbb{Z}\right\}\cup\left\{\frac{2m+1}{2j}:\ m,j\in\mathbb{Z}\right\} \tag{2.1}$$

This result was complemented by Gerver, who showed [10]:

Theorem 2 Riemann's function is differentiable at any rational of the set

$$S_1=\left\{\frac{2j+1}{2m+1}:\ m,j\in\mathbb{Z}\right\} \tag{2.2}$$

Hardy's proof of Theorem 1 depended on the Jacobi theta function. In contrast, Gerver used elementary methods for the proof of Theorem 2.

Hardy's method has an historical interest, and we shall sketch it here. His method is a historical present to modern techiques from wavelet analysis.

We shall give a proof of Theorem 2 later, in Section $\boxed{3}$.

Suppose that R were differentiable at x_0, then, formally, "

$$(2.2) \qquad i\pi \sum_{n=1}^{\infty} e^{i\pi n^2 x_0} = R'(x_0)".$$

Of course, this series is divergent and the argument breaks down. On the other hand, if we employ a summability method to the divergent series, the argument can become rigorous. This is the case for Abel summability, as the reader can find in any good treatise about trigonometric series. Thus, the right interpretation of (2.2) could be

$$(2.3) \qquad i\pi \sum_{n=1}^{\infty} e^{i\pi n^2 x_0} = R'(x_0) \quad (A),$$

i.e., $\displaystyle\lim_{y\to 0^+} \sum_{n=1}^{\infty} e^{i\pi n^2 (x_0+iy)}$ would exist. So, for

Theorem 1 one should show the non-existence of these limits at the underlying points x_0. This leads to the study of the boundary behavior of the analytic function

$$(2.4) \qquad F(z) = \sum_{n=1}^{\infty} e^{i\pi n^2 z}, \quad \operatorname{Im} z > 0,$$

when approaching the real axis.

This can be done in a straightforward manner when $x_0 = 0$, an element of the set (2.1). In fact,

$$F(iy) = \frac{1}{\sqrt{y}} \sum_{n=1} \sqrt{y}\, e^{-\pi(\sqrt{y}\,n)^2} \sim \frac{1}{\sqrt{y}} \int_0^{\infty} e^{-\pi t^2}\, dt$$

$$= \frac{1}{\sqrt{\pi y}} \int_0^\infty e^{-t^2}\,dt = \frac{\Gamma(\frac{1}{2})}{2\sqrt{\pi y}} = \frac{1}{2\sqrt{y}}, \quad y \to 0^+$$

This shows that (2.3) is not satisfied and therefore R is non-differentiable at 0. In order to show that neither $\operatorname{Im} R$ nor $\operatorname{Re} R$ are differentiable at 0, we have to strengthen the argument. It is enough to observe that if either of them is differentiable at 0, a version of Abel's theorem for formally differentiated series would give us the existence of

$$\lim_{z\to 0} \operatorname{Im} F(z), \text{ resp. }, \lim_{z\to 0^+} \operatorname{Re} F(z), \text{ non-tangentially}$$

i.e., when z runs over cones $C_M = \{z \in \mathbb{C} : |\operatorname{Re} z| \le M \operatorname{Im} z\}$. But, this does not hold, because a similar argument shows that

$$(2.5) \quad F(z) \sim \frac{1}{2}\sqrt{\frac{i}{z}}, \quad \text{as } z \to 0, \text{ non-tangentially.}$$

This argument applies everywhere below, we thus make no further comments about $\operatorname{Re} R$ and $\operatorname{Im} R$.

In order to extend this argument to other points of S_0, Hardy proceeded as follows. He slightly modified (2.4) and rather considered the Jacobi-theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{i\pi z n^2} = 2F(z) + 1, \quad \operatorname{Im} z > 0.$$

The advantage of ϑ over F is that it has nice transformation formulas over certain modular transformations. Consider

$$T_1(z) = z+1 \quad \text{and} \quad T_2(z) = -\frac{1}{z}, \quad \operatorname{Im} z > 0.$$

The generate the so-called modular group, which lives invariant the upper half-plane and the real axis and consists of the Möbius transformations

$$\frac{mz+j}{\nu z+l}, \quad m,j,\nu,l \in \mathbb{Z} \text{ and } ml - j\nu = 1.$$

We are interested in the subgroup generated by $T_1^2(z) = z+2$ and T_2, it is called the ϑ-group, denoted as G_ϑ. We have the simple transformations

$$(2.6) \quad \vartheta(T_1^2(z)) = \vartheta(z) \quad \text{and} \quad \vartheta(T_2(z)) = \sqrt{-iz}\,\vartheta(z)$$

The first formula is trival, while the second one follows from the Poisson summation formula [7].

$$(2.7) \quad \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n)$$

applied to $f(t) = e^{i\pi z t^2}$ and $\hat{f}(\omega) = \sqrt{\frac{1}{-iz}}\, e^{\frac{-i\pi w^2}{z}}$.

Let us come back to the proof of Theorem 1. It turns out that S_0 is the orbit of 0 under G_ϑ

and S_1 that of 1. Since

$$(2.8) \qquad \theta(z) \sim \sqrt{\frac{i}{z}}, \quad z \to 0 \text{ non-angularly},$$

as follows from the behavior of $F(z)$ (2.5), one can use the θ-group together with (2.6) to move (2.8) through $S_0 = G_\theta \cdot 0$ and conclude that

$$\theta(x_0 + z) \sim \frac{C_{x_0}}{\sqrt{z}}, \quad z \to 0 \text{ non-tangentially},$$

at any point $x_0 \in S_0$, for some constant $C_{x_0} \neq 0$. The details are left to the reader.

In other to show the non-differentiability at the irrationals, Hardy made use of deeper result of himself and Littlewood. They showed in [13], by considering diophantine approximation of the irrational point x_0 by elements of the set S_0, that there is a constant $M = M_{x_0} > 0$ such that

$$|\operatorname{Im} \theta(x_0 + iy)| > \frac{M_{x_0}}{y^{\frac{1}{2}}}, \quad |\operatorname{Re} \theta(x_0 + iy)| > \frac{M_{x_0}}{y^{\frac{1}{2}}}$$

for infinitely many values of y, as $y \to 0^+$.

This concludes the sketch of the proof of Theorem 1.

Let us finish with a comment. Why does this argument fail in $S_1 = G_\theta \cdot 1$?

Well, we come back to (2.5) for $z = iy$.

Using the Euler-Maclaurin formula, it is easy to improve one of our formulas to

$$F(iy) = \frac{1}{2\sqrt{y}} - \frac{1}{2} + O(1), \quad y \to 0^+.$$

Now,

$$F(1+z) = \sum_{n \text{ even}} e^{i\pi n^2 z} - \sum_{n \text{ odd}} e^{i\pi n^2 z}$$

$$= 2F(4z) - F(z)$$

Thus,
$$F(1+iy) = -1 + \frac{1}{2\sqrt{y}} + \frac{1}{2} - \frac{1}{2\sqrt{y}} + O(1) = -\frac{1}{2}$$

Thus,
$$\sum_{n=1}^{\infty} e^{i\pi n^2} = -\frac{1}{2} \qquad (A).$$

Moving 1 to any point of S_1 with the θ-group and making use of (2.6), one can show that

$$(2.9) \qquad \sum_{n=1}^{\infty} e^{i\pi x_0 n^2} = -\frac{1}{2} \quad (A), \quad \forall x_0 \in S_1 = G_\theta \cdot 1.$$

In view of (2.3), (2.9) suggests that $R'(x_0) = \frac{-i\pi}{2}$, which is the case as shown by Gerver [10].

It should be observed that the arguments applied by Hardy have a marked "Abelian" character. They also tells the "irregularity" of Riemann's function. On the other hand, Theorem 2 is about the regularity of R. Thus, to show that $R'(x_0)=\frac{-i\pi}{2}$ from (2.4) one would have to "reverse" the Abelian result, that is, to have some sort of "Tauberian" machinary for this kind of problems. In [14], Holschneider and Tchamitchian did so by means of Tauberian type theorems for the wavelet transform. The same approach was followed by Jaffard, who gave a complete picture of the properties of R [16]. Some new oscillatory properties were found recently in [22].

3 Hölderian Properties - Oscillations

A more precise measurement of point regularity is provided by the Hölder exponent. Recall that a function f is Hölder continuous at x_0 of order $\alpha \leq 1$ if

$$(3.1) \quad |f(x_0+h)-f(x_0)| = O(|h|^{\alpha}) \text{ as } h \to 0.$$

One writes $f \in C^{\alpha}(x_0)$. This can also be defined for $1<\alpha$, in such a case one replaces $f(x_0)$ in (3.1) by a certain polynomial. One defines the pointwise Hölder exponent of f at x_0 as

$$\alpha_f(x_0) = \sup\{\beta : f \in C^{\beta}(x_0)\}.$$

In 1991, Duistermaat [5] gave an upper bound for the pointwise Hölder exponent of Riemann's function

$$R(x)=\sum_{n=1}^{\infty}\frac{e^{i\pi n^2 x}}{n^2}.$$

Set

(3.2) $\alpha(x_0)=\alpha_R(x_0)=\sup\{\beta : R\in C^{\beta}(x_0)\}.$

In order to get a bound of (3.2), Duistermaat refined the "wavelet method" of Hardy. The result depends on how good x_0 is approximated by convergents of its continued fraction that belong to the set of non-differentiability S_0.

Let $\left\{\frac{p_n}{q_n}\right\}_{n=1}^{\infty}$ be the convergents of x_0. Set

$$\tau(x_0)=\sup\left\{\tau : \left|x_0-\frac{p_n}{q_n}\right|\le\frac{1}{q_n^{\tau}},\ \text{for infinitely many } \frac{p_n}{q_n}\in S_0\right\}$$

i.e., infinitely many such that p_n and q_n are not both odd.

Then, Duistermaat showed:

Theorem 3: $\alpha(x_0)\le\frac{1}{2}+\frac{1}{2\tau(x_0)}$. //

As in Hardy's method, Duistermaat used "Abelian" arguments that led to the "irregularity" of R.

In 1996, Jaffard successfully applied a Tauberian type result for the wavelet transform and finally set an end to the problem of the determination of the pointwise regularity of Riemann's function. He showed

Theorem 4 The pointwise Hölder exponent of Riemann's function is indeed

$$\alpha(x_0) = \frac{1}{2} + \frac{1}{2\tau(x_0)} \quad //$$

The case $x_0 \in S_1 = \left\{ \frac{2j+1}{2m+1} : j, m \in \mathbb{Z} \right\}$ is of interest.

Here Holschneider and Tchamitchian showed that $R \in C^{3/2}(x_0)$. Their result was sharpened by Jaffard and Meyer [17], they showed that R has a chirp at x_0, which describes the oscillations of R:

$$R(x) = U(x) + (x-x_0)^{\frac{3}{2}} T_+\left(\frac{1}{x-x_0}\right), \qquad x > x_0$$

$$R(x) = U(x) + (x-x_0)^{\frac{3}{2}} T_-\left(\frac{1}{|x-x_0|}\right), \qquad x < x_0$$

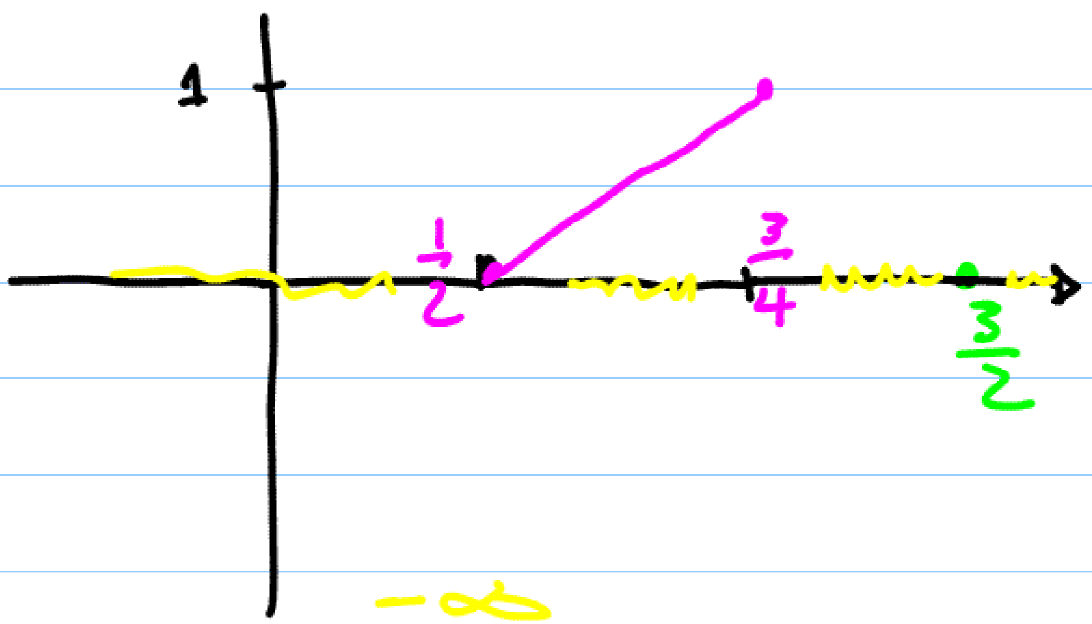
where T_+ and T_- are periodic functions with vanishing integrals (typical examples: sin and cos).

4 Spectrum of Singularities.

The spectrum of singularity of a function f is a function to each $\beta \in \overline{\mathbb{R}}$ associates the Hausdorff dimension of the set of points where f has pointwise Hölder exponent equal to β. Denote by d the spectrum of singularities of Riemann's function. Jaffard showed

Theorem 5 The spectrum of singularities of Riemann's function is

$$d(\alpha) = \begin{cases} 4\alpha - 2, & \text{if } \alpha \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ 0, & \text{if } \alpha = \frac{3}{2} \\ -\infty & ; \text{ elsewhere.} \end{cases}$$



Thus, the Hölder singularities of R are located on a big collection of sets of different dimentions, and R is then a "multi-fractal" function.

Let us recall the definition of Hausdorff dimension. First,

$$M(\varepsilon, d) = \inf \sum_i (\operatorname{diam} A_i)^d$$

where the A_i run over all coverings of A by a numerable

collection of sets with diameter less than ε.
The d-Hausdorff measure of A is

$$\mathrm{Mes}_d(A) = \lim_{\varepsilon \to 0^+} \sup M(\varepsilon, d).$$

Finally, the Hausdorff dimension of A is

$$D(A) = \inf \{ d : \mathrm{Mes}_d(A) = 0 \}$$

$$= \sup \{ d : \mathrm{Mes}_d(A) = \infty \}.$$

$\boxed{5}$ Concluding Remarks

Let us start by pointing out that the existence sets of "smoothness" points for Riemann's function is a consequence of the type of "lacunarity" introduced by n^2, but also choice of coefficients has much to do with this behavior. If one considers

$$\sum_{n=1}^{\infty} \frac{c_n}{n^2} e^{i\pi n^2 x}$$

where $c_n = \pm 1$ is a Ramadacher series [18], then one would obtain a random function which would be almost surely nowhere $C^{\frac{1}{2}}$.

In addition, most of the methods depend on the nice properties of the functions

e^{iu^2}, e^{-u^2}, for example reflected in the automorphic properties of the ϑ function (2.6). If one wishes to study the general case

$$\sum_{n=1}^{\infty} \frac{e^{in^{\alpha}x}}{n^{\beta}}$$

such properties are not available for $e^{iu^{\alpha}}, e^{-u^{\alpha}}$, $\alpha \neq 2$. And hence new methods have to be employed.

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