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Simple Proofs of the No-Where Differentiability for the Weierstrass Function

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$\boxed{0}$ Introduction

In 1872 Weierstrass gave his famous example of a no-where differentiable function.
Let $b \geq a > 1$ and define

$$W(t) = \sum_{n=0}^{\infty} \frac{1}{a^n} \cos b^n t \,, \qquad t \in \mathbb{R}$$

clearly W is continuous over $\mathbb{R}$. Weierstrass showed that W is no-where differentiable if

$$\frac{b}{a} > 1 + \frac{3\pi}{2} \,, \quad b \text{ is an odd integer.}$$

Thereafter, many mathematicians tried to remove this constrains over b and a, but it was Hardy [1] in 1916 who first showed that W is non-differentiable at any point whenever $b \geq a$, the last condition being optimal.

Theorem 1. (Hardy [1])
For any $b \geq a > 1$, the functions

$$W(t) \quad \text{and} \quad S(t) = \sum_{n=0}^{\infty} a^{-n} \sin b^n t$$

are continuous and bounded but have no points of differentiability //

Hardy proof was not simple at all, it took around 15 pages of his paper!

The aim of this talk is to present simple proofs of Theorem 1. The proofs to be given are motivated by a recent article by Johnsen [1] which just appeared this month. We also reinterpret Johnsen's proof within another method developed by R. Estrada and myself in [7], our method leads to stronger results (see Theorem 3 and its corollaries below).

1 Preliminaries on Fourier Transforms

We shall make use of the Fourier transform, defined as

$$\hat{\phi}(u) = \int_{-\infty}^{\infty} \phi(t) e^{iut} \, dt , \quad u \in \mathbb{R},$$

whenever the integral makes sense.
The Fourier transform is very well behaved when acting over the Schwartz class of smooth rapidly decreasing function [5], which we now define.
We say that a smooth function ϕ belongs to the space $S(\mathbb{R})$ if any derivative of ϕ decreases faster than any polynomial, namely, for any $k, m \in \mathbb{N}$

$$\lim_{t\to\infty} \frac{\phi^{(k)}(t)}{(1+|t|)^m} = 0 .$$

Using the rapid decrease of the elements of $S(\mathbb{R})$, it is not difficult to show [5] that $\phi \in S(\mathbb{R}) \Rightarrow \hat{\phi} \in S(\mathbb{R})$. Moreover, the Fourier transform has inverse

$$\phi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(u) e^{-iut}\, du , \quad \phi \in S(\mathbb{R}).$$

We remark that:

(1.1) The Fourier transform is an isomorphism on $S(\mathbb{R})$.

$\boxed{2}$ Simple Direct Proof of Theorem 1

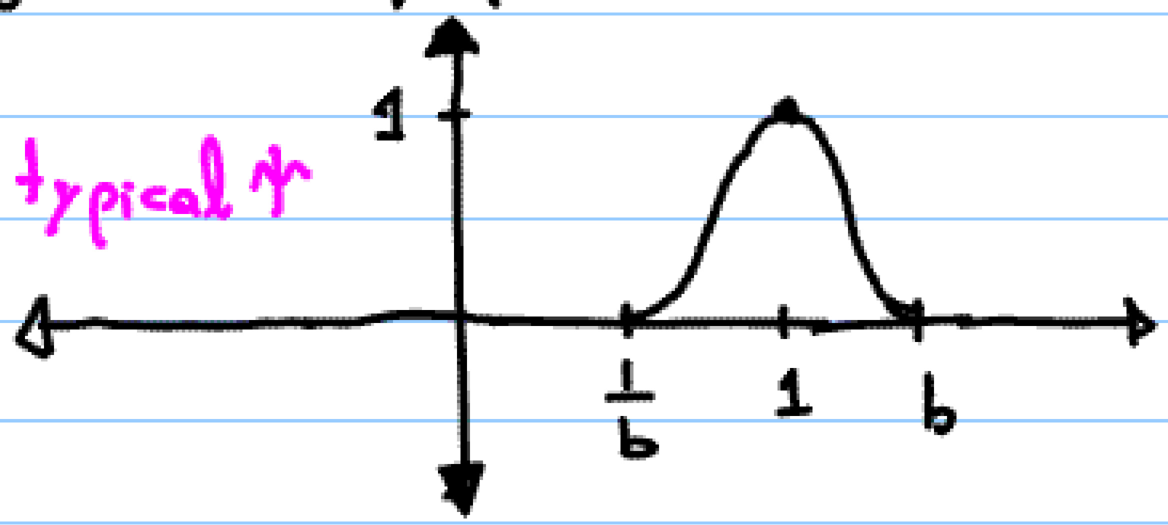
We first make some preparations for the proof. We choose a smooth function ψ such that:

(2.1) $\psi(u) = 0$ if $u \notin (\frac{1}{b}, b)$; $\psi(1) = 1$

Remark 1: A typical ψ is

$$\psi(u) = \begin{cases} \psi(u) = 0 & \text{if } u \notin [b^{-1}, b] \\ \psi(u) = \exp\left((u-b)^{-1}(b^{-1}-u)^{-1} + bb^{-1}\right) \end{cases}$$

Graph of typical ψ



Clearly $\psi \in S(\mathbb{R})$ and by (1.1)

(2.2) $\exists \phi \in S(\mathbb{R}): \psi(u) = \int_{-\infty}^{\infty} \phi(t) e^{iut} dt = \hat{\phi}(u)$

Observe also that

(2.3) $\psi(b^{n-k}) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$

(2.4) $\int_{-\infty}^{\infty} t \phi'(t) dt = \phi(t)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} \phi(t) dt = 0 - \hat{\phi}(0) = -\psi(0) = 0$

For didactical purposes, we analyze first the no-where differentiability of the function:

(2.5) $G(t) = \sum_{n=0}^{\infty} \frac{1}{a^n} e^{ib^n t}$; $b \geq a > 1$

$\boxed{2.1}$ Proof of the Non-Differentiability of G

The proof follows easily from the following formula. Here $t_0 \in \mathbb{R}$ is a fixed number.

Lemma 1 For ψ defined in (2.1), we have

(2.6) $a^{-k} b^k e^{ikt_0} = i \int_{-\infty}^{\infty} t \phi'(t) \frac{G(t_0 + b^{-k} t) - G(t_0)}{t b^{-k}} dt$

Proof:

By (2.2), (2.3), and (2.4):

$$a^{-k} b^k e^{ib^k t_0} = \sum_{n=0}^{\infty} a^{-n} b^n e^{ib^n t_0} \psi(b^{n-k})$$

$$= \sum_{n=0}^{\infty} a^{-n} b^n \int_{-\infty}^{\infty} \phi(t) e^{i(b^n t_0 + b^{n-k} t)} dt \quad \left(\begin{array}{c}\text{integrating} \\ \text{by parts}\end{array}\right)$$

$$= b^{-k} i \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \phi'(t)\, a^{-n} e^{i(b^n(b^{-k}t + t_0))} dt$$

$$= i \int_{-\infty}^{\infty} \phi'(t) \frac{G(t_0 + b^{-k}t) - G(t_0)}{b^{-k}} dt \quad \left(\begin{array}{c}\text{interchange} \\ \Sigma \text{ and } \int, \text{ and} \\ \text{then apply} \\ (2.4)\end{array}\right)$$

Let us now proof the non-differentiability of G at t_0:

Assume the contrary; namely, $G'(t_0)$ exists.

Taking $k \to \infty$ in the right hand side of (2.6), we see that

$$\left(\frac{b}{a}\right)^k e^{ikt_0} \longrightarrow i G'(t_0) \int_{-\infty}^{\infty} t \phi'(t)\, dt = 0$$

in contradiction to our assumption $\left(\frac{b}{a}\right) \geq 1$

Remark 2:

At a first look Lemma 1 seems merely a tricky formula; moreover, the introduction of ϕ and ψ is apparently artificial as well. However, in $\boxed{4}$ we give a natural interpretation of them.

$\boxed{2.2}$ Proof of No-Where Differentiability for W and S

We need the following lemmas.

Lemma 2:

$$(2.7) \quad a^{-k} b^k e^{ib^k t_0} = 2i \int_{-\infty}^{\infty} t\phi'(t) \frac{W(t_0 + b^{-k}t) - W(t_0)}{b^{-k}t} dt$$

Proof: Since $\psi(u) = 0$ for $u < 0$

$$a^{-k} b^k e^{ib^k t_0} = \sum_{n=0}^{\infty} a^{-k} b^k \left(\psi(b^{k-n}) e^{ib^n t_0} - \psi(-b^{k-n}) e^{-ib^n t_0}\right)$$

$$= \sum_{n=0}^{\infty} a^{-n} b^{n} \int_{-\infty}^{\infty} \phi(t) \left(e^{i(b^n t_0 + b^{k-n} t_0} - e^{-i(b^n t_0 + b^{k-n} t_0)} \right) dt$$

$$= 2i \sum_{n=1}^{\infty} a^{-n} b^{n} \int_{-\infty}^{\infty} \phi(t) \sin(b^n t_0 + b^{k-n} t) \, dt = \cdots$$

and continue as in the proof of Lemma 1 . . .

Lemma 2*:

$$a^{-k} b^{k} e^{i b^k t_0} = -2 \int_{-\infty}^{\infty} t \phi'(t) \frac{S(t_0 + b^{-k} t) - S(t_0)}{b^{-k} t} dt$$

Proof:

$$a^{-k} b^{k} e^{i b^k t_0} = \sum_{n=0}^{\infty} a^{-n} b^{n} \left(\psi(b^{k-n}) e^{i b^n t_0} + \psi(-b^{k-n}) e^{-i b^n t_0} \right)$$

$$= 2 \sum_{n=1}^{\infty} a^{-n} b^{n} \int_{-\infty}^{\infty} \phi(t) \cos(b^n t_0 + b^{k-n} t) \, dt = \cdots$$

The no-where differentiability of W and S follows now as in the last three lines of the corresponding result for G

3 Lacunary Series.

We may show a more general result for series with exponential gaps.

Definition 1 Let $\{b_n\}_{n=0}^{\infty}$ be an increasing sequence of positive numbers. We say $\{b_n\}_{n=0}^{\infty}$ is lacunary (in the sense of Hadamard) if $\exists \alpha$ and $n_0 \in \mathbb{N}$

$$1 < \alpha < \frac{b_{n+1}}{b_n}, \quad \forall n \geq n_0.$$

Remark 3: If $n > n_0$, $b_n > \alpha^{n-n_0} b_{n_0}$, therefore, $b_n \longrightarrow \infty$, as $n \to \infty$. So, we may assume

Remark 4: Observe that given λ sufficiently large ($\lambda > \alpha b_j$ for all $j \leq n_0$ is enough)

There exist at most one n_λ such that

$$\text{(3.1)} \qquad \frac{1}{\alpha^{\frac{1}{2}}} \leq \frac{b_{n_\lambda}}{\lambda} < \alpha^{\frac{1}{2}}$$

Proof: Suppose that n and $n+k$ have both this property.

Then $n > n_0$. So we would have

$$b_{n+k} < \lambda \alpha^{\frac{1}{2}} \text{ while } \frac{1}{b_n} \leq \frac{\alpha^{\frac{1}{2}}}{\lambda}$$

$$\Rightarrow \alpha^k \leq \frac{b_{n+k}}{b_n} < \alpha \quad \Rightarrow\Leftarrow$$

Bearing in mind Remark 4, we notice that for k large enough

$$\text{(3.2)} \qquad \psi\left(\frac{b_n}{b_k}\right) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases},$$

where ψ is defined as in (2.1) but with $b = \alpha^{\frac{1}{2}}$.
So, proceeding indentically as in [2], we obtain

Theorem 2: Let $\{b_n\}_{n=0}^{\infty}$ lacunary and $\{a_n\}_{n=0}^{\infty}$ a sequence of complex numbers such that

$$\sum_{n=0}^{\infty} |a_n| < \infty .$$

If

$$b_n |a_n| \not\longrightarrow 0 \quad \text{as } n \to \infty,$$

then $\tilde{G}(t) = \sum_{n=0}^{\infty} a_n e^{ib_n t}$, $\tilde{W}(t) = \sum_{n=0}^{\infty} a_n \cos b_n t$

and $\tilde{S}(t) = \sum_{n=0}^{\infty} a_n e^{ib_n t}$ are continuous but no-where differentiable

[4] A Generalized Function Approch

In [6], we gave a simple proof of Theorem 2, We actually showed something stronger (see Theorem 3 below). Our methods were based on the theory of Schwartz distributions [5]
I present here a more elementary approach without even making reference to distributions; though, we will mimic them with the introduction of a more modest space, We point out the idea of elementary treatments of generalized functions is in the spirit of Silva's axiomatic approach to distributions. We also use ideas from Łojasiewicz [4]

I will restrict my attention to series of type

$$\sum_{n=0} c_n e^{ib_n t} ;$$

but similar arguments can be easily adapted (as we did in [2]) to show the corresponding results for series of cosine or sine type.

4.1 A Space of Formal Series

We always assume that $B=\{b_n\}_{n=0}^{\infty}$ is an increasing sequence of positive numbers having the property: $\exists m \in \mathbb{N}$ such that

$$(*) \qquad \sum_{n=0}^{\infty} \frac{1}{b_n{}^m} < \infty$$

Definition 1 A formal series

$$f(t) = \sum_{n=0}^{\infty} c_n e^{ib_n t}$$

is said to belong to the space $T_B(\mathbb{R})$ if $\exists l$ and $M>0$ such that

$$|c_n| \leq M b_n^l \quad , \forall n$$

Notice that $f(t)$ can be very divergent. So f does not necessarily come from a classical function. On the other hand, if $\sum |c_n| < \infty$, then the series is absolutely convergent and we identify $f(t)$ with the continuous and bounded function represented by the series.

Observed also that $T_B(\mathbb{R})$ is a vector space.

We define the following operations

$\boxed{4.1.1}$ **Translation and Dilation** if $f \in T_B(\mathbb{R})$, $t_0 \in \mathbb{R}$, $\varepsilon \in_d \mathbb{R}$,

$$f(t_0+\varepsilon t) = \sum_{n=0} e^{it_0 b_n} c_n e^{i\varepsilon b_n t} \in T_{\varepsilon B} \; ; \; \varepsilon B = \{\varepsilon b_n\}$$

$\boxed{4.1.2}$ **Derivatives:** $f'(t) = \sum_{n=0}^{\infty} i c_n b_n e^{i b_n t}$

if $f(t) = \sum_{n=0}^{\infty} c_n e^{ib_n t}$; in particular any f is the K-derivative of a formal series which represent a continuous and bounded function F, namely,

$$F(t) = (-i)^k \sum_{n=0}^{\infty} \frac{c_n}{b_n^k} e^{ib_n t}, \quad F^{(k)}(t) = f(t)$$

where $k = l+m$ as in Definition 1 and $(*)$

$\boxed{4.1.3}$ **Action of $T(\mathbb{R})$ on $S(\mathbb{R})$**

Given $f(t)$ we define its action on $\phi \in S(\mathbb{R})$ by

$$\langle f(t), \phi(t) \rangle = \sum_{n=0}^{\infty} c_n \int_{-\infty}^{\infty} \phi(t) e^{ib_n t} dt. \tag{4.1}$$

So, $\langle \; , \; \rangle : T_B(\mathbb{R}) \times S(\mathbb{R}) \longrightarrow \mathbb{C}$ is bilinear.

This allows one to see $T_B(\mathbb{R}) \subseteq S^*(\mathbb{R})$, the algebraic dual space of $S(\mathbb{R})$

Example 1: $\langle f(t_0+\varepsilon t), \phi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{ib_n t_0}\hat{\phi}(\varepsilon b_n)$.

Lemma 3: $\langle f'(t), \phi(t)\rangle = -\langle f(t), \phi'(t)\rangle$

Proof: Integrate by parts the integral in the definition (4.1).

Example 2 If $\sum |c_n| < \infty \Rightarrow \langle f(t), \phi(t)\rangle = \int_{-\infty}^{\infty} \phi(t) f(t)\, dt$

4.1.4 Point values

Given $t_0 \in \mathbb{R}$ and $f \in \mathcal{T}_B(\mathbb{R})$, we say the generalized point value of $f(t_0) \in \mathbb{C}$ exists if for each $\phi \in S(\mathbb{R})$

$$\lim_{\varepsilon\to 0} \langle f(t_0+\varepsilon t), \phi(t)\rangle = \int_{-\infty}^{\infty} \phi(t)\, dt. \tag{4.2}$$

Example 2: Let $g(t) = \sum_{n=0}^{\infty} a_n e^{ib_n t_0}$,

where $\sum_{n=0}^{\infty} |a_n|$, thus, g is continuous everywhere.

If g is differentiable at $t_0 \in \mathbb{R}$, then the generalized point value of the formal series $g'(t)$, in the sense of (4.2), coincides with $g'(t_0)$. **Proof**

$$\lim_{\varepsilon\to 0^+} \langle g'(t_0+\varepsilon t), \phi(t)\rangle = \lim_{\varepsilon\to 0^+} -\frac{1}{\varepsilon}\int_{-\infty}^{\infty} g(t_0+\varepsilon t)\phi'(t)dt$$

$$= -\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} \frac{g(t_0+\varepsilon t)-g(t_0)}{\varepsilon t} t\phi'(t)dt = g'(t_0)\int_{-\infty}^{\infty}\phi(t)dt$$

Remark 5: An arbitrary $f\in T_B(\mathbb{R})$ may not have generalized point values at any $t_0\in\mathbb{R}$.

$\boxed{4.2}$ A formula related to Point values and Lemma 1

Lemma 4: Suppose that $f\in T_B(\mathbb{R})$ then $f(t_0)$ exists $\Longleftrightarrow$ for each $\varphi\in S(\mathbb{R})$

$$\lim_{\lambda\to\infty}\sum_{n=0}^{\infty} c_n e^{ib_n t_0}\varphi\left(\frac{b_n}{\lambda}\right) = f(t_0)\varphi(0) \qquad (4.3)$$

Proof: By definition and Example 1, $f(t_0)$ exists iff

$$\lim_{\lambda\to\infty}\sum_{n=0}^{\infty} a_n e^{ib_n t_0}\hat{\phi}(\varepsilon b_n) = f(t_0)\hat{\phi}(0).$$

changing $\varepsilon\longleftrightarrow\frac{1}{\lambda}$ and $\hat{\phi}\longleftrightarrow\varphi$ we get (4.5)

Formula (4.3) and (3.2) fully justify now why the arguments of $\boxed{2}$ and $\boxed{3}$ work!

Indeed, just replace f by $\tilde{G}$ in (4.3), that is, $c_n = a_n b_n$; use then Example 3 an choose

↑ as in (3.2) (with $b = \alpha^{\frac{1}{2}}$, where $b_{n+1} > \alpha b_n$).

4.3 Characterization of Point Values of Lacunary Series.

Theorem 3: Let $\{b_n\}_{n=0}^{\infty}$ lacunary and

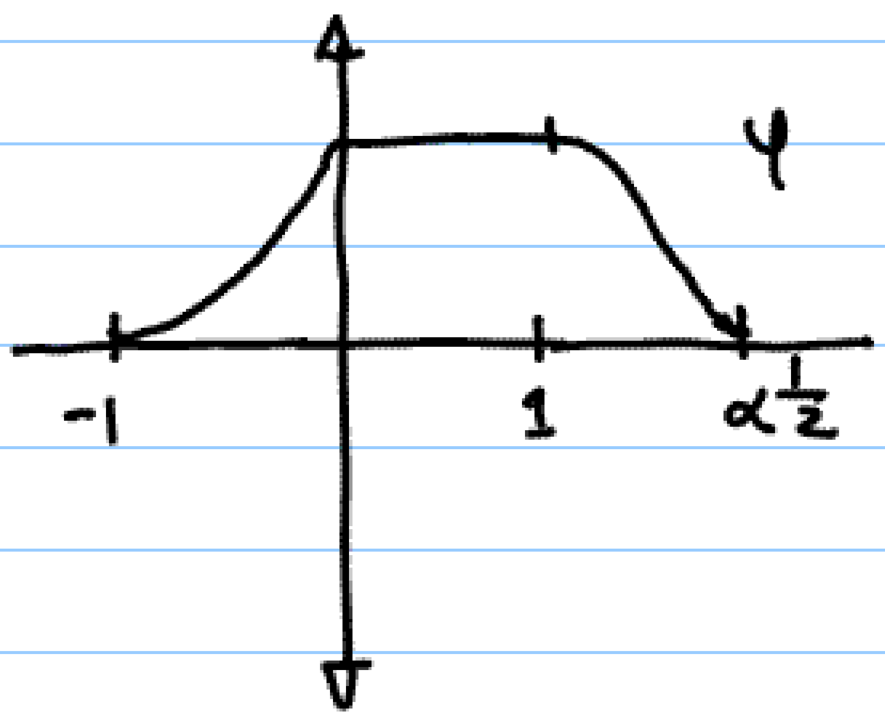
$$f(t) = \sum_{n=0}^{\infty} c_n e^{ib_n t} \in T_B(\mathbb{R}) ;$$

then the generalized pt. value $f(t_0)$ exists iff

$$\sum_{n=0}^{\infty} c_n e^{ib_n t_0} = f(t_0)$$

Proof: By $\boxed{4.2}$ $f(t_0)$ exists iff (4.3) holds
Let α such that $\alpha \le \frac{b_{n+1}}{b_n}$ now, we can select $\varphi \in S(\mathbb{R})$ such that

$$\varphi(u) = 1 \ , \ u \in [0,1] \text{ and } \varphi(u) = 0 \text{ if } u \notin [-1, \alpha^{\frac{1}{2}}]$$



So using $\lambda = n_k$ in (4.3) and using Remark 4, we get

$$f(t_0) = \phi(0) f(t_0) = \lim_{k\to\infty} \sum_{n=0}^{\infty} c_n e^{ib_n t_0} \varphi\left(\frac{b_n}{b_k}\right)$$

$$= \lim_{k\to\infty} \sum_{n=0}^{k} c_n e^{ib_n t_0} + \sum_{1<\frac{b_n}{b_k}<\alpha} c_n e^{ib_n t_0}$$

$$= \lim_{k\to\infty} \sum_{n=0}^{k} c_n e^{ib_n t_0}$$

Corollary 1: Let $f(t) = \sum_{n=0}^{\infty} c_n e^{ib_n t}$, with $\{b_n\}_{n=0}^{\infty}$ lacunary:
if $c_n \not\to 0 \Rightarrow f(t_0)$ does not exist.

Corollary 2: Let $\{b_n\}_{n=0}^{\infty}$ lacunary and $\{a_n\}_{n=0}^{\infty}$ a sequence of complex numbers such that

$$\sum_{n=0}^{\infty} |a_n| < \infty .$$

Set

$$g(t) = \sum_{n=0}^{\infty} a_n e^{ib_n t} .$$

If

$$b_n |a_n| \not\to 0 \text{ as } n \to \infty,$$

then the generalized point value $g'(t_0)$ does not exist at any point $t_0 \in \mathbb{R}$, in the sense of (4.2). In particular, g is continuous and bounded but no-where differentiable.

Proof: $g'(t) = \sum i a_n b_n e^{ib_n t}$, apply Corollary 1 with $c_n = i a_n b_n$.

5 Final Remarks

For another simple proof of Theorems 1 and Theorem 2, the reader can consult the book by Holschneider [2]. The proof given there is based on ideas from wavelet analysis; however, a close examination of the arguments shows that the wavelet approach is nothing but the approach followed here with simply a different language.

One can also study Hölder continuity properties of the Weierstrass function, see for instance [1, 2, 3, 7], or simply use a modification of the method presented here!

References:

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