Spaces of functions with nearly optimal time-frequency decay

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1 Introduction

In this lecture we are interested in functions $f:\mathbb{R}\to\mathbb{C}$ such that they and their Fourier transform are simultaneously small at ∞, in the sense that $\left(\hat{f}(\xi)=\int_{-\infty}^{\infty}f(x)e^{-i\xi x}dx\right)$

$$(1)\quad |f(x)|\lesssim e^{-rx^2} \quad\text{and}\quad |\hat{f}(\xi)|\lesssim e^{-r\xi^2}.$$

A classical uncertainty principles of Hardy tells us that r cannot be arbitrary if we would like f to be non-trivial:

Theorem 1 (Hardy, 1933) If (1) holds with $r>\frac{1}{2}$, then $f\equiv 0$. Furthermore, if (1) holds with $r=\frac{1}{2}$, then $f(x)=C\cdot e^{-\frac{x^2}{2}}$ for some $C\in\mathbb{C}$. //

Proof. This result is directly derive from the Phragmén-Lindelöf principle. Suppose first that (1) holds with $r=\frac{1}{2}$. By Fourier inversion, the bound $|\hat{f}(\xi)|\lesssim e^{-\frac{\xi^2}{2}}$ implies that f extends to an entire function

$$f(z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\xi)e^{iz\cdot\xi}d\xi,$$

so that
$$|f(x+iy)| \lesssim \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2} - y\cdot\xi}\, d\xi \lesssim e^{\frac{y^2}{2}} \lesssim e^{\frac{|z|^2}{2}}$$
So, $F(z) = f(z)\cdot e^{\frac{z^2}{2}}$ is bounded on both the imaginary and real axis. Tweaking a bit the Phragmén-Lindelöf principle, one deduces that F is bounded on $\mathbb{C}$ and Liouville's theorem implies that F is constant. If (1) holds with $r > \frac{1}{2}$, we must have $f(z) = 0$. ////

Due to this limitation imposed by Hardy's uncertainty principle, if we want to obtain a non-trivial function space out of (1), it is natural to consider the Fréchet space
$$\mathcal{H}(\mathbb{R}) = \{ f : (1) \text{ holds } \forall\, 0<r<\tfrac{1}{2} \}.$$
Clearly $\mathcal{H}(\mathbb{R}) \subset S(\mathbb{R})$, the Schwartz space of rapidly decreasing smooth functions. It is rich enough, it contains e.g. every Hermite function
$$h_n(x) = (-1)^n \pi^{-\frac{1}{4}} (2^n n!)^{-\frac{1}{2}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}[e^{-x^2}].$$
A related space is the Gelfand-Shilov space
$$S_{\frac{1}{2}}(\mathbb{R}) = \{ f : (1) \text{ holds for some } r>0 \}$$

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1 Sketch of proof from e.g. Folland and Sitaram, The Uncertainty Principle: A Mathematical Survey, J. Fourier Anal. App. (1997).

Interestingly, $S_{\frac{1}{2}}(\mathbb{R})$ admits a characterization in terms of Hermite coefficients[2]. We recall that every $f \in L^2(\mathbb{R})$ can be written as

$$f = \sum H(f,n) h_n, \quad H(f,n) = (f, h_n)_{L^2(\mathbb{R})}$$

Theorem 2 (Zhang, 1963) Let $f \in L^2(\mathbb{R})$

$$f \in S_{\frac{1}{2}}(\mathbb{R}) \Leftrightarrow \exists \lambda > 0 \text{ s.t. } |H(f,n)| \lesssim e^{-\lambda n}.$$

It is natural to ask whether a similar Hermite coefficient characterization holds for $\mathcal{H}(\mathbb{R})$. In fact, it is been recently conjectured that

Conjecture 1 (Gumber, Toft, 2023)

$$f \in \mathcal{H}(\mathbb{R}) \Leftrightarrow \forall \lambda > 0 \quad |H(f,n)| \lesssim e^{-\lambda n}$$

Gumber and Toft actually showed a weaker statement in terms of the fractional Fourier transform

$$\forall \lambda > 0, \ |H(f,n)| \lesssim e^{-\lambda n} \Leftrightarrow \forall t, \forall r \in (0, \tfrac{1}{2}) \ |\mathcal{F}_r f(x)| \lesssim e^{-t x^2}.$$

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[2] This can be very much generalized, see Vučković, V. J. Pseudo-Diff. Oper. Appl. (2016). In particular if $P(x,D_x)$ is a (normal) elliptic polynomial and $\{\varphi_n\}$ is an orthonormal basis of eigenfunctions of $P(x,D_x)$, then $f = \sum c_n \varphi_n \in L^2$ satisfies $f \in S_{\frac{1}{2}}(\mathbb{R}) \Leftrightarrow \exists \lambda > 0 \text{ s.t. } |c_n| \lesssim e^{-\lambda n}$.

2 The characterization of $\mathcal{H}(\mathbb{R})$

We have recently given an affirmative answer to Conjecture 1. Our result passes throught the STFT, the Bargmann transform, and careful application of a version of the Phragmén-Lindelöf principle.

We fix the Gaussian window ϕ, normalized as

$$\phi(x) = h_0(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}.$$

Given $f \in L^2(\mathbb{R})$, its STFT is

$$V_\phi f(x,\xi) = \int_{-\infty}^{\infty} f(x)\phi(x-t) e^{-ix\xi} dx, \quad (x,\xi) \in \mathbb{R}^2$$

while the Bargmann transform of f is the entire function

$$\mathfrak{B}f(z) = \pi^{-\frac{1}{4}} \int_{-\infty}^{\infty} f(t) e^{-\frac{1}{2}(z^2+t^2)+\sqrt{2}\, t\cdot z} dt, \quad z \in \mathbb{C}.$$

These integral transforms are connected by

$$V_\phi(t,\xi) = e^{-\frac{1}{4}(t^2+\xi^2)} e^{-\frac{i}{2}t\xi} \mathfrak{B}\left(\frac{\bar{z}}{\sqrt{2}}\right), \quad z = t + i\xi \in \mathbb{C}.$$

We then have the following result.

Theorem 3 (Neyt, Toft, V., 2024) Let $f \in L^2(\mathbb{R})$. The following are equivalent.

i) $f \in \mathcal{H}(\mathbb{R})$

ii) $(\forall \lambda > 0)\left(|H(f,h)| \lesssim e^{-\lambda h}\right)$

iii) $(\forall r>0)\left(|B(z)| \lesssim e^{r|z|^2}\right)$

iv) $(\forall r>0)\left(|B(z)| \lesssim e^{\frac{x^2}{2}+ry^2} \text{ and } |B(z)| \lesssim e^{\frac{y^2}{2}+rx^2}\right)$.

Comments:

① ii) $\Leftrightarrow$ iii) is standard to establish, making use of $Bh_n(z)=\frac{z^n}{\sqrt{n!}}$. iii) $\Rightarrow$ iv) is of course trivial.

② iii) $\Rightarrow$ i) is based on the observation

$$\pi^{-\frac{1}{4}} e^{-\frac{|x|^2}{8}} f\left(\frac{x}{2}\right) = f\left(\frac{x}{2}\right)\phi\left(x-\frac{x}{2}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_\phi f(x,\xi) e^{\frac{ix\xi}{2}} d\xi.$$

③ i) $\Rightarrow$ iv) is an easy computation, but this intermediate step makes clear that the actual problem is being able to deduce the stronger bound iii) from iv).

④ The idea to show iii) from iv) is that if $0<\theta<\frac{\pi}{2}$, then $|B(z)| \lesssim e^{\frac{x^2}{2}+\frac{ry^2}{2}}$ would lead to iii) for all θ sufficiently close to $\frac{\pi}{2}$. On $0<\arg z<\theta$ for a fixed θ, one can use Phragmen-Lindelöf after a tweaking.

Let us indeed make this more precise. Let $r<1$ and let θ_0 such that if $\theta_0 \le \theta \le \frac{\pi}{2}$, $\cos\theta \le r$ and $\frac{1}{2}\sin\theta$ the bound $|B(z)| \lesssim e^{\frac{x^2}{2} + \frac{r}{2}y^2}$ implies $(z = |z|e^{i\theta})$

$$|B(z)| \lesssim e^{\frac{|z|^2\cos^2\theta}{2} + \frac{r}{2}|z|^2} \lesssim e^{r|z|^2}$$

on $\theta_0 \le \arg z \le \frac{\pi}{2}$. On the sector $0 \le \arg z \le \theta_0$ $\left(= \frac{\pi}{\sigma},\ \sigma > 2\right)$, the bounds $|B(z)| \lesssim e^{\frac{1}{2}|z|^2}$, and

$$B(|z|e^{i\theta_0}) \lesssim e^{r|z|^2},$$

$$B(x) \lesssim e^{r|z|^2},$$

allow us to use the classical Phragmén-Lindelöf principle on sectors

3 <u>Subspaces of $\mathcal{H}(\mathbb{R})$</u>. Let $\omega : [0,\infty) \to [0,\infty)$ be a weight function ($=$ nondecreasing) such that $\omega(t) = O(t^2)$. We consider the subspace $\mathcal{H}_\omega(\mathbb{R})$ of $\mathcal{H}(\mathbb{R})$ consisting of f such that $\forall r > 0$

$$|f(x)| \lesssim e^{-\frac{x^2}{2} + r\,\omega(|x|)} \quad \text{and} \quad |\hat{f}(\xi)| \lesssim e^{-\frac{\xi^2}{2} + r\,\omega(|\xi|)}.$$

<u>Theorem 4</u> (Neyt, Toft, V., 2024).
Suppose that the weight function ω satisfies that $\varphi(t) := \omega(e^t)$ is convex and

$$\int_x^\infty \frac{\omega(t)}{t^3}\,dt = O\left(\frac{\omega(x)}{x^2}\right), \quad x \to \infty.$$

Then $f \in L^2(\mathbb{R})$ satisfies

$$f \in \mathcal{H}_\omega(\mathbb{R}) \Leftrightarrow (\forall \lambda > 0)\left(|H(f,n)| \lesssim \sqrt{n!}\, e^{-\frac{1}{\lambda}\varphi^*(\lambda n)}\right)$$

<u>Examples:</u> The following choices of ω lead to the equivalence between $f \in \mathcal{H}_\omega(\mathbb{R})$ and the following decay for Hermite coefficients.

(1) $\omega_s(t) = \log(1+t)^{\frac{1}{1-2s}}, \quad 0 \le s < \frac{1}{2}$

$$\longrightarrow \quad |H(f,n)| \lesssim e^{-\lambda n^{\frac{1}{2s}}}$$

(2) $\omega_\sigma(t) = t^{\frac{2\sigma}{\sigma+1}} \longrightarrow |H(f,n)| \lesssim \dfrac{1}{\lambda^n (n!)^{\frac{1}{2\sigma}}}$.

$\boxed{4}$ A weighted Phragmén-Lindelöf principle

Let $S(\alpha,\beta) = \{ z : \alpha \le \arg z < \beta \}$. We set $\rho = \beta - \alpha$.

Suppose that ω is a weight function that satisfies

(α) $\omega(2t) = O(\omega(t))$

(β_σ) $\displaystyle\int_1^\infty \frac{\omega(t)}{t^{\sigma+1}}\,dt < \infty.$ $\sigma > 0$

<u>Theorem 5</u>: The following conditions are equivalent:

(I) $\displaystyle\int_x^\infty \frac{\omega(t)}{t^{\sigma+1}}\,dt = O\left(\frac{\omega(x)}{x^\sigma}\right)$

(II) There is $A > 0$ s.t. for any sector with $0 < \rho \le \frac{\pi}{\sigma}$ and $\lambda > 0$: $\exists B > 0$ if F is analytic and $S(\alpha,\beta)$ with continuous extension to $\overline{S(\alpha,\beta)}$ s.t.

$$|F(z)| \lesssim_\varepsilon e^{\varepsilon |z|^\sigma}, \quad z \in S(\alpha,\beta), \ \forall \varepsilon > 0.$$

and

$$|F(z)| \le M \cdot e^{\lambda \omega(|z|)}, \quad z \in \partial S(\alpha,\beta),$$

then

$$|F(z)| \le B \cdot M\, e^{A \lambda \omega(|z|)}$$

Here A and B are absolute: $A = A_{\sigma,\omega}$ and $B = B_{\sigma,\omega,\lambda}$.