On the density hypothesis for L-functions associated with holomorphic cusp forms.

(by Jasson Vindas, 10-4-2024)

1 The Riemann hypothesis:

The distribution of prime numbers is intimately connected with the properties of the Riemann zeta function, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \operatorname{Re} s > 1,$$

and extended by analytic continuation as a meromorphic function with a single simple pole at $s=1$, so that

$$\zeta(s) - \frac{1}{s-1} \text{ is entire.}$$

The Riemann hypothesis claims that the only zeros of ζ are located at:

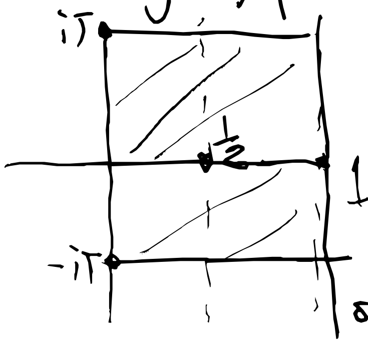
① $s = -2, -4, -6, \ldots$ (trivial zeros)

② $\operatorname{Re} s = \frac{1}{2}$.

The claim ① follows from the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s),$$

while $\{ s \in \mathbb{C} : \operatorname{Re} s = \frac{1}{2} \}$ is the so-called critical line. It was shown by Hardy that ζ has infinitely many zeros on this critical line. In fact, it is even known that the number of zeros $N(T)$ of ξ lying on the critical strip $0 < \operatorname{Re} s < 1$ and having imaginary part $|\operatorname{Im} s| \leq T$



is $$N(T) = \frac{T}{\pi} \log \frac{T}{2\pi e} + O(\log T),$$

$$T \to \infty,$$

a formula stated by Riemann (1859) and shown by von Mangoldt (1905).

The RH is one of the greatest unsolved* problems in mathematics. It is equivalent to the following quantified form of the PNT. Let $\pi(x)$ be the number of prime numbers $\leq x$. Then RH holds $\Leftrightarrow$ for each $\varepsilon > 0$

$$\pi(x) = \int_2^x \frac{du}{\log u} + O_\varepsilon\left(x^{\frac{1}{2} - \varepsilon}\right).$$

Here are some of the records for zero free regions of ξ

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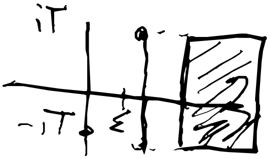
* David Hilbert once said: If I would wake up after sleeping 1000 years, my first question would be: has RH already been proved?

zero free region $s=\sigma+it$.	Record
$\sigma \geq 1 - \dfrac{c}{\log(1+\lvert t\rvert)}$	de la Vallée-Poussin 1897
$\sigma \geq 1 - \dfrac{\log\log(e+\lvert t\rvert)}{\log(1+\lvert t\rvert)}$	Littlewood 1922
$\sigma \geq 1 - \dfrac{c}{(\log(1+\lvert t\rvert))^{2/3}(\log\log(e+\lvert t\rvert))^{1/3}}$	Vinogradov-Korobov 1958 (Best known)

$\boxed{2}$ Zero-density estimates: There is also a great interest in zero-density estimates. Denote as $N(\sigma,T)$ the number of zeros of ζ on the rectangle $|\mathrm{Im}\, s|<T$ and $\sigma \leq \mathrm{Re}\, s \leq 1$



In 1937, Ingham connected estimates of the form

$$(1)\quad N(\sigma,T) \ll_\varepsilon T^{c(1-\sigma)+\varepsilon}, \quad \frac{1}{2} \leq \sigma \leq 1,$$

with the behavior of primes in short intervals. Using (1) and the so-called exact von Mangoldt formula**

$$\sum_{x \leq p \leq x+h} \log p \sim h \quad , \quad x \to \infty , \quad h >> x^{1+\varepsilon-\frac{1}{c}} .$$

This estimate when $c=2$ is of the same quality as if one would assume the RH. Since this is the case for several other arithmetic results, the (conjectural) inequality

$$(DH) \quad N(\sigma,T) <<_\varepsilon T^{2(1-\sigma)+\varepsilon} \qquad \tfrac{1}{2} \leq \sigma \leq 1$$

is known as the density hypothesis.

$\boxed{3}$ The density hypothesis for the Riemann zeta function.

The (DH) is out of reach of current methods. Nevertheless there has been a lot of progress maximizing its range of validity. Let $\sigma_0 \geq \frac{1}{2}$ be such that

$$N(\sigma,T) <<_\varepsilon T^{2(1-\sigma)+\varepsilon} \qquad \sigma_0 \leq \sigma \leq 1.$$

We have the following results:

** If ψ is the Chebyshev function,

$$\psi(x) = x - \sum_{\zeta(\rho)=0} \frac{x^\rho}{\rho} - \log(2\pi)$$

σ_0	Record
0.9	Montgomery, 1969
$0.8333\ldots$	Huxley, 1972
$0.8076\ldots$	Ramachandra, 1975
0.8	Huxley, 1975
$11/14 = 0.7857\ldots$	Jutila, 1977
$25/32 = 0.78125$	Bourgain, 2000

$\boxed{4}$ Dirichlet L-functions. Let χ be a Dirichlet character mod q, that is,

$\chi : \mathbb{N} \longrightarrow \mathbb{C}$ such that

1. χ is completely multiplicative
2. χ is q-periodic
3. $\chi(n) = 0 \Leftrightarrow (n, q) \neq 1$.

It is associated L-function is

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

Let $N_\chi(\sigma,T)$ be the number of zeros in

$$[\sigma, 1]+i[-T,T].$$

The density hypothesis estimate

$$(2) \qquad \sum_{\chi \bmod q} N_\chi(\sigma,T) \ll (qT)^{2(1-\sigma)+\varepsilon}, \qquad \sigma_0\leq\sigma\leq 1$$

has also been very much studied for L-functions. The best record is due to Heath-Brown (1979)

<u>Theorem</u> (2) holds with $\sigma_0 = 15/19 = 0.7894\ldots$ //

We note that Bourgain (2002) applied his so-called dicothomy method (developed to get his current best record for the Riemann zeta function) to Dirichlet L-functions, but this time he only could recover $\sigma_0 = 15/19$.

$\boxed{5}$ <u>L-functions associated with holomorphic cusp forms</u>.

We have recently applied the dichotomy approach to obtain the current record for the range of validity for the density hypothesis estimate for L-functions associated to certain holomorphic cusp forms. We introduce some notations to state our result.

Let $\mathbb{H}$ be the upper half-plane. A holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called a holomorphic form of weight $k \in \mathbb{N}$ (w.r.t. the full $SL(2,\mathbb{Z})$) if

① For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

$$f(\gamma(z)) = (cz+d)^k f(z) \; ; \; \gamma(z) = \frac{az+b}{cz+d}.$$

② $f(z) = O(1)$ as $\operatorname{Im} z \longrightarrow \infty$.

Taking $\gamma(z) = z+1$, f is 1-periodic and therefore always has Fourier series expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi i z}.$$

We denote M_k the space of modular forms of weight k. The form is called a (holomorphic) cusp form if $a_f(0)=0$, we write $f\in M_{k,0}$. We are interested in those for which their coefficients are multiplicative functions, i.e.,

$$(3) \quad a_f(n\cdot m)=a_f(m)\cdot a_f(n), \quad (n,m)=1.$$

In order to characterize this property Hecke introduce a family of operators $(n=1,2,\dots)$

$$T_n: M_k\to M_k \quad \text{and} \quad T_n: M_{k,0}\to M_{k,0}$$

such that every simultaneous eigenvector of T_n, $n=1,2,\dots$ satisfies (3). We called such forms (Hecke) eigenforms.

Here

$$(T_n f)(z)=n^{k-1}\sum_{d/n} d^{-k}\sum_{b=0}^{d-1} f\left(\frac{nz+bd}{b^2}\right).$$

From now on $\boxed{\text{we fix}}$ a holomorphic Hecke eigenform of weight $k\in\mathbb{N}$, and we normalize $a_f(1)=1$.

<u>Lemma:</u> (Deligne***, 1974) $|a_f(n)|\le n^{\frac{k-1}{2}}d(n)$, with $d(n)=\sum_{d/n}1$. ///

We define $\lambda_f(n) = \dfrac{a_n}{n^{\frac{k-1}{2}}}$ & the (normalised) L-function associated with f is

$$L(s,f) = \sum_{n=1}^{+\infty} \frac{\lambda_f(n)}{n^s} \left(= \prod_p \frac{1}{\left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)} \right)$$

Hecke showed that $L(s,f)$ extends as entire function to $\mathbb{C}$, while Good (1982) showed the second moment estimate

$$\int_0^T \left| L\left(\tfrac{1}{2}+it, f\right)\right|^2 dt \ll_\varepsilon T \log T$$

If $N_f(\sigma,T)$ is the number of zeros of f on $[\sigma,1] + i[-T,T]$, the following summarises what is known for the range of validity for

$$N_f(\sigma,T) \ll_\varepsilon T^{2(1-\sigma)+\varepsilon}, \quad \sigma_0 \leq \sigma \leq 1$$

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(1) This was shown in the context of Deligne's proof of Weil conjecture: the Riemann hypothesis for the zeta function of a non-singular projective algebraic variety over $\mathbb{F}_q$

Γ_0	Record
$53/60 = 0.8833\ldots$	Ivić, 1989
$1407/1601 = 0.8788\ldots$	Chen, Debruyne, V. 2024

Our result (to appear in Rev. Mat. Iberoam.) actually applies to L-functions $L(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$:

- satisfying a Ramanujan conjecture for its coefficients (i.e., multiplicativity, Euler type identity and good bounds on coefficients
- satisfying polynomial bounds on suitable half-planes.
- satisfying a second moment type estimate $\ll T^{1+\varepsilon}$