SMOOTH AND ANALYTIC FACTORIZATION THEOREMS FOR LIE GROUP REPRESENTATIONS

ANDREAS DEBROUWERE, JASSON VINDAS

Introduction: Historical overview (Lecture 1 (1st part), August 3, 2026)

1.1. The weak and strong factorization property.

Let M be a (left) module over an algebra A. M is said to have the **weak factorization property** (WFP) over A if

$$M = \operatorname{span}(M \cdot A).$$

M is said to have the **strong factorization property** (SFP) over A if

$$M = M \cdot A.$$

We say that A has the WFP/SFP if A has this property when considered as a module over itself. Of course, these notions are only interesting if A is non-unital.

We will be interested in factorization properties of convolution modules over function algebras and generalizations of such results involving group representations.

1.2. The convolution algebra $(L^1(G), *)$.

The first factorization result seems to go back to Salem:

Theorem 1.1 (Salem, 1945)**.** *$(L^1(\mathbb{T}), *)$ has the SFP.*

By using Fourier series, Theorem 1.1 is equivalent to the following **division problem**: For all $f \in L^1(\mathbb{T})$ there are $f_1, f_2 \in L^1(\mathbb{T})$ such that

$$\widehat{f}(n) = \widehat{f_1}(n)\widehat{f_2}(n), \qquad \forall n \in \mathbb{Z},$$

where $\widehat{f}(n)$ are the Fourier coefficients of f. This was actually Salem's original statement and what he proved. The same approach will be used in Lecture 2 when studying the SFP for other convolution algebras over the circle and, more generally, compact groups.

Rudin showed an analogue of Theorem 1.1 on the real line:

Theorem 1.2 (Rudin, 1957)**.** *$(L^1(\mathbb{R}), *)$ has the SFP.*

Following Salem, Rudin used the Fourier transform to prove his result.

Cohen generalized Rudin's result to general locally compact groups. Before we state his result, we review some basic facts about topological groups. A **topological group** is a group G endowed with a topology such that the group actions

$$G \times G \to G,\ (g, h) \to g \cdot h, \qquad G \to G,\ g \mapsto g^{-1},$$

are continuous. A **(locally) compact group** is a topological group whose underlying topology is (locally) compact.

It is a general fact that every locally compact group G admits a Radon measure μ that is left-invariant, meaning that for all $g \in E$ and $U \subseteq E$ open

$$\mu(gU) = \mu(U).$$

Moreover, such a measure is unique up to multiplication by a positive constant. Once a normalization has been fixed, we call μ the **(left) Haar measure** on G and simply denote it by $\mathrm{d}g$.

All integrals over G are taken with respect to the Haar measure. In particular, we can define the Lebesgue spaces $L^p(G)$, $1 \leq p < \infty$, in the usual way. In terms of integrals, the left-invariance of the Haar measure means that for all $f \in L^1(G)$ and $g \in G$

$$\int_G f(gh)\mathrm{d}h = \int_G f(h)\mathrm{d}h.$$

Example 1.3. (1) $G = (\mathbb{R}^n, +)$. The Haar measure is equal to the Lebesgue measure.
(2) $G = (\mathbb{T}^n, +)$. The Haar measure is equal to the Lebesgue measure.
(3) $G = (\mathbb{R}_+, \cdot)$. The Haar measure is equal to dx/x, with dx the Lebesgue measure.

The **(left) convolution** of $f_1, f_2 \in L^1(G)$ is defined as

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g)\mathrm{d}h$$

Then, $f_1 * f_2 \in L^1(G)$ and $\|f_1 * f_2\|_{L^1(G)} \leq \|f_1\|_{L^1(G)} \|f_2\|_{L^1(G)}$. Thus, $(L^1(G), *)$ is a Banach algebra.

We can now state Cohen's result.

Theorem 1.4 (Cohen, 1959)**.** *Let G be a locally compact group. $(L^1(G), *)$ has the SFP.*

The method via the Fourier transform breaks down for generally locally compact groups (the Fourier transform can be defined but is not very useful for our purposes). Instead, Cohen showed an abstract factorization result about Banach algebras. Let A be a Banach algebra. A net $(a_\lambda)_{\lambda \in \Lambda}$ is called a **bounded (left) approximate identity** if $(a_\lambda)_{\lambda\in\Lambda}$ is bounded, meaning that $\sup_\lambda \|a_\lambda\|_A < \infty$, and for all $a \in A$

$$a = \lim_\lambda a_\lambda \cdot a.$$

Cohen then showed:

Theorem 1.5 (Cohen, 1959)**.** *Every Banach algebra A that possesses a bounded approximate identity has the SFP.*

Since $(L^1(G), *)$ has a bounded approximate identity (take e.g. $(|U|^{-1}\chi_U)_{U\in\mathcal{U}}$, where $\mathcal{U}$ is a neigbhorhood base of the identity, $|U|$ is the Haar measure of U and χ_U is the indicator function of U), Theorem 1.4 is a direct consequence of Theorem 1.5.

Hewitt generalized Theorem 1.5 to Banach modules:

Theorem 1.6 (Hewitt, 1964). *Let M be a Banach module over a Banach algebra A. Suppose that A has a bounded approximate identity and*

$$M = \overline{\operatorname{span}(M \cdot A)}.$$

Then, M has the SFP over A.

Let G be a locally compact group. Theorem 1.6 implies that $L^p(G)$, $1 \leq p < \infty$, has the SFP over $(L^1(G), *)$.

Theorem 1.6 is nowadays called the **Cohen-Hewitt factorization theorem**. As far as I know, it is the only abstract factorization result.

1.3. **Ehrenpreis' problem.**

We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of compactly supported smooth functions on $\mathbb{R}^n$. It is the test function space of the space of Schwartz distributions. Note that $(\mathcal{D}(\mathbb{R}^n), *)$ is an algebra.

During his study of convolution operators, Ehrenpreis raised the following problem:

Problem 1.7 (Ehrenpreis, 1960). Does $(\mathcal{D}(\mathbb{R}^n), *)$ has the WFP/SFP?

The solution to Ehrenpreis' problem is somewhat surprising and has the following history:

- Rubel, Squires, Taylor (1978): $(\mathcal{D}(\mathbb{R}^n), *)$ has the WFP. For $n \geq 3$, $(\mathcal{D}(\mathbb{R}^n), *)$ does not have the SFP.
- Dixmier, Malliavin (1978): $(\mathcal{D}(\mathbb{R}^2), *)$ does not have the SFP.
- Yulmukhametov (1999): $(\mathcal{D}(\mathbb{R}), *)$ does have the SFP.

The Paley-Wiener-Schwartz theorem asserts that the Fourier image of $\mathcal{D}(\mathbb{R}^n)$ consists precisely of the entire functions F on $\mathbb{C}^n$ for which there exists $A > 0$ such that for all $N \in \mathbb{N}$ there is $C > 0$

$$|F(z)| \leq C\frac{e^{A|\operatorname{Im} z|}}{(1 + |z|)^N}, \qquad \forall z \in \mathbb{C}^n.$$

Hence, Ehrenpreis' problem is equivalent to the following division problem: Can every entire function satisfying the above bounds can be written as the product of two entire functions satisfying the same bounds? It is this formulation of the problem that the above authors studied. It also indicates the difference between the cases $n = 1$ and $n > 1$: Entire functions, and in particular their zero sets, in one and several variables behave quite differently.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all smooth functions f on $\mathbb{R}^n$ such that for all $\alpha \in \mathbb{N}^n$ and $N \in \mathbb{N}$ there is $C > 0$

$$|f^{(\alpha)}(x)| \leq C\frac{1}{(1 + |x|)^N}, \qquad \forall x \in \mathbb{R}^n.$$

We call $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing smooth functions. It is the test function space of the space of Schwartz tempered distributions. Note that $(\mathcal{S}(\mathbb{R}^n), *)$ is an algebra.

Theorem 1.8 (Miyazaki, 1960). *$(\mathcal{S}(\mathbb{R}^n), *)$ has the SFP.*

Since the Fourier transform is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$, Theorem 1.8 is equivalent to the fact that every rapidly decreasing smooth function can be written as the product of two rapidly decreasing smooth functions. This division problem is much easier than the one related to Ehrenpreis' problem as it is about smooth (and not entire) functions.

1.4. **The Dixmier-Malliavin weak factorization theorem.**

Let E be Banach space and denote by $L(E)$ the space of bounded linear operators on E. We write $GL(E)$ for the subspace of $L(E)$ consisting of isomorphisms on E.

Let G be a locally compact group and E a Banach space. A **representation** π of G on E is a group isomorphism $\pi : G \to (GL(E), \circ)$, that is, for all $g, h \in G$

$$\pi(gh) = \pi(g) \circ \pi(h).$$

Given $v \in E$, we define its **orbit** as the function

$$\gamma_v : G \to E, \quad \gamma_v(g) = \pi(g)v.$$

The representation π is called **strongly continuous** if γ_v is continuous on G for all $v \in V$.

We denote by $C_c(G)$ the space of compactly supported continuous functions on G. $(C_c(G), *)$ is an algebra. The representation π induces a natural action of $C_c(G)$ on E:

$$\Pi : C_c(G) \to L(E), \quad \Pi(f)v = \int_G f(g)\gamma_v(g)\mathrm{d}g,$$

where the integral should be interpreted as an E-valued Bochner (or Riemann) integral. $\Pi : (C_c(G), *) \to (L(E), \circ)$ is an algebra homomorhpism, meaning that for all $f_1, f_2 \in C_c(G)$

$$\Pi(f_1 * f_2) = \Pi(f_1) \circ \Pi(f_2).$$

Hence, Π endows E with a natural module structure over $(C_c(G), *)$. Π is sometimes called the **integrated representation**.

Example 1.9. The **left-regular representation** π_L of G on $L^p(G)$, $1 \leq p < \infty$, is given by left-translation:

$$(\pi_L(g)f)(h) = f(g^{-1}h), \qquad f \in L^p(G), g, h \in G.$$

The integrated representation is nothing else but convolution: For all $f_1 \in C_c(G)$ and $f_2 \in L^p(G)$

$$\Pi_L(f_1)f_2 = f_1 * f_2.$$

We will be interested in locally compact groups with an additional structure. A **(real) Lie group** is a group G endowed with a (real) smooth manifold structure such that the group actions

$$G \times G \to G,\ (g, h) \to g \cdot h, \qquad G \to G,\ g \mapsto g^{-1},$$

are smooth. Every Lie group is a locally compact group and thus admits a Haar measure. The examples in Example 1.3 are Lie groups.

Let G be a Lie group and π be a strongly continuous representation of G on a Banach space E. The space E^∞ of **smooth vectors** of π consists of al $v \in E$ whose orbit γ_v is a smooth E-valued function on G. It holds that $\Pi(C_c(G))E^\infty \subseteq E^\infty$. Hence, E^∞ is a module over $(C_c(G), *)$. We denote by $\mathcal{D}(G)$ the space of compactly supported smooth functions on G. Then, $(\mathcal{D}(G), *)$ is an algebra. Hence, E and E^∞ are modules over $(\mathcal{D}(G), *)$ as well.

Example 1.10. Consider the (left) regular representation of $\mathbb{R}^n$ on $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$. The associated space E^∞ of smooth vectors is equal to

$$\mathcal{D}_{L^p}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) \mid f^{(\alpha)} \in L^p(\mathbb{R}^n),\, \forall \alpha \in \mathbb{N}^n\}.$$

These spaces were already considered by Schwartz.

We are ready to state the celebrated Dixmier-Malliavin weak factorization theorem.

Theorem 1.11 (Dixmier-Malliavin, 1978)**.** *Let G be a Lie group and π be a strongly continuous representation of G on a Banach space E. Then, E^∞ has the WFP over $(\mathcal{D}(G), *)$, that is,*

$$E^\infty = \operatorname{span}(\Pi(\mathcal{D}(G))E^\infty).$$

Dixmier and Malliavin first treated the case $G = \mathbb{R}$ by using infinite-order differential operators (more precisely, parametrices). Next, they reduced the general case to this special one by a nice inductive argument involving exponential coordinates of the second kind. These ideas will return in Lecture 3.

Remark 1.12. Let n be the dimension of G. The original proof of Dixmier and Malliavin implies that every element in E^∞ can be written as the sum of 2^n elements in $\Pi(\mathcal{D}(G))E^\infty$. In 1980, Malliavin improved this result by showing that every element in E^∞ can be written as the sum of two elements in $\Pi(\mathcal{D}(G))E^\infty$.

Smooth and analytic factorization theorems for Lie group representations

(by Andreas Debrouwere and Jasson Vindas)

Minicourse delivered at University of Costa Rica on August 3, 4, and 6, 2026

Lecture 1 (2nd Part): August 3, 2026.

$\boxed{1.2}$ Introduction: The Dixmier-Malliavin strong factorization theorem.

Let $\pi : G \longrightarrow GL(E)$ be a strongly continuous representation of the (real) Lie group G on a Banach space E. Dixmier and Malliavin showed that the space of smooth vectors has the strong factorization property for certain special classes of Lie groups. The first of which is the class of compact Lie groups. Notice that there is no longer a distinction between compactly supported smooth functions and smooth functions in that case.

Theorem 1.2.1 (Dixmier-Malliavin) If G is a compact

Lie group, then the smooth vectors of a strongly continuous representation π have the strong factorization property, that is,

$$E^\infty = \pi(C^\infty(G))E^\infty. ///$$

We shall give a prove of Theorem 1.2.1 in Lecture 2.

They also gave a strong factorization theorem for the class of so-called simply connected nilpotent Lie groups. We need some preliminaries to state their result.

A fundamental object in Lie theory is the Lie algebra of a Lie group. Let $\mathfrak{X}(G)$ be the space of smooth vector fields on G, i.e., a smooth mapping $X: G \to T(G)$ so that the tangent vector $X_g \in T_g G$, $g \in G$. We can define the Lie algebra $\mathfrak{g}$ of G as the vector space of all left-invariant (smooth) vector fields on G.

Given $x \in G$, write $l_x : G \to G$ for the diffeomorphism $l_x(g) = x \cdot g$. We say $X \in \mathfrak{g}$ if $(l_x)_* X = X \circ l_x$ for all $x \in G$, that is, if

$$X_{x\cdot g} = d_g l_x(X_g), \quad \forall x, g \in G.$$

In particular X is completely determine by its value at the identity $X_e \in T_e(G)$ as

(1.2.1) $$X_g = d_e l_x(X_e),$$

nd this identity induces the natural isomorphism $T_e(G) \longrightarrow \mathfrak{g}$. In what follows we identify $\mathfrak{g}$ with $Te(G)$ nd sometimes write X for both the left invariant vector field nd its value at e, a distinction will be made only if not clear from the context.

Let finally be $\exp(tX) = F(t)$ by the flow of X with initial point e, that is, the unique solution of the ODE:

$$\begin{cases} \dfrac{dF}{dt} = X_{F(t)} \\ F(e) = X_e \end{cases}$$

The map $\exp : \mathfrak{g} \longrightarrow G$ is called the exponential map.

$$X \longmapsto \exp(X).$$

Any $X \in \mathfrak{g}$ c. be viewed as a linear differential operator $X : C^\infty(G) \longrightarrow C^\infty(G)$ via

$[X(f)](g) = X_g(f) = \frac{d}{dt}\Big|_{t=0} f(g\exp(tX))$. The Lie bracket is then the commutator, i.e., $[\,,\,]: \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is $[X,Y] = X\circ Y - Y\circ X$, given to $\mathfrak{g}$ the structure of a Lie algebra.

<u>Definition 1.2.2</u> A Lie algebra is called nilpotent if the sequence $\mathfrak{g}_0 = \mathfrak{g} \supseteq \mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}] \supseteq \dots \supseteq \mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}_{k-1}]$ eventually arrives to $\{0\}$. A Lie group is nilpotent if its Lie algebra is nilpotent. ////

It is a general fact that, up to Lie group isomorphism, a simple connected nilpotent Lie group is essentially $\mathbb{R}^n$ with a polynomial law multiplication (the map $x\odot y = P(x,y)$ being an $\mathbb{R}^n$-valued polynomial function). Upon identifying a nilpotent Lie group $G \cong \mathbb{R}^n$, the Schwartz space on it is

$$S(G) = \Big\{ \varphi\in C^\infty(\mathbb{R}^n) : \sup_{x\in\mathbb{R}} (|x|+1)^k \,|\varphi^{(\alpha)}_{(x)}| < \infty,\ \forall k\in\mathbb{N},\ \alpha\in\mathbb{N}^n \Big\}.$$

We also point out that the standard Lebesgue measure is then a Haar measure on G.

Let us come back to factorization. When π is bounded, in the sense that its operator norm is uniformly bounded, namely,

$$(1.2.1) \qquad \sup_{g\in G} \|\pi(g)\|_{\mathcal{L}_b(E)} < \infty,$$

we can define the bounded operator $\Pi(f): E \to E$ for any $f \in L^1(G)$ via

$$\Pi(f)v = \int_G f(g)\pi(g)v\,dg, \quad v \in E.$$

If G is nilpotent and $f \in \mathcal{S}(G)$, one easily checks that $\Pi(f): E^\infty \to E^\infty$.

<u>Theorem 1.2.3</u> (Dixmier-Malliavin). Let π be a bounded strongly continuous representation of a simply connected nilpotent Lie group on a Banach space E. Then E^∞ has the strong factorization property with respect to $\mathcal{S}(G)$, that is,

$$E^\infty = \Pi(\mathcal{S}(G))E^\infty. \quad ///$$

Example 1.2.4: The most basic example of a nilpotent Lie group is the Heisenberg group $\mathbb{H} = \mathbb{R}^3$ provided with the multiplication

$$(x_1, y_1, t_1)\cdot(x_2, y_2, t_2) = (x_1+x_2, y_1+y_2, t_1+t_2+\frac{1}{2}(x_1y_2 - y_1x_2))$$

One checks that its Lie algebra is spanned by the left-invariant vector fields

$$X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial t}$$

and $T = \frac{\partial}{\partial t}$.

Example 1.2.5: Let $G = (\mathbb{R}^n, +)$ be a simply connected nilpotent Lie group. We just write $g_1 \circ g_2 = g_1 g_2$. We consider $E^\infty = \mathcal{D}_{L^p}(\mathbb{R}^n) = \{f \in C^\infty(G) : Df \in L^p(G), \forall D \in U(\mathfrak{g})\}$, which are the smooth vectors of the left-regular representation π_L acting on the Banach space $E = L^p(\mathbb{R}^n)$ as

$$(\pi_L(g)f)(x) = f(g^{-1}x).$$

Here $U(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra, which we may identify with the algebra of left-invariant differential operators on G.
Theorem 1.2.3 then tells us that given $f \in D_{L^p}(G)$ (assuming $p \in [1, \infty)$), there is $\varphi \in S(G)$ and $\psi \in D_{L^p}(G)$ s.t.

$$f(g) = \int_G \varphi(x)\, \psi(x^{-1} g)\, dx. \quad ///$$

Questions related to the strong factorization theorems.

① Do they hold for more general Lie groups?

② In case of non-compact Lie groups, can we reduce the size of the convolution algebra?

③ Can we factor other spaces of regular vectors?
A class of interest is the analytic vectors: $v \in E$ is called analytic if $g \in G \longmapsto \pi(g) v \in E$ is a real analytic function. We write E^{ω} for the space of analytic vectors of π.

$\boxed{2}$ Factorization on compact Lie groups.

We now study Dixmier-Malliavin strong factorization theorems for compact Lie group representations. We fix a Banach space E with norm $\| \ \|$.

$\boxed{2.1}$ The torus.

We consider first the case of $G = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, where the problem is rather easy to solve. We recall that if $f \in L^2(\mathbb{T})$, we can write it as the L^2-convergent Fourier series

$$(2.1) \qquad f(z) = \sum_{-\infty}^{\infty} \hat{f}(n) z^n$$

Here

$$(2.2) \qquad \hat{f}(n) = \int_{\mathbb{T}} f(z) z^{-n} dz = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta,$$

so that dz is the normalized Haar probability measure of $\mathbb{T}$. We call $f \longmapsto \hat{f} = \mathcal{F}(f)$ the Fourier transform and $\mathcal{F} : L^2(\mathbb{T}) \longrightarrow l^2(\mathbb{Z})$ is then an isometry.

One can use the decay of Fourier coefficients to characterize smoothness and real analyticity of vector-valued functions.

We first notice that we also have an isometry $\mathcal{F}: L^2(\mathbb{T}, E) \longrightarrow \ell^2(\mathbb{Z}, E)$, where we simply extend (2.1) and (2.2) to E-valued functions.

We denote as $\mathcal{A}(\mathbb{T}, E)$ the space of real analytic functions on $\mathbb{T}$ with values in E.

<u>Lemma 2.1</u> Let $f \in L^2(\mathbb{T}, E)$.

(i) $f \in C^\infty(\mathbb{T}, E) \Leftrightarrow (\forall k) \left(\sup_{n \in \mathbb{Z}} \|\hat{f}(n)\| (1+|n|)^k < \infty \right)$

(ii) $f \in \mathcal{A}(\mathbb{T}, E) \Leftrightarrow (\exists R > 1, C > 0) \left(\|\hat{f}(n)\| \le C \cdot R^{-|n|} \right)$

<u>Proof.</u> Integrating by parts, $n \neq 0$,

$$\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta = \frac{1}{in} \int \frac{d}{d\theta}[f(e^{i\theta})] e^{-in\theta} d\theta$$

$$= \cdots = \left(\frac{1}{in}\right)^k \int_0^{2\pi} \frac{d^k}{d\theta^k} [f(e^{i\theta})] e^{-in\theta} \, d\theta,$$

whence the direct implication follows at once. The converse follows from the Fourier series expansion (2.1).

Notice that $f \in \mathcal{A}(\mathbb{T})$ means that (2.1) has holomorphic extension to ring $\frac{1}{R} \le |z| \le R$. Using the familiar formula for the convergence radius of power series, we obtain (ii). ///

Remark 2.2. We write $s(\mathbb{Z}, E)$ for the space of E-valued sequences satisfying the bounds from Lemma 2.1 (i), while $\mathfrak{k}(\mathbb{Z}, E)$ for the E-valued sequences from Lemma 2.1 (ii). If $E = \mathbb{C}$, we simply write $s(\mathbb{Z})$ and $\mathfrak{k}(\mathbb{Z})$ for them.

Theorem 2.3 (Strong factorization on $\mathbb{T}$).
Let π be a strongly continuous representation of $\mathbb{T}$ on a Banach space E. Then

$$E^\infty = \Pi(C^\infty(\mathbb{T}))E^\infty \text{ and } E^\omega = \Pi(\Lambda(\mathbb{T}))E^\omega.$$

Proof. v in either E^∞ or E^ω. We set $f(z) = \pi(z)v$.

Assume first $v \in E^\infty$. We write $\hat{f}(n) = a_{-n} v_n$ where $a_n = \sqrt{\|\hat{f}(-n)\|}$ and $v_n = \frac{\hat{f}(n)}{a_{-n}}$ if $\hat{f}(n) \neq 0$ and 0 otherwise. Clearly $a_n \in s(\mathbb{Z})$ and $v_n \in s(\mathbb{Z}, E)$. Thus, we can find $h_0 \in C^\infty(\mathbb{T})$ and $h_1 \in C^\infty(\mathbb{T}, E)$ s.t. $a_n = \hat{h}_0(n)$ and $v_n = \hat{h}_1(n)$ and obtain the factorization

$$f(w) = \int_{\mathbb{T}} h_0(w^{-1}z)\, h_1(z)\, dz$$

So,

$$v = \int_{\mathbb{T}} h_0(z)\, h_1(z)\, dz.$$

If we show

$$(2.3) \quad h_1(z) = \pi(z)\, h_1(0),$$

we would obtain $v = \pi(h_0)(h_1(0))$ and will be done. Let us thus prove (2.3). We have

$$\pi(z)\, h_1(0) = \sum_{n \in \mathbb{Z}} \frac{\pi(z) \hat{f}(n)}{a_{-n}}$$

and

$$h_1(z) = \sum_{n \in \mathbb{Z}} z^n \frac{\hat{f}(n)}{a_{-n}}.$$

Therefore, (2.3) holds if $\pi(z)\hat{f}(n) = z^n \hat{f}(n)$. For it,

$y = zw$, $dy = dw$

$$\pi(z)\hat{f}(n) = \pi(z) \int_{\mathbb{T}} w^{-n} f(w)\, dw = \int_{\mathbb{T}} w^{-n}\, \pi(z \cdot w) v\, dw$$

$$= z^n \int_{\mathbb{T}} y^{-n} \pi(y) v\, dy = z^n \hat{f}(n),$$

as claimed.

Assume now $v \in E^{\omega}$. If we decompose exactly as before $\hat{f}(h) = a_{-n} Y_n$, we now obtain $h_0 \in \mathcal{A}(\mathbb{T})$ and $h_1 \in \mathcal{A}(\mathbb{T}, E)$. Thus the same argument as before yields the factorization theorem. ///

2.2 Compact Lie groups.

We shall now extend Theorem 2.3 to general compact Lie groups. At the level of ideas, the proof of the extension is basically the same as that of Theorem 2.3 but we need to introduce the necessary machinery for it. We fix in this section a compact Lie group G, with normalized Haar measure: $\int_G dg = 1$.

2.2.1 Background on compact Lie group representations

Let ξ and ξ' be (strongly continuous) representations of G on Hilbert spaces $\mathcal{H}_\xi$ and $\mathcal{H}_{\xi'}$. They are unitary if $\pi(g)$ and $\pi'(g)$ are unitary operators for all g.

They are called (unitary) equivalent if there is an isometry

$U : \mathcal{H}_\xi \longrightarrow \mathcal{H}_{\xi'}$ such that $U \xi(g) U^{-1} = \xi'(g)$, $\forall g \in G$. A representation is called unitary irreducible if it is unitary and the only closed subspaces $X \subseteq \mathcal{H}$ such that $\xi(g) X \subseteq X$, $\forall g \in G$ are

$X = \mathcal{H}_\xi$ or $X = \{0\}$.

Remark 2.4 Given a strongly continuous representation on a Hilbert space, there is always an inner product making ξ unitary: $\langle v, w \rangle_\xi = \int_G \langle \xi(g)v, \xi(g)w \rangle dg$.

The following property is fundamental. We first remark that any irreducible representation of a compact Lie group must satisfy $\dim \mathcal{H}_\xi < \infty$. We denote as $\hat{G}$ the set of equivalence classes of unitary irreducible representations. We write $d_\xi = \dim \mathcal{H}_\xi$. If $f \in L^2(G)$, we write $\hat{f}$ for the operator-valued function

$$\hat{f}(\xi) = \Xi(f), \quad \xi \text{ unitary irreducible},$$

that is

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_G f(g)\, \xi(g)\, dg.$$

Each $\mathcal{L}(\mathcal{H}_\xi)$ can be made into a Hilbert space via $\langle A, B \rangle_{\mathcal{L}(\mathcal{H}_\xi)} = \mathrm{Tr}(AB^*)$, which gives rise to the Hilbert space norm

$$\|A\|_{HS} = \sqrt{\operatorname{Tr}(AA^{*})}.$$

We write $l_2(\hat{G}) = \{ T = (T_\xi)_{[\xi]\in\hat{G}} \in \prod \mathcal{L}(H_\xi)$

(here we make of choice of a representative of $\xi \in [\xi]$)

$: \|T\|_2^2 = \sum_{[\xi]\in\hat{G}} d_\xi \|T_\xi\|_{HS}^2 < \infty \}.$

The following is a version of the celebrated Peter-Weyl theorem:

<u>Theorem 2.4</u> $\mathcal{F}: L^2(G) \longrightarrow l_2(\hat{G})$ is an isometric isomorphism. Its inverse is

$$\mathcal{F}^{-1}(T)(g) = \sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}[\xi^{*}(g) \circ T_\xi].$$ ///

<u>Corollary 2.5</u> For any $f \in L^2(G)$,

$$f(g) = \sum_{[\xi]\in\hat{G}} d_\xi \operatorname{Tr}(\xi^{*}(g)\cdot \hat{f}(\xi)).$$ ///

What we have told so far applies for compact groups. The Lie group structure comes into play if we involve the Laplace-Beltrami operator (with respect to a bi-invariant Riemannian metric).

Let $\{e_j : 1 \leq j \leq d_\xi\}$ be an orthonormal basis of $\mathcal{H}_\xi$. We call the $G \to \mathbb{C}$ functions

$$\xi_{i,j}(g) = (e_j, \xi(g) e_i)_{\mathcal{H}_\xi}$$

are called the matrix coefficients and it can be shown (this is known as Schur's orthogonality relations) that $\{\xi_{i,j}\}_{i,j,\xi}$ is an orthogonal system. In fact it follows from Corollary 2.5 that $\left\{ \frac{\xi_{i,j}}{\sqrt{d_\xi}} \right\}_{\substack{1 \leq i,j \leq d_\xi \\ [\xi] \in \widehat{G}}}$ is an orthonormal basis of $L^2(G)$.

Using the so-called Schur's lemma for unitary irreducible representations, one shows:

<u>Lemma 2.6</u>: There is $\lambda_\xi \geq 0$ such that

$$\Delta \xi_{i,j} = -\lambda_\xi \xi_{i,j}, \quad 1 \leq i,j \leq d_\xi$$

In fact $\{\lambda_\xi\}_{[\xi] \in \widehat{G}}$ is the sequence of eigenvalues of $-\Delta$. ///

If we combine Lemma 2.6 with Corollary 2.5 we formally obtain,

$$\Delta^p f(g) = \sum_{[\xi] \in \hat{G}_\xi} d_\xi (-\lambda_\xi)^p \, Tr\left(\xi^*(g) \circ \hat{f}(\xi)\right).$$

The following result is known as Weyl's asymptotic formula:

<u>Lemma 2.7</u>: $N(\lambda) = \sum_{\lambda_\xi \leq \lambda} d_\xi^2 \sim \frac{\lambda^{n/2}}{n(2\pi)^n} |S_{n-1}|$,

where $n = \dim G$ and $|S_{n-1}|$ is the surface area of the unit sphere in $\mathbb{R}^n$. ///

The relevance of Lemma 2.7 for us is the following corollary:

<u>Corollary 2.8</u> We have:

$$(2.4) \qquad \sum_{[\xi] \in \hat{G}} \frac{d_\xi^2}{(1+\lambda_\xi)^n} < \infty.$$

<u>Proof.</u> The partial sums of (2.4) are

$$\int_{0^-}^{\lambda} (1+t)^{-n}\, dN(t) = \frac{-N(\lambda)}{(1+\lambda)^n} + n\int_0^\lambda \frac{N(t)}{(1+t)^{n+1}}\, dt$$

which is bounded $\lambda \to \infty$. ////

2.2.2 Characterisation E-valued smooth and real analytic functions on G.

We can use the Peter-Weyl theorem to characterise smoothness and analyticity via the decay of the Fourier transform with respect to the eigenvalues of $-\Delta$.

Recall $C^\infty(G, E)$ and $\mathcal{A}(G,E)$ stand for the spaces of C^∞ and real analytic E-valued functions.

The next lemma allows us to reduce the analysis of E-valued functions to the scalar valued case.

Lemma 2.9 Let $f : G \to E$.

(i) $f \in C^\infty(M;E) \iff \langle v', f\rangle \in C^\infty(G)$ $\forall\, v' \in E'$ (the space of continuous linear functionals on E).

(iii) $f \in \mathcal{L}(G;E) \iff \langle v', f \rangle \in \mathcal{L}(G)\ \forall v' \in E'$.

Let us now extend the Fourier transform to vector-valued functions. We consider the space $E \otimes \mathcal{L}(\mathcal{H}_\xi)$ (if operator of $\mathcal{L}(\mathcal{H}_\xi)$ are thought of as matrices, the elements of $E \otimes \mathcal{L}(\mathcal{H}_\xi)$ can be thought of as matrices whose entries are elements of E). Each $v' \in E'$ induces the map $\langle v', \cdot \rangle = v' \otimes Id_{\mathcal{L}(\mathcal{H}_\xi)}$

$$E \otimes \mathcal{H}_\xi \longrightarrow \mathbb{C} \otimes \mathcal{H}_\xi = \mathcal{H}_\xi.$$

We then define the norm ($A \in E \otimes \mathcal{L}(\mathcal{H}_\xi)$)

$$(2.5) \quad \|A\|_{HS,E} = \sup_{\|v'\|_{E'} \le 1} \|\langle v', A \rangle\|_{HS}.$$

The Fourier transform of $f \in L^2(G,E)$

$$(2.6) \quad \hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_G f(x) \otimes \xi(x)\, dx \in E \otimes \mathcal{L}(\mathcal{H}_\xi)$$

If we write

$$\ell^2(\hat{G},E) = \Big\{ T = (T_\xi)_{[\xi] \in \hat{G}} : \sum_{[\xi] \in \hat{G}} d_\xi \|T_\xi\|^2_{HS,E} < \infty \Big\}$$

we obtain an isometry again

$$\mathcal{F}: L^2(G,E) \longrightarrow \ell^2(\widehat{G},E)$$

and the formula from Corollary 2.5 remains valid.
The norm of $\ell^2(\widehat{G},E)$ is

$$\| T \|^2_{\ell^2(\widehat{G},E)} = \sum d_\xi \| T_\xi \|^2_{HS,E}.$$

<u>Theorem 2.10</u> Let $f \in L^2(G,E)$.

(i) $f \in C^\infty(G,E)$ iff $\forall k \in \mathbb{N}$ $\sup\limits_{[\xi]\in\widehat{G}} (1+\lambda_\xi)^k \|\widehat{f}(\xi)\|_{HS,E} < \infty$.

(ii) $f \in \mathcal{A}(G,E)$ iff $(\exists R>0)\left(\sup\limits_{[\xi]\in\widehat{G}} R^{\lambda_\xi} \|\widehat{f}(\xi)\|_{HS,E} < \infty\right)$.

<u>Proof.</u> (i) If $f \in C^\infty(G,E)$, we have $\Delta^k f \in L^2(G,E)$ $\forall k \in \mathbb{N}$. Therefore, $\widehat{\Delta^k f} \in \ell^2(\widehat{G},E)$ for each $k \in \mathbb{N}$. In particular

$$\sup_{[\xi]\in\widehat{G}} \| \widehat{\Delta^k f} \|_{HS,E} < \infty.$$

Now using (2.6) and Lemma 2.6, we obtain

(2.7) $\widehat{\Delta^k f}(\xi) = (-\lambda_\xi)^k \widehat{f}(\xi)$, which yields the direct implication at once.

For the converse, the general theory of elliptic equations and the E-valued version of the Peter-Weyl theorem imply that it is enough to show that for each $p \in \mathbb{N}$ $\widehat{\Delta^p f} \in l^2(\widehat{G}, E)$. Using our hypothesis, we can find $C > 0$ s.t.

$$\|\hat{f}(\xi)\|^2_{HS,E} \leq \frac{C}{(1+\lambda_\xi)^{n+2p}}.$$

By (2.7),

$$\sum_{[\xi]\in\widehat{G}} d_\xi \, \|\widehat{\Delta^p f}(\xi)\|^2_{HS,E} \leq \sum_{[\xi]\in\widehat{G}} d_\xi^2 \, C \frac{\lambda_\xi^{2p}}{(1+\lambda_\xi)^{n+2p}}$$

$$< \infty,$$

in view of Corollary 2.8. //

(ii) This part is a particular case of a classical result of Seeley for the characterization of real analytic functions in terms of Fourier coefficients with respect to eigenexpansions (20) related to elliptic analytic PDO on compact manifolds. It can actually be shown as (i) once one admits the following characterization (a particular case of a result of Kotake and Narasimham) of $\mathcal{A}(G,E)$ in terms of iterates of the Laplace-Beltrami operator (or more generally elliptic analytic PDO):

$$f \in \mathcal{A}(G,E) \Leftrightarrow (\exists R>0)\left(\sup_{\substack{p \in \mathbb{N} \\ g \in G}} \frac{\|\Delta^p f\|_{L^2(G,E)}}{R^p p^p} < \infty \right).$$

We leave modification of the argument with the aid the latter iterate characterization as an exercise for the reader. ///

$\boxed{2.2.3}$ The factorization theorem

We now show Theorem 1.2.1 and its extension to analytic vectors. We state both results in the following theorem

Theorem 2.11 Let π be a strongly continuous representation of a compact Lie group G on a Banach space.

Then

$$E^\infty = \Pi(C^\infty(G))E^\infty \text{ and } E^\omega = \Pi(\mathcal{A}(G))E^\omega. \quad ///$$

We use the following lemma, whose proof is left to the reader.

<u>Lemma 2.12</u> $\widehat{f * g}(\xi) = \hat{f}(\xi) \circ \hat{g}(\xi)$, for each $f \in L^1(G, E)$, $g \in L^1(G)$. ///

<u>Proof of Theorem 2.11</u> Using all tools we have introduced, the proof is essentially the same as that of Theorem 2.3. Let us sketch $E^\omega = \Pi(\mathcal{A}(G))E^\omega$, the other case is similar.

Let $v \in E^\omega$. We consider $f(g) = \pi(g)v \in \mathcal{A}(G, E)$, and consider the factorization of matrices

$$\hat{f}(\xi) = T_\xi \circ (a_\xi \cdot Id_{H_\xi}), \quad [\xi] \in \hat{G};$$

where $T_\xi = \dfrac{\hat{f}(\xi)}{\|\hat{f}(\xi)\|_{HS,E}^{1/2}}$ and $a_\xi = \|\hat{f}(\xi)\|_{HS,E}^{1/2}$.

(T_ξ being 0 if $\hat{f}(\xi) = 0$). Using the (E-valued) Peter-Weyl theorem and Theorem 2.10, we find $h_0 \in \mathcal{A}(G)$ and $h_1 \in \mathcal{A}(G, E)$ s.t.

$$\hat{h}_0(\xi) = a_\xi \text{ and } \hat{h}_1(\xi) = T_\xi.$$

We get, by Lemma 2.12,

$$V = \int_G h_0(g) h_1(g)\, dg.$$

As in the proof of Theorem 2.3, one needs to check that $h_1(g) = \pi(g) h_1(0)$. For it,

$$h_1(g) = \sum_{[\xi] \in \hat{G}} d_\xi \operatorname{Tr}\left(\xi^*(g) \cdot \frac{\hat{f}(\xi)}{d_{\xi^*}}\right)$$

On the other hand,*

$$\pi(g) \hat{f}(\xi) = \pi(g) \int_G \pi(x) v \otimes \xi(x)\, dx$$

$$= \int_G \pi(y) v \otimes \xi(g^{-1}y)\, dy$$

$$= \xi^*(g) \hat{f}(\xi).$$

Therefore,

$$\pi(g) h_1(0) = \sum_{[\xi] \in \hat{G}} d_\xi \operatorname{Tr}\left(\pi(g) \hat{f}(\xi)\right) = h_1(g)$$ ////.

---

* we abuse notation, e.g. $\pi(g) \hat{f}(\xi) = \pi(g) \otimes Id_{L(H_\xi)}(\hat{f}(\xi))$