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Complex Tauberian theorems for Laplace transforms and power series

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$\boxed{1}$ Introduction:

A Tauberian theorem allows one to extract asymptotic information for a function or a sequence from asymptotic information of a transform, typically a Laplace transform or a power series. Here are two samples of Tauberian theorems.

Theorem 1 (Littlewood, 1911; Hardy-Littlewood 1913). Suppose that $F(r) := \sum_{n=0}^{\infty} c_n r^n$ converges for $|r|<1$ and that

$$\lim_{r\to 1^-} F(r) = \gamma.$$

If $-\frac{M}{n} < c_n$ for some $M>0$, then $\sum_{n=0}^{\infty} c_n = \gamma$.

Theorem 2 (Karamata, 1931). Suppose $S \nearrow$ such that

$$F(\sigma) \doteq \int_{0^-}^{\infty} e^{-\sigma x}\, dS(x) \text{ exists for } \sigma>0.$$

If $F(\sigma) \sim \frac{a}{\sigma^\alpha}$ as $\sigma \to 0^+$, where $a>0$,

then

$$S(x) \sim \frac{a}{\Gamma(\alpha+1)} x^\alpha, \quad x \to +\infty.$$

These are examples of real Tauberian theorems as only 'real' information on the transform is needed to draw information on the original sequence or function. Let us mention that Theorem 1 is contained in Theorem 2. For a modern account on Tauberian theory, we refer to Korevaar's book 'Tauberian theory. A century of developments', 2004. We mention that Tauberian theory is a useful tool in many areas of math, such as number theory or spectral theory.

$\boxed{2}$ Complex Tauberian theorems: Two prototype theorems

Tauberian theorems in which information in the complex domain is used are called Complex Tauberians. This will be the subject of this talk. We will focus on two sorts of statements, usually labeled as Fatou-Riesz theorems and Wiener-Ikehara theorems. Although the Fatou-Riesz theorem preceeds the Wiener-Ikehara theorem, we

② start our discussion with the later one. The W-I theorem was initially deviced for giving a short proof of the prime number theorem (PNT), we discuss this below.

Theorem 3 (Wiener-Ikehara, first version 1931). Suppose that $S\nearrow$ on $[0,\infty)$ and that

$$\mathcal{L}\{dS;s\}=\int_{0^-}^{\infty}e^{-sx}dS(x)\left(=s\int_0^{\infty}S(x)e^{-sx}dx\right)$$

converges for $\operatorname{Re}s>1$. If there is a constant $A\ (\geq 0)$ s.t.

$$G(s)=\mathcal{L}\{dS;s\}-\frac{A}{s-1}$$ has "good boundary behavior"

on $\operatorname{Re}s=1$, then

$$(1)\qquad S(x)\sim Ae^{x}\quad(x\to\infty)$$ //

"good boundary" behavior can mean:

1. analytic continuation beyond $\operatorname{Re}s=1$ (Ikehara)
2. continuous extension to $\operatorname{Re}s\geq 1$ (Wiener)
3. L^1_{loc}-boundary behavior: $\lim_{\sigma\to 1^-}G(\sigma+it)$ converges in $L^1(I)$ for every finite interval $I\subseteq\mathbb{R}$. (Hard to trace back where this was first stated).
4. local pseudofunction boundary behavior (Korevaar, 2005) - I will explain later what this means.

The condition 4. is minimal in the sense that if (1) holds then G must satisfy it. We can add a new condition (Debruyne-V, 2016)

5. local pseudofunction behavior except in a "small set" (where an additional condition must be satisfied).

If one is interested only in a proof of the PNT, analytic continuation suffices:

Deduction of the PNT: Recall the PNT asserts that

$$\pi(x):=\sum_{p\leq x}1\sim\frac{x}{\log x},$$ where p runs over prime numbers.

3 We will take for granted:

(a) PNT $\Leftrightarrow$ $\psi(x) = \sum_{n\leq x} \Delta(n) \sim x$, where $\Delta(n) = \begin{cases} \log p, & n = p^M \\ 0 & \text{otherwise} \end{cases}$

(b) $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ has analytic continuation to $\operatorname{Re} s > 0$ except for a simple pole with residue $(= 1)$ at $s = 1$.

(c) $\sum_{n=1}^{\infty} \frac{\Delta(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$

(d) $(s-1)\zeta(s)$ has no zeros on a region containing $\operatorname{Re} s \geq 1$ (Hadamard - De la Vallée Poussin, 1896).

Let us start by adapting Theorem 3 to Dirichlet series.

<u>Corollary 1</u>: Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges $\operatorname{Re} s > 1$.

If $\sum_{n=1}^{\infty} \frac{a_n}{n^s} - \frac{A}{s-1}$ has "good boundary behavior" on $\operatorname{Re} s = 1$, then

$$\sum_{n\leq y} a_n \sim Ay.$$

<u>Proof</u>: Set $S(x) = \sum_{n\leq e^x} a_n = \sum_{\log n \leq x} a_n \Rightarrow dS(x) = \sum_{n=1}^{\infty} a_n \delta(x - \log n)$

and $\mathcal{L}\{dS; s\} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$. Thus $S(x) \sim Ae^x$ //

Now, for the PNT: From (d) we get that $-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} = -\frac{d}{ds}\left((s-1)\zeta(s)\right)$ has analytic extension beyond $\operatorname{Re} s = 1$, the result follows by combining (c) with Corollary 1 ($\Rightarrow \psi(x) \sim x$) and then using (a). ///

<u>We come back to Tauberian theorems</u>: In his very influential 1906 paper Fatou showed the following theorem

<u>Theorem 4</u>: Suppose $F(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < 1$.

If $c_n \to 0$ and F has analytic continuation to a neighborhood of $z = 1$, then $\sum_{n=0}^{\infty} c_n = F(1)$.

(4) Marcel Riesz gave 3 proofs of this theorem (1909, 1911, 1916).

Several generalizations of Theorem 4 have been considered in the literature. The following theorem was proved by Newman (1980) proving a "short" way to the PNT.

<u>Theorem 5</u> Suppose that a_n is bonded $F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ (which of course is convergent for $\operatorname{Re} s > 1$) has analytic continuation to $\operatorname{Re} s \geq 1$, then $\sum_{n=1}^{\infty} \frac{a_n}{n} = F(1)$.

<u>Connection with the PNT</u>: It is well known (Landau) that the PNT is equivalent to $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$, where μ is the Möbius function. Also, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$ and, since $\zeta(s)$ has a pole at $s=1$, this Dirichlet series has analytic continuation to $\operatorname{Re} s = 1$. By Theorem 5, $\sum_{n=0}^{\infty} \frac{\mu(n)}{n} = \frac{1}{\zeta(0)} = 0$ ///

Theorem 5 is actually contained in the following theorem of Ingham (1935, 1936 with addendum of Korevaar in 2002)

<u>Theorem 6</u>: Suppose τ is slowly decreasing, which means

$$\tau(x+h) - \tau(x) \geq -\varepsilon \quad \text{for } x \gg_{\varepsilon} 1 \text{ and } 0 < h < \delta_{\varepsilon}.$$

If there is a constant b s.t.

$$\left(\int_0^{\infty} e^{-sx} \tau(x)\,dx - \frac{b}{s} = \right) \mathcal{L}\{\tau; s\} - \frac{b}{s} \text{ has } L^1_{loc}\text{-boundary behavior on } \operatorname{Re} s = 0,$$

then $\lim_{x \to +\infty} \tau(x) = b$.

- If τ is 'very' slowly decreasing which means that the δ above does not depend on ε, the same conclusion holds if we just ask $\mathcal{L}\{\tau; s\} - \frac{b}{s}$ to have L^1_{loc}-behavior near $s=0$.

This theorem has important consequences in semigroup and operator theory. It can be shown it contains all previous theorems stated in this section when L^1_{loc}-boundary behavior is used.

(5) [3] The Katznelson-Tzafriri theorem:

Katznelson and Tzafriri gave the following theorem (1986)

Theorem 7: Suppose $F(z) = \sum_{n=0}^{\infty} c_n z^n$ converges for $|z|<1$.

If F has analytic continuation at every point of $\partial\mathbb{D}\setminus\{1\}$ and $s_n := \sum_{j=0}^{n} c_j = O(1)$, then $c_n \to 0$. ($\partial\mathbb{D}$ unit circle in the complex plane)

The theorem holds true if the coefficients take values in a Banach space. This was used to show the following theorem, which is one of the cornerstones in the asymptotic theory of operators:

$T \in \mathcal{L}(X)$ is power-bounded if $\sup_{n\geq 0} \|T^n\| < \infty$, where X is a Banach space. If this is the case $sp(T) \subseteq \overline{\mathbb{D}}$.

Theorem 8: Let T power-bounded operator. Then

$$\lim_{n\to+\infty} \|T^{n+1} - T^n\| = 0 \iff sp(T)\cap\partial\mathbb{D} \subseteq \{1\}.$$

Proof "$\Rightarrow$": By contraposition if $\exists \lambda \in sp(T)\cap\partial\mathbb{D}$ with $\lambda \neq 1$, then

$$\|T^n - T^{n+1}\| \geq r(T^n - T^{n+1}) \qquad (r \text{ denotes spectral radius})$$

But, setting $f(z) = z^n - z^{n+1}$, by functional calculus

$$sp(T^n - T^{n+1}) \supseteq f(sp(T)). \text{ So, } \|T^n - T^{n+1}\| \geq |f(\lambda)| = |1-\lambda| > 0.$$

"$\Leftarrow$" $\lambda I - T$ is invertible for all $|\lambda|\geq 1$, $\lambda \neq 1$, we obtain that $g(z) = \sum_{n=0}^{+\infty} T^n z^n$ is analytic on $\partial\mathbb{D}\setminus\{1\}$, the same is true for

$$F(z) = (1-z)g(z) = I + \sum_{n=1}^{\infty}(T^n - T^{n+1})z^n$$

$$\Rightarrow \|T^n - T^{n+1}\| \longrightarrow 0.$$

[4] Local pseudo-function behavior

All Tauberian theorems we have stated so far are valid if the boundary condition is replaced by local pseudo function boundary behavior.

(6) <u>Definition 1</u>: A tempered distribution $f \in S'(\mathbb{R})$ is called a (global) pseudofunction if $\hat{f} \in L^{\infty}(\mathbb{R})$ and $\lim_{|x| \to +\infty} \hat{f}(x) = 0$. We write $f \in PF(\mathbb{R})$.

A distribution $f \in D'(U)$ is a local pseudofunction on U if for every open subset V with compact closure $\overline{V} \subseteq U$, $\exists g_V \in PF(\mathbb{R})$ s.t. $f = g_V$ on V. We write $f \in PF_{loc}(U)$.

Equivalently, $f \in PF_{loc}(U)$ iff $\forall \varphi \in D(U)$, $\varphi . f \in PF(\mathbb{R})$.

Let G be analytic in $\operatorname{Re} s > \alpha$. G has distributional boundary values on $\operatorname{Re} s = \alpha$ if $\exists g \in D'(\mathbb{R})$ s.t.

$$\lim_{\sigma \to \alpha^+} G(\sigma + it) = g(t) \text{ in } D'(\mathbb{R}).$$

If $g \in PF_{loc}(\mathbb{R})$, we say that G has local pseudofunction boundary behavior on $\operatorname{Re} s = \alpha$. We have shown the following extension of Theorem 6:

<u>Theorem 8</u>: Let Υ be slowly decreasing, vanishing on $(-\infty, 0)$ and have convergent Laplace transform on $\operatorname{Re} s > 0$. Let $\beta_1 \leq \cdots \leq \beta_M \in [0,1)$ and $k_1, \ldots, k_M \in \mathbb{Z}_+$.

(i) If $\mathcal{L}\{\Upsilon; s\} - \frac{a}{s^2} - \sum_{n=1}^{N} \frac{b_n}{s - it_n} - \sum_{n=1}^{M} \frac{c_n + d_n \log^{k_n}\left(\frac{1}{s}\right)}{s^{\beta_n + 1}}$

($t_n \in \mathbb{R}$) has local pseudofunction boundary behavior on $\operatorname{Re} s = 0$ then

$$\Upsilon(x) = ax + \sum_{n=1}^{N} b_n e^{it_n x} + \sum_{n=1}^{M} x^{\beta_n} \left(\frac{c_n}{\Gamma(\beta_n + 1)} + d_n \sum_{j=0}^{k_n} D_j(\beta_n + 1) \log^{k_n - j} x \right) + o(1),$$

where $D_j(\omega) = \frac{d^j}{dy^j}\left(\frac{1}{\Gamma(y)}\right)\Big|_{y=\omega}$.

(ii) Suppose there is a closed null set $E \subseteq \mathbb{R}$ s.t.:

(7) (I) $\mathcal{L}\{\tau;s\} - \sum_{n=1}^{N} \frac{b_n}{s_n - it_n}$ $(t_n \in \mathbb{R})$ has local pseudofunction boundary behavior on $i(\mathbb{R}\setminus E)$.

(II) $\forall t \in E \ \exists M_t > 0$ s.t.

$$\sup_{x>0} \left| \int_0^x \tau(u) e^{-itu} du \right| < M_t,$$

(III) $E \cap \{t_1, \dots, t_N\} = \phi$.

Then $\tau(x) = \sum_{n=1}^{N} b_n e^{it_n x} + o(1)$. ///

Theorem 8 can be used to given the following extension of Korevaar's version of the W-I (2005).

Theorem 9: Let $S \nearrow$ have convergent Laplace-Stieltjes transform for $\operatorname{Re} s > \alpha \geq 0$. Suppose there is a closed null set E, constants $r_0, r_1, \dots, r_N \in \mathbb{R}$, $\theta_1, \dots, \theta_N \in \mathbb{R}$, and $t_1, \dots, t_N > 0$ s.t. (I) $\mathcal{L}\{dS;s\} - \frac{r_0}{s-\alpha} - \sum_{n=1}^{N} r_n \left(\frac{e^{i\theta_n}}{s-\alpha-it_n} + \frac{e^{-i\theta_n}}{s-\alpha+it_n} \right)$ admits local pseudofunction boundary behavior on $\alpha + i(\mathbb{R}\setminus E)$.

(II) $E \cap \{0, t_1, \dots, t_N\} = \phi$, and

(III) $\sup_{x>0} \left| \int_0^x e^{-\alpha u - itu} dS(u) \right| < M_t < \infty$, $\forall t \in E$.

Then,

$$S(x) = e^{\alpha x} \left(\frac{r_0}{\alpha} + 2 \sum_{n=1}^{N} \frac{r_n \cos\left(t_n x + \theta_n - \arctan\left(\frac{t_n}{\alpha}\right)\right)}{\sqrt{\alpha^2 + t_n^2}} + o(1) \right), \quad x \to \infty$$

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⑧ Finally, Theorem 8 also contains the following generalization of Theorem 7:

<u>Theorem 10</u>: Let $F(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic in $\mathbb{D}$. Suppose there is a closed subset $E \subseteq \partial\mathbb{D}$ of null (linear) measure s.t. F has local pseudofunction boundary behavior on $\partial\mathbb{D} \setminus E$, while for each $e^{i\theta} \in E$

$$\sum_{n=0}^{N} c_n e^{in\theta} = O_\theta(1)$$

Then $c_n = o(1)$ and $\sum_{n=0}^{\infty} c_n e^{in\theta_0}$ converges at every point where there is a constant $F(e^{i\theta_0})$ s.t.

$$\frac{F(z) - F(e^{i\theta_0})}{z - e^{i\theta_0}}$$

has pseudofunction boundary behavior in a neighborhood of $z = e^{i\theta_0}$; moreover,

$$\sum_{n=0}^{\infty} c_n e^{in\theta_0} = F(e^{i\theta_0})$$

The proofs of the above stated theorems can be found in the recent preprint:

G. Debruyne, J. Vindas, Complex Tauberian theorems for Laplace transforms with local pseudofunction boundary behavior, arXiv 1604.05069