On an open question for Beurling primes concerning the relation between the PNT and the mean-value of the Möbius function.

(Ghent, 14-3-2018)

$\boxed{1}$ Introduction: In classical number theory there are several asymptotic relations that are considered to be equivalent to the PNT. Some of them involve the Möbius function.

$$\mu(n)=\begin{cases} 1 & n=1 \\ (-1)^r & n \text{ is squarefree and } n=p_1\cdots p_r \\ 0 & \text{otherwise.} \end{cases}$$

It is thus very well known that the following three statements are deducible one another via relatively simple elementary (= real variable) methods:

(1) PNT: $\pi(x)=\sum_{p\leq x} 1 \sim \frac{x}{\log x}$

(2) $M(x)=\sum_{n\leq x}\mu(n)=o(x)$.

(3) $m(x)=\sum_{n\leq x}\frac{\mu(n)}{n}=o(1)$.

Equivalences between such kinds of asymptotic relations are called $\boxed{\text{PNT equivalences}}$. For Beurling primes these PNT equivalences do not longer hold in general (but sometimes implications hold true if additional conditions are assumed).

W.B. Zhang was the first to observe failures in 1987:

$M(x)=o(x) \not\Rightarrow$ (in general) the PNT in Beurling context.

or

$m(x)=o(1)$. The following question for Beurling primes is open:

$\boxed{\text{Open question}}$: Prove or disprove "PNT $\Rightarrow M(x)=o(x)$".

We do not know how to answer this question but we would like to discuss some related points here.

2 Beurling number system: We adopt a general definition here.

Given two measures (always on $[0,\infty)$, vanishing on $[0,1)$), their (multiplicative) convolution is determined via

$$\int_{1^-}^{x} d\alpha * d\beta = \iint_{u\cdot v \leq x} d\alpha(u)\, d\beta(v) = \int_{1^-}^{x} \beta\left(\frac{x}{u}\right) \cdot d\alpha(u)$$

$$= \int_{1^-}^{x} \alpha\left(\frac{x}{v}\right) d\beta(v)$$

Def. 1 A g-number system is a pair of non-decreasing (right-continuous) functions Π and N s.t. $\Pi(1)=0$, $N(1)=1$ and

$$\boxed{dN = \exp^*(d\Pi)} \quad (1)$$

i.e.,

$$dN = \delta_1 + d\Pi + \frac{d\Pi^{2*}}{2!} + \cdots + \frac{d\Pi^{n*}}{n!} + \cdots \quad //$$

(Dirac delta concentrated at 1)

We are mostly interested in N and Π growth $\ll x^m$ for some m. This would ensure convergence of their Mellin-Stieltjes transform so that (1) becomes equivalent to

$$\zeta(s) := \int_{1^-}^{\infty} x^{-s} dN(x) = \exp\left(\int_{1}^{\infty} x^{-s} d\Pi(x)\right). \quad (2)$$

Example 1: Let $1 < p_1 \leq p_2 \leq \cdots \leq p_k \longrightarrow \infty$, set $n_0 = 1 < n_1 \leq n_2 \leq \cdots$ for the semigroup generated by $\{p_k\}$ and 1.

and $\Pi(x) = \sum_{p_k^\alpha \leq x} \frac{1}{\alpha} \left(= \sum_{\alpha=1}^{\infty} \frac{\pi(x^{\frac{1}{\alpha}})}{\alpha}\right)$.

Then,

$$\exp\left(\int_{1^-}^{\infty} x^{-s} d\Pi(x)\right) = \exp\left(\sum_{p_k} \sum_{\alpha=1}^{\infty} \frac{1}{\alpha p_k^{\alpha s}}\right) = \exp\left(-\sum_{p_k} \log\left(1 - \frac{1}{p_k^s}\right)\right)$$

$$= \prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{p_k^s}} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{p_k^s} + \frac{1}{p_k^{2s}} + \cdots\right) = \sum_{k=0}^{\infty} \frac{1}{n_k^s}.$$

Therefore,

$$N(x) = \sum_{n_k \leq x} 1. \quad //$$

Other number-theoretic functions can be defined

Definition 2. The measures dM and dm are defined as:

- dM is the convolution inverse of dN, i.e. $dM * dN = \delta_1$ (equivalently, $dM = \exp^*(-d\Pi)$.
- dm is defined via $m(x) = \int_{1^-}^{x} \frac{dM(u)}{u}$.

Remark 1.: The Mellin transform version of the definition of M is (see (2)):

$$\boxed{\int_{1^-}^{\infty} x^{-s}\, dM(x) = \frac{1}{\zeta(s)}} \quad (3).$$

Example 2. For the number system of Example 1,

$$M(x) = \sum_{n_k \leq x} \mu(n_k),$$

with μ defined analogously as for the ordinary integers.

Example 3. The most simple number system is the continuous one with

$$\boxed{d\Pi_0(u) = \frac{1 - \frac{1}{u}}{\log u}\, du.}$$

To find dN_0, we calculate the zeta function:

$$\log \zeta_0(s) = \int_1^{\infty} (u^{-s} - u^{-s-1}) \frac{du}{\log u} = \log s - \log(s-1)$$

$$\Rightarrow \zeta_0(s) = \frac{s}{s-1} = 1 + \frac{1}{s-1} = \int_{1^-}^{\infty} x^{-s}\, dN_0(x) \Rightarrow \boxed{dN_0(x) = \delta_1 + dx}$$

i.e.,

$$\boxed{N_0(x) = x}, \quad x \geq 1.$$

The dM_0 can also be computed.

$$\int_{1^-}^{\infty} x^{-s} dM_0(x) = \frac{1}{\zeta_0(s)} = \frac{s-1}{s} = 1 - \frac{1}{s}$$

$$\Rightarrow \boxed{dM_0 = \delta_1 - \frac{du}{u}} \Rightarrow \boxed{M_0(x) = 1 - \log x} = (\vartheta(x))$$

Thus $m(x) = \int_{1^-}^{x} \delta_1 - \frac{du}{u^2} = 1 + \frac{1}{x} - 1 = \frac{1}{x}$, $x \geq 1$.

$\boxed{3}$ Positive results for a PNT $\Rightarrow M(x)$?

First we remark:

Proposition 1: $m(x) = O(1) \Rightarrow M(x) = O(x)$.

Proof. $dm(u) = \frac{dM(u)}{u} \Rightarrow M(x) = \int_{1^-}^{x} u\, dm(u) = x m(x) - \int_1^x m(u) du$
$= O(x)$.

Now, we have been able to show the following results in collaboration with G. Debruyne and H. Diamond. The first result has the PNT as hypothesis

Theorem 1 (Debruyne-Diamond-V.)
The PNT + the condition $N(x) << x \Rightarrow M(x) = O(x)$. //

Our second result uses a Chebyshev inequality as part of its hypotheses and very much improves results by W.-B. Zhang

Theorem 2. (Debruyne-Diamond-V.)
The estimate $\Pi(x) << \frac{x}{\log x}$ + the side condition $N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right)$
where $a > 0$ and $\gamma > \frac{1}{2}$ imply $M(x) = O(x)$.

We (= the same people as before) have also been able to show that the Chebyshev bound by itself is not strong enough to ensure $M(x) = \Theta(x)$.

Proposition 2. There is a number system such that

① $\pi(x) \ll \dfrac{x}{\log x}$

② $N(x) \ll x$

But:

③ $M(x) = \Omega(x)$ (i.e. $M(x) = \Theta(x)$ does not hold).

Naturally, one proves Proposition 2 by exhibiting an example.

Example 3: We do not give technical details but rather list properties of the number system. It is given by

$$\pi(x) = \sum_{2^{k+\frac{1}{2}} \le x} \frac{2^{k+\frac{1}{2}}}{k}$$

One verifies:

$$\limsup_{x \to \infty} \frac{\pi(x) \log x}{x} = 2 \log 2.$$

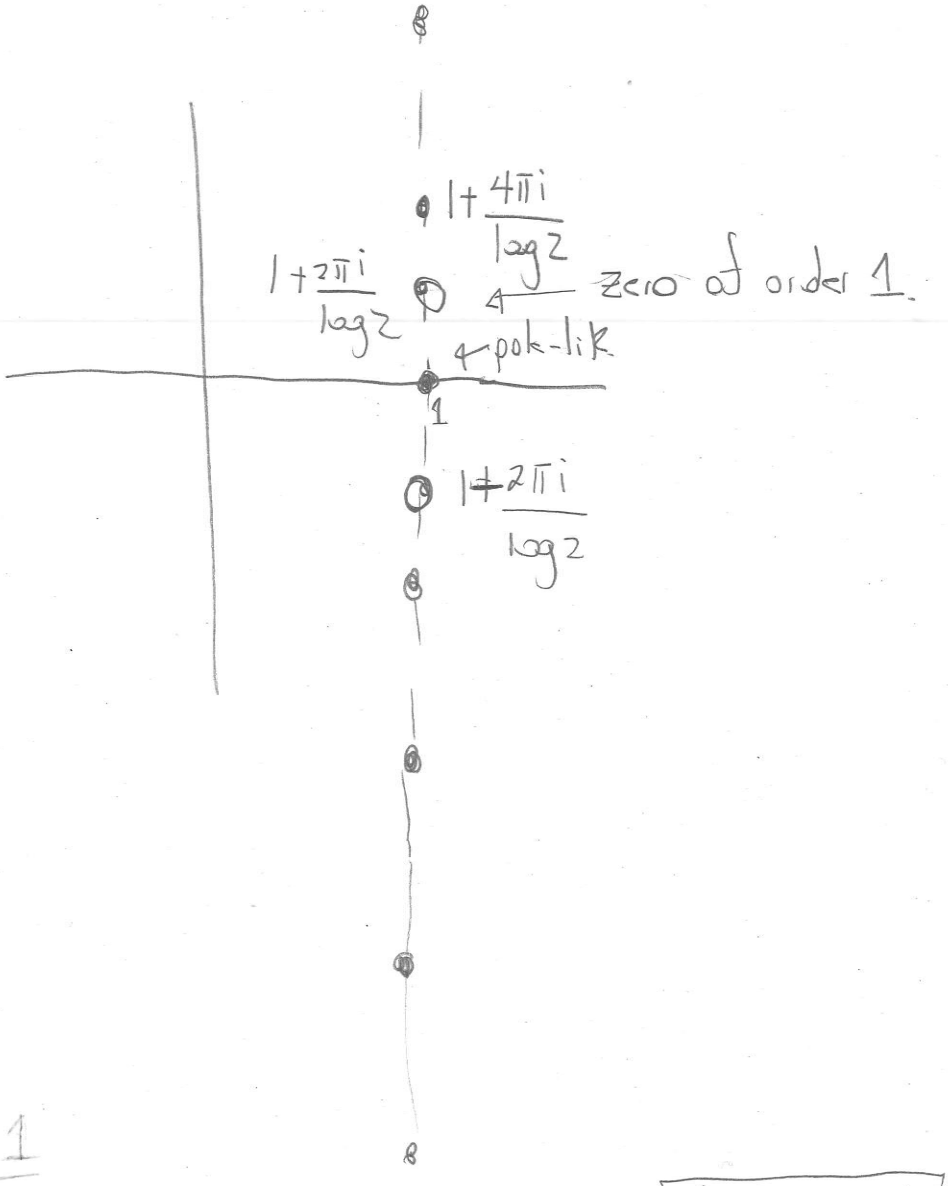
Thus ① holds.

The zeta function is

$$\log \zeta(s) = \sum_{k=1}^{\infty} \frac{2^{-(k+\frac{1}{2})(s-1)}}{k} = -2^{\frac{-(s-1)}{2}} \log\left(1 - \frac{1}{2^{s-1}}\right)$$

$$\zeta(s) \sim \frac{1}{(s-1)\log 2} \qquad s \to 1$$

Also $4\pi i/\log 2$-periodic.



Ocurrence of zero in $\zeta(s)$ for $\mathrm{Re}\, s=1 \Rightarrow$ $\boxed{M(x)=\Omega(x).}$

The latter gives $\boxed{m(x)=\Omega(1)}$.

Interestingly, we have shown via a Tauberian analysis that

$$\boxed{m(x)=O(1)}$$

Finally, the occurrence of the poles others than at $s=1$ give a measure of the wobble of N. In fact, an oscilation theorem (essentially due to Ingham in 1942) give quantitave wobble:

$$1.51 < \varliminf \sup \frac{N(x)}{x}$$

$$\varliminf \inf \frac{N(x)}{x} < 1.37.$$

And the Graham-Vaaler version of the Wiener-Ikehara theorem yields.

$$1.20 \leq \varliminf \inf \frac{N(x)}{x} < 1.37 < 1.51 < \varliminf \sup \frac{N(x)}{x} \leq 1.71 \; //$$

$\boxed{4}$ More about "$PNT \not\Rightarrow M(x) = O(x)$".

If one attempts to show that PNT by itself does not neccesarily yields $M(x) = O(x)$, Theorem 1 tells that

$$\frac{N(x)}{x} \text{ must be unbounded.} \quad (4)$$

There are conditions in the literature ensuring that (4) and the PNT simultaneously hold. One example is given by the following proposition of Diamond and Zhang (see their book on Beurling generalized numbers, p. 60, Proposition 6.11), here we use a special case).

Recall $d\Pi_0$ stands for the "basic" number system from Example 3.

<u>Proposition 3.</u> Let $d\Pi(u) = d\Pi_0(u) + \frac{du}{f(u)\log u} \chi_{[A,\infty)}(u)$ $(A \geq e)$ be such that

$$\int_A^\infty \frac{du}{u f(u) \log u} = \infty, \text{ with } f(x) \geq 1.$$

Then $\overline{\lim_{x\to+\infty}} \frac{N(x)}{x} = \infty$ //

If additionally, $f(x) \to \infty$. The $d\Pi$ satisfies the PNT as easily seen ($\Pi(x) \sim \frac{x}{\log x}$).
This suggested J.-P. Kahane and Saias that

$$(5)\quad d\Pi(u) = \frac{1-\frac{1}{u}}{\log u}\,du + \frac{\chi_{[A,\infty)}(u)}{\log u \log\log u}\,du \quad (A = e^e)$$

might be a numbersystem such that the PNT holds but $M(x) = o(x)$ fails. For some time we tried to show their conjecture (which would answer the main question we are discussing in this lecture). However, I eventually realized that $M(x) = o(x)$ actually holds. Moreover, an analysis based on elementary convolution identities and Koromotoc type real Tauber theorems can be performed to show:

Proposition 4: For the number system (5)
$$m(x) = o\left(\frac{1}{\log\log x}\right) \text{ and } M(x) = o\left(\frac{x}{\log\log x}\right) //$$

The convolution technique used to show Proposition 4 inspired me to prove the following theorem, which basically tells a large class of examples constructed by perturbation of $d\Pi_0$ that have $M(x) = o(x)$.

Theorem 3. Suppose Π can be written as $\Pi(x) = \frac{x}{\log x} + E_1(x) + E_2(x)$, $x \geq 1$,
where (a) $E_2(x) = O\left(\frac{x}{\log x}\right)$

(b) E_2 is monotone

(c) $\int^{\infty} \frac{|dE_1(x)|}{x} < \infty$

Then, $m(x) = o(1)$
($\Rightarrow M(x) = o(x)$) ///