Wavelets and Gelfand-Shilov spaces

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In this talk we discuss approximation properties of MRA and wavelets in the so-called Gelfand-Shilov spaces.

I will talk about:

1. Some classes of ‘highly regular’ MRA and wavelets.
2. Their connection with Gevrey and Gelfand-Shilov spaces.
3. Approximation properties of these highly regular MRA and wavelets.
4. Some mapping properties of the wavelet transform.

The talk is based on collaborative works with Dušan Rakić, Stevan Pilipović, and Nenad Teofanov.

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Orthonormal wavelets

A function $\psi \in L^2(\mathbb{R})$ is called an orthonormal wavelet if

$$\psi_{n,m}(x) = 2^{m/2}\psi(2^m x - n), \qquad n, m \in \mathbb{Z},$$

is an orthonormal basis of $L^2(\mathbb{R})$. Given any $f \in L^2(\mathbb{R})$ we have the wavelet series expansion

$$f = \sum_{n,m \in \mathbb{Z}} c_{n,m}\psi_{n,m},$$

where

$$c_{n,m} = \int_{-\infty}^{\infty} f(x)\overline{\psi}_{n,m}(x)\mathrm{d}x.$$

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Multiresolution Analysis

One effective way to construct wavelets is via the next concept.

Definition

A multiresolution analysis (MRA) is an increasing sequence $\{V_m\}_{m\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $\bigcap_{m\in\mathbb{Z}} V_m = \{0\}$ and $\bigcup_{m\in\mathbb{Z}} V_m$ is dense in $L^2(\mathbb{R}^d)$;

(ii) $f(x) \in V_m \Leftrightarrow f(2x) \in V_{m+1}, \ m \in \mathbb{Z}$;

(iii) $f(x) \in V_0 \Leftrightarrow f(x-n) \in V_0, \ n \in \mathbb{Z}^d$;

(iv) there exists $\phi \in L^2(\mathbb{R}^d)$ such that $\{\phi(x-n)\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis of V_0.

The function ϕ from (iv) is called a scaling function for the MRA.

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Construction of wavelets from an MRA

Let $\{V_m\}_{m\in\mathbb{Z}}$ be an MRA with scaling function ϕ.

- Write $V_1 = V_0 \oplus W_0$.
- $V_{m+1} = V_m \oplus W_m$ with $W_m = \{f : f(2^{-m}x) \in W_0\}$.
- $V_m = W_{m-1} \oplus W_{m-2} \oplus \dots$.
- $L^2(\mathbb{R}) = \bigoplus_{m\in\mathbb{Z}} W_m$.

Conclusion: We get an orthonormal wavelet if we find $\psi \in W_0$ such that $\{\psi(x-n)\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis of W_0.

$\phi(x/2) \in V_{-1}$ has expansion in terms of $\{\phi(x-n)\}_{n\in\mathbb{N}}$. Fourier transforming we find a 2π-periodic function m_0 with

$$\widehat{\phi}(2\xi) = m_0(\xi)\widehat{\phi}(\xi).$$

Characterization: $\{\psi(x-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis of W_0 iff there is a 2π-periodic function $\nu \in L^2[-\pi,\pi]$ such that

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Regularity of wavelets: smoothness vs decay

- MRA and wavelets are effective to approximate functions,
- and, in turn, to describe a large number of function and distribution spaces.
- This effectiveness: related to regularity properties of scaling function and wavelet.
- By regularity we mean: smoothness and decay.
- There is however a trade-off between smoothness and decay.

Here a well-known example of this interplay leading to conflicts:

There is no orthonormal wavelet ψ sharing simultaneously these two properties:

1. $\psi(x) \ll e^{-c|x|}$ for some $c > 0$.
2. $\psi \in C^{\infty}(\mathbb{R})$, with all derivatives being bounded.

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What we cannot get!

Let us fix $\psi \in L^1(\mathbb{R})$ with the second property from the last statement, that is,

$$\psi^{(n)} \in L^\infty(\mathbb{R}), \qquad n = 0, 1, 2, \ldots . \tag{1}$$

We consider the decay (for a positive weight function ω):

$$\psi(x) \ll e^{-\omega(|x|)}, \tag{2}$$

Under certain standard regularity assumptions ω, one shows:

If there is an orthonormal wavelet ψ satisfying (1) and (2) then

$$\int_1^\infty \frac{\omega(x)}{x^2} < \infty. \tag{3}$$

Conclusion: No wavelets with (1) and (2) such that (3) diverges

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What we can try to do!

Due to the constrains we have discussed so far, we might try to find smooth ψ (with bounded derivatives) with decay

$$\psi(x) \ll e^{-\omega(x)}, \qquad \text{where } \int_1^\infty \frac{\omega(x)}{x^2} < \infty.$$

First try

$\omega(x) = n \log x$, so that $\psi(x) \ll |x|^{-n}$.

This works! Actually, Meyer did better in 1985 and found an orthonormal wavelet $\psi \in \mathcal{S}(\mathbb{R})$. For future reference:

Properties of orthonormal wavelets $\psi \in \mathcal{S}(\mathbb{R})$

1. ψ is an MRA wavelet.
2. $\int_{-\infty}^{\infty} x^n \psi(x) dx = 0$, $n = 0, 1, \ldots$

We write $\mathcal{S}_0(\mathbb{R})$ for the subspace of $\mathcal{S}(\mathbb{R})$ consisting of functions whose all moments vanish.

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Properties of orthonormal wavelets $\psi \in \mathcal{S}(\mathbb{R})$

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Define the class $\mathcal{E}^{\{M_n\}}[a,b]$ of smooth functions such that

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One may assume $m_n = M_{n+1}/M_n$ increases (Cartan-Gorny theorem).

Hadamard's problem, 1912

Characterize $\{M_n\}_{n\in\mathbb{N}}$ such that $\mathcal{E}^{\{M_n\}}[a,b]$ contains non-trivial compactly supported functions in (a,b) (= non-quasianalyticity).

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Suppose $m_n = M_{n+1}/M_n$ is increasing. Then, $\mathcal{E}^{\{M_n\}}[a,b]$ is non-quasianalytic iff $\sum_{n=0}^{\infty} 1/m_n < \infty$.

Under 'standard assumptions', one adapts the Dziubański-Hernández construction to find a band-limited orthogonal wavelet with decay $\psi(x) \ll e^{-M(|x|)}$, where $M(x) = \sup_{n\in\mathbb{N}} \log_+(x^n/M_n)$

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- The Dziubański-Hernández wavelets belong to $\mathcal{F}(G^t_c(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transform.
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Gelfand-Shilov spaces

- The Dziubański-Hernández wavelets belong to $\mathcal{F}(G_c^t(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transform.
- Elements of $\mathcal{F}(G_c^t(\mathbb{R}))$ are determined by global estimates

$$|x^m f^{(n)}(x)| \ll B^{n+m}(m!)^t \qquad x \in \mathbb{R}.$$

Definition

Let $t, s \geq 0$. The space $\mathcal{S}_t^s(\mathbb{R})$ consists of all Schwartz functions such that, for some B,

$$|x^m f^{(n)}(x)| \ll B^{n+m}(n!)^s(m!)^t.$$

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Some properties of Gelfand-Shilov spaces

- The the family $\mathcal{S}_t^s(\mathbb{R})$ is increasing with respect to s and t.
- $\mathcal{F} : \mathcal{S}_t^s(\mathbb{R}) \to \mathcal{S}_s^t(\mathbb{R})$ is an isomorphism.
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$$|f(x)| \ll e^{-c|x|^{1/t}} \quad \text{and} \quad |\widehat{f}(\xi)| \ll e^{-c|\xi|^{1/s}}.$$

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$$|f(x + iy)| \ll e^{-c|x|^{1/t}} \quad \text{for } |y| < h$$

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Gelfand-Shilov regular MRA and wavelets

If ψ is a Dziubański-Hernández wavelet with $\psi(x) \ll e^{-c|x|^{1/\rho_2}}$, then $\psi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R})$ for all $\rho_1 \geq 0$. They are examples of

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Remark

It should be by now clear that $\rho_2 \leq 1$ is not admissible here.

Open question

Every (ρ_1, ρ_2)-regular is an MRA wavelet. Does it arise from a (ρ_1, ρ_2)-regular MRA?

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Convergence of multiresolution expansions

Theorem

Let $\{V_m\}_{m\in\mathbb{Z}}$ be a (ρ_1,ρ_2)-regular MRA with orthogonal projections

$$E_m : L^2(\mathbb{R}) \to V_m$$

and set $\sigma = \rho_1 + \rho_2 - 1$. Let $s \geq \sigma$ and $t \geq \rho_2$. Then,

$$\lim_{m\to\infty} E_m f = f \ \text{ in } \ \mathcal{S}_t^s(\mathbb{R}),$$

for each $f \in \mathcal{S}_t^{s-\sigma}(\mathbb{R})$.

There is a loss of regularity measured by $\sigma > 0$. We wonder

1. Is σ optimal? We conjecture so ...
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Write $(\mathcal{S}_t^s)_0(\mathbb{R}) = \{f \in \mathcal{S}_t^s(\mathbb{R}) : \int_{-\infty}^{\infty} x^n f(x)dx = 0,\ n = 0, 1, \dots\}$.

A (ρ_1, ρ_2)-regular wavelet automatically satisfies $\psi \in (\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R})$.

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Let $\psi \in (\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R})$ be a (ρ_1, ρ_2)-regular orthonormal wavelet. Set $\sigma = \rho_1 + \rho_2 - 1$ and consider $s > \sigma$ and $t > \sigma + 1$.

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The wavelet transform: distribution case

We consider the wavelet transform

$$\mathcal{W}_{\psi} f(b,a) = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\overline{\psi}\left(\frac{x-b}{a}\right)\,\mathrm{d}x.$$

Denote $\mathbb{H} = \{(b,a): \ a > 0\}$. The wavelet synthesis operator is

$$\mathcal{M}_{\psi} F(x) = \iint_{\mathbb{H}} F(b,a)\psi\left(\frac{x-b}{a}\right)\frac{\mathrm{d}b\mathrm{d}a}{a^2}.$$

The space of highly localized functions on $\mathbb{H}$ is

$$\mathcal{S}(\mathbb{H}) = \{F \in C^{\infty}(\mathbb{H}) : F(b,a) \ll (1+|b|)^{-n}(a+1/a)^{-n},\ \forall n > 0\}.$$

For a wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$, one gets continuity of

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$$\mathcal{W}_\psi f(b,a) = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\overline{\psi}\left(\frac{x-b}{a}\right)\,\mathrm{d}x.$$

Denote $\mathbb{H} = \{(b,a) : \; a > 0\}$. The wavelet synthesis operator is

$$\mathcal{M}_\psi F(x) = \iint_{\mathbb{H}} F(b,a)\psi\left(\frac{x-b}{a}\right)\frac{\mathrm{d}b\mathrm{d}a}{a^2}.$$

The space of highly localized functions on $\mathbb{H}$ is

$$\mathcal{S}(\mathbb{H}) = \{F \in C^\infty(\mathbb{H}) : F(b,a) \ll (1+|b|)^{-n}(a+1/a)^{-n}, \; \forall n > 0\}.$$

For a wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$, one gets continuity of

$$\mathcal{W}_\psi : \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}(\mathbb{H}) \quad \text{and} \quad \mathcal{M}_\psi : \mathcal{S}(\mathbb{H}) \to \mathcal{S}_0(\mathbb{R}),$$

which yields a wavelet transform theory for distributions.

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The wavelet transform in Gelfand-Shilov spaces

Let s, t, τ_1, τ_2. Define $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H})$ as the space of smooth functions satisfying estimates

$$\partial_a^m \partial_b^n F(b,a) \ll_m B^n (n!)^s \exp\left(-c\left(a^{1/\tau_1} + a^{-1/\tau_2} + |b|^{1/t}\right)\right)$$

We have made a thorough analysis of wavelet transforms on Gelfand-Shilov spaces. In their simplest forms, our results yield:

Theorem

Let $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R})$ where $\rho_1 \geq 0$ and $\rho_2 > 1$. Set $\sigma = \rho_1 + \rho_2 - 1$. If $s > \sigma$ and $t > \sigma + 1$, the wavelet mappings

$$\mathcal{W}_\psi : (\mathcal{S}^{s-\sigma}_{t-\sigma})_0(\mathbb{R}) \to \mathcal{S}^s_{t,t-\rho_2,s-\rho_1}(\mathbb{H})$$

and

$$\mathcal{M}_\psi : \mathcal{S}^s_{t,t-\rho_2,s-\rho_1}(\mathbb{H}) \to (\mathcal{S}^s_t)_0(\mathbb{R})$$

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Some references

For more details about the subject of this talk, see my joint articles with D. Rakić, S. Pilipović, and N. Teofanov:

1. *The wavelet transforms in Gelfand-Shilov spaces,* Collect. Math. 67 (2016), 443–460.
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For details on the construction of wavelets of subexponential decay, see e.g.:

- J. Dziubański, E. Hernández, *Band-limited wavelets with subexponential decay*, Canad. Math. Bull. 41 (1998), 398–403.
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