Fórmula de inversión de Fourier y caracterización de valores puntuales de distribuciones temperadas

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- What does f at x_0 mean?
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- What is the meaning of $\int_{-\infty}^{\infty} f\hat{(t)}e^{-itx_0}dt$?
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- The evaluation of f at a test function ϕ is denoted by

$$\langle f(x), \phi(x) \rangle$$

Distributional Point Values

Lojasiewicz defined the value of a distribution $f \in \mathcal{D}'$ at the point x_0 as the limit

$$f(x_0) = \lim_{\varepsilon \longrightarrow 0} f(x_0 + \varepsilon x)\,,$$

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In terms of test funcions, it means that for all $\phi \in \mathcal{D}$

$$\lim_{\varepsilon \to 0} \frac{1}{\epsilon} \left\langle f(x), \phi\left(\frac{x - x_0}{\epsilon}\right)\right\rangle = f(x_0) \int_{-\infty}^{\infty} \phi(x)dx.$$

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- **Notation:** If $f \in \mathcal{D}'$ has a value γ at x_0, we say that $f(x_0) = \gamma$ in $\mathcal{D}'$. The meaning of $f(x_0) = \gamma$ in $\mathcal{S}'$, ..., must be clear.

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- **Remark:** R.Estrada has shown that if $f \in \mathcal{S}'$, then $f(x_0) = \gamma$ in $\mathcal{D}'$ implies $f(x_0) = \gamma$ in $\mathcal{S}'$.

Characterization of Distributional Point Values

Lojasiewicz showed that $f(x_0) = \gamma$ is equivalent to the existence of $n \in \mathbb{N}$, and a primitive of order n of f which is continuous in a neighborhood of x_0 and satisfies

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In particular if f is locally integrable and x_0 is a Lebesgue point of f, then f has a distributional point value at x_0

Cesaro limits and Cesaro summability

We say that

$$\lim_{n\to\infty} a_n = \gamma \ \ (C,1)$$

if

$$\lim_{n\to\infty} \frac{a_0 + a_1 + ... + a_{n-1} + a_n}{n+1} = \gamma$$

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Remark: (C, k) summability implies Abel summability.

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Remark: (C,k) summability implies Abel summability.
Remark: There are others equivalent methods for considering (C,k) summability which are more adequate to our analysis, namely Riesz Typical means.

Cesaro integrability of integrals and measures

Let f be locally integrable, we say that

$$\int_0^\infty f(t)dt = \gamma \ (C,k),$$

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$$\lim_{x\to\infty} \int_0^x f(t)\left(1-\frac{t}{x}\right)^k dt = \gamma$$

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Remark: The last can be also expressed as $F_{k+1}(x) = \gamma\frac{x^k}{k!} + o(x^k)$ as $x \to \infty$, where F_{k+1} is a $k+1$-iterated primitive with support in $[0,\infty)$.

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Remark: (R, λ_n, k) for series falls into this case by considering the borel measure $\sum_{n=0}^\infty c_n \delta(x - \lambda_n)$.

Summability of Fourier Series

G.Walter proved the following:

Theorem 1 *Let f be a periodic distribution (and hence tempered) with Fourier Series*

$$\sum_{n=0}^{\infty} c_n e^{inx}.$$

Then, $f(x_0) = \gamma$ *in* $\mathcal{S}'$ *implies that*

$$\sum_{n=0}^{\infty} c_n e^{inx_0} = \gamma \ (C, k),$$

for some $k \in \mathbb{N}$*.*

Proof

Lemma *Let* f *be a* 2π*-periodic distribution with Fourier series* $\sum_{-\infty}^{\infty} c_n e^{inx}$*, then* $f(x_0) = \gamma$ *iff*

$$\sum_{-\infty}^{\infty} c_n e^{inx_0} \psi(\epsilon x) = \gamma \psi(0) + o(1) \; as \; \epsilon \rightarrow 0,$$

for each $\psi \in \mathcal{S}$.

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Lemma *Let f be a 2π-periodic distribution with Fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$, then $f(x_0) = \gamma$ iff*

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for each $\psi \in \mathcal{S}$.

- We come back to the situation of our Theorem, let $g(x) = \sum_{n=0}^{\infty} c_n \delta(x - n)$. We have $\epsilon g\left(\frac{x}{\epsilon}\right) - \gamma = o(1)$ as $\epsilon \to 0$ in the weak topology of $\mathcal{D}'$.

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Schwartz'argument: Let $\{f_j\}$ be a sequence in $\mathcal{D}'$ such that $f_j \to 0$ in the weak topology of $\mathcal{D}'$. Then given a compact set K there exist $k \in \mathbb{N}$ and sequence F_j of continuous functions defined in an open neighborhood of K, V, such that $F_j \to 0$ uniformly on K and $F_j^{(k)} = f_j$ on distributionally on V.

Summability of Fourier Series

Moreover, he also showed

Theorem 2 *Let f be a periodic distribution (and hence tempered) with Fourier Series*

$$\sum_{-\infty}^{\infty} c_n e^{inx}.$$

If $f(x_0) = \gamma$ *in* $\mathcal{S}'$*, then for some* $k \in \mathbb{N}$

$$\lim_{N\to\infty} \sum_{-N}^{N} c_n e^{inx_0} = \gamma \ (C, k).$$

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Some remarks

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- Under certain assumtions on the conjutated series, G.Walter gave a sort of converse of this result.
- If we only assume the (C,k)-summability of the symmetric partial sums, the converse is far from being true as shown by

$$2\sum_{n=1}^{\infty}\frac{sin(nx)}{n} = -i\sum_{n\neq 0}\frac{e^{inx}}{n}.$$

at $x = 0$

Characterization of Point Values

R.Estrada has characterized the distributional point values of periodic distribution in terms of the summability of their Fourier Series.

Theorem 3 *Let $f \in \mathcal{S}'$ be a periodic distribution of period 2π and let $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ be its Fourier series. Let $x_0 \in \mathbb{R}$. Then*

$$f(x_0) = \gamma \ in \ \mathcal{D}'$$

if and only if there exists k such that

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} a_n e^{inx_0} = \gamma \ (\mathrm{C}, k)$$

for each $a > 0$.

Needed for a generalization

The last Theorem admits a generalization to tempered distribution which "looks" like

$$f(x_0) = \lim_{x \to \infty} \int_{-x}^{ax} \hat{f}(t) e^{-itx_0} dt \ (C).$$

Cesaro behavior of Distributions

Let $f \in \mathcal{D}'$ and and $\alpha \in \mathbb{R} - \{-1, -2, -3, ...\}$, then we say that

$$f(x) = O\left(x^{\alpha}\right) \ (C, N) \ as \ x \to \infty,$$

if there exists $N \in \mathbb{N}$ such that every primitive F of order N, is an ordinary function(locally integrable) for large arguments and satisfies the ordinary order relation,

$$F(x) = p(x) + O\left(x^{\alpha+N}\right) \ as \ x \to \infty$$

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Note that if $\alpha > -1$, then the polynomial p is irrelevant.

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- The definitions when $x \to -\infty$ are clear.
- One can define the limit at ∞ in the Cesàro sense for distribution. We say that $f \in \mathcal{D}'$ has a limit L at infinity in the Cesàro sense and write

$$\lim_{x \to \infty} f(x) = L \ (\mathrm{C}) \,,$$

if $f(x) = L + o(1)\ (\mathrm{C})$, as $x \to \infty$.

Parametric Behavior

The Cesàro behavior of a distribution f at infinity is related to the parametric behavior of $f(\lambda x)$ as $\lambda \to \infty$(To be interpreted in the weak sense, i.e evaluating at test functions)

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In fact, one can show that if $\alpha > -1$, then $f(x) = O\left(x^{\alpha}\right)$ (C) as $x \to \infty$ and $f(x) = O\left(|x|^{\alpha}\right)$ (C) as $x \to -\infty$ if and only if

$$f(\lambda x) = O\left(\lambda^{\alpha}\right) \ as \ \lambda \to \infty\,,$$

Special values of distributional evaluations

Definition 1 *Let* $g \in \mathcal{D}'$, *and* $k \in \mathbb{N}$. *We say that the evaluation* $\langle g(x), \phi(x) \rangle$ *exists in the* $\mathrm{e.v.}$ *Cesàro sense, and write*

$$\mathrm{e.v.}\, \langle g(x), \phi(x) \rangle = \gamma \quad (\mathrm{C}, k), \tag{1}$$

if for some primitive G *of* $g\phi$ *and* $\forall a > 0$ *we have*

$$\lim_{x \to \infty} (G(ax) - G(-x)) = \gamma\ (\mathrm{C}, k).$$

If g *is locally integrable then we write (1) as*

$$\mathrm{e.v.} \int_{-\infty}^{\infty} g(x)\, \phi(x)\, \mathrm{d}x = \gamma \quad (\mathrm{C}, k).$$

Remark: *In this definition the evaluation of* g *at* ϕ *does not have to be defined, we only require that* $g\phi$ *is well defined.*

Example

Suppose that $\{\lambda_n\}$ is positive increasing sequence. If $g \in \mathcal{S}'$ is given by $g(x) = \sum_{n=0}^{\infty} a_n \delta(x - \lambda_n)$, then

$$\mathrm{e.v}\left\langle \sum_{n=0}^{\infty} a_n \delta(x - \lambda_n), 1 \right\rangle = \gamma\left(C, k\right)$$

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if and only if

$$\sum a_n = \gamma \ \ (R, \lambda_n, k)$$

Wawak weak integrability of distributions

Wawak has defined the integral of a distribution with support on $[0,\infty)$ (1990) as follows, we say that

$$\int_0^\infty f(t)dt = \gamma\ (W)$$

if for a the primitive of f, F,with support in $[0,\infty)$ such that we have

$$\lim_{x\to\infty} F(x) = \gamma\ (C)$$

The notion of evaluations in the e.v. sense generalizes Wawak weak integrability.

Pointwise Inversion Formula

Now, we characterize the point values of a distribution in $\mathcal{S}'$ by using Fourier transforms.

Theorem 4 *Let $f \in \mathcal{S}'$. We have $f(x_0) = \gamma$ in $\mathcal{S}'$ if and only if there exists a $k \in \mathbb{N}$ such that*

$$\frac{1}{2\pi}\mathrm{e.v.}\left\langle \hat{f}\left(t\right), e^{-ix_0 t}\right\rangle = \gamma \quad (C, k)\,,$$

which in case $\hat{f}$ is locally integrable means that

$$\frac{1}{2\pi}\mathrm{e.v.}\int_{-\infty}^{\infty} \hat{f}(t)e^{-ix_0 t}\mathrm{d}t = \gamma \quad (C, k)\,.$$

Consequences

Estrada Theorem on Fourier Series follows at one by looking at the form of the Fourier transforms of periodic distributions. Moreover,

Theorem 5 *Let* $\{\lambda_n\}_{n=0}^{\infty}$ *be an increasing sequence of positive real numbers. Let*

(2)
$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{isgn(n)\lambda_{|n|}x} \; in \; \mathcal{S}'.$$

Then, $f(x_0) = \gamma \; in \; \mathcal{D}'$,
if and only if there exists $k \in \mathbb{N}$ *such that*

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} a_n e^{isgn(n)\lambda_{|n|}x_0} = \gamma\left(\mathrm{R}, \lambda_n, k\right),$$

for each $a > 0$.

Consequences

The inversion formula proves a Wawak's theorem.

Theorem 6 *Let* $f \in \mathcal{S}'$. *Suppose that* $supp\ \hat{f} \subseteq [0, \infty)$. *We have* $f(x_0) = \gamma$ *in* $\mathcal{S}'$ *if and only if*

$$\frac{1}{2\pi} \int_0^{\infty} f(t) e^{-ix_0 t} dt = \gamma \ (W)$$

Example

Suppose that

$$f(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n,m} e^{-ig(n,m)x},$$

then, $f(x_0) = \gamma$ in $\mathcal{S}'$ iff there is a k such that

$$\lim_{x\to\infty} \sum_{g(n,m)\le x} a_{n,m}\left(1 - \frac{g(n,m)}{x}\right)^k = \gamma.$$

Order of Point Values

Definition 2 *We say that* $f(x_0) = \gamma$ *in* $\mathcal{D}'$ *has order* k, *if* k *is the mimimum integer such that there exists a primitive of order* k *of* f, F, *such that* F *is locally integrable in a neighborhood of* x_0 *and F satisfies*

$$\lim_{x \to x_0} \frac{n!F(x)}{(x - x_0)^n} = \gamma.$$

Remark: Lojasewicz had defined the order of the point value in a different way, but I propose this new definition to be consistent with the following Theorems.

Order of inversion Formula

Theorem 7 *Let $f \in \mathcal{S}'$. Suppose that there exists a $m \in \mathbb{N}$, such that for every m-primitive h of f, i.e., $h^{(m)} = f$, h is measurable and $h(x) = O\left(|x|^{m-1}\right)$. Let m_0 be the smallest natural number with this property. If f has a distributional point value γ at x_0, whose order is n, then*

$$\frac{1}{2\pi} e.v. \left\langle \hat{f}(x), e^{-ix_0 x} \right\rangle = \gamma \ (C, k+1),$$

where $k = max\{m_0, n\}$.

Two Remarkable Cases

Define

$$\phi_a^\beta(t) = (1+t)^\beta \chi_{[-1,0]}(t) + \left(1 - \frac{t}{a}\right)^\beta \chi_{[0,a]}(t).$$

Two Remarkable Cases

Theorem 8 *Let f be a distribution with compact support. Suppose that $f(x_0) = \gamma$ in $\mathcal{D}'$ with order k. Let $\beta > k$. Then for each $a > 0$*

$$\lim_{x \to \infty} \frac{1}{2\pi} \int_{-x}^{ax} \hat{f}(t) e^{-ix_0 t} dt = \gamma \ (C, \beta)$$

or which is the same

$$\lim_{x \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_a^\beta \left(\frac{t}{x} \right) \hat{f}(t) e^{-ix_0 t} dt = \gamma,$$

Moreover, these relations hold uniformly for a in compact subsets of $(0, \infty)$.

Two Remarkable Cases

Theorem 9 *Let f be a 2π-periodic distribution, with Fourier series $\sum_{-\infty}^{\infty} c_n e^{ixn}$. If $f(x_0) = \gamma$ in $\mathcal{D}'$ with order k. Let $\beta > k$. Then for each $a > 0$,*

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} c_n e^{ix_0 n} = \gamma \ (C, \beta),$$

or equivalently

$$\lim_{x\to\infty} \sum_{-x\leq n\leq ax} \phi_a^{\beta}\left(\frac{n}{x}\right) c_n e^{ix_0 n} = \gamma.$$

Moreover, these relations hold uniformly for a in compact subsets of $(0, \infty)$.

Order of Point Value

Theorem 10 *Let $f \in \mathcal{S}'$. Suppose that*

$$\frac{1}{2\pi} e.v. \left\langle \hat{f}(x), e^{-ixx_0} \right\rangle = \gamma \ (C, k);$$

then, $f(x_0) = \gamma$ in $\mathcal{S}'$ f is the derivative of order $k+1$ of a locally integrable function and the order of $f(x_0)$ is less or equal to $k+2$.