

Factorization theorems in Denjoy-Carleman classes associated to representations of $(\mathbb{R}^d, +)$

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Introduction

Factorization theorems in modules over function algebras is an important subject with a long tradition in mathematical analysis.

A module \mathcal{M} over a non-unital algebra \mathcal{A} is said to have the **strong factorization** property if

$$\mathcal{M} = \mathcal{A} \cdot \mathcal{M} = \{a \cdot m \mid a \in \mathcal{A}, m \in \mathcal{M}\}.$$

It is said to have the **weak factorization** property if

$$\mathcal{M} = \text{span}(\mathcal{A} \cdot \mathcal{M}).$$

We will present some new results about strong factorization:

- 1 A strong factorization theorem of Dixmier-Malliavin type for ultradifferentiable vectors of representations of $(\mathbb{R}^d, +)$.
- 2 We have established the strong factorization property for many families of convolution modules of ultradifferentiable functions. We will give some examples.

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Factorization in classical function algebras

- Factorization theorems on \mathbb{T} go back to Salem and Zygmund.
- Rudin showed (1957-1958): $L^1(\mathbb{R}^d) = L^1(\mathbb{R}^d) * L^1(\mathbb{R}^d)$.
- Cohen (1959) extended this result to the function algebra of a locally compact abelian group G ,

$$L^1(G) = L^1(G) * L^1(G).$$

- Hewitt (1964) used Cohen technique to prove a general factorization theorem for Banach modules.
- Cohen-Hewitt type factorization theorems also hold for various Fréchet modules.
- Essential hypothesis: existence of bounded approximative units on the algebra under consideration.
- Many locally convex algebras **do not** have bounded approximative units. Examples: many basic algebras of smooth functions occurring in analysis.

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Does $\mathcal{D}(\mathbb{R}^d)$ factorize as $\mathcal{D}(\mathbb{R}^d) = \mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d)$?

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$$\mathcal{D}(\mathbb{R}^d) = \text{span}(\mathcal{D}(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d))$$

- If $d \geq 3$, they also showed that $\mathcal{D}(\mathbb{R}^d)$ **does not** have the strong factorization property.
- Dixmier and Malliavin (1979): negative answer for $d = 2$.
- Yulmukhametov (1999): in contrast $\mathcal{D}(\mathbb{R}) = \mathcal{D}(\mathbb{R}) * \mathcal{D}(\mathbb{R})$ holds.
- Several authors have independently shown (Miyazaki; Petzeltová and P. Vrbová; Dixmier and Malliavin; Voigt; ...)

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Factorization on Lie groups

- Let G be a real connected Lie group.
- Dixmier and Malliavin showed (1979) that

$$\mathcal{D}(G) = \text{span}(\mathcal{D}(G) * \mathcal{D}(G))$$

and, when additionally G is nilpotent,

$$\mathcal{S}(G) = \mathcal{S}(G) * \mathcal{S}(G).$$

(hereafter: convolution = left convolution)

- Let E be a locally convex Hausdorff (sequentially complete) space and denote as $GL(E)$ its group of isomorphisms.
- A group homomorphism $\pi : G \rightarrow GL(E)$ such that

$$G \times E \rightarrow E, \quad (g, e) \mapsto \pi(g)e$$

is separately continuous is a representation of G on E .

- We call $e \in E$ a **smooth vector** if its orbit mapping

$$G \rightarrow E \quad g \mapsto \pi(g)e, \quad \text{belongs to } C^\infty(G; E).$$

- E^∞ is the subspace of smooth vectors.



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Dixmier-Malliavin factorization theorems

- A representation of G on a sequentially complete lCHs E induces an action of the convolution algebra $\mathcal{D}(G)$ on the smooth vectors,

$(f, e) \mapsto \Pi(f)e, \quad \mathcal{D}(G) \times E^\infty \rightarrow E^\infty, \quad \text{where}$

$$\Pi(f)e = \int_G f(g)\pi(g)e \, dg \in E$$

- If E is Banach and the representation is bounded, the action extends to $\mathcal{S}(G)$ and we can regard E^∞ as a module over $\mathcal{S}(G)$.

Theorem

If E is a Fréchet space, E^∞ has the **weak** factorization property w.r.t. $\mathcal{D}(G)$, that is, $E^\infty = \text{span}(\Pi(\mathcal{D}(G))E^\infty)$.

Theorem

If E is a Hilbert space, the representation is unitary, and G is nilpotent, then E^∞ has the **strong** factorization property w.r.t. $\mathcal{S}(G)$.

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Analytic factorization of Lie group representations

- $e \in E$ is an **analytic vector** if $g \mapsto \pi(g)e$ is an analytic mapping.
- E^ω : subspace of analytic vectors.
- A representation is called an F -representation if
 - E is a Fréchet space;
 - there is a basis of continuous seminorms $(p_n)_{n \in \mathbb{N}}$ such that for each n the action $G \times (E, p_n) \rightarrow (E, p_n)$ is continuous.
- For F -representations, we get an action of the algebra of exponentially rapidly decreasing analytic functions $\mathcal{A}(G)$ on E^ω .

Theorem (Gimperlein, Krötz, and Lienau (2012))

For F -representations, E^ω has the **weak** factorization property w.r.t. $\mathcal{A}(G)$, that is, $E^\omega = \text{span}(\Pi(\mathcal{A}(G))E^\omega)$.

Conjecture

They have conjectured that one might even have $E^\omega = \Pi(\mathcal{A}(G))E^\omega$.



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Analytic factorization for $(\mathbb{R}^d, +)$

The convolution algebra $\mathcal{A}(\mathbb{R}^d)$ consists of real analytic functions f admitting holomorphic extension to $\mathbb{R}^d + i] - h, h[^d$ for some $h > 0$ and satisfying

$$\sup_{|\operatorname{Im} z| \leq h} e^{n|\operatorname{Re} z|} |f(z)| < \infty, \quad \text{for each } n \in \mathbb{N}.$$

Theorem (Debrouwere, Prangoski, and V. (2021))

For F -representations of \mathbb{R}^d , E^ω has the **strong** factorization property w.r.t. $\mathcal{A}(\mathbb{R}^d)$, that is, $E^\omega = \Pi(\mathcal{A}(\mathbb{R}^d))E^\omega$.

Our results hold for more general representations than F -representations:

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The convolution algebra $\mathcal{A}(\mathbb{R}^d)$ consists of real analytic functions f admitting holomorphic extension to $\mathbb{R}^d + i] - h, h[^d$ for some $h > 0$ and satisfying

$$\sup_{|\operatorname{Im} z| \leq h} e^{n|\operatorname{Re} z|} |f(z)| < \infty, \quad \text{for each } n \in \mathbb{N}.$$

Theorem (Debrouwere, Prangoski, and V. (2021))

For F -representations of \mathbb{R}^d , E^ω has the **strong** factorization property w.r.t. $\mathcal{A}(\mathbb{R}^d)$, that is, $E^\omega = \Pi(\mathcal{A}(\mathbb{R}^d))E^\omega$.

Our results hold for more general representations than F -representations:

- projective generalized proto-Banach representations;
- inductive generalized proto-Banach representations.

Projective and inductive generalized proto-Banach representations

Definition

A representation (π, E) is said to be a **projective generalized proto-Banach** representation if

$$\forall p \in \text{csn}(E) \exists q_p \in \text{csn}(E) \exists \kappa_p > 0 \forall x \in \mathbb{R}^d \forall e \in E : \\ p(\pi(x)e) \leq e^{\kappa_p |x|} q_p(e)$$

$\mathfrak{B}(E)$ stands for the collection of non-empty absolutely convex closed bounded subsets of E and for $B \in \mathfrak{B}(E)$ we denote by E_B the subspace of E spanned by B .

Definition

(π, E) is an **inductive generalized proto-Banach** representation if

$$\forall B \in \mathfrak{B}(E) \exists A_B \in \mathfrak{B}(E) \exists \kappa_B > 0 \forall x \in \mathbb{R}^d \forall e \in E_B : \\ \|\pi(x)e\|_{E_{A_B}} \leq e^{\kappa_B |x|} \|e\|_{E_B}.$$

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Denjoy-Carleman classes

Consider a log-convex sequence $M = (M_p)_p$ of positive numbers and set $m_p = M_p/M_{p-1}$. We assume:

- (M.2) there are $C_0, H > 0$ such that $M_{p+q} \leq C_0 H^{p+q} M_p M_q$;
- (M.2)* there are $Q, p_0 \in \mathbb{N}$ such that $2m_p \leq m_{Qp}$ for all $p \geq p_0$.

Prototypical example: $M_p = (p!)^\sigma$, with $\sigma > 0$.

- a vector $e \in E$ is **ultradifferentiable** of class $[M]$ if its orbit mapping w.r.t. the representation is (bornologically) ultradifferentiable of class $[M]$.
- $[M]$ is the common notation for both the Beurling (M) and $\{M\}$ Roumieu cases of ultradifferentiability.
- $E^{[M]}$ denotes the space of ultradifferentiable vectors of class $[M]$ of a representation.
- Note that $E^\omega = E^{\{p!\}}$.

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Factorization theorem for ultradifferentiable vectors

For $h > 0$, we define the Fréchet space

$$\mathcal{K}^{M,h}(\mathbb{R}^d) = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^d) \mid \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{h^{|\alpha|} |\partial^\alpha \varphi(x)| e^{n|x|}}{M_{|\alpha|}} < \infty, \quad \forall n \in \mathbb{N}\}.$$

We set

$$\mathcal{K}^{(M)}(\mathbb{R}^d) = \varprojlim_{h \rightarrow \infty} \mathcal{K}^{M,h}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{K}^{\{M\}}(\mathbb{R}^d) = \varinjlim_{h \rightarrow 0^+} \mathcal{K}^{M,h}(\mathbb{R}^d).$$

If $M_p = p!$, then $\mathcal{A}(\mathbb{R}^d) = \mathcal{K}^{\{M\}}(\mathbb{R}^d)$.

Theorem (Debrouwere, Prangoski, and V. (2021))

Let (π, E) be either a projective or an inductive generalized proto-Banach representation of $(\mathbb{R}^d, +)$ on a sequentially complete lchS E . Then, $E^{[M]}$ has the strong factorization property w.r.t. $\mathcal{K}^{[M]}(\mathbb{R}^d)$

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Factorization of modules of ultradifferentiable functions

- Our factorization theorem implies the strong factorization property for many concrete families of modules of ultradifferentiable functions.

Example:

- Let $\omega : \mathbb{R}^d \rightarrow (0, \infty)$ be a continuous weight function satisfying

$$\sup_{x \in \mathbb{R}^d} \frac{\omega(x + \cdot)}{\omega(x)} \in L_{loc}^\infty(\mathbb{R}^d).$$

- Consider $E = L_\omega^p = \{f \mid \omega \cdot f \in L^p(\mathbb{R})\}$ if $1 \leq p < \infty$.
- The ultradifferentiable vectors are (w.r.t. regular representation)

$$E^{(M)} = \mathcal{D}_E^{(M)} = \varprojlim_{h \rightarrow \infty} \mathcal{D}_E^{\{M\}, h} \quad \text{and} \quad E^{\{M\}} = \mathcal{D}_E^{\{M\}} = \varinjlim_{h \rightarrow 0^+} \mathcal{D}_E^{\{M\}, h},$$

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For more details, see our article:

- A. Debrouwere, B. Prangoski, J. Vindas, *Factorization in Denjoy-Carleman classes associated to representations of $(\mathbb{R}^d, +)$* , J. Funct. Anal. 280 (2021), Article 108831.

Related works on factorization theorems for representations:

- J. Dixmier, P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. **102** (1978), 307–330.
- H. Gimperlein, B. Krötz, C. Lienau, *Analytic factorization of Lie group representations*, J. Funct. Anal. **262** (2012), 667–681.
- H. Glöckner, *Continuity of LF-algebra representations associated to representations of Lie groups*, Kyoto J. Math. 53 (2013), 567–595.