Wavelets and Gelfand-Shilov spaces

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In this talk we discuss approximation properties of MRA and wavelets in the so-called Gelfand-Shilov spaces.

I will talk about:

1. Some classes of ‘highly regular’ MRA and wavelets.
2. Their connection with Gevrey and Gelfand-Shilov spaces.
3. Approximation properties of these highly regular MRA and wavelets.
4. Some mapping properties of the wavelet transform.

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Regularity of wavelets: smoothness vs decay

- MRA and wavelets are effective to approximate functions,
- and, in turn, to describe a large number of function and distribution spaces.
- This effectiveness: related to regularity properties of scaling function and wavelet.
- By regularity we mean: smoothness and decay.
- There is however a trade-off between smoothness and decay.

Here a well-known example of this interplay leading to conflicts:

There is no orthonormal wavelet ψ sharing simultaneously these two properties:

1. $\psi(x) \ll e^{-c|x|}$ for some $c > 0$.
2. $\psi \in C^{\infty}(\mathbb{R})$, with all derivatives being bounded.

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What we cannot get!

Let us fix $\psi \in L^1(\mathbb{R})$ with the second property from the last statement, that is,

$$\psi^{(n)} \in L^\infty(\mathbb{R}), \qquad n = 0, 1, 2, \dots . \tag{1}$$

We consider the decay (for a positive weight function ω):

$$\psi(x) \ll e^{-\omega(|x|)}, \tag{2}$$

Under certain standard regularity assumptions ω, one shows:

If there is an orthonormal wavelet ψ satisfying (1) and (2) then

$$\int_1^\infty \frac{\omega(x)}{x^2} < \infty. \tag{3}$$

Conclusion: No wavelets with (1) and (2) such that (3) diverges

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What we can try to do!

Due to the constrains we have discussed so far, we might try to find smooth ψ (with bounded derivatives) with decay

$$\psi(x) \ll e^{-\omega(x)}, \qquad \text{where } \int_1^\infty \frac{\omega(x)}{x^2} < \infty.$$

First try

$\omega(x) = n \log x$, so that $\psi(x) \ll |x|^{-n}$.

This works! Actually, Meyer did better in 1985 and found an orthonormal wavelet $\psi \in \mathcal{S}(\mathbb{R})$. For future reference:

Properties of orthonormal wavelets $\psi \in \mathcal{S}(\mathbb{R})$

1. ψ is an MRA wavelet.
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Better decay: the Dziubański-Hernández wavelets

We now try $\psi(x) \ll e^{-c|x|^{1/t}}$. To match the integral constrain:

$$\int_1^\infty |x|^{-(2-1/t)} < \infty, \qquad \text{i.e.,} \quad t > 1.$$

To make progress, note Meyer's wavelets $\psi \in \mathcal{S}(\mathbb{R})$ satisfy:

- It is of Lemarié-Meyer type: $\widehat{\psi}$ has compact support.
- Since ψ is band-limited, $\psi \in \mathcal{S}(\mathbb{R})$ iff $\widehat{\psi} \in C^\infty(\mathbb{R})$.
- The latter achieved by taking smooth 'bell functions'.

A real Paley-Wiener type theorem, $t > 1$

A band-limited function g satisfies $g(x) \ll e^{-c|x|^{1/t}}$ iff $\widehat{g}$ belongs to the Gevrey class $G^t(\mathbb{R})$.

Theorem (Dziubański-Hernández)

Given $t > 1$ and $c > 0$, there is a band-limited orthonormal wavelet with

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Gevrey classes

- The Gevrey functions generalize real analytic functions.
- A function f is real analytic in I iff for each compact subinterval there are A and C such that

$$\sup_{x\in[a,b]} |f^{(n)}(x)| \leq CA^n n!, \qquad n\in\mathbb{N}.$$

Definition

$f\in G^t(I)$ if $\sup_{x\in[a,b]} |f^{(n)}(x)| \leq CA^n (n!)^t$ on each $[a,b]\subset I$.

- Gevrey classes naturally arise in the analysis of PDE.
- If $t<1$, $G^t(\mathbb{R})$ consists of entire functions.
- If $t>1$, an example of $f\in G^t(\mathbb{R})$ is ($\alpha = 1/(t-1)$)

$$f(x) = e^{-(x+1)^{-\alpha}-(1-x)^{-\alpha}} \text{ if } |x|\leq 1 \qquad \text{and otherwise } f(x)=0.$$

Conclusion: $G^t(\mathbb{R})$ contains non-trivial compactly supported functions if $t>1$, we write $G^t_c(\mathbb{R}) = G^t(\mathbb{R})\cap C^\infty_c(\mathbb{R})$.

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The Denjoy-Carleman theorem

Define the class $\mathcal{E}^{\{M_n\}}[a,b]$ of smooth functions such that

$$\sup_{x\in[a,b]} |f^{(n)}(x)| \leq CA^n M_n \qquad \text{(for some } C, A).$$

One may assume $m_n = M_{n+1}/M_n$ increases (Cartan-Gorny theorem).

Hadamard's problem, 1912

Characterize $\{M_n\}_{n\in\mathbb{N}}$ such that $\mathcal{E}^{\{M_n\}}[a,b]$ contains non-trivial compactly supported functions in (a,b) (= non-quasianalyticity).

Denjoy-Carleman theorem

Suppose $m_n = M_{n+1}/M_n$ is increasing. Then, $\mathcal{E}^{\{M_n\}}[a,b]$ is non-quasianalytic iff $\sum_{n=0}^{\infty} 1/m_n < \infty$.

Under 'standard assumptions', one adapts the Dziubański-Hernández construction to find a band-limited orthogonal wavelet with decay $\psi(x) \ll e^{-M(|x|)}$, where $M(x) = \sup_{n\in\mathbb{N}} \log_+(x^n/M_n)$

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Gelfand-Shilov spaces

- The Dziubański-Hernández wavelets belong to $\mathcal{F}(G^t_c(\mathbb{R}))$, where $\mathcal{F}$ stands for the Fourier transform.
- Elements of $\mathcal{F}(G^t_c(\mathbb{R}))$ are determined by global estimates

$$|x^m f^{(n)}(x)| \ll B^{n+m}(m!)^t \qquad x \in \mathbb{R}.$$

Definition

Let $t, s \geq 0$. The space $\mathcal{S}^s_t(\mathbb{R})$ consists of all Schwartz functions such that, for some B,

$$|x^m f^{(n)}(x)| \ll B^{n+m}(n!)^s(m!)^t.$$

- Introduced by Gelfand-Shilov in connection with PDEs.
- $\mathcal{S}^s_t(\mathbb{R}) \subset G^s(\mathbb{R})$, so s measures Gevrey regularity.
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- If $t > 0$, $\mathcal{S}^1_t(\mathbb{R})$ consists of functions f that can be extended analytically to some horizontal strip around $\mathbb{R}$ where it satisfies

$$|f(x+iy)| \ll e^{-c|x|^{1/t}} \quad \text{for } |y| < h$$

- If $s, t > 0$ and $s < 1$, then $f \in \mathcal{S}^s_t(\mathbb{R})$ iff f is entire and satisfies

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Gelfand-Shilov regular MRA and wavelets

If ψ is a Dziubański-Hernández wavelet with $\psi(x) \ll e^{-c|x|^{1/\rho_2}}$, then $\psi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R})$ for all $\rho_1 \geq 0$. They are examples of

Definition

Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An orthonormal wavelet ψ is (ρ_1, ρ_2)-regular if $\psi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R})$.

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Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An MRA is called (ρ_1, ρ_2)-regular if it possesses a scaling function $\phi \in \mathcal{S}^{\rho_1}_{\rho_2}(\mathbb{R})$.

Remark

It should be by now clear that $\rho_2 \leq 1$ is not admissible here.

Open question

Every (ρ_1, ρ_2)-regular is an MRA wavelet. Does it arise from a (ρ_1, ρ_2)-regular MRA?

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Convergence of multiresolution expansions

Theorem

Let $\{V_m\}_{m\in\mathbb{Z}}$ be a (ρ_1, ρ_2)-regular MRA with orthogonal projections

$$E_m : L^2(\mathbb{R}) \to V_m$$

and set $\sigma = \rho_1 + \rho_2 - 1$. Let $s \geq \sigma$ and $t \geq \rho_2$. Then,

$$\lim_{m\to\infty} E_m f = f \ \ in \ \ \mathcal{S}_t^s(\mathbb{R}),$$

for each $f \in \mathcal{S}_t^{s-\sigma}(\mathbb{R})$.

There is a loss of regularity measured by $\sigma > 0$. We wonder

1. Is σ optimal? We conjecture so ...
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1. Is σ optimal? We conjecture so ...
2. Are there special classes of MRA that avoid the loss of regularity?

Convergence of multiresolution expansions

Theorem

Let $\{V_m\}_{m\in\mathbb{Z}}$ be a (ρ_1,ρ_2)-regular MRA with orthogonal projections

$$E_m : L^2(\mathbb{R}) \to V_m$$

and set $\sigma = \rho_1 + \rho_2 - 1$. Let $s \geq \sigma$ and $t \geq \rho_2$. Then,

$$\lim_{m\to\infty} E_m f = f \ \ in \ \ \mathcal{S}_t^s(\mathbb{R}),$$

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Convergence of wavelet expansions

Write $(\mathcal{S}^s_t)_0(\mathbb{R}) = \{f \in \mathcal{S}^s_t(\mathbb{R}) : \int_{-\infty}^{\infty} x^n f(x)dx = 0, \ n = 0, 1, \dots\}$.

A (ρ_1, ρ_2)-regular wavelet automatically satisfies $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R})$.

Theorem

Let $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R})$ be a (ρ_1, ρ_2)-regular orthonormal wavelet. Set $\sigma = \rho_1 + \rho_2 - 1$ and consider $s > \sigma$ and $t > \sigma + 1$.

If $f \in (\mathcal{S}^{s-\sigma}_{t-\sigma})_0(\mathbb{R})$, then

$$f = \sum_{n,m} \langle f, \overline{\psi}_{n,m} \rangle \, \psi_{n,m} \quad \text{converges in the space } (\mathcal{S}^s_t)_0(\mathbb{R}).$$

- Again we have lost regularity and the same questions as before make sense ...
- Our arguments here rely on mapping properties of wavelet transforms.

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The wavelet transform: distribution case

We consider the wavelet transform

$$\mathcal{W}_{\psi} f(b,a) = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\overline{\psi}\left(\frac{x-b}{a}\right)\,\mathrm{d}x.$$

Denote $\mathbb{H} = \{(b,a) : \ a > 0\}$. The wavelet synthesis operator is

$$\mathcal{M}_{\psi} F(x) = \iint_{\mathbb{H}} F(b,a)\psi\left(\frac{x-b}{a}\right)\frac{\mathrm{d}b\mathrm{d}a}{a^2}.$$

The space of highly localized functions on $\mathbb{H}$ is

$$\mathcal{S}(\mathbb{H}) = \{F \in C^{\infty}(\mathbb{H}) : F(b,a) \ll (1+|b|)^{-n}(a+1/a)^{-n}, \ \forall n > 0\}.$$

For a wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$, one gets continuity of

$$\mathcal{W}_{\psi} : \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}(\mathbb{H}) \quad \text{and} \quad \mathcal{M}_{\psi} : \mathcal{S}(\mathbb{H}) \to \mathcal{S}_0(\mathbb{R}),$$

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The wavelet transform in Gelfand-Shilov spaces

Let s, t, τ_1, τ_2. Define $\mathcal{S}^{s}_{t,\tau_1,\tau_2}(\mathbb{H})$ as the space of smooth functions satisfying estimates

$$\partial_a^m \partial_b^n F(b,a) \ll_m B^n (n!)^s \exp\left(-c\left(a^{1/\tau_1} + a^{-1/\tau_2} + |b|^{1/t}\right)\right)$$

We have made a thorough analysis of wavelet transforms on Gelfand-Shilov spaces. In their simplest forms, our results yield:

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Some references

For more details about the subject of this talk, see my joint articles with D. Rakić, S. Pilipović, and N. Teofanov:

1. *The wavelet transforms in Gelfand-Shilov spaces,* Collect. Math. 67 (2016), 443–460.
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