WAVELET EXPANSIONS OF DISTRIBUTIONS

JASSON VINDAS

ABSTRACT. These are lecture notes of a talk at the School of Mathematics of the National University of Costa Rica. The aim is to present a wavelet expansion theory for tempered distributions. It is shown that, for suitable orthogonal wavelets, the wavelet expansion of a tempered distribution converges to its proyection in the quotient of $\mathcal{S}'(\mathbb{R})$ modulo the space of polynomials; we also characterize bounded sets and convergence in such a quotient space via wavelet coefficients.

1. INTRODUCTION

Wavelet theory is a powerful tool in analysis. It has shown to be of importance in areas such as time-frequency analysis and approximation theory, among others. The existent applications of wavelet methods in functional analysis are very rich. Wavelet analysis can also be used to provide intrinsic characterizations of important function and distribution spaces [10]. Our goal in this note is to study wavelet expansions of tempered distributions. The present paper is based on a joint work with K. Saneva [15].

Previous attempts to study wavelet expansions of general tempered distributions have been based on approximations by finitely regular multiresolution analysis [23, 12, 18]; consequently, they are only applicable to finite order distributions. In theory and practice, it is difficult to determine the order of a distribution, it is then desirable to have a theory independent of it. In this paper we develope a distribution wavelet expansions theory within the framework of the space of highly time-frequency localized test functions over the real line $\mathcal{S}_0(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ and its dual space $\mathcal{S}_0'(\mathbb{R})$, i.e., the quotient of the space of tempered distributions modulo polynomials. Such an approach has been taken in [4] to study the wavelet transform of distributions. It is proved that the wavelet expansion of a tempered distributions converges to its projection in $\mathcal{S}_0'(\mathbb{R})$. Using these ideas we characterize boundedness and convergence in the space $\mathcal{S}_0'(\mathbb{R})$.

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It should be metioned that the results discussed herein were obtained in order to study local and and non-local asymptotic properties of Schwartz distributions via wavelet expansions; indeed, in [15], we provide Abelian and Tauberian type results relating the asymptotic behavior of tempered distributions with the asymptotics of wavelet coefficients. For other results in this direction the reader can consult the references [4, 5, 6, 11, 12, 13, 14, 17, 20, 21, 22, 23].

2. Preliminaries and Notations

The set $\mathbb{H}$ denotes the upper half-plane, that is, $\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$; we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1. Spaces of Functions and Distributions.

The Schwartz spaces of test functions and distributions on the real line $\mathbb{R}$ are denoted by $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$, respectively; the space of rapidly decreasing smooth functions and its dual, the space of tempered distributions, are denoted by $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$. We refer to [16] for the well known properties of these spaces.

Following [4], we define the space of highly time-frequency localized functions over the real line as the set of those elements of $\phi \in \mathcal{S}(\mathbb{R})$ for which all the moments vanish, i.e.,

$$\int_{-\infty}^{\infty} x^n \phi(x) \mathrm{d}x = 0 \ , \quad \forall n \in \mathbb{N}_0 \ . \tag{2.1}$$

The space of highly time-frequency localized functions over the real line will be denoted by $\mathcal{S}_0(\mathbb{R})$, provided with the relative topology inhered from $\mathcal{S}(\mathbb{R})$. Observe that $\mathcal{S}_0(\mathbb{R})$ is closed subspace of $\mathcal{S}(\mathbb{R})$. Its dual space is $\mathcal{S}_0'(\mathbb{R})$. Notice that there exists a well-defined continuous linear projector from $\mathcal{S}'(\mathbb{R})$ to $\mathcal{S}_0'(\mathbb{R})$ as the transpose of the trivial inclusion from the closed subspace $\mathcal{S}_0(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$. This map is surjective due to the Hanh-Banach theorem; however, there is no continuous right inverse for it [2]. The kernel of this projection is the space of polynomials; hence, the space $\mathcal{S}_0'(\mathbb{R})$ can be regarded as the quotient space of $\mathcal{S}'(\mathbb{R})$ by the space of polynomials. We do not want to introduce a notation for this map, so if $f \in \mathcal{S}'(\mathbb{R})$, we will keep calling by f the restriction of f to $\mathcal{S}_0(\mathbb{R})$.

The corresponding space of highly localized function over $\mathbb{H}$ is denoted by $\mathcal{S}(\mathbb{H})$. It consists of those smooth functions Φ on $\mathbb{H}$ for which

$$\sup_{(b,a)\in\mathbb{H}} \left(a + \frac{1}{a}\right)^m (1+|b|)^n \left| \frac{\partial^{k+l}\Phi}{\partial a^k \partial b^l}(b,a) \right| < \infty \ ,$$

for all $m, n, k, l \in \mathbb{N}_0$. The canonical topology of this space is defined in the standard way [4].

2.2. **The Wavelet Transform of Distributions.** The *wavelet transform* of $f \in \mathcal{S}'(\mathbb{R})$ with respect to $\psi \in \mathcal{S}_0(\mathbb{R})$ is the C^∞-function on $\mathbb{H}$ defined by

$$\mathcal{W}_\psi f(b,a) := \langle f(b+ax), \bar{\psi}(x)\rangle = \left\langle f(t), \frac{1}{a}\bar{\psi}\Big(\frac{t-b}{a}\Big)\right\rangle, \ (b,a) \in \mathbb{H}\ . \tag{2.2}$$

Note that $\mathcal{W}_\psi : \mathcal{S}_0(\mathbb{R}) \longmapsto \mathcal{S}(\mathbb{H})$ is continuous linear map [4]. For an arbitrary tempered distribution $f \in \mathcal{S}'(\mathbb{R})$, one can verify that $\mathcal{W}_\psi f$ is a function of slow growth on $\mathbb{H}$, that is, it satisfies an estimate of the form

$$|\mathcal{W}_\psi f(b,a)| \leq O\left(\left(a + \frac{1}{a}\right)^m (1+|b|)^n\right)\ , \tag{2.3}$$

for some $m, n \in \mathbb{N}_0$.

Naturally, the wavelet transform (2.2) may be considered for f seen merely as an element of $\mathcal{S}_0'(\mathbb{R})$. The reader can find a complete distribution wavelet transform theory based on the spaces $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}_0'(\mathbb{R})$ in Holschneider's book [4].

2.3. **Orthogonal Wavelets.** We shall merely recall some concepts of the theory of orthonormal wavelet bases in $L^2(\mathbb{R})$. In particular, we are not concerned with the construction of wavelets. A detailed introduction to this theory can be found in [1, 8, 23] while a more comprehensive treatment in [3, 10].

An *orthonormal wavelet* on $\mathbb{R}$ is a function $\psi \in L^2(\mathbb{R})$ such that the family $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where $\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)$, $m, n \in \mathbb{Z}$. So, any $f \in L^2(\mathbb{R})$ can be written as

$$f = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} (f, \psi_{m,n})_{L^2(\mathbb{R})}\, \psi_{m,n} \tag{2.4}$$

with convergence in $L^2(\mathbb{R})$-norm. The series representation of f in (2.4) is called a wavelet series. We will denote the wavelet coefficients of f with respect to the ortonormal wavelet ψ by $c^\psi_{m,n}(f)$, i.e.,

$$c^\psi_{m,n}(f) = (f, \psi_{m,n})_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} f(x)\bar{\psi}_{m,n}(x)\mathrm{d}x,\ \ m, n \in \mathbb{Z}. \tag{2.5}$$

Note that the relation between the wavelet coefficients and the wavelet transform of f is given by

$$c^\psi_{m,n}(f) = 2^{-\frac{m}{2}}\mathcal{W}_\psi f\left(n2^{-m}, 2^{-m}\right). \tag{2.6}$$

Since we are interested in tempered distributions, we will only use orthonormal wavelets which are elements of $\mathcal{S}(\mathbb{R})$. It is well known that every orthonormal wavelet from $\mathcal{S}(\mathbb{R})$ must belong to the space $\mathcal{S}_0(\mathbb{R})$ [3, Cor.3.7, p.75]. The existence of such wavelets was proved by Lemarié and Meyer [7, 9]. Indeed, Meyer constructed in [9] orthonormal wavelets $\psi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\psi} \in \mathcal{D}(\mathbb{R})$, arising from a Littlewood-Paley MRA [10, p.25]; in [7], they found the corresponding multidimensional wavelets of this type.

3. Wavelet Expansions Theory on $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}'_0(\mathbb{R})$

In this section we provide a wavelet expansion theory for the spaces $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}'_0(\mathbb{R})$. We will show convergence of the wavelet series on these spaces. We shall always assume that the orthogonal wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$. Therefore, it makes sense to consider the wavelet coefficients of $f \in \mathcal{S}'_0(\mathbb{R})$, defined as usual by

$$c^{\psi}_{m,n}(f) := \left\langle f, \bar{\psi}_{m,n} \right\rangle \ . \tag{3.1}$$

We shall also use wavelet expansions to characterize boundedness and convergence on $\mathcal{S}'_0(\mathbb{R})$, provided with the strong dual topology; for this purpose, we describe below a natural isomorphisms of $\mathcal{S}_0(\mathbb{R})$ with a certain space of sequences identified with the wavelet coefficients. To describe the topology on $\mathcal{S}_0(\mathbb{R})$, we use the following family of seminorms

$$\|\phi\|^{\mathcal{S}_0}_l := \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \left|\phi^{(k)}(x)\right| \ , \quad l \in \mathbb{N}_0 \ .$$

3.1. Convergence of Wavelet Expansions in $\mathcal{S}_0(\mathbb{R})$.

We start by estimating the wavelet coefficients of functions from $\mathcal{S}_0(\mathbb{R})$.

Lemma 3.1. *Let $\phi \in \mathcal{S}_0(\mathbb{R})$ and $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then, given β, $\gamma > 0$ there exists $l \in \mathbb{N}_0$ and a constant $C > 0$ such that*

$$\left|c^{\psi}_{m,n}(\phi)\right| \le C \, \|\phi\|^{\mathcal{S}_0}_l \, (|n|+1)^{-\beta} \left(2^m + \frac{1}{2^m}\right)^{-\gamma} \ , \quad \forall \phi \in \mathcal{S}_0(\mathbb{R}) \ . \tag{3.2}$$

Proof. The proof follows from the relation (2.6) and the fact that $\mathcal{W}_\psi : \mathcal{S}_0(\mathbb{R}) \longmapsto \mathcal{S}(\mathbb{H})$ is a continuous linear map. Therefore, given $k, j \in \mathbb{N}_0$, $k > j$, we have the existence of an integer l and a constant $C_{j,k} > 0$ such that

$$|c^{\psi}_{m,n}(\phi)| \le C_{j,k} \, \|\phi\|^{\mathcal{S}_0}_l \left(1 + \frac{|n|}{2^m}\right)^{-j} \left(\frac{1}{2^m} + 2^m\right)^{-k}.$$

From the following inequality

$$\left(1+\frac{|n|}{2^m}\right)^{-j} = \begin{cases} \dfrac{2^{mj}}{(2^m+|n|)^j} \le \dfrac{2^{mj}}{(1+|n|)^j}, & m \ge 0, \\[2ex] \dfrac{1}{(1+\frac{|n|}{2^m})^j} \le \dfrac{1}{(1+|n|)^j}, & m < 0, \end{cases}$$
$$\le \frac{1}{(1+|n|)^j}\left(2^m + \frac{1}{2^m}\right)^j ,$$

we obtain

$$|c^{\psi}_{m,n}(\phi)| \le C_{j,k} \, \|\phi\|^{\mathcal{S}_0}_l \, (1+|n|)^{-j} \left(\frac{1}{2^m} + 2^m\right)^{-(k-j)} \ . \tag{3.3}$$

Relation (3.2) follows by taking $j > \beta$ and $k > j + \gamma$. □

We now show convergence of the wavelet series in topology of $\mathcal{S}_0(\mathbb{R})$.

Theorem 3.2. *Let $\phi \in \mathcal{S}_0(\mathbb{R})$ and $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then ϕ can be expanded as*

$$\phi = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(\phi)\psi_{m,n} \ , \tag{3.4}$$

with convergence in $\mathcal{S}_0(\mathbb{R})$.

Proof. Observe that the fast decrease of the wavelet coefficients (3.2) gives us that the representation (3.4) converges uniformly to ϕ; by the same reason (3.4) can be differentiated and we still get uniform convergence. To show convergence in $\mathcal{S}_0(\mathbb{R})$, we should prove that for each $l \in \mathbb{N}_0$

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \left\| \phi - \sum_{|m|\le M} \sum_{|n|\le N} c^{\psi}_{m,n}(\phi)\psi_{m,n} \right\|^{\mathcal{S}_0}_{l} = 0$$

i.e.,

$$\lim_{\substack{M\to\infty \\ N\to\infty}} \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \Big| \sum_{|m|>M} \sum_{|n|>N} c^{\psi}_{m,n}(\phi)(\psi_{m,n}(x))^{(k)} \Big| = 0 \ .$$

We have then

$$\begin{aligned}
&\sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \Big| \sum_{|m|>M} \sum_{|n|>N} c^{\psi}_{m,n}(\phi)(\psi_{m,n}(x))^{(k)} \Big| \\
&= \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \Big| \sum_{|m|>M} \sum_{|n|>N} c^{\psi}_{m,n}(\phi)(2^{\frac{m}{2}}\psi(2^m x - n))^{(k)} \Big| \\
&\le \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \sum_{|m|>M} \sum_{|n|>N} 2^{m(k+1/2)} |c^{\psi}_{m,n}(\phi)| \, |\psi^{(k)}(2^m x - n)| \\
&\le O(1) \sup_{x\in\mathbb{R},\, 0\le k\le l} (1+|x|)^l \sum_{|m|>M} \sum_{|n|>N} \left(2^m + \frac{1}{2^m}\right)^{k+1} \frac{|c^{\psi}_{m,n}(\phi)|}{(1+|2^m x - n|)^l} \\
&\le O(1) \sup_{x\in\mathbb{R}} (1+|x|)^l \sum_{|m|>M} \sum_{|n|>N} \left(2^m + \frac{1}{2^m}\right)^{l+1} \frac{|c^{\psi}_{m,n}(\phi)|}{(1+|2^m x - n|)^l} \ .
\end{aligned}$$

If we now use the elementary inequality

$$\frac{1+|x|}{1+|x-y|} \le 1 + |y| \ ,$$

we obtain that

$$\begin{aligned}\frac{1}{(1+|2^m x-n|)^l}&=\frac{1}{(1+2^m|x|)^l}\frac{(1+2^m|x|)^l}{(1+|2^m x-n|)^l}\\&\leq\frac{(1+|n|)^l}{(1+2^m|x|)^l}\\&\leq\begin{cases}\dfrac{2^{-ml}(1+|n|)^l}{(1/2^m+|x|)^l}\,, & m<0\,,\\ \dfrac{(1+|n|)^l}{(1+|x|)^l}\,, & m\geq 0\,,\end{cases}\\&\leq\frac{(1+|n|)^l}{(1+|x|)^l}\left(2^m+\frac{1}{2^m}\right)^l.\end{aligned}$$

Therefore, from the last two inequalities, we get

$$\sup_{x\in\mathbb{R},\,0\leq k\leq l}(1+|x|)^l\Big|\sum_{|m|>M}\sum_{|n|>N}c^{\psi}_{m,n}(\phi)(\psi_{m,n}(x))^{(k)}\Big|$$
$$\leq O(1)\sum_{|m|>M}\sum_{|n|>N}|c^{\psi}_{m,n}(\phi)|\,(1+|n|)^l\left(2^m+\frac{1}{2^m}\right)^{2l+1}.$$

Finally, the rapid decay obtained in Lemma 3.1 for the wavelet coefficients implies that the last term tends to 0. Indeed, it is enough to choose $\beta=l+2$ and $\gamma=2l+2$ in (3.2) to ensure that the term in the last inequality is less than $O(N^{-1}2^{-M})$.

□

We obviously obtain the next corollary from Theorem 3.2.

Corollary 3.3. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then, the linear span of $\{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is dense in the space $\mathcal{S}_0(\mathbb{R})$.*

3.2. **Convergence of Wavelet Expansions in $\mathcal{S}'_0(\mathbb{R})$.** For the convergence of wavelet series expansions in the space $\mathcal{S}'_0(\mathbb{R})$, we first show the following lemma.

Lemma 3.4. *Let $\psi\in\mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. The wavelet coefficients of $f\in\mathcal{S}'_0(\mathbb{R})$ satisfy an estimate*

$$|c^{\psi}_{m,n}(f)|\leq M(|n|+1)^{\beta}\Big(\frac{1}{2^m}+2^m\Big)^{\gamma}\tag{3.5}$$

for some $\beta,\gamma,M>0$.

Proof. It follows from the growth properties of $\mathcal{W}_{\psi}f$ on $\mathbb{H}$ that the wavelet coefficients satisfy an estimate of the form (2.3); the same argument used in the proof of Lemma 3.1 shows an estimate of the form (3.5) for the wavelet coefficients. □

From Theorem 3.2, Lemma 3.1, and Lemma 3.4, we easily obtain the ensuing convergence result.

Theorem 3.5. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then, the wavelet expansion series of $f \in \mathcal{S}_0'(\mathbb{R})$,*

$$f = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(f)\psi_{m,n} \ , \tag{3.6}$$

converges in (the strong dual topology of) $\mathcal{S}_0'(\mathbb{R})$.

Proof. If we show weak convergence of (3.6), then the strong convergence would follow from it and the Banach-Steinhauss theorem [19]. Let $\phi \in \mathcal{S}_0(\mathbb{R})$. Since $\bar{\psi}$ is also an orthonormal wavelet, we have from Theorem 3.2

$$\phi = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^{\bar{\psi}}_{m,n}(\phi)\bar{\psi}_{m,n} \ ,$$

with convergence in $\mathcal{S}_0(\mathbb{R})$. Using Lemma 3.1 and Lemma 3.4, we obtain the convergence of the wavelet series with coefficients $c^{\psi}_{m,n}(\mathrm{f})$. Moreover,

$$\begin{aligned}\langle f,\phi\rangle &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^{\bar{\psi}}_{m,n}(\phi)\left\langle f,\bar{\psi}_{m,n}\right\rangle = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(f)\left\langle \psi_{m,n},\phi\right\rangle \\ &= \left\langle \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(f)\psi_{m,n},\ \phi \right\rangle \ ,\end{aligned}$$

which shows (3.6).

□

Corollary 3.6. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. Then, for $f \in \mathcal{S}_0'(\mathbb{R})$ and $\phi \in \mathcal{S}_0(\mathbb{R})$*

$$\langle f,\phi\rangle = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c^{\psi}_{m,n}(f)c^{\bar{\psi}}_{m,n}(\phi) \ . \tag{3.7}$$

3.3. **The Space of Diadic Rapidly Decreasing Sequences.** We shall say that a double sequence $\{c_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is of diadic rapid decrease if

(3.8)

$$\|\{c_{m,n}\}\|^{\mathcal{W}}_l := \sup_{(m,n)\in\mathcal{Z}^2} |c_{m,n}|\,(1+|n|)^l \left(2^m + \frac{1}{2^m}\right)^l < \infty \ , \text{ for all } l \in \mathbb{N}_0 \ .$$

We denote the space of all sequences satisfying (3.8) by $\mathcal{W}(\mathbb{Z}^2)$, we call it the space of *diadic rapidly decreasing* sequences. The canonical Fréchet space topology in $\mathcal{W}(\mathbb{Z}^2)$ is defined by means of the seminorms (3.8). Its dual is $\mathcal{W}'(\mathbb{Z}^2)$, the space of diadic slowly increasing sequences. One readily verifies that this dual space is canonically identificable with those sequences satisfying

(3.9)

$$\left\|\{c'_{m,n}\}\right\|^{\mathcal{W}'}_{-l} := \sup_{(m,n)\in\mathcal{Z}^2} |c_{m,n}|\,(1+|n|)^{-l} \left(2^m + \frac{1}{2^m}\right)^{-l} < \infty \ , \text{ for some } l \in \mathbb{N}_0 \ .$$

So, we obtain the following isomorphisms.

Proposition 3.7. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. The linear map $c^{\psi} : \mathcal{S}_0(\mathbb{R}) \longmapsto \mathcal{W}(\mathbb{Z}^2)$ which takes $\phi \mapsto \left\{c^{\psi}_{m,n}(\phi)\right\}_{(m,n)\in\mathbb{Z}^2}$ is an isomorphism of Fréchet spaces.*

Proof. The continuity of the map follows directly from Lemma 3.1. The map is injective because $\{\psi_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is an orthonormal basis of $L^2(\mathbb{R})$. That it is onto can be established as in the proof of Lemma 3.4, being its inverse $\{c_{m,n}\} \mapsto \sum\sum c_{m,n}\psi_{m,n}$. Finally, one shows easily that the inverse is continuous, for instance, applying the open mapping theorem [19]. □

Proposition 3.8. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. The map which takes $f \in \mathcal{S}'_0(\mathbb{R})$ to its wavelet coefficients is an isomorphism of $\mathcal{S}'_0(\mathbb{R})$ onto $\mathcal{W}'(\mathbb{Z}^2)$ for the strong dual topologies.*

Proof. It is enough to observe that its inverse is the transpose of □

Remark 3.9. We can describe $\mathcal{W}'(\mathbb{Z}^2)$ as an inductive limit of an increasing sequence of Banach spaces. For each $l \in \mathbb{N}_0$, set

$$\mathcal{W}_{-l}(\mathbb{Z}^2) := \left\{\left\{c'_{m,n}\right\} : \left\|\left\{c'_{m,n}\right\}\right\|^{\mathcal{W}'}_{-l} < \infty\right\}$$

with norm $\|\cdot\|^{\mathcal{W}'}_{-l}$; then,

$$\mathcal{W}'(\mathbb{Z}^2) = \bigcup_{l\in\mathbb{N}_0} \mathcal{W}_{-l}(\mathbb{Z}^2) = \operatorname{ind}\lim_{l\in\mathbb{N}_0} \mathcal{W}_{-l}(\mathbb{Z}^2)\ .$$

3.4. **Characterization of Boundedness and Convergence in $\mathcal{S}'_0(\mathbb{R})$ through Wavelet Coefficients.** We now use Proposition 3.8 and Remark 3.9 to obtain a characterization bounded sets in $\mathcal{S}'_0(\mathbb{R})$ in terms of localization of wavelet coefficients; note that because of the Banach-Steinhaus theorem weak boundedness is equivalent to strong boundedness [19].

Corollary 3.10. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. A subset $\mathfrak{B} \subset \mathcal{S}'_0(\mathbb{R})$ is (strongly) weakly bounded in $\mathcal{S}'_0(\mathbb{R})$ if and only if there exist constants $C, \beta, \gamma > 0$ such that*

$$|c^{\psi}_{m,n}(f)| \leq C(|n|+1)^{\beta}\left(2^m + \frac{1}{2^m}\right)^{\gamma}\ , \quad \forall f \in \mathfrak{B}\ . \tag{3.10}$$

Proof. By Proposition 3.8, $\mathfrak{B}$ is bounded if and only if $\left\{c^{\psi}(f) : f \in \mathfrak{B}\right\}$ is bounded in $\mathcal{W}'(\mathbb{Z}^2)$, and since the latter is the inductive limit of the Banach spaces $\mathcal{W}_{-l}(\mathbb{Z}^2)$, it holds if and only if that set lies in one of these spaces and is bounded in a $\|\ \|^{\mathcal{W}'}_{-l}$ norm, which is obvioulsy equivalent to (3.10). □

As a corollary of Corollary 3.10, we characterize convergent sequences. We state this result in the next theorem.

Theorem 3.11. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be an orthonormal wavelet. A net $\{f_\lambda\}_{\lambda\in\mathbb{R}}$ is (strongly) weakly convergent ($\lambda \to \infty$) in $\mathcal{S}'_0(\mathbb{R})$ if and only each of the following limits exist*

$$\lim_{\lambda\to\infty} c^{\psi}_{m,n}(f_\lambda) = a_{m,n} < \infty\ , \tag{3.11}$$

and there exist constants $\lambda_0, C, \beta, \gamma > 0$ *such that*

$$|c^{\psi}_{m,n}(f_\lambda)| \leq C(|n|+1)^{\beta}\Big(2^m + \frac{1}{2^m}\Big)^{\gamma}\ , \quad \forall\ \lambda \geq \lambda_0\ . \tag{3.12}$$

In such a case the limit functional, $\lim_{\lambda\to\infty} f_\lambda = g$*, satisfies* $c^{\psi}_{m,n}(g) = a_{m,n}$.

Proof. Assume (3.11) and (3.12). By Corollary 3.10, relation (3.12), and the Banach-Steinhaus theorem $\{f_\lambda\}_{\lambda\in\mathbb{R}}$ is strongly bounded in $\mathcal{S}'_0(\mathbb{R})$; on the other hand, as a consequence of (3.11), it is weakly convergent on the the linear span of $\{\psi_{m,n}\}$, which turns out to be dense in $\mathcal{S}_0(\mathbb{R})$ (Corollary 3.3), hence the net is weakly convergent. The Montel property [19, p.358] of $\mathcal{S}_0(\mathbb{R})$ shows now that the net is in fact strongly convergent. Conversely, the weak convergence gives directly (3.11) while (3.12) is a consequence of Corollary 3.10. □

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Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281 Gebouw S22, 9000 Ghent, Belgium

E-mail address: jvindas@math.lsu.edu