Characterising and constructing codes using finite geometries

Lins Denaux

September 29, 2023

PhD Defence

GHENT UNIVERSITY

Overview

1 Points & hyperplanes

- Known results in the plane, $q = p$
- Known results for $d > 2$
- Known results in the plane, $q > p$
- Known results for $n > 2$, $q > p$
- Minimal small weight codewords

2 Strong blocking sets

- Motivation
- Known results
- Construction methods
- Six lines
- Seven planes

3 Saturating sets

- Motivation
- Known results
- The inspiration
- Two different approaches
- The spark
- A monstrous construction

Points & hyperplanes

The code $\mathcal{C}_{d-1}(d, q)$
Vector space over $\mathbb{F}_p$ spanned by the rows of the incidence matrix of hyperplanes and points in $\mathrm{PG}(d, q)$. Vectors = '**codewords**'.

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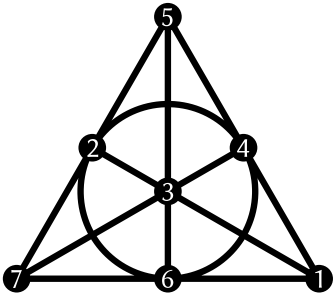
$$\text{hyperplanes}\left\{\overbrace{\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}}^{\text{points}}\right.$$



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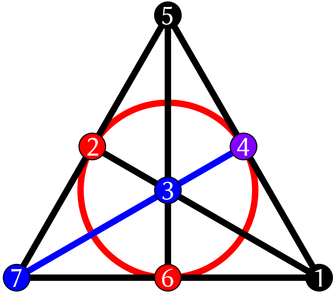
points (columns) / hyperplanes (rows)

1	1	1	0	0	0	0
1	0	0	1	1	0	0
1	0	0	0	0	1	1
0	1	0	1	0	1	0
0	1	0	0	1	0	1
0	0	1	1	0	0	1
0	0	1	0	1	1	0



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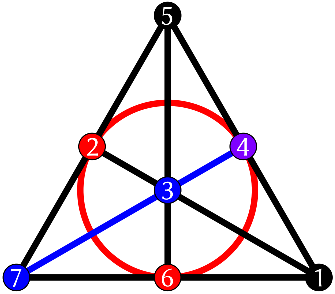
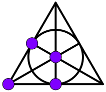


red + blue = (0 1 1 0 0 1 1) = 

Points & hyperplanes

Known results in the plane, $q = p$

Points & hyperplanes

Characterised up till $\mathrm{wt}(c) \leqslant 4p - 22$ (Szőnyi and Weiner [17])

$\mathrm{wt(c)} \leqslant \max\{3p + 1, 4p - 22\}$

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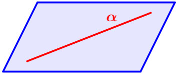
wt(c) $= p + 1$



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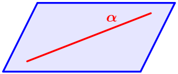
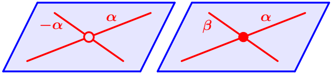


$\mathrm{wt}(c) = 2p\ (+1)$



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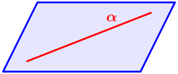

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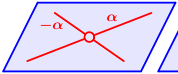

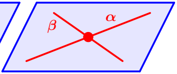

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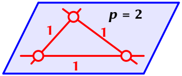

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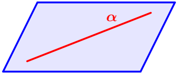

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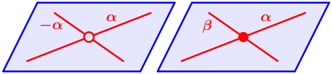

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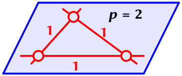

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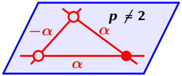

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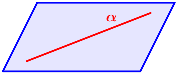

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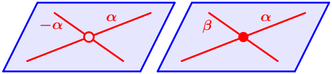

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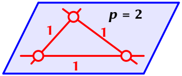

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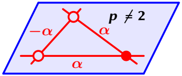

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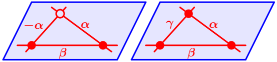

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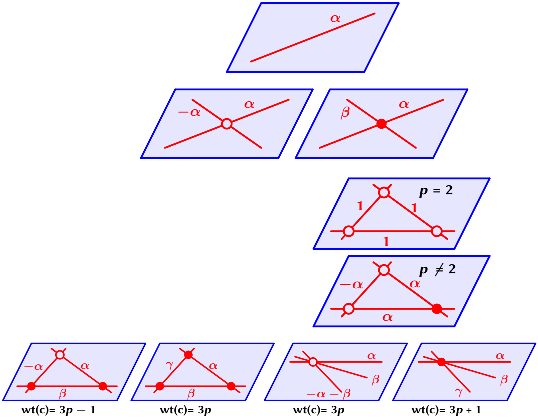

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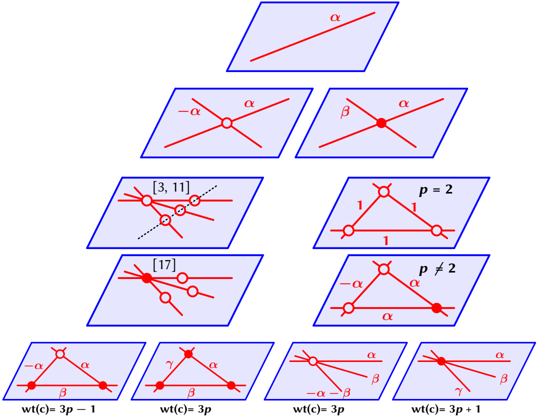

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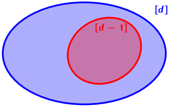

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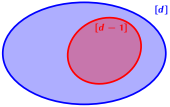

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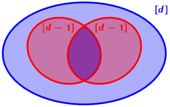

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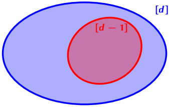

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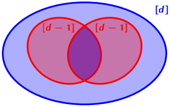
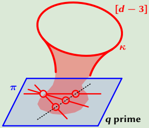


New result: characterisation up to

$$\mathrm{wt}(c) \leqslant \left(4 - \mathcal{O}\left(\frac{1}{q}\right)\right)\theta_{d-1}$$



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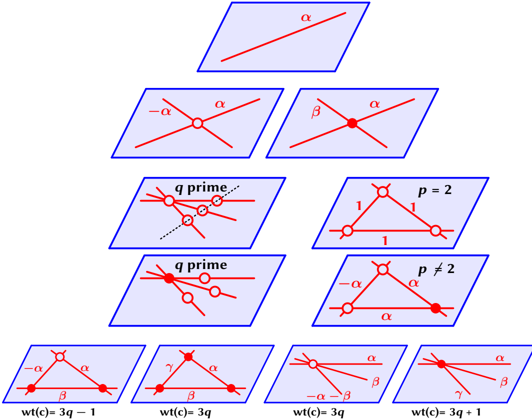

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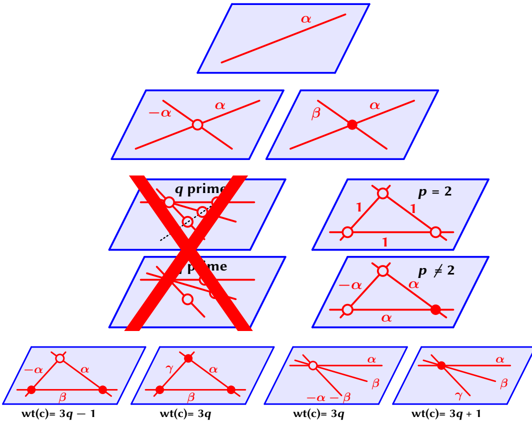

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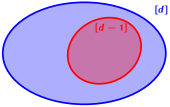

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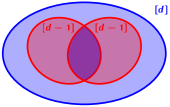

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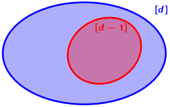

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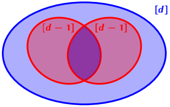

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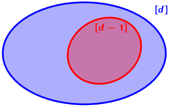

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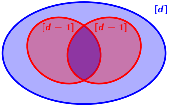
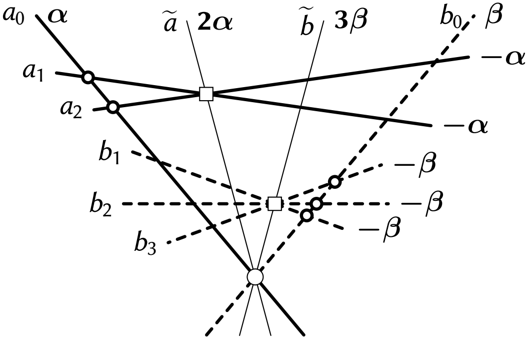


New result: characterisation up to

$$\mathbf{wt(c)} \lesssim \frac{1}{2^{d-2}}\sqrt{q}\,\theta_{d-1} \quad \rightarrow \quad \mathbf{wt(c)} \lesssim \sqrt{q}\,\theta_{d-1}$$

A codeword c is *minimal* if for any c' with $\mathrm{supp}(c') \subseteq \mathrm{supp}(c)$ there exists an $\alpha \in \mathbb{F}_p$ such that $c' = \alpha c$.

Suppose that $3\alpha + 4\beta = 0$.



$$\mathbb{H}_c^1 = \{\{a_0, a_1, a_2\}, \{\widetilde{a}\}, \{b_0, b_1, b_2, b_3\}, \{\widetilde{b}\}\}$$

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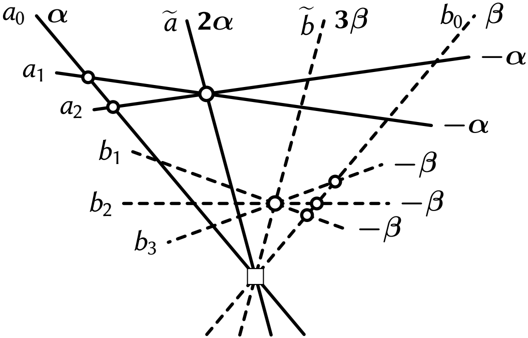

$$\mathbb{H}_c^2 = \{\{a_0, a_1, a_2, \widetilde{a}\}, \{b_0, b_1, b_2, b_3, \widetilde{b}\}\}$$

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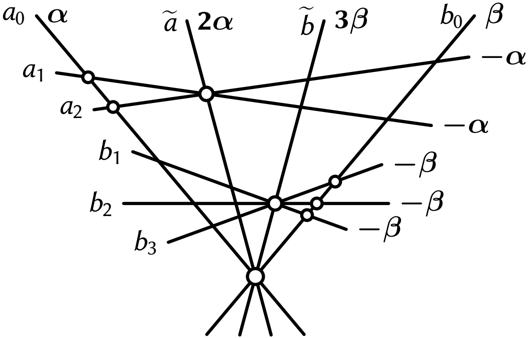

$$\mathbb{H}_c^3 = \{\{a_0, a_1, a_2, \widetilde{a}, b_0, b_1, b_2, b_3, \widetilde{b}\}\} = \mathbb{H}_c^\infty$$

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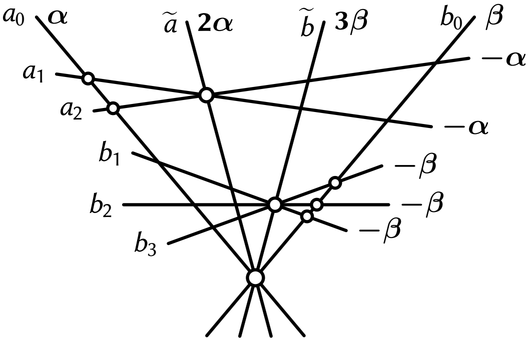

New result

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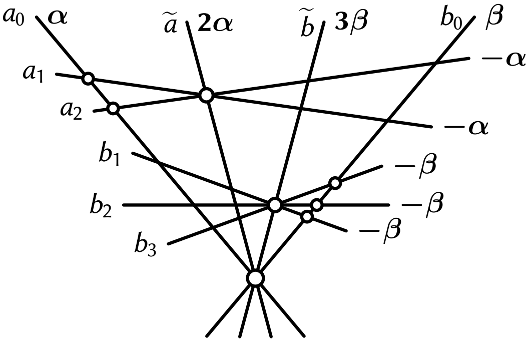

New result

Let $c \in \mathcal{C}_{d-1}(d, q)$, $\mathrm{wt}(c)$ small. $|\mathbb{H}_{\boldsymbol{c}}^{\infty}| = \mathbf{1} \Rightarrow \boldsymbol{c}$ **minimal**.

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Let $k \in \{0, \ldots, d\}$.

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An *$(d-k+1)$-fold k-blocking set* of $\mathrm{PG}(d,q)$ is a point set that meets every $(d-k)$-dimensional space in at least $d-k+1$ points.

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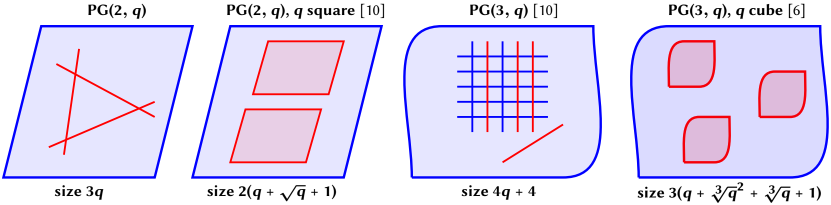

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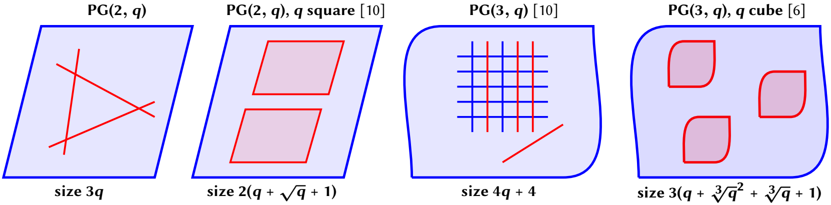

A *higgledy-piggledy set of k-spaces* is a set $\mathcal{K}$ of k-spaces such that the point set $\cup\mathcal{K}$ is a strong k-blocking set.

Let $k \in \{0, \ldots, d\}$.

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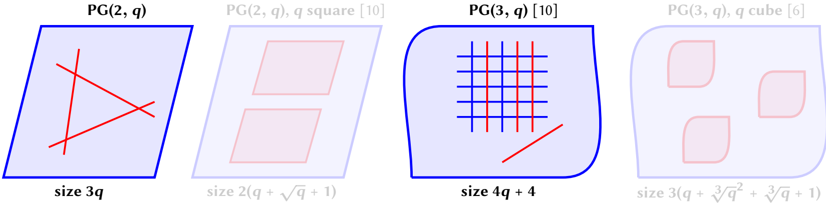

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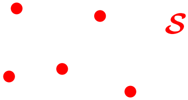
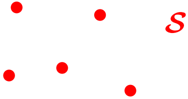
2 Strong blocking sets



strong block. set w.r.t. hyperplanes, $\mathcal{S} := \{P_1, \ldots, P_{|\mathcal{S}|}\}$.



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$$\begin{array}{ccccccc} P_1 & P_2 & P_3 & & P_i & & P_{|\mathcal{S}|} \end{array}$$

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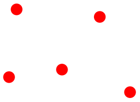
Strong blocking sets



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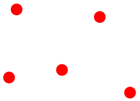
$$

coordinates of P_i



$\mathcal{S}$

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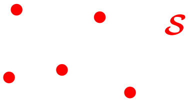
coordinates of P_i

Theorem (Alfarano, Borello and Neri [1];
Tang, Qiu, Liao and Zhou [18]).
$\rightarrow$ the generator matrix of a **minimal** linear $[|\mathcal{S}|, d+1]_q$-code!

Strong blocking sets



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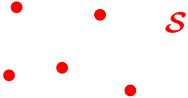
Goal: Finding small strong blocking sets.

Strong blocking sets



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Goal: Finding small ~~strong blocking sets~~ higgledy-piggledy sets.

Theorem (Fancsali and Sziklai [14]).
If $q \geqslant d + \lfloor \frac{d}{2} \rfloor$, then
a higg.-pigg. line set contains at least $d + \lfloor \frac{d}{2} \rfloor$ lines.

Theorem (Fancsali and Sziklai [14]; Héger and Nagy [15]).
If $q \geqslant d + \lfloor \frac{d}{2} \rfloor$, then
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If $q \geqslant 2d - 1$, then
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Strong blocking sets

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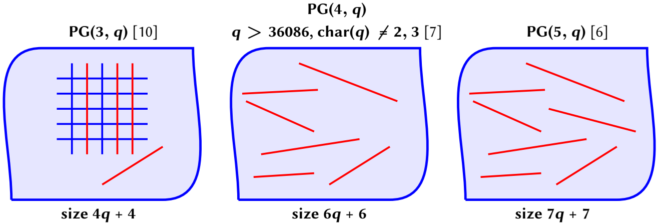

Theorem (Fancsali and Sziklai [13]).
A set $\mathcal{K}$ of k-spaces, $|\mathcal{K}| \leqslant q$, is a higg.-pigg. set
$\Leftrightarrow$ no $(d-k-1)$-space meets all its elements.

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Projection
e.g. consider one line parity case.

Dualisation
$k \to d-k-1$
(e.g. lower bound).

Field reduction
only if $d+1$ is composite.

Coordinates
disjoint six lines in $\mathrm{PG}(4,q)$ [7].

Probability
asymptotic bounds, see e.g. [15].

Elem. geometry
intersecting six lines, see next slide.

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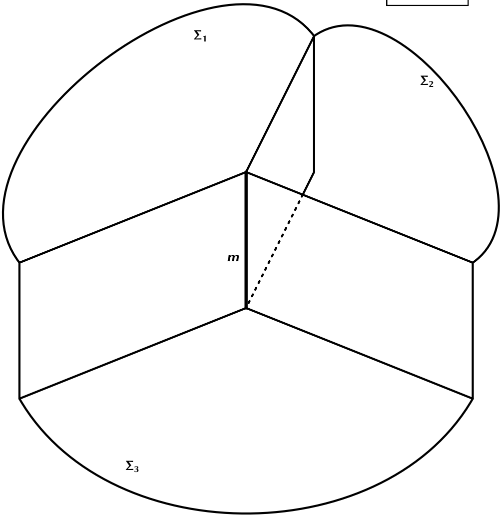
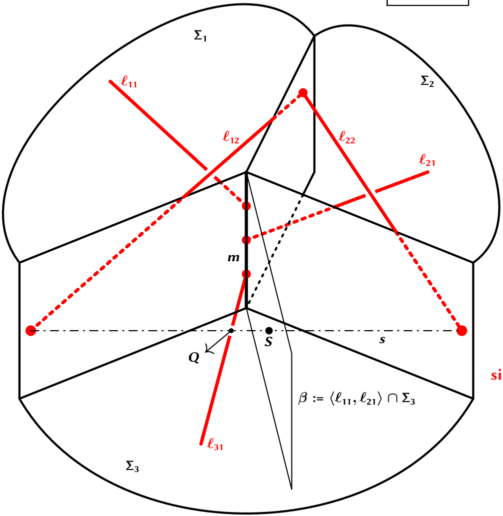
PG(4, q)



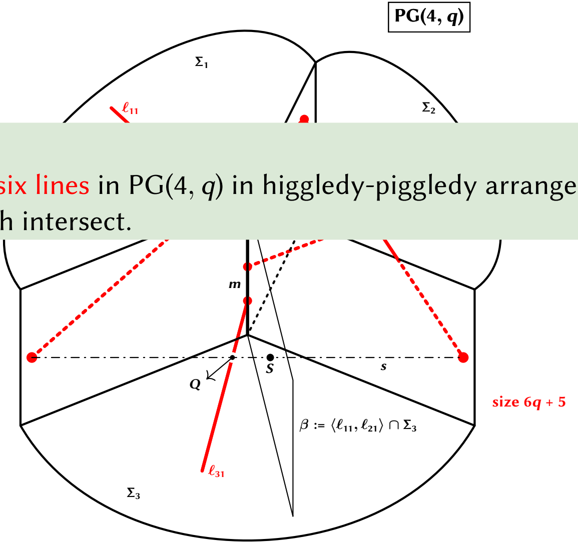
Strong blocking sets

Six lines

PG(4, q)

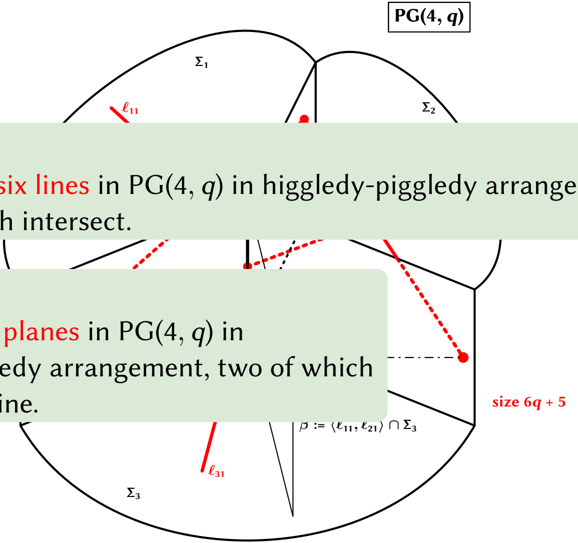


size $6q + 5$



New result

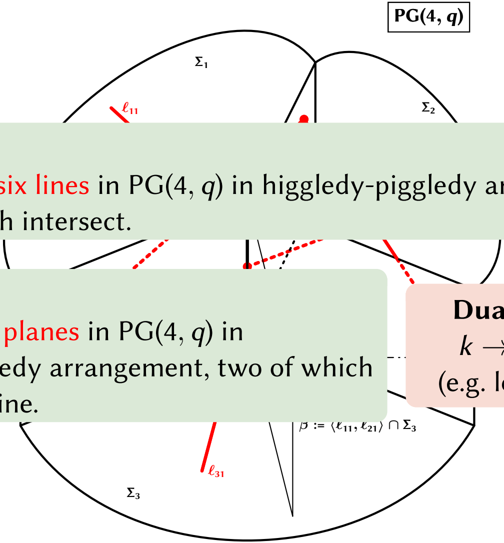
There exist six lines in PG(4, q) in higgledy-piggledy arrangement, two of which intersect.



New result
There exist six lines in PG$(4, q)$ in higgledy-piggledy arrangement, two of which intersect.

New result
There exist six planes in PG$(4, q)$ in higgledy-piggledy arrangement, two of which intersect in a line.

Six lines



New result
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$k \to d - k - 1$
(e.g. lower bound).

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Theorem (Fancsali and Sziklai [13]).
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Projection
e.g. consider one line parity case.

Dualisation
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Field reduction
only if $d+1$ is composite.

Coordinates
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Definition (Linear set).
A *linear set* is a point set $\mathcal{P} \subseteq \mathrm{PG}\left(r-1, q^t\right)$ s.t.
$(\exists \pi \in \mathrm{PG}(rt-1, q))(\mathcal{F}_{r,t,q}(P) \cap \pi \neq \emptyset \Leftrightarrow P \in \mathcal{P})$

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Field reduction
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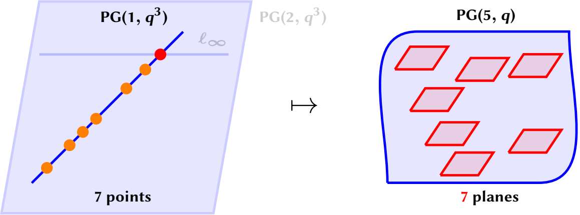
Theorem.
If $\mathcal{P} \subseteq \mathrm{PG}(r-1, q^t)$ is not contained in a (non-triv.) linear set, then $\mathcal{F}_{r,t,q}(\mathcal{P})$ is a hig.-pig. set of $(t-1)$-spaces in $\mathrm{PG}(rt-1, q)$.

Seven planes

Strong blocking sets

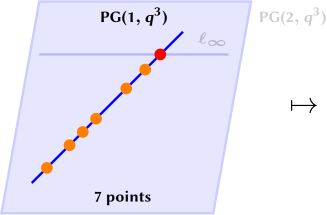
Seven planes



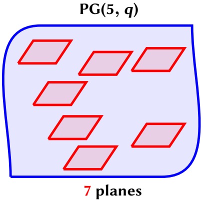
Theoretical lower bound:
7 planes.

Strong blocking sets

Seven planes



$\mapsto$



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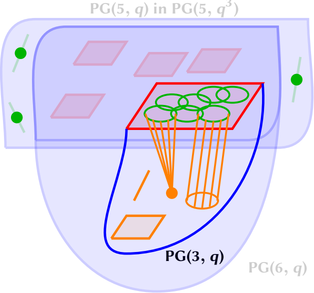
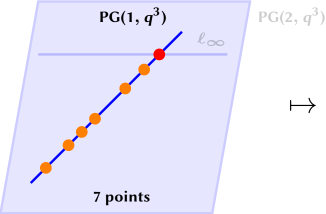


- Bundle of conics determined by 3 points
- $\mathbb{F}_q$-lines $\to$ affine lines
- clubs with head at infinity $\to$ affine planes
- clubs without head at infinity $\to$ cones
- scattered $\mathbb{F}_q$-linear sets $\to$ hyperbolic quadrics

Strong blocking sets

Seven planes



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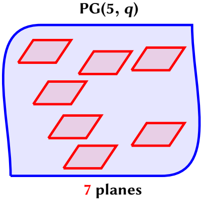
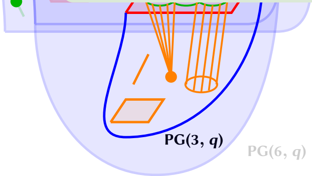


Theoretical lower bound:
7 planes.

New result
There exist seven planes in PG(5, q) in higgledy-piggledy arrangement.



- clubs with head at infinity → affine planes
- clubs without head at infinity → cones
- scattered $\mathbb{F}_q$-linear sets → hyperbolic quadrics

Saturating sets

Let $\varrho \in \{0, 1, \ldots, d\}$.

A ***ϱ-saturating set $\mathcal{S}$*** of $\mathrm{PG}(d, q)$: point set such that

- any point of $\mathrm{PG}(d, q)$ lies in span of $\leqslant \varrho + 1$ points of $\mathcal{S}$.

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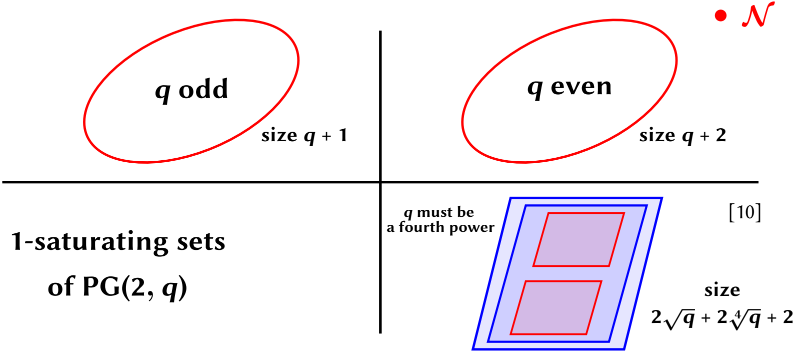
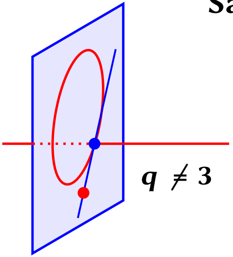
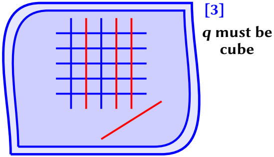


Saturating sets of PG(3, q)

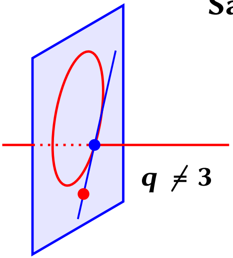


- **1**-saturating set of PG(3, q) [8].
- Size: $2q + 1$.



- **2**-saturating set of PG(3, q) [10].
- Size: $4\sqrt[3]{q} + 4$.

Saturating sets of PG(3, q)



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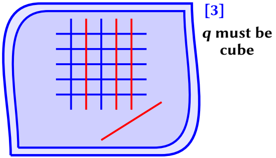

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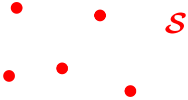
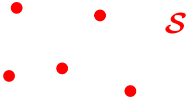
- Size: $4\sqrt[3]{q} + 4$.

$\boldsymbol{s_q(d, \varrho)}$:= size of a smallest ϱ-saturating set of PG(d, q).



ϱ-saturating set of $\mathrm{PG}(d, q)$, $\mathcal{S} := \{P_1, P_2, P_3, \dots, P_{|\mathcal{S}|}\}$.



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$$
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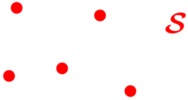
Saturating sets



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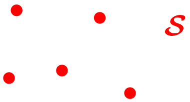
coordinates of P_i

Saturating sets

Motivation



ϱ-saturating set of $\mathrm{PG}(d, q)$, $\mathcal{S} := \{P_1, P_2, P_3, \ldots, P_{|\mathcal{S}|}\}$.

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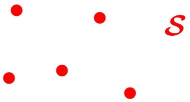
coordinates of P_i

PC-matrix of a $[|\mathcal{S}|, |\mathcal{S}| - d - 1]_q$ $(\varrho + 1)$-*covering code*!

Any vector of $\mathbb{F}_q^{|\mathcal{S}|}$ lies within Hamming distance $\varrho + 1$ of a codeword.

Saturating sets



ϱ-saturating set of PG(d, q), $\mathcal{S} := \{P_1, P_2, P_3, \dots, P_{|\mathcal{S}|}\}$.

$$\begin{array}{ccccccc} P_1 & P_2 & P_3 & & P_i & & P_{|\mathcal{S}|} \\ \end{array}$$

$$\begin{pmatrix} x_{10} & x_{20} & x_{30} & \cdots & x_{i0} & \cdots & x_{|\mathcal{S}|0} \\ x_{11} & x_{21} & x_{31} & \cdots & x_{i1} & \cdots & x_{|\mathcal{S}|1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1d} & x_{2d} & x_{3d} & \cdots & x_{id} & \cdots & x_{|\mathcal{S}|d} \end{pmatrix}$$

coordinates of P_i

PC-matrix of a $[|\mathcal{S}|, |\mathcal{S}| - d - 1]_q$ $(\varrho + 1)$-*covering code*!

Any vector of $\mathbb{F}_q^{|\mathcal{S}|}$ lies within Hamming distance $\varrho + 1$ of a codeword.

Goal: Finding good upper bounds for $s_q(d, \varrho)$.

Saturating sets

PG(2, q): *LOTS of research!*

- Strongly $\sim$ to complete caps.
- Often computer searches.
- Nice survey in [9]. (Davydov & Östergård, 2000)

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$$s_q(d, \varrho) \lesssim \binom{d+1}{\varrho} q^{\frac{d-\varrho}{\varrho+1}}$$

if q is a $(\varrho+1)^{th}$ power.

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$$s_q(d, 1) \lesssim 2q^{\frac{d-1}{2}} \sqrt{\ln(q)}$$

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$$s_q(d, \varrho) \gtrsim \varrho \cdot q^{\frac{d-\varrho}{\varrho+1}}.$$

~~Hypothesis~~ Desperate wish

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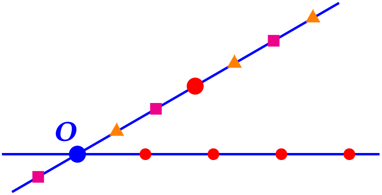
- Let q be square.



Theorem (Davydov, 1995 [8])
Let q be square. Then

$$s_q(2,1) \leqslant 3\sqrt{q} - 1.$$

Keep in mind

$$s_q(2,1) \quad \gtrsim \quad \sqrt{\boldsymbol{q}}.$$

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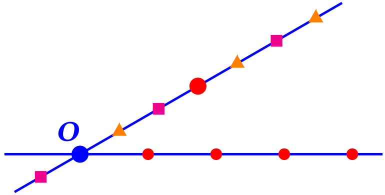

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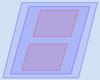

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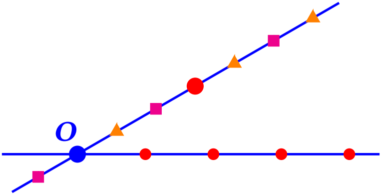

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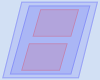

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3 Saturating sets

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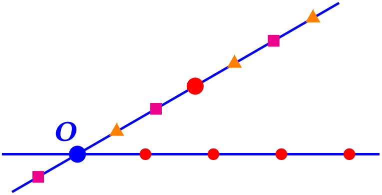

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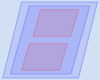

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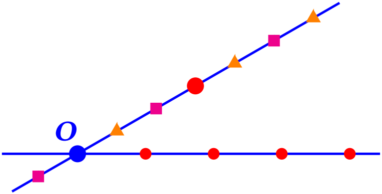

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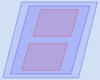

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If q is a $(\varrho + 1)^{\text{th}}$ power: two possible paths to take

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Path of the single subgeometry
Strong blocking set approach

Saturating sets

Two different approaches

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Path of the single subgeometry
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- Strong $d - \varrho$-blocking sets in $\mathrm{PG}\left(d, \sqrt[\varrho+1]{q}\right)$.

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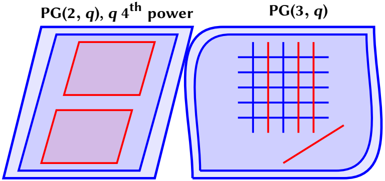

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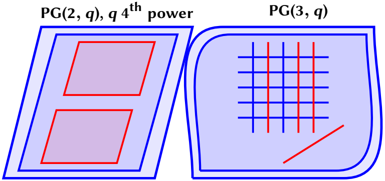

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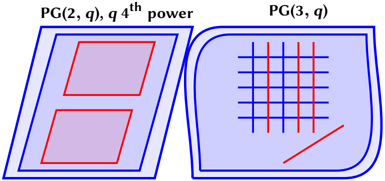

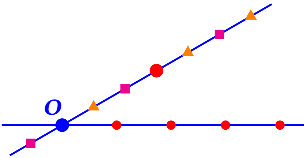

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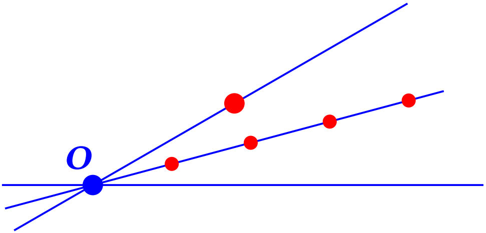

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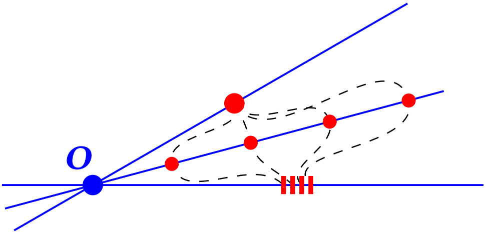

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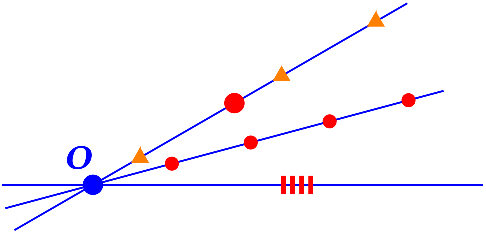

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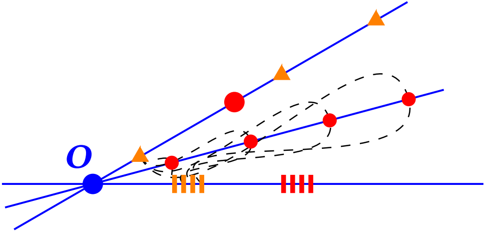

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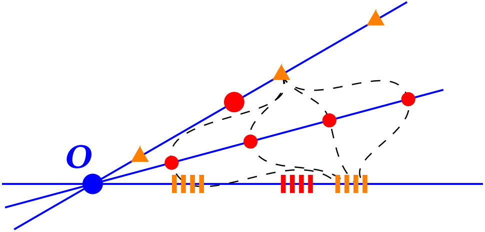

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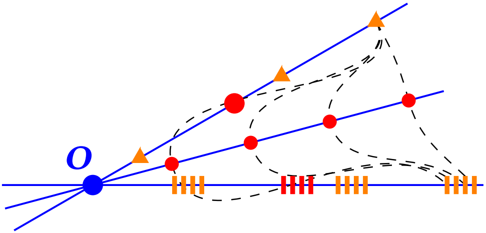

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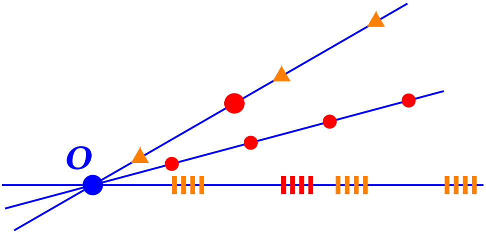

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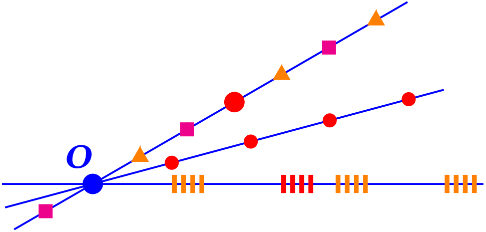

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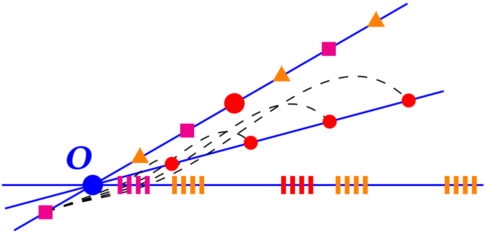

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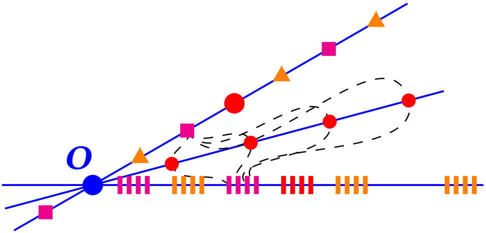

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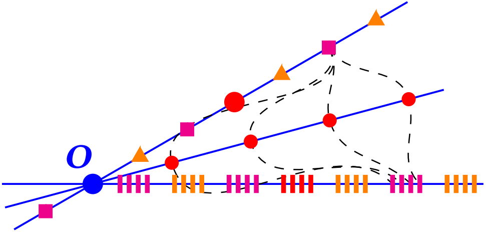

Saturating sets

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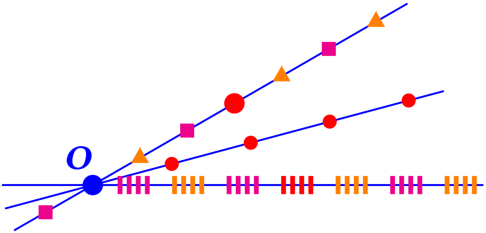

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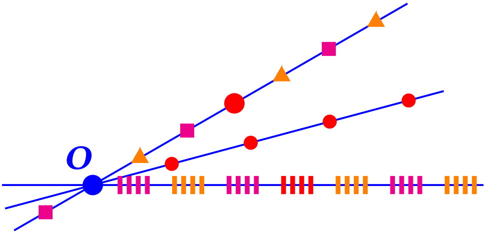

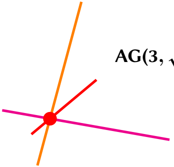

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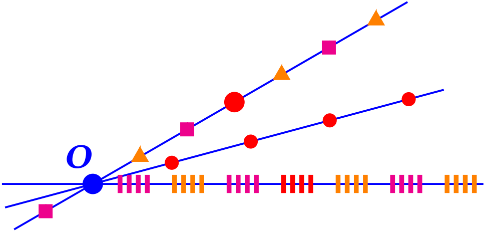

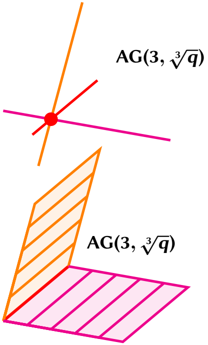

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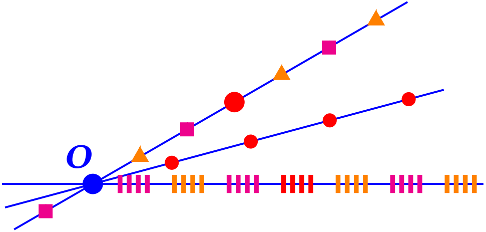

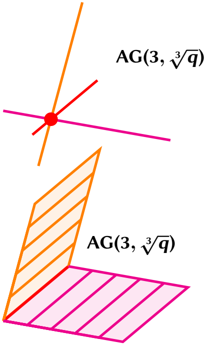

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Let q be cube. Then

$$s_q(3, 2) \leqslant 6\sqrt[3]{q} - 3.$$

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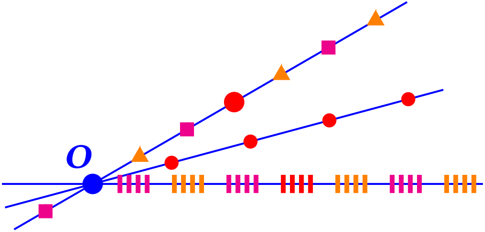

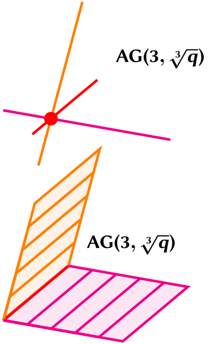

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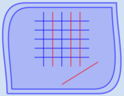

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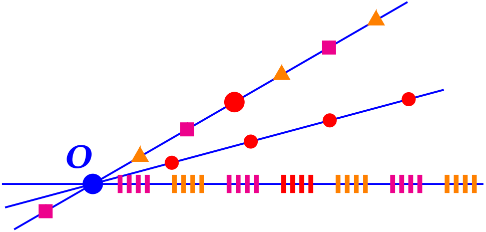

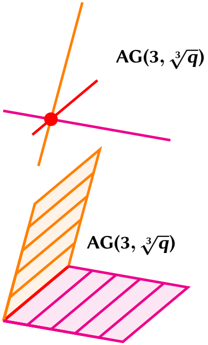

“This last construction looks promising!” - LD, 2019

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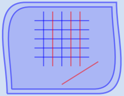
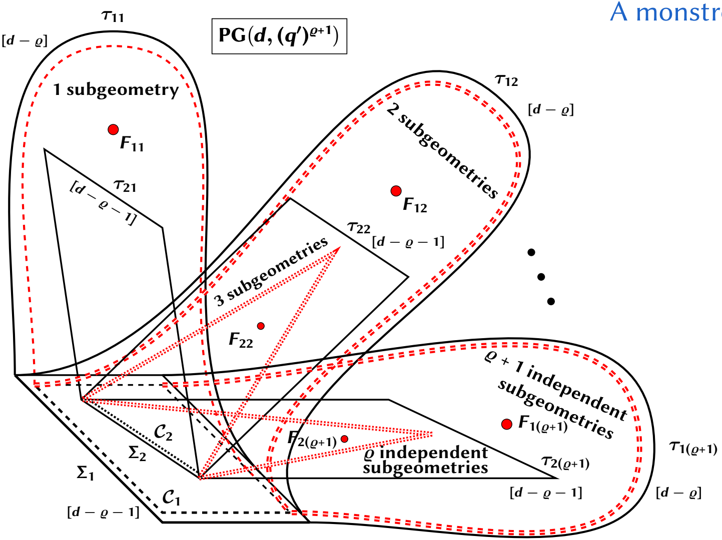


Saturating sets

A monstrous construction



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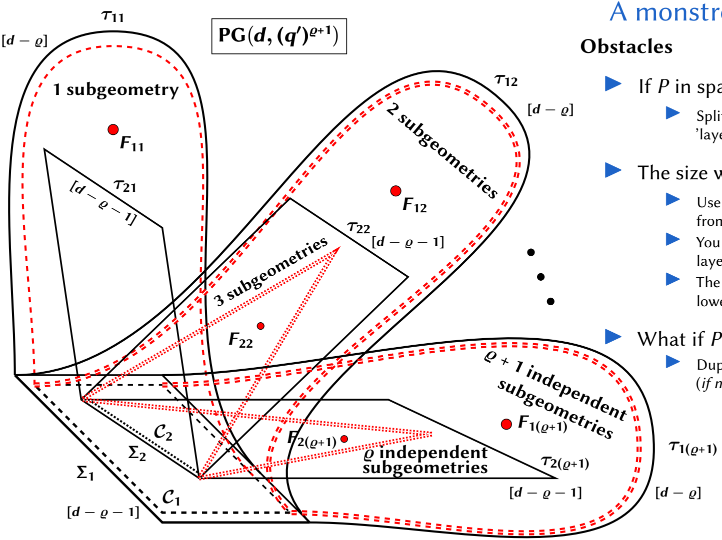

Obstacles

- If P in span of $< (\varrho + 1)$ petals?
  - Split petals and add multiple 'layers' in each petal!
- The size will get expon. big...
  - Use the subgeometries from the petal above!
  - You only need $\min\{\varrho, d - \varrho\}$ layers, and not in *all* petals!
  - The number of subgeometries in lower layers can be reduced!
- What if $P \in \Sigma_1$?
  - Duplicate construction! (*if necessary*)

New result

Let $0 < \varrho < d$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(d,\varrho) \leqslant \sum_{i=1}^{k(d,\varrho)} \left(\frac{(\varrho+1)(\varrho+2)}{2} (q')^{d+1-i(\varrho+1)} \right) + \sum_{i=1}^{k(d,\varrho)-1} \sum_{j=1}^{\varrho-1} \tilde{a}(\varrho,j)(q')^{d+1-i(\varrho+1)-j}$$
$$+ \sum_{j=1}^{\ell(d,\varrho)-1} \overline{a}(d,\varrho,j)(q')^{\ell(d,\varrho)-j} - \tilde{c}(d,\varrho) - \overline{c}(d,\varrho)$$
$$+ \delta_{q'=2} \cdot \left((2^{\varrho-1}-1) \cdot \sum_{i=1}^{k(d,\varrho)-1} \left(2^{d-\varrho+2-i(\varrho+1)} \right) + 2^{\ell(d,\varrho)} - 2 \right),$$

with

- $k(d,\varrho) := \left\lceil \frac{d-\varrho}{\varrho+1} \right\rceil$,
- $\ell(d,\varrho) := \left(d \pmod{\varrho+1} \right) + 1$,
- $\tilde{a}(\varrho,j) := \frac{\varrho(\varrho+2j+1)-j(3j+1)}{2}$,
- $\overline{a}(d,\varrho,j) := \frac{\ell(d,\varrho)\left(2\varrho-\ell(d,\varrho)+2j+1\right)-j(3j+1)}{2}$,
- $\tilde{c}(d,\varrho) := \left(k(d,\varrho)-1\right)\frac{\varrho^2(\varrho+1)}{2}$,
- $\overline{c}(d,\varrho) := \frac{\varrho(\varrho+1)+\ell(d,\varrho)\left(\ell(d,\varrho)-1\right)\left(2\varrho-\ell(d,\varrho)+1\right)}{2}$,
- $\delta_{q'=2} := \begin{cases} 1 & \text{if } q'=2, \\ 0 & \text{if } q' \neq 2. \end{cases}$

New result

Let $1 < \varrho < d$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(d, \varrho) \leqslant \frac{(\varrho+1)(\varrho+2)}{2}(q')^{d-\varrho} + \varrho(\varrho+1)\left((q')^{d-\varrho-1} + \cdots + q' + 1\right).$$

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$$s_q(d,\varrho) \leqslant \frac{(\varrho+1)(\varrho+2)}{2}(q')^{d-\varrho} + \varrho(\varrho+1)\left((q')^{d-\varrho-1} + \cdots + q' + 1\right).$$

Somewhat new result

$$s_q(d,\varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{d-\varrho}{\varrho+1}}.$$

~~Hypothesis~~ Desperate wish

$$s_q(d,\varrho) \quad \lesssim \quad \varrho \cdot q^{\frac{d-\varrho}{\varrho+1}},$$

for all d, $\varrho \leqslant d$ and ∞-many q.

New result

Let $1 < \varrho < d$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(d,\varrho) \leqslant \frac{(\boldsymbol{\varrho+1})(\boldsymbol{\varrho+2})}{\mathbf{2}}(q')^{d-\varrho} + \varrho(\varrho+1)\left((q')^{d-\varrho-1} + \cdots + q' + 1\right).$$

Somewhat new result

$$s_q(d,\varrho) \quad \gtrsim \quad \boldsymbol{\varrho \cdot q^{\frac{d-\varrho}{\varrho+1}}}.$$

~~Hypothesis~~ Desperate wish

$$s_q(d,\varrho) \quad \lesssim \quad \boldsymbol{\varrho \cdot q^{\frac{d-\varrho}{\varrho+1}}},$$

for all d, $\varrho \leqslant d$ and ∞-many q.

Davydov et al., 2011 [10]

$$\boldsymbol{s_q(d,\varrho) \lesssim \binom{d+1}{\varrho} q^{\frac{d-\varrho}{\varrho+1}}}$$

if q is a $(\varrho+1)^{th}$ power.

A monstrous construction

New result

Let $1 < \varrho < d$ and let $q = (q')^{\varrho+1}$ for any prime power q'. Then

$$s_q(d,\varrho) \leqslant \frac{(\varrho+1)(\varrho+2)}{2}(q')^{d-\varrho} + \varrho(\varrho+1)\left((q')^{d-\varrho-1} + \cdots + q' + 1\right).$$

Somewhat new result

$$s_q(d,\varrho) \quad \gtrsim \quad \varrho \cdot q^{\frac{d-\varrho}{\varrho+1}}.$$

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$$s_q(d,\varrho) \lesssim \binom{d+1}{\varrho} q^{\frac{d-\varrho}{\varrho+1}}$$

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$$s_q(d,\varrho) \quad \lesssim \quad \varrho \cdot q^{\frac{d-\varrho}{\varrho+1}},$$

for all d, $\varrho \leqslant d$ and ∞-many q.

Fin.

Thank you for your attention. Are there any questions?

Q & A

Fin.ite geometry is awesome!

Thank you for your attention. Are there any
questions?

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