On the smallest maximal partial ovoids and spreads of the generalized quadrangles $W(q)$ and $Q(4, q)$

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Abstract

We present results on the size of the smallest maximal partial ovoids and on the size of the smallest maximal partial spreads of the generalized quadrangles $W(q)$ and $Q(4, q)$.

Key words: generalized quadrangle, partial ovoid, partial spread

1 Introduction

A *finite generalized quadrangle* $\mathrm{GQ}(s, t)$ is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ consisting of two non-empty disjoint sets P and B, consisting respectively of points and lines, such that:

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(1) every line is incident with $s+1$ points and every point is incident with $t+1$ lines,
(2) two distinct points are incident with at most one common line, and two distinct lines are incident with at most one common point, and
(3) for every non-incident point-line pair (r, L), there exists a unique line M and a unique point r' such that $r \,\mathrm{I}\, M \,\mathrm{I}\, r' \,\mathrm{I}\, L$.

We call the pair (s, t) the *order* of this GQ(s, t). We denote collinear points x and y by $x \sim y$, and concurrent lines L and M by $L \sim M$.

The thick *classical* finite generalized quadrangles are respectively the non-singular 4-dimensional parabolic quadrics $Q(4, q)$ of order (q, q), the non-singular 5-dimensional elliptic quadrics $Q^-(5, q)$ of order (q, q^2), the non-singular 3- and 4-dimensional Hermitian varieties $H(3, q^2)$ and $H(4, q^2)$ of respective orders (q^2, q) and (q^2, q^3), and the non-singular finite generalized quadrangle $W(q)$ of order (q, q) consisting of the points of $PG(3, q)$ and of the totally isotropic lines of a symplectic polarity η.

A *spread* of a GQ(s, t) is a set of lines partitioning the point set of this generalized quadrangle. A *partial spread* of a GQ(s, t) is a set of pairwise disjoint lines of this generalized quadrangle. A partial spread is called *maximal* when it is not contained in a larger partial spread. An *ovoid* $\mathcal{O}$ of a GQ(s, t) is a set of points such that every line of this generalized quadrangle shares exactly one point with $\mathcal{O}$. A *partial ovoid* $\mathcal{O}$ of a GQ(s, t) is a set of points such that every line of this generalized quadrangle shares at most one point with $\mathcal{O}$. A partial ovoid is called *maximal* when it is not contained in a larger partial ovoid.

A spread and an ovoid of a GQ(s, t) have size $st + 1$.

A lot of attention has been paid to the (non-)existence of spreads and ovoids in finite generalized quadrangles [18,19]. Similarly, a lot of research has already been done on partial spreads and partial ovoids of size $st + 1 - d$, with small deficiency d, with special emphasis on the extendability of such partial spreads and partial ovoids to spreads and ovoids [4,12].

Recently, special attention has been paid to the smallest maximal partial ovoids and to the smallest maximal partial spreads of finite generalized quadrangles.

A maximal partial ovoid in a GQ(s, t) always must have size greater than or equal to $s + 1$ and a maximal partial spread in a GQ(s, t) must have size greater than or equal to $t + 1$.

In [1], Aguglia, Ebert and Luyckx studied the smallest maximal partial spreads of $Q^-(5, q)$ =GQ(q, q^2). They prove that the minimal size for such a maximal partial spread is equal to $t+1 = q^2+1$ if and only if q is even, and in this case, this maximal partial spread is equal to a spread of a subquadrangle $Q(4,q)$. For q odd, they prove that a maximal partial spread of $Q^-(5,q)$ must have size larger than q^2+2.

Since $Q^-(5,q)$ is dual to the generalized quadrangle $H(3,q^2)$, the analogous results on maximal partial ovoids for $H(3,q^2)$ are valid.

Ebert and Hirschfeld studied the smallest maximal partial spreads of $H(3,q^2)$ [10]. They prove that every maximal partial spread has size at least $2q+1$, and for $q \geq 4$, at least size $2q+2$. Their results translate into results on the smallest maximal partial ovoids of $Q^-(5,q)$.

In [8], Cimráková and Fack present computer results obtained for the spectra of sizes of maximal partial ovoids in $Q^-(5,q)$ and $H(3,q^2)$, including values for small sizes.

We contribute to this study for the two thick finite classical generalized quadrangles $W(q)$ and $Q(4,q)$. We note that $W(q)$ is dual to $Q(4,q)$, and that $Q(4,q)$ and $W(q)$ are self-dual if and only if q is even [14].

In [4,13], a (large) maximal partial ovoid of size q^2-q+1 in $W(q)$, q even, is constructed and it is proven that no partial ovoids with sizes larger than q^2-q+1 and smaller than q^2+1 exist. We present in this article a maximal partial ovoid of size q^2-2q+3 of $W(q)$, q even. The motivation for paying special attention to maximal partial ovoids of size q^2-2q+3 follows from the fact that computer searches seem to indicate that no maximal partial ovoids of size larger than q^2-2q+3 and smaller than q^2-q+1 exist in $W(q)$, q even; see also Table 1.

A *blocking set* of $\mathrm{PG}(n,q)$ is a set of points having a non-empty intersection with every hyperplane of $\mathrm{PG}(n,q)$. A *blocking set* is called *trivial* when it contains a line of $\mathrm{PG}(n,q)$. A *blocking set* is called *minimal* when none of its proper subsets still is a blocking set.

In our study, blocking sets in $\mathrm{PG}(2,q)$ and in $\mathrm{PG}(3,q)$ will play an important role.

In a generalized quadrangle, for a set A of points, the notation $A^\perp$ denotes the set of points collinear with every point of A. For two non-collinear points x and y of a generalized quadrangle, the set $\{x,y\}^{\perp\perp}$ is called the *hyperbolic line* defined by x and y. We note that for the generalized quadrangle $W(q)$, the hyperbolic lines $\{x,y\}^{\perp\perp}$ coincide with the projective lines xy of $\mathrm{PG}(3,q)$ which are not totally isotropic with respect to the symplectic polarity η.

2 Small maximal partial ovoids in $W(q)$

Theorem 2.1 *The smallest maximal partial ovoids of $W(q)$ have size $q+1$ and consist of the point sets of the hyperbolic lines of $W(q)$.*

Proof. Consider $W(q)$ in its natural representation in $\mathrm{PG}(3,q)$ described by the symplectic polarity η, then it follows that every maximal partial ovoid $\mathcal{O}$ of $W(q)$ must be a blocking set of $\mathrm{PG}(3,q)$ with respect to the planes of $\mathrm{PG}(3,q)$. Namely, if there is a plane π skew to $\mathcal{O}$, then the point π^η extends $\mathcal{O}$ to a larger partial ovoid, which contradicts the maximality of $\mathcal{O}$. Since, from the result of Bose and Burton [3], the smallest blocking set of this type consists of the $q+1$ points of a line, the theorem follows. □

Corollary 2.2 (1) *The smallest maximal partial spreads of $Q(4,q)$ have size $q+1$ and consist of the lines of a regulus of* $\mathrm{PG}(3,q)$.

(2) *The smallest maximal partial spreads of $W(q)$, q even, have size $q+1$ and consist of the lines of a regulus of* $\mathrm{PG}(3,q)$.

(3) *The smallest maximal partial ovoids of $Q(4,q)$, q even, have size $q+1$ and consist of the point sets of conics having the nucleus of $Q(4,q)$ as their nucleus.*

Now that we have classified the smallest maximal partial ovoids of $W(q)$, we focus on results on the second smallest maximal partial ovoids of $W(q)$. Since the preceding proof shows that such a maximal partial ovoid must be a blocking set with respect to the planes of $\mathrm{PG}(3,q)$, the planar non-trivial blocking sets are obvious candidates for such maximal partial ovoids. However, these are easily excluded.

Theorem 2.3 *A maximal partial ovoid $\mathcal{O}$ of $W(q)$, different from a hyperbolic line, cannot be a planar blocking set.*

Proof. Suppose that $\mathcal{O}$ is a planar blocking set, lying in the plane π of $\mathrm{PG}(3,q)$. Let $r=\pi^\eta$. Then $r \notin \mathcal{O}$. But since $|\mathcal{O}| > q+1$, there is at least one totally isotropic line through r in π containing more than one point of $\mathcal{O}$; we have a contradiction. □

Lemma 2.4 *A maximal partial ovoid $\mathcal{O}$ of $W(q)$ is a minimal blocking set with respect to the planes of* $\mathrm{PG}(3,q)$.

Proof. It follows from the preceding proofs that $\mathcal{O}$ is a blocking set with respect to the planes of $\mathrm{PG}(3,q)$. Assume that it is not minimal. Suppose that the point r of $\mathcal{O}$ is not essential as point of $\mathcal{O}$, considered as blocking set with respect to the planes of $\mathrm{PG}(3,q)$. Then every plane through r contains a second point of $\mathcal{O}$. So also the plane r^η contains a second point r' of $\mathcal{O}$. Then the totally isotropic line rr' contains at least two points of $\mathcal{O}$. This is impossible. $\square$

We now use results on the minimal blocking sets with respect to planes of $\mathrm{PG}(3,q)$. The first result is of Bruen.

Theorem 2.5 (Bruen [5]) *The smallest non-trivial blocking sets with respect to planes of* $\mathrm{PG}(3,q)$ *are equal to the smallest planar non-trivial blocking sets of* $\mathrm{PG}(2,q)$.

Theorem 2.3 shows us that the second smallest maximal partial ovoids of $W(q)$ cannot be equal to the smallest non-trivial minimal blocking sets with respect to planes of $\mathrm{PG}(3,q)$. So for the second smallest maximal partial ovoids of $W(q)$, we need to focus on the second smallest non-trivial minimal blocking sets with respect to planes of $\mathrm{PG}(3,q)$. This allows us to obtain a considerably stronger result in some specific cases. We will first use the following two theorems from [15].

Let $s(q)$ denote the cardinality of the second smallest non-trivial minimal blocking sets in $\mathrm{PG}(2,q)$.

Theorem 2.6 (Storme and Weiner [15, Theorem 4.9]) *Let K be a blocking set of* $\mathrm{PG}(3,q^2)$, *$q = p^h$, $p > 3$ prime, $h \geq 1$, of cardinality smaller than or equal to $s(q^2)$. Then K contains a line or a planar blocking set of* $\mathrm{PG}(3,q^2)$.

Theorem 2.7 (Storme and Weiner [15, Theorem 5.9 and 5.10]) *A minimal blocking set of* $\mathrm{PG}(3,q^3)$, *$q = p^h$, $p \geq 7$ prime, $h \geq 1$, of size at most $q^3 + q^2 + q + 1$ is one of the following:*

- *a line,*
- *a Baer-subplane if q is a square,*
- *a minimal planar blocking set of size $q^3 + q^2 + 1$,*
- *a minimal planar blocking set of size $q^3 + q^2 + q + 1$,*
- *a subgeometry* $\mathrm{PG}(3,q)$.

Corollary 2.8 *The second smallest maximal partial ovoids $\mathcal{O}$ of $W(q^2)$, $q = p^h$, $p > 3$ prime, $h \geq 1$, contain at least $s(q^2) + 1$ points. If $q = p > 2$, then $\mathcal{O}$ contains at least $3(p^2 + 1)/2 + 1$ points.*

Proof. This follows immediately from Theorem 2.6 and the fact that $s(p^2) = 3(p^2 + 1)/2$ if $p > 2$ (see e.g. [16]). $\square$

Corollary 2.9 *The second smallest maximal partial ovoids $\mathcal{O}$ of $W(q^3)$, $q = p^h$, $p \geq 7$ prime, $h \geq 1$, contain at least $q^3 + q^2 + q + 1$ points. If $|\mathcal{O}| = q^3 + q^2 + q + 1$, then $\mathcal{O}$ consists of the point set of a subgeometry* $\mathrm{PG}(3,q)$ *of*

PG$(3, q^3)$.

An open problem regarding maximal partial ovoids of $W(q^3)$ is whether $W(q^3)$ effectively has maximal partial ovoids equal to a subgeometry PG$(3, q)$.

Finally in the case when $q = p$ prime, we can use the result of Blokhuis [2] which states that every non-trivial planar blocking set of PG$(2, p)$ contains at least $3(p + 1)/2$ points.

Corollary 2.10 *Let $\mathcal{O}$ be a second smallest maximal partial ovoid of $W(p)$, p prime. Then $|\mathcal{O}| \geq 3(p + 1)/2 + 1$.*

Remark 2.11 (1) The preceding results can be translated into results on maximal partial spreads of $Q(4, q)$, on maximal partial spreads of $W(q)$, q even, and on maximal partial ovoids of $Q(4, q)$, q even.

(2) To conclude this section on the size of the second smallest maximal partial ovoids of $W(q)$, we note that an example of a maximal partial ovoid of size $2q + 1$ can be obtained by taking all points except one point r on a hyperbolic line L in PG$(3, q)$, together with one arbitrary point (not collinear with one of the remaining points of L) from each of the $q + 1$ lines of $W(q)$ through r.

3 Small maximal partial spreads in $W(q)$

The only cases we have not yet discussed are the smallest maximal partial ovoids of $Q(4, q)$, q odd, and the smallest maximal partial spreads of $W(q)$, q odd. Since $W(q)$ is dual to $Q(4, q)$, we concentrate on maximal partial spreads of $W(q)$, q odd.

Recall that when q is an odd prime power, $\{L_1, L_2, L_3\}^\perp \in \{0, 2\}$ for every triad of skew lines of $W(q)$ (see e.g. [14]). We will use a counting technique from [11] to prove the following theorem. In the following theorem, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Theorem 3.1 *Suppose that S is a maximal partial spread of $W(q)$, q odd. Then $|S| \geq \lceil 1,419q \rceil$.*

Proof. Suppose that $|S| = x$. Then there are exactly $D := q^3 + q^2 + q + 1 - x$ lines of $W(q)$ not belonging to S. Let n_i, $i = 1, \ldots, q + 1$, denote the number of such lines intersecting exactly i lines of the partial spread S. By counting in two ways the pairs (L, M), where L is a line not belonging to S, where M

is a line belonging to S, and where $L \sim M$, we obtain

$$\sum_i in_i = x(q+1)q.$$

For the triples (L_1, L_2, M), where $L_1 \neq L_2$ are lines belonging to S, where M is a line not belonging to S and where $L_1 \sim M \sim L_2$, we obtain

$$\sum_i \binom{i}{2} n_i = \binom{x}{2}(q+1),$$

and for the quadruples (L_1, L_2, L_3, M), where L_1, L_2, L_3 are distinct lines belonging to S, where M is a line not belonging to S, and where $M \sim L_m$, $m = 1, 2, 3$, we obtain

$$\sum_i \binom{i}{3} n_i \leq \binom{x}{3} 2.$$

Consider the polynomial $P(i) := (i - r_1)(i - r_2)(i - r_3)$ and the coefficients a_0, a_1, a_2, a_3 such that $P(i) = a_3\binom{i}{3} + a_2\binom{i}{2} + a_1 i + a_0$. We see that $a_3 = 6$, $a_2 = -2(r_1 + r_2 + r_3) + 6$, $a_1 = r_1r_2 + r_1r_3 + r_2r_3 - (r_1 + r_2 + r_3) + 1$, and $a_0 = -r_1r_2r_3$. Henceforth,

$$\sum_i P(i)n_i = a_3 \sum_i \binom{i}{3} n_i + a_2 \sum_i \binom{i}{2} n_i + a_1 \sum_i in_i + a_0 \sum_i n_i.$$

¿From this, using $a_3 > 0$, it follows that

$$\sum_i P(i)n_i \leq 2a_3\binom{x}{3} + (q+1)a_2\binom{x}{2} + q(q+1)a_1x + a_0(q^3+q^2+q+1-x). \quad (1)$$

If we choose coefficients r_1, r_2, r_3 in such a way that $P(i)n_i \geq 0$ for every $i \in \{1, \ldots, q+1\}$, then $\sum_i P(i)n_i \geq 0$ and consequently x has to be such that the right hand side of Equation (1) is greater than or equal to 0. We will select $r_1 = 1$, $r_2 = a$, and $r_3 = a + 1$, with $a \in \mathbb{N}$ to be determined. We obtain $a_0 = -a^2 - a$, $a_1 = a + a^2$, $a_2 = -4a + 2$, and $a_3 = 6$.

In order to obtain a bound of the form $x \geq cq$, we substitute $x = cq$ in the right hand side of Equation (1), and we obtain a polynomial of degree 3 in q. As we want our bound to be valid for general q, the coefficient $g = ca^2 + ca - 2ac^2 - a^2 - a + c^2 + 2c^3$ of q^3 has to be less than or equal to 0. For $c = 1.419$, we find that the solutions in a of $g = 0$ are $3.99\ldots$ and $4.612\ldots$ So if we choose $a = 4$, it readily follows that $x \geq \lceil 1.419q \rceil$. □

Remark 3.2 The result of the previous theorem can be slightly improved to $x \geq \lceil 1.419q + b \rceil$ for certain $b > 0$, by substituting $x = 1.419q + b$ and $a = 4$ in the right hand side of Equation (1), and by solving for the greatest b for which the obtained polynomial in q is still negative. The expression for b obtained in this way is a tedious formula in q, but it can easily be computed by computer for given q. For example, in the cases $q = 7, 9, 11$, this increases the smallest theoretical value of x by one to 11, 14 and 17, respectively. It should however be noted that b is extremely small with respect to q.

4 Computer results

In this section, we present results obtained by computer searches implementing the exhaustive and heuristic search techniques described in [9]. All programs are written in Java and the results are obtained on a 1.6Ghz Pentium processor running Linux.

4.1 Maximal partial ovoids in $W(q)$

In Table 1, we give results for maximal partial ovoids in $W(q)$. For each value of q, we list the sizes for which the heuristic search found maximal partial ovoids of that given size. The notation $a..b$ means that a maximal partial ovoid of that size has been found for all values in the interval $[a, b]$.

For $q = 2, 3, 4, 5$, exhaustive search confirmed that the spectrum found by the heuristic is complete. Note that the largest value found for $W(5)$ and $W(7)$ is indeed the size of the largest maximal partial ovoid – this was confirmed by exhaustive search.

The results in Table 1 confirm the result from Theorem 2.1 that the smallest maximal partial ovoids have size $q + 1$. For the cases presented here, we also observe that maximal partial ovoids of size $2q + 1$ were always found, while no maximal partial ovoids with sizes between $q + 1$ and $2q + 1$ were found. As indicated in Remark 2.11, an example of a maximal partial ovoid of size $2q+1$ can be obtained by taking all points except one point r on a hyperbolic line L in $W(q)$, together with one arbitrary point (not collinear with one of the remaining points of L) from each of the $q + 1$ lines of $W(q)$ through r.

Moreover, our results show the existence of a maximal partial ovoid of size $3q - 1$, for all values of q considered. Such a maximal partial ovoid can be constructed in the following way if $q \geq 4$.

Let x and y be two non-collinear points of $W(q)$ and consider the two hyperbolic lines $H_1 := \{x, y\}^\perp$ and $H_2 := \{x, y\}^{\perp\perp}$. Define $\mathcal{O}_1$ to be the set $H_2 \setminus \{x, y\}$. Let z be any point on H_1 and H any hyperbolic line, distinct from H_1, through z in the plane x^η (here η is the symplectic polarity defining

q	Spectrum found		
2*	**3,5**		
3*	**4,7**		
4*	**5,9,**		**11**,13,17
5*	**6,11,**	12,**14**..18	
7	**8,15,**	17..**20**..33	
8	**9,17,**	21..**23**..47,49,	**51**,57,65
9	**10,19,**	25..**26**..51	
11	**12,23,**	28..**32**..70	
13	**14,27,**	**38**..92	
16	**17,33,**	**47**,49,51..163,165,	**227**,241,257
17	**18,35,**	**50**..129	
19	**20,39,**	**56**..150	
23	**24,47,**	**68**..70,72..190	
25	**26,51,**	**74**..76,78,80..203	
27	**28,55,**	**80**..236	

Table 1
Spectrum of sizes for maximal partial ovoids of $W(q)$, for small values of q. For $q = 2, 3, 4, 5$, the complete spectrum was obtained by exhaustive search. For larger values of q, the results are obtained by heuristic search. For $q = 5, 7$, the size of the largest partial ovoid was determined by exhaustive search.

$W(q)$). Choose any point $u \in H \setminus \{z\}$ and define $\mathcal{O}_2 := (H \setminus \{u, z\}) \cup \{v\}$, where v is any point on xu distinct from x and u, and with v not contained in H_1. On the line yz of $W(q)$, there is a unique point p collinear with all points of H. Since $q \geq 4$, it is possible to choose a point $o \in yz \setminus \{y, p, z\}$ that is not collinear with v. If w is the point of H_1 on the line uv, then every point (with exception of y) of the line yw is collinear with a point of $\mathcal{O}_2$. On each of the $q - 1$ totally isotropic lines through y distinct from yz and yw, there is a unique point, lying on the line $u^\eta \cap y^\eta = wp$, collinear with no point of $\mathcal{O}_2 \cup \{o\}$. Denote by $\mathcal{O}_3$ the set $(wp \setminus \{w, p\}) \cup \{o\}$. Then $\mathcal{O} := \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$ is a maximal partial ovoid of size $3q - 1$. We check the maximality. Since $\mathcal{O}_1 = H_2 \setminus \{x, y\}$, only points of x^η and y^η could extend $\mathcal{O}$ to a larger partial ovoid. In x^η, since $\mathcal{O}_2 := (H \setminus \{u, z\}) \cup \{v\}$, only the points of xz could extend $\mathcal{O}$ to a larger partial ovoid. Similarly, in y^η, only points of yw could extend $\mathcal{O}$. A detailed check shows that no points of xz or yw extend $\mathcal{O}$ to a larger partial ovoid.

For q even, our computer searches also find a maximal partial ovoid of size $q^2 - q + 1$ and no maximal partial ovoids with sizes larger than $q^2 - q + 1$ and smaller than $q^2 + 1$, as the results of [4] and [13] show. We also observed the existence of a maximal partial ovoid with size $q^2 - q + 1 - (q - 2) = q^2 - 2q + 3$, and we found no maximal partial ovoids with size larger than $q^2 - 2q + 3$ and smaller than $q^2 - q + 1$.

We can describe in a compact way a geometric construction for maximal partial ovoids of sizes $q^2 - q + 1$ and $q^2 - 2q + 3$ of $W(q)$, q even. We explain the construction on $Q(4, q)$ (recall that q is even and so $Q(4, q) \cong W(q)$). First notice that $\left|C^{\perp}\right| \in \{1, q + 1\}$ for any conic C in $Q(4, q)$. From this we see that if we consider a conic C in an elliptic quadric $\mathcal{O} := Q^-(3, q) \subset Q(4, q)$, then necessarily $C^{\perp}$ is a unique point c. It is easily seen that $(\mathcal{O} \cup \{c\}) \setminus C$ is a maximal partial ovoid of size $q^2 - q + 1$. Now let $\mathcal{O}$ be an elliptic quadric of $Q(4, q)$ and suppose that C_1 and C_2 are two conics of $\mathcal{O}$, with $|C_1 \cap C_2| = 2$. Clearly the points $c_1 := C_1^{\perp}$ and $c_2 := C_2^{\perp}$ are not collinear (since $|C_1 \cap C_2| = 2$). If $q > 2$, it follows easily that $(\mathcal{O} \cup \{c_1, c_2\}) \setminus (C_1 \cup C_2)$ is a maximal partial ovoid of size $q^2 - 2q + 3$.

4.2 Maximal partial ovoids in $Q(4, q)$, q odd

In Table 2, we give results for maximal partial ovoids in $Q(4, q)$, q odd. For each value of q, we list the value of the lower bound (LB) from Theorem 3.1 and Remark 3.2, and the sizes for which our program found maximal partial ovoids of that given size. The notation $a..b$ means that for all values in the interval $[a, b]$, a maximal partial ovoid of that size has been found.

For $q = 3, 5$, we confirmed by exhaustive search that the spectrum found is complete. For $q = 7, 9$, we confirmed by exhaustive search for some sizes (also given in the table) that no maximal partial ovoid of that size exists.

In spite of the fact that the theoretical lower bounds are linear in q, these results rather seem to indicate a quadratic lower bound.

In all cases our heuristic finds an ovoid (of size $q^2 + 1$). For $q = 3, 5, 7, 11$, a maximal partial ovoid of size $q^2 - 1$ is found; for $q = 9$, it is confirmed by exhaustive search that no such maximal partial ovoid exists; for larger values of q, no such maximal partial ovoids were found by the heuristic.

For all values of q considered, the largest (resp. second largest, for the cases $q = 3, 5, 7, 11$) size for a maximal (strictly) partial ovoid found by the heuristic search is $q^2 - q + 2$.

q	LB	Spectrum found (by heuristics)	Non-existence (exhaustive search)
3*	5	5,8,10	all other values
5*	8	13..20,**22**,**24**,26	all other values
7	11	14,17..42,**44**,**48**,50	10,11,43,45,46,47,49 (still open: 12,13,15,16)
9	14	22..68,70,73,**74**,82	79,**80**
11	17	28,30..106,109..110,**112**,**120**,122	
13	19	41..42,44..136,138,140,146,148,**158**,170	
17	25	67..218,220..224,226,228..230, 232..238,240,244,246..248,258,260,**274**,290	
19	27	84..118,122..275,278,280,282..286,294, 296,298,300,310,312,326,328,**344**,362	

Table 2
Spectrum of sizes for maximal partial ovoids of $Q(4,q)$, for small values of q. For $q = 3,5$, the complete spectrum was obtained by exhaustive search. For larger values of q, the results are obtained by heuristic search. For $q = 7,9$, the non-existence of maximal partial ovoids of certain sizes was confirmed by exhaustive search.

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