Generalized lax Veronesean embeddings of projective spaces

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Abstract

We classify all embeddings $\theta : \mathsf{PG}(n,\mathbb{K}) \longrightarrow \mathsf{PG}(d,\mathbb{F})$, with $d \geq \frac{n(n+3)}{2}$ and $\mathbb{K},\mathbb{F}$ skew fields with $|\mathbb{K}| > 2$, such that θ maps the set of points of each line of $\mathsf{PG}(n,\mathbb{K})$ to a set of coplanar points of $\mathsf{PG}(d,\mathbb{F})$, and such that the image of θ generates $\mathsf{PG}(d,\mathbb{F})$. It turns out that $d = \frac{1}{2}n(n+3)$ and all examples "essentially" arise from a similar "full" embedding $\theta' : \mathsf{PG}(n,\mathbb{K}) \longrightarrow \mathsf{PG}(d,\mathbb{K})$ by identifying $\mathbb{K}$ with subfields of $\mathbb{F}$ and embedding $\mathsf{PG}(d,\mathbb{K})$ into $\mathsf{PG}(d,\mathbb{F})$ by several ordinary field extensions. These "full" embeddings satisfy one more property and are classified in [4]. They relate to the quadric Veronesean of $\mathsf{PG}(n,\mathbb{K})$ in $\mathsf{PG}(d,\mathbb{K})$ and its projections from subspaces of $\mathsf{PG}(d,\mathbb{K})$ generated by sub-Veroneseans (the points corresponding to subspaces of $\mathsf{PG}(n,\mathbb{K})$), if $\mathbb{K}$ is commutative, and to a degenerate analogue of this, if $\mathbb{K}$ is noncommutative.

1 Introduction

The goal of this paper is to relax the conditions in a recent result of Thas & Van Maldeghem [4]. This result concerns a classification of full generalized Veronesean embeddings of projective spaces, which, as the name suggests, relate strongly to the quadric Veroneseans of these projective spaces. Given the importance of these objects in classical

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algebraic geometry and finite geometry, it is a worthwhile job to do. Let us sketch the situation.

According to [4], a "full" *generalized Veronesean embedding* is an embedding $\theta : \mathsf{PG}(n,q) \longrightarrow \mathsf{PG}(d,q)$, with $d \geq \frac{n(n+3)}{2}$, such that θ maps the set of points of each line of $\mathsf{PG}(n,q)$ to a set of coplanar points of $\mathsf{PG}(d,q)$ and such that the image of θ generates $\mathsf{PG}(d,q)$. The image in $\mathsf{PG}(d,q)$ is then called a *(full) generalized Veronesean.* It is shown in [4] that $d = \frac{n(n+3)}{2}$ and that each such embedding θ is constructed as follows. Let $\alpha : \mathsf{PG}(n,q) \longrightarrow \mathsf{PG}(d,q)$ be the ordinary quadric Veronesean map, and let U be an i-dimensional subspace of $\mathsf{PG}(n,q)$, with $-1 \leq i \leq n-1$. Put $d' = \frac{i(i+3)}{2}$. Then the image of U under α spans a d'-dimensional subspace V of $\mathsf{PG}(d,q)$. Let W be a $(d-d'-1)$-dimensional subspace of $\mathsf{PG}(d,q)$ skew to V and let $\theta' : U \to V$ be a (full) generalized Veronesean embedding of U (defined inductively). Then define $\theta : \mathsf{PG}(n,q) \longrightarrow \mathsf{PG}(d,q)$ as $\theta(x) = \theta'(x)$ for $x \in U$, and $\theta(x) = \langle \alpha(x), V\rangle \cap W$ for $x \in \mathsf{PG}(n,q) \setminus U$. This embedding is called a *(full)* $(i+1)$*-Veronesean embedding*, and the subspace U will be referred to as the *lid* of the embedding. Hence a (full) 0-Veronesean embedding is an ordinary quadric Veronesean embedding and has empty lid.

The inductive definition implies that we can look at the lid of the lid, i.e., the lid of the embedding induced by the lid, and we can refer to this as the *second order lid.* Similarly, we can now define the ℓth order lid, for any positive integer ℓ. It is clear that there is a unique positive integer ℓ_0 such that the (ℓ_0+1)st order lid is empty. Then we call the ℓ_0th order lid the *ultimate lid* of the embedding, and ℓ_0 is called the *depth* of the embedding (0 for a 0-Veronesean). The (ℓ_0-1)st order lid is called the *pre-ultimate lid.* For an i-Veronesean embedding, the depth is at most equal to i.

The main result of [4] says that any (full) generalized Veronesean embedding is a (full) i-Veronesean embedding, for some suitable i.

In the proof, one encounters lines L of $\mathsf{PG}(d,q)$ containing the image of all points of some line of $\mathsf{PG}(n,q)$ but one. By finiteness it then follows that exactly one point z of L does not belong to the embedding. The set of such points is shown to be a subspace and the proof then continues, heavily relying on this observation. In the infinite case, an additional condition implies the same thing (see below). The question is: what happens if we do not have uniqueness of the point z above. This occurs if we substitute $\mathsf{PG}(d,q)$ with $\mathsf{PG}(d,r)$ in the above definitions to obtain a "lax" generalized Veronesean embedding. In principle, we have the inequality $q \leq r$ in mind, and expect $\mathsf{GF}(q)$ to be a subfield of $\mathsf{GF}(r)$, but our curiosity asks to also allow for the case $q > r$ and see what happens.

In the infinite case, the above results hold substituting $\mathsf{GF}(q)$ with an arbitrary field $\mathbb{K}$ (in the noncommutative case, though, there is no notion of Veronesean, and in this case only a (full) n-Veronesean embedding of $\mathsf{PG}(n,\mathbb{K})$ exists, which is defined directly but inductively as follows: choose an affine space $\mathsf{AG}(n,\mathbb{K})$ in $\mathsf{PG}(n,\mathbb{K})$, embed it in a natural way in a new n-dimensional projective space over $\mathbb{K}$, and take the direct sum of the latter with a (full) $(n-1)$st Veronesean embedding of the $(n-1)$-dimensonal projective space $\mathsf{PG}(n,\mathbb{K}) \setminus \mathsf{AG}(n,\mathbb{K})$), and adding the following condition. Let θ be an embedding of $\mathsf{PG}(n,\mathbb{K})$ in $\mathsf{PG}(d,\mathbb{K})$, with the above restrictions. Then we additionally require that,

(*) for each line L of $\mathsf{PG}(n,\mathbb{K})$, and each point $x \in \theta(L)$, whenever the map $y \mapsto \langle x, y\rangle$, $y \in \theta(L) \setminus \{x\}$, is injective, then there is a unique line T of $\mathsf{PG}(d,\mathbb{K})$ in $\langle\theta(L)\rangle$ through x such that $T \cap \theta(L) = \{x\}$.

In the present paper we remove this additional condition and consider projective spaces over two fields. Hence we consider maps $\theta : \mathsf{PG}(n,\mathbb{K}) \longrightarrow \mathsf{PG}(d,\mathbb{F})$, with $\mathbb{K}$ and $\mathbb{F}$ (skew) fields, such that collinear points are mapped onto coplanar ones. These are called *lax generalized Veronesean embeddings*. For a given embedding $\theta : \mathsf{PG}(n,\mathbb{K}) \longrightarrow \mathsf{PG}(d,\mathbb{F})$, we will identify from now on each point of $\mathsf{PG}(n,\mathbb{K})$ with its image under θ. For θ a lax generalized Veronesean embedding we call the image under θ a *lax generalized Veronesean* in the projective space $\mathsf{PG}(d,\mathbb{F})$, but also in the generalized projective space generated by the image (and which consists of the direct sum of $\ell+1$ proper subspaces, with ℓ the depth of the embedding). We can then formulate our main result as follows.

Main Result. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \in)$ be isomorphic to the geometry of points and lines of $\mathsf{PG}(n,\mathbb{K})$, $n \geq 2$, $|\mathbb{K}| > 2$, with $\mathcal{P} \subseteq \mathsf{PG}(d,\mathbb{F})$, $\langle\mathcal{P}\rangle = \mathsf{PG}(d,\mathbb{F})$, $d \geq \frac{n(n+3)}{2}$, and such that every member L of $\mathcal{L}$ is a subset of points of a plane in $\mathsf{PG}(d,\mathbb{F})$. Then $d = \frac{1}{2}n(n+3)$ and either $\mathbb{K}$ is isomorphic to a subfield of $\mathbb{F}$ and there exist ℓ pairwise disjoint subspaces $\mathsf{PG}(d_j,\mathbb{K}_j)$ of $\mathsf{PG}(d,\mathbb{F})$, with $\mathbb{K}_j \cong \mathbb{K}$ a subfield of $\mathbb{F}$, $1 \leq j \leq \ell$, such that $\mathcal{P}$ is an i-Veronesean in the direct sum of all subspaces $\mathsf{PG}(d_j,\mathbb{K}_j)$, for j running through $\{1,2,\ldots,\ell\}$, or*

- *the same thing holds, except that the ultimate or pre-ultimate lid is a line of $\mathsf{PG}(n,\mathbb{K})$ and not all points of that line belong to any plane over some subfield isomorphic to $\mathbb{K}$, or*

- *$|\mathbb{K}| = 3$, $n = 2$, the ultimate or pre-ultimate lid is a line L of $\mathsf{PG}(2,\mathbb{K})$ spanning some plane π of $\mathsf{PG}(5,\mathbb{F})$, and the points of $\mathsf{PG}(2,\mathbb{K})$ not on L are contained in a subplane π' of $\mathsf{PG}(5,\mathbb{F})$ defined over a (commutative) subfield $\mathbb{K}'$ of $\mathbb{F}$, with $\mathbb{K}'$ containing nontrivial cubic roots of unity.*

As in the full case, the case $|\mathbb{K}| = 2$ is a true exception, since every injective mapping from $\mathsf{PG}(n,2)$ into $\mathsf{PG}(d,\mathbb{F})$, with $d \geq \frac{1}{2}n(n+3)$, is a lax generalized Veronesean embedding as soon as the image set generates $\mathsf{PG}(d,\mathbb{F})$. And this can be achieved whenever $\mathsf{PG}(d,\mathbb{F})$ has at least $2^{n+1}-1$ points and $d \leq 2^{n+1}-2$.

But notice that, unlike the full case, also the case $|\mathbb{K}| = 3$ and $n = 2$ is a special case now with an unusual behaviour. This is due to the fact that affine planes of order 3 can be embedded in projective spaces over fields with characteristic unequal to 3 admitting nontrivial cubic root of unity, see Section 2.

The remainder of the paper is devoted to the proof of the Main Result. A lot of arguments of the full case can be used in the lax case, but there are two main obstacles that have no analogue in the full case. Firstly, we must identify appropriate subfields of $\mathbb{F}$ isomorphic to $\mathbb{K}$ and define suitable subspaces over these subfields the direct sum of which contains "almost all" points of $\mathcal{S}$. Secondly, we must prove that the embedding in that direct sum is full, i.e., we must verify the additional condition (*) for all lines of $\mathcal{S}$.

For the sake of convenience, we will call a generalized Veronesean embedding of $\mathcal{S}$ in $\mathsf{PG}(d,\mathbb{F})$, $d = \frac{1}{2}n(n+3)$, as explained in the statement of the Main Result, but distinct from the last case $|\mathbb{K}| = 3$ and $n = 2$, a *lax i-Veronesean*.

The proof of the Main Result is by induction on n, and in Section 3 we start with the case $n = 2$. But first we prove a lemma on lax embeddings of affine planes (spaces) in Desarguesian projective planes (spaces).

2 Affine planes in projective planes

Embeddings of affine planes in projective planes have been investigated by Limbos [2] in her PhD thesis. Since these results are not published elsewhere, we provide a short argument for the following lemma.

Lemma 1 *Let $\mathcal{A} = (\mathcal{P},\mathcal{L},\in)$ be an affine plane with $\mathcal{P} \subseteq \mathsf{PG}(2,\mathbb{F})$, $\mathbb{F}$ a skew field, such that every member L of $\mathcal{L}$ is a subset of a line of $\mathsf{PG}(2,\mathbb{F})$, and such that different members of $\mathcal{L}$ define different lines of $\mathsf{PG}(2,\mathbb{F})$. Then $\mathcal{A}$ is Desarguesian and either the lines of $\mathsf{PG}(2,\mathbb{F})$ corresponding to all lines of $\mathcal{A}$ belonging to an arbitrary parallel class meet in a unique point of $\mathsf{PG}(2,\mathbb{F})$ and then the projective closure of $\mathcal{A}$ is canonically embedded in $\mathsf{PG}(2,\mathbb{F})$ (consequently there is a subfield $\mathbb{K}$ of $\mathbb{F}$ such that $\mathcal{A}$ is an affine plane arising from some subplane $\mathsf{PG}(2,\mathbb{K})$ of $\mathsf{PG}(2,\mathbb{F})$), or $\mathcal{A}$ has order 2, or $\mathcal{A}$ has order 3 and $\mathbb{F}$ contains nontrivial cubic roots of unity.*

Proof If $\mathcal{A}$ is not Desarguesian, then with a standard argument one can find a non-closing Desargues configuration in $\mathcal{A}$, contradicting the fact that this configuration should close inside $\mathsf{PG}(2,\mathbb{F})$. Now let $\mathcal{A}$ be different from the projective planes of order 2 or 3. Choose three parallel lines L_1, L_2, L_3 in $\mathcal{A}$ and a line $M \in \mathcal{L}$ meeting all three of these in points of $\mathcal{A}$, say, x_1, x_2, x_3, respectively. For any pair of points (z_1, z_2) on L_1, $z_1 \neq x_1 \neq z_2$, we can find a point y on L_2 such that none of the lines yz_1 and yz_2 of $\mathcal{A}$ is parallel to M in $\mathcal{A}$. Say they meet M in y_1, y_2, respectively. Then the self-projectivity σ_2 of L_1 defined by first projecting L_1 onto L_2 from y_1 followed by projecting L_2 onto L_1 from y_2 fixes x_1 and maps z_1 onto z_2. It can be easily checked with an elementary calculation that σ_2 is given by left multiplication with respect to a suitable coordinate system (putting x_1 in the origin). Likewise, there is such a projectivity σ_3 similarly defined by considering L_3 instead of L_2. We may assume that for both projectivities a common coordinate system on L_1 was chosen. The projectivity $\sigma_2\sigma_3^{-1}$ fixes x_1, z_1 and z_2, and since it is given by left multiplication, it fixes all points of L_1. Hence the extension of $\sigma_2\sigma_3^{-1}$ to $\mathsf{PG}(2,\mathbb{F})$ must also fix all points of the line L' containing all points of L. This now easily implies that $L_1' \cap L_2' = L_1' \cap L_3'$, with L_i' the line of $\mathsf{PG}(2,\mathbb{F})$ containing all points of L_i, $i = 1, 2, 3$, and so each point at infinity of $\mathcal{A}$ defines a unique point of $\mathsf{PG}(2,\mathbb{F})$. It remains to show that all such points at infinity are collinear in $\mathsf{PG}(2,\mathbb{F})$. But this follows directly by the dual argument, or, alternatively, by a standard argument using Desargues' theorem. This implies the lemma for $\mathcal{A}$ not of order 2 or 3.

If $\mathcal{A}$ has order 3, then a straightforward calculation yields the result. $\square$

We also observe:

Lemma 2 *Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \in)$ be the point-line geometry of an affine space of dimension $n \geq 3$ with $\mathcal{P} \subseteq \mathsf{PG}(n,\mathbb{F})$, $\mathbb{F}$ a skew field, such that every member L of $\mathcal{L}$ is a subset of a line of $\mathsf{PG}(2,\mathbb{F})$, such that different members of $\mathcal{L}$ define different lines of $\mathsf{PG}(n,\mathbb{F})$, and such that $\mathcal{P}$ generates $\mathsf{PG}(n,\mathbb{F})$. Then the lines of $\mathsf{PG}(n,\mathbb{F})$ corresponding to all lines of $\mathcal{A}$ belonging to an arbitrary parallel class meet in a unique point of $\mathsf{PG}(n,\mathbb{F})$ and then the projective closure of $\mathcal{A}$ is canonically embedded in $\mathsf{PG}(n,\mathbb{F})$; consequently there is a subfield $\mathbb{K}$ of $\mathbb{F}$ such that $\mathcal{A}$ is an affine space arising from some subspace $\mathsf{PG}(n,\mathbb{K})$ of $\mathsf{PG}(n,\mathbb{F})$).*

Proof It suffices to show that the three lines L_1', L_2', L_3' of $\mathsf{PG}(n,\mathbb{F})$ containing the points of three parallel non-coplanar lines L_1, L_2, L_3, respectively, of $\mathcal{A}$ meet in a point x of $\mathsf{PG}(n,\mathbb{F})$. But this follows immediately since L_1', L_2', L_3' are not coplanar (by a dimension argument), but they are pairwise coplanar, and hence x is the intersection of the three planes of $\mathsf{PG}(n,\mathbb{F})$ spanned by the respective pairs $\{L_1', L_2'\}, \{L_2', L_3'\}, \{L_1', L_3'\}$. $\square$

3 Lax generalized Veronesean embeddings of projective planes

In the present section, we assume that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \in)$ is isomorphic to $\mathsf{PG}(2, \mathbb{K})$ with $\mathcal{P} \subseteq \mathsf{PG}(d, \mathbb{F})$, $\langle \mathcal{P} \rangle = \mathsf{PG}(d, \mathbb{F})$, $d \geq 5$, and such that every member L of $\mathcal{L}$ is a subset of points of a plane in $\mathsf{PG}(d, \mathbb{F})$, which we denote by π_L if it is unique; if it is not unique, then π_L is the intersection of all such planes (and so π_L is the line of $\mathsf{PG}(d, \mathbb{F})$ containing all points of L).

We denote the line of $\mathsf{PG}(d, \mathbb{F})$ spanned by two points $a, b \in \mathcal{P}$ by $\langle a, b \rangle$, while the line of $\mathcal{S}$ through a, b is denoted by ab. More generally, we use the symbol $\langle A \rangle$ to denote the subspace of $\mathsf{PG}(d, \mathbb{F})$ generated by the elements of A.

We will assume that $|\mathbb{K}| > 3$.

Recall the following lemma from [4].

Lemma 3 *Let S_1, S_2, S_3 be three sets, of at least three lines each, in $\mathsf{PG}(m, \mathbb{F})$, $m \geq 3$, such that each member of S_i meets every member of S_j in a unique point, for $i \neq j$, for all $i, j \in \{1, 2, 3\}$. Then there are distinct indices $i, j \in \{1, 2, 3\}$ such that either all lines of $S_i \cup S_j$ are contained in a plane, or they contain a common point.*

Also, the proof of the next lemma can be taken over from [4] (and is elementary anyway).

Lemma 4 *If L, M are two distinct lines of $\mathcal{S}$, meeting in the point $z \in \mathcal{P}$, and $x \in \mathcal{P}$ is a point off $L \cup M$ not contained in $\langle \pi_L, \pi_M \rangle$, then every point $y \in \mathcal{P}$ off xz is contained in the space $W := \langle \pi_L, \pi_M, x \rangle$.*

We now prove the Main Result for $n = 2$ and $|\mathbb{K}| > 3$. As in [4], we distinguish two cases.

(i) First suppose that for every line L of $\mathcal{S}$, the set of points $\mathcal{P} \setminus L$ generates $\mathsf{PG}(d, \mathbb{F})$. This, combined with Lemma 4, implies that $d = 5$ and every pair of lines of $\mathcal{S}$ generates a 4-space of $\mathsf{PG}(5, \mathbb{F})$. This, in turn, implies that the projection of $\mathcal{P} \setminus xy$ from the line $\langle x, y \rangle$, with $x, y \in \mathcal{P}$ is injective on the set of lines of $\mathcal{S}$ through x or y. Hence, if $\langle x, y \rangle$ contained a third point z of $\mathcal{S}$, then Lemma 3 would lead to a contradiction. Hence, $\mathcal{P}$ is a cap and every line is a plane arc. This, in turn, implies that the projection of $\mathcal{P} \setminus xy$ from π_{xy} onto a suitable plane π is injective, and since this projection forms an affine plane, and since $|\mathbb{K}| > 3$, we see that there is a subfield $\mathbb{K}'$ of $\mathbb{F}$ isomorphic to $\mathbb{K}$ such that this projection coincides with all points of a projective subplane of π except for one line. Hence, for every line $K \in \mathcal{L}$ and for every point $z \in \mathcal{P}$ on K, the lines $\langle z, u\rangle$, with $u \in K \setminus \{z\}$ form an affine line pencil over the subfield $\mathbb{K}'$ of $\mathbb{F}$. If we do this for two different points z_1, z_2 of K, then the unique projective subplane π' containing the respective affine line pencils contains K, and there are unique tangents of K in π' at z_1 and z_2. Varying z_1 and z_2 in K we obtain the same projective plane, and unique tangent lines in π' at every point of K; hence K is an oval in π'. Note also that π' is isomorphic to $\mathsf{PG}(2,\mathbb{K})$ and defined over the subfield $\mathbb{K}'$ of $\mathbb{F}$. Now choose three lines $L_1, L_2, L_3 \in \mathcal{L}$ not incident with a common point of $\mathcal{S}$ and set $y_j = L_j \cap L_3$, $j = 1, 2$. Let π'_i be the plane over $\mathbb{K}'$ containing L_i as an oval, $i = 1, 2, 3$. Since $\langle L_1, L_2\rangle$ is 4-dimensional, the planes π'_1 and π'_2 generate a unique 4-dimensional subspace ξ' over $\mathbb{K}'$. Choose an arbitrary point $y_3 \in L_3 \setminus \{y_1, y_2\}$ and let K_0 be a line of $\mathcal{S}$ containing y_3, but distinct from L_3, and assume also that K_0 meets L_1 and L_2 in two distinct points, say u_1 and u_2, respectively. Let L'_i be the projection of L_i from $\langle K_0\rangle$ onto a plane π_0 skew to $\langle K_0\rangle$, $i = 1, 2, 3$; we may assume that π_0 is contained in $\langle\xi'\rangle$. Then L'_3 is contained in the subplane π'_0 of π_0 over $\mathbb{K}'$ defined by the point set $L'_1 \cup L'_2$ (use Lemma 1). Also, the projection of ξ' from $\langle K_0\rangle$ is contained in π'_0 as our assumptions imply that this projection coincides with the projection of ξ' onto π_0 from $\langle u_1, u_2\rangle$. It easily follows now that each secant line of L_3 containing y_3 also contains a point of the line $\langle y_1, y_2\rangle_{\mathbb{K}'}$ of ξ', and hence such a point is contained in π'_3. Hence $\pi'_3 \cap \xi'$ is a line in both ξ' and π'_3. Consequently L_1, L_2, L_3 are contained in a unique subspace $\mathsf{PG}(5,\mathbb{K}')$, generated by $\pi'_1 \cup \pi'_2 \cup \pi'_3$. Since $|\mathbb{K}| > 3$, every point v of $\mathcal{S} \setminus (L_1 \cup L_2 \cup L_3)$ is the intersection of two planes π_{M_1} and π_{M_2}, $M_1, M_2 \in \mathcal{L}$, each of which intersects $L_1 \cup L_2 \cup L_3$ in three distinct noncollinear points, and hence belongs to $\mathsf{PG}(5,\mathbb{K}')$. Consequently v belongs to $\mathsf{PG}(5,\mathbb{K}')$ and so $\mathcal{P} \subseteq \mathsf{PG}(5,\mathbb{K}')$. Since the same space $\mathsf{PG}(5,\mathbb{K}')$ is obtained starting from three other lines, we see that all lines are plane ovals in $\mathsf{PG}(5,\mathbb{K}')$ and hence Condition (*) is satisfied and we can apply the Main Result—General Version of [4] to conclude that $\mathcal{P}$ is a Veronesean embedding of $\mathsf{PG}(2,\mathbb{K})$ in $\mathsf{PG}(5,\mathbb{K}')$; in fact, $\mathcal{P}$ is a 0-Veronesean in $\mathsf{PG}(5,\mathbb{K}')$.

(ii) Now suppose that for some line L of $\mathcal{S}$, the set of points $\mathcal{P} \setminus L$ generates a proper subspace W of $\mathsf{PG}(d,\mathbb{F})$. Since $U := \langle L\rangle$ is at most 2-dimensional, the codimension c of W is at most 3.

(a) Suppose $c = 3$. Then $W \cap U = \emptyset$, and for every line $M \neq L$ of $\mathcal{S}$ the plane π_M meets W in a line L_M, and only one point of M does not belong to L_M. Hence the set of lines L_M, for $M \neq L$, forms the affine plane arising from $\mathcal{S}$

by deleting L. It follows that W is 2-dimensional and $d = 5$. Since $|\mathbb{K}| > 3$, there is a subfield $\mathbb{K}'$ of $\mathbb{F}$ and a unique subplane $\mathsf{PG}(2, \mathbb{K}')$ of W such that all lines of $\mathsf{PG}(2, \mathbb{K}')$ but one are of the form L_M, $M \in \mathcal{L}$. Hence, in this case, the Main Result follows.

(b) Suppose $c = 2$. Then $W \cap U$ is a point x, and if x does not belong to $\mathcal{P}$, then all lines of $\mathcal{S}$ distinct from L again meet W in a line, and arguing as in (a) we deduce $d = 4$, a contradiction. Hence $x \in \mathcal{P}$. We can choose three points z_1, z_2, z_3 on L distinct from x (this is also possible if $|\mathbb{K}| = 3$), and considering the intersection with W of the nine planes π_M obtained by choosing three lines $M \in \mathcal{L}$, $M \neq L$, through each of z_1, z_2, z_3, we see that Lemma 3 implies that W is a plane, and so $d = 4$, a contradiction. Hence Case (b) does not occur.

(c) Suppose $c = 1$. If $U \setminus W$ contains two points y, z of $\mathcal{S}$, then the planes corresponding to the lines of $\mathcal{S}$ through y, z (distinct from L) meet W in lines which contain all points of $\mathcal{S}$ not on L. It easily follows that W has at most dimension 3, hence $d \leq 4$, a contradiction. Hence there is a unique point $x \in \mathcal{P}$ not contained in W. For every line $M \in \mathcal{L}$ through x we have that $\pi_M \cap W$ is a line. Let $y \in \mathcal{P} \setminus \{x\}$. Let K_1, K_2 be two lines of $\mathcal{S}$ through y different from xy. Suppose $X := \langle \pi_{K_1}, \pi_{K_2} \rangle$ is at most 3-dimensional. Then $\langle X, x \rangle$ is a proper subspace of $\mathsf{PG}(d, \mathbb{F})$, and Lemma 4 ensures that it contains all points of $\mathcal{S}$ except possibly those of the line xy. Hence $\langle X, x \rangle \cap W$, which has dimension at most 3, contains all points of $\mathcal{S}$ except possibly those of xy, hence we are in a previous case. So we may assume that X has dimension 4, for arbitrary K_1, K_2. If we project $\mathcal{P} \setminus xy$ from y onto some suitable hyperplane of W, then the projections of the lines of $\mathcal{S}$ through x and y are part of two complementary reguli (since the dimension of that hyperplane is at least 3), and so $d = 5$. Also, if we consider a line $K \in \mathcal{L}$ neither through x nor y, then the projection of its points not on xy lie on the intersection of a plane and a hyperbolic quadric, hence the points of $K \setminus xy$ form an arc. It follows easily that K is an arc in π_K (by re-choosing y). This now implies that the projection of $\mathcal{P} \setminus xy$ from π_{xy} onto a suitable plane of W is injective. As before, this projection is an affine plane isomorphic to $\mathsf{AG}(2, \mathbb{K})$. It follows easily that, for every line M of $\mathcal{S}$ through x, the intersection $W \cap \pi_M$ is an affine line over a subfield $\mathbb{K}'$ isomorphic to $\mathbb{K}$. Likewise, for every line $K \in \mathcal{L}$ not through x, and for every point $z \in \mathcal{P}$ on K, the lines $\langle z, u \rangle$, with $u \in K \setminus \{z\}$, form an affine line pencil over the very same subfield $\mathbb{K}'$ of $\mathbb{F}$. As in (i) above, this implies that K is an oval (even a conic, as follows from our previous arguments) in a subplane of π_K over $\mathbb{K}'$. Considering now the affine lines over $\mathbb{K}'$ arising from two lines of $\mathcal{S}$ through x, and the plane over $\mathbb{K}'$ corresponding with some line of $\mathcal{S}$ not through x, we can construct, similarly as in (i), a projective subspace $\mathsf{PG}(4, \mathbb{K}')$ of W containing $\mathcal{P} \setminus \{x\}$. Adding x, there is a subspace $\mathsf{PG}(5, \mathbb{K}')$ containing $\mathcal{P}$ such that Condition (*) is satisfied. We can now apply the Main Result—General Version of [4].

4 Lax generalized Veronesean embeddings of $PG(2, 3)$

In this section, we consider the case $\mathbb{K} = \mathsf{GF}(3)$ and $n = 2$. It is easy to check that the arguments in the previous section lead here, too, to the following three cases; we also remark that $d = 5$.

Case I. *For every line L of $\mathcal{S}$, the set of points $\mathcal{P} \setminus L$ generates* $\mathsf{PG}(5, \mathbb{F})$.

In this case, which corresponds to Case (i) of Section 3, the lines of $\mathcal{S}$ are plane arcs in $\mathsf{PG}(5, \mathbb{F})$. We use coordinates which are determined up to right multiples, and we consider indices modulo 13. Let $\{p_1, p_2, \ldots, p_{13}\}$ be the set of points of $\mathsf{PG}(2, 3)$, with line set $\{\{p_i, p_{i+1}, p_{i+3}, p_{i+9}\} : i = 1, 2, \ldots, 13\}$. Without loss of generality, we can assign the following coordinates:

$$\begin{array}{lll} p_1(1,0,0,0,0,0), & p_3(0,0,0,0,0,1), & p_{13}(1,1,0,0,0,1), \\ p_9(0,1,0,0,0,0), & p_2(0,0,0,0,1,0), & p_4(1,0,1,0,1,0), \\ p_{10}(0,0,1,0,0,0), & p_{12}(0,0,0,1,0,0), & p_5(0,1,a,1,0,0), \end{array}$$

with $a \in \mathbb{F}$. Since p_{11} is collinear with p_2, p_3 and p_5, there are constants $b, c \in \mathbb{F}$ so that the coordinates of p_{11} are $(0, 1, a, 1, b, c)$. Likewise, using the line $\{p_3, p_4, p_6, p_{12}\}$, there are constants $d, e \in \mathbb{F}$ so that the coordinates of p_6 are $(1, 0, 1, d, 1, e)$, and using $\{p_2, p_8, p_{12}, p_{13}\}$, there are constants $f, g \in \mathbb{F}$ so that p_8 has the coordinates $(1, 1, 0, f, g, 1)$.

Expressing that p_4, p_8, p_9, p_{11} are collinear gives the equivalent conditions $c = -a$, $f = -a^{-1}$ and $a - ga = b$. Similarly, $p_6, p_{10}, p_{11}, p_{13}$ collinear implies $d = b = -1$ and $c + e = 1$, and p_1, p_5, p_6, p_8 collinear means $f = 1 - da$, $g = -a$ and $ea = -1$. All this implies

$$\begin{array}{lll} b = -1, & d = -1, & f = 1 + a, \\ c = -a, & e = 1 + a, & g = -a, \end{array}$$

with $1 + a + a^2 = 0$. But expressing that p_7 is the intersection of the planes $\langle p_2, p_6, p_9 \rangle$ and $\langle p_3, p_8, p_{10} \rangle$ we obtain $f = d = -1$, hence $a = -2$ and so $3 = 0$. It follows easily that $\mathcal{P}$ lies in a 5-dimensional subspace $\mathsf{PG}(5, 3)$ over the prime field of $\mathbb{F}$ (of order 3). Hence $\mathcal{P}$ is a 0-Veronesean in the subspace $\mathsf{PG}(5, 3)$ of $\mathsf{PG}(5, \mathbb{F})$.

Case II. *There are three lines $L_1, L_2, L_3 \in \mathcal{L}$ not containing a common point such that $(L_1 \cup L_2 \cup L_3) \setminus (L_1 \cap L_2)$ is contained in a 4-space $\mathsf{PG}(4, \mathbb{F})$, such that $L_i \setminus (L_1 \cap L_2)$, $i = 1, 2$, is contained in a line of $\mathsf{PG}(4, \mathbb{F})$, and such that L_3 is an arc in π_{L_3}, which is entirely contained in $\mathsf{PG}(4, \mathbb{F})$.*

This case corresponds to Case (ii)(c) of Section 3.

With the same notation as above, we can take without loss of generality,

$$\begin{array}{lll} p_1(0,0,0,0,0,1), & p_2(0,0,0,1,0,0), & p_3(1,0,0,0,0,0), \\ p_4(0,0,1,1,0,0), & p_5(0,0,0,0,1,0), & p_9(0,1,0,0,0,0), \\ p_{10}(0,0,1,0,0,0), & p_{12}(0,1,1,0,1,0), & p_{13}(1,1,0,0,0,0). \end{array}$$

Expressing that p_7 belongs to the plane $\langle p_4, p_5, p_{13} \rangle$, p_{11} belongs to the plane $\langle p_2, p_3, p_5 \rangle$, and p_{12} belongs to the line $\langle p_7, p_{11} \rangle$, there exists a constant a such that p_7 has coordinates $(1, 1, 1, 1, a, 0)$ and p_{11} has coordinates $(1, 0, 0, 1, a-1, 0)$. Also, one calculates that p_8 has coordinates $(1, a, a-1, a, a-1, 0)$, as the intersection of $\langle p_4, p_9, p_{11} \rangle$ and $\langle p_2, p_{12}, p_{13} \rangle$. Expressing that p_8 belongs to $\langle p_3, p_7, p_{10} \rangle$, we see that $a^2 - a + 1 = 0$. Expressing that p_6 is the intersection of the plane $\langle p_2, p_7, p_9 \rangle$ with the line $\langle p_5, p_8 \rangle$, we obtain $a = 2$. This now implies that $\mathbb{F}$ has characteristic 3, that all points of $\mathcal{S}$ are contained in a subspace $\mathsf{PG}(5, 3)$ over the prime field of $\mathbb{F}$ (of order 3) and that $\mathcal{P}$ is a 1-Veronesean in $\mathsf{PG}(5, 3)$.

Case III. *There is a unique line $L \in \mathcal{L}$ all points of which lie outside a certain plane π of $\mathsf{PG}(5, \mathbb{F})$, while π contains all other points of $\mathcal{S}$.*

This case corresponds to Case (ii)(a) of Section 3.

The affine plane of order 3 arising from $\mathcal{S}$ by deleting L is embedded in π, and so the last case of the Main Result follows from Lemma 1.

This takes care of the case $\mathbb{F} = \mathsf{GF}(3)$ and $n = 2$, which we will also need in the next section for the induction argument.

5 Lax generalized Veronesean embeddings of projective spaces of dimension at least 3

Here we assume that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \in)$ is isomorphic to $\mathsf{PG}(n, \mathbb{K})$, $n > 2$, with $\mathcal{P} \subseteq \mathsf{PG}(d, \mathbb{F})$, $\langle \mathcal{P} \rangle = \mathsf{PG}(d, \mathbb{F})$, $d \geq \frac{1}{2}n(n+3)$, and such that every member L of $\mathcal{L}$ is a subset of points of a plane in $\mathsf{PG}(d,\mathbb{F})$, which we again denote by π_L if it is unique; if it is not unique, then π_L is again the line of $\mathsf{PG}(d,\mathbb{F})$ containing all points of L.

We use the same notation as before to distinguish lines of $\mathcal{S}$ from lines of $\mathsf{PG}(d,\mathbb{F})$: we denote the line of $\mathsf{PG}(d,\mathbb{F})$ spanned by two points $a,b \in \mathcal{P}$ by $\langle a,b\rangle$, while the line of $\mathcal{S}$ through a,b is denoted by ab. More generally, we use the symbol $\langle A\rangle$ to denote the subspace of $\mathsf{PG}(d,\mathbb{F})$ generated by the elements of A, and we use $\langle A\rangle_{\mathcal{S}}$ to denote the subspace of $\mathcal{S}$ spanned by A.

We will assume that $|\mathbb{K}| > 2$ as for $\mathbb{K} = \mathsf{GF}(2)$ every injective map from $\mathcal{S}$ to $\mathsf{PG}(d,\mathbb{F})$ such that the image of $\mathcal{S}$ spans $\mathsf{PG}(d,\mathbb{F})$ is a lax generalized Veronesean embedding, for every d.

Our proof proceeds by induction on n. The result for $n = 2$ has been proved in Section 3, and we assume that the result is true for any generalized Veronesean embedding of $\mathsf{PG}(n',\mathbb{K})$ in $\mathsf{PG}(d',\mathbb{F})$, with $n' < n$ and $d' \geq \frac{1}{2}n'(n'+3)$. For $|\mathbb{K}| = 3$, we of course only assume what we have proved in Section 4.

We first state some facts from [4] the proof of which can be taken over verbatim in our lax case.

Proposition 1 *If $d \geq \frac{1}{2}n(n+3)$, then $d = \frac{1}{2}n(n+3)$ and every i-dimensional subspace U of $\mathcal{S}$, $i \leq n-1$, generates in $\mathsf{PG}(d,\mathbb{F})$ a subspace of dimension $\frac{1}{2}i(i+3)$. Hence the induction hypothesis implies that U is a lax ℓ-Veronesean, for some nonnegative integer $\ell \leq i$. In particular, for every line $L \in \mathcal{L}$ holds that π_L is 2-dimensional.*

Now, as in the full case, we introduce the following notions. Let $L \in \mathcal{L}$ be arbitrary. Then we say that L is a *semiaffine line* if there is a unique point x on L such that $\langle L \setminus \{x\}\rangle$ is 1-dimensional. The point x is called a *lid point*, or the *lid of L*. The line L is called a *box for x*. Clearly, the lid of a semiaffine line is unique, but a lid point can have several boxes. The *lid* of $\mathcal{S}$ is the set of the lid points of all semiaffine lines.

We will denote by $\mathfrak{L}$ the lid of $\mathcal{S}$. The following proposition is proved in [4] for the full case, but the proof easily holds without this restriction.

Proposition 2 *The set $\mathfrak{L}$ is a proper subspace of $\mathcal{S}$. Also, if a line $L \in \mathcal{L}$ intersects $\mathfrak{L}$ in a unique point x, then L is a box for x.*

Note that every line $L \in \mathcal{L}$ disjoint from $\mathfrak{L}$ is an arc in π_L; this follows from the case $n = 2$.

We now first treat a special case.

Proposition 3 *If $\mathfrak{L}$ is a hyperplane, then $\mathcal{S}$ is a lax n-Veronesean.*

Proof By Proposition 1, the space $\langle\mathfrak{L}\rangle$ has dimension $\frac{1}{2}(n-1)(n+2)$, and by induction, $\mathfrak{L}$ is a lax i-Veronesean, $0 \leq i \leq n-1$, say in a generalized subspace W of dimension $\frac{1}{2}(n-1)(n+2)$. Also, Proposition 2 combined with the lax 2-Veronesean structure of any plane of $\mathcal{S}$ not contained in $\mathfrak{L}$, and Lemma 2, imply that the inclusion map $\iota : \mathcal{S} \setminus \mathfrak{L} \subseteq \mathsf{PG}(d, \mathbb{F})$ induces an isomorphism between affine spaces. Since $\frac{1}{2}n(n+3) = 1 + \frac{1}{2}(n-1)(n+2) + n$, we see that the subspace of $\mathsf{PG}(d, \mathbb{F})$ generated by the image of ι and the one generated by $\mathfrak{L}$ are disjoint. Hence the result follows now from the induction hypothesis (note that this also holds for $|\mathbb{K}| = 3$ as in this case the characteristic of $\mathbb{F}$ is 3 by the property of the inclusion map ι above inducing an isomorphism). $\square$

We can now finish the proof of our Main Result.

In view of the previous proposition, we may assume $0 \leq m < n-1$, with $m = \dim \mathfrak{L}$. If we consider a plane of $\mathcal{S}$ meeting $\mathfrak{L}$ in at most one point (this is possible in view of $m < n-1$), then we see that, by Sections 3 and 4, the characteristics of $\mathbb{K}$ and $\mathbb{F}$ are equal, and $\mathbb{K}$ is commutative. Now let H_1 and H_2 be two hyperplanes of $\mathcal{S}$ containing $\mathfrak{L}$. The previous remark and the induction hypothesis imply that H_1 and H_2 are both lax $(m+1)$-Veroneseans. Note that $\langle H_1\rangle$ and $\langle H_2\rangle$ both have dimension $\frac{1}{2}(n-1)(n+2)$, and the dimension of $\langle H_1 \cap H_2\rangle$ is $\frac{1}{2}(n-2)(n+1)$. It follows that the dimension of $\langle H_1, H_2\rangle$ is at most $\frac{1}{2}n(n+3)-1$. So we can choose a point $x \in \mathcal{P}$ outside $H_1 \cup H_2$ which does not belong to $\langle H_1, H_2\rangle$. Since every line L not meeting $\mathfrak{L}$ is an arc in π_L, we see that all points of $\mathcal{P}$, except possibly those lying in the hyperplane H_3 of $\mathcal{S}$ generated by x and $H_1 \cap H_2$, are contained in $\langle H_1, H_2, x\rangle$. Since there are at least four hyperplanes of $\mathcal{S}$ through $H_1 \cap H_2$, we can interchange the roles of x and a point $y \in \mathcal{P}$ not contained in $H_1 \cup H_2 \cup H_3$ and obtain that $\mathcal{P} \subseteq \langle H_1, H_2, x\rangle$. This also implies that $\dim\langle H_1, H_2\rangle = \frac{1}{2}n(n+3) - 1$ and $\langle H_1\rangle \cap \langle H_2\rangle = \langle H_1 \cap H_2\rangle$ has dimension $\frac{1}{2}(n-2)(n+1)$.

Note that a similar argument (used inductively) as in the previous paragraph shows that $\mathsf{PG}(d, \mathbb{F})$ is generated by $\langle H_1\rangle$ and $n+1$ points of $\mathcal{S}$ outside H_1, in general position viewed as points of $\mathcal{S}$. Similarly for H_2.

The induction hypothesis implies that $H_i \setminus \mathfrak{L}$, $i = 1, 2$, is contained in a subspace W_i of $\mathsf{PG}(d, \mathbb{F})$ of dimension $\frac{1}{2}(n-1)(n+2) - \frac{1}{2}m(m+3) - 1$ defined over some subfield $\mathbb{K}_i$ of $\mathbb{F}$ isomorphic to $\mathbb{K}$, and such that this embedding is isomorphic to the projection from a subspace U_i (generated by a sub-Veronesean induced by an m-dimensional subspace of $\mathsf{PG}(n-1, \mathbb{K})$) of the full Veronesean embedding of $\mathsf{PG}(n-1, \mathbb{K})$ minus the subspace U_i. Moreover, the intersection $W_1 \cap W_2$ has dimension $\frac{1}{2}(n-2)(n+1) - \frac{1}{2}m(m+3) - 1$, implying W_1 and W_2 generate (over $\mathbb{F}$) a subspace of dimension $\frac{1}{2}n(n+3) - \frac{1}{2}m(m+3) - 2$. Now consider an arbitrary plane π of $\mathcal{S}$ through x meeting $H_1 \cap H_2$ in a point (which might or might not belong to $\mathfrak{L}$). Then our arguments in Section 3 (especially those leading to the lax 0- and 1-Veronesean) imply that π is contained in a unique 5-dimensional subspace $\mathsf{PG}(5, \mathbb{K}')$ of $\mathsf{PG}(d, \mathbb{F})$ defined over a subfield $\mathbb{K}'$ of $\mathbb{F}$ isomorphic to $\mathbb{K}$ (use the fact that there is a line $L \in \mathcal{L}$ through x in π skew to $\mathfrak{L}$ and that $\mathsf{PG}(5, \mathbb{K}')$ is determined by L and the intersections of π with the H_i, $i = 1, 2$). Moreover, we see that $\langle \pi \rangle \cap \langle H_i \rangle$ is 2-dimensional (indeed, this follows immediately from the fact that, by the previous paragraph, $\langle \pi \rangle$, $\langle H_i \rangle$ and $n - 2$ well-chosen additional points of $\mathcal{S}$ generate $\mathsf{PG}(d, \mathbb{F})$) and the points of $\mathcal{S}$ in this plane are contained in a subplane over both $\mathbb{K}'$ and $\mathbb{K}_i$. We infer from the fullness of the embedding that $\mathbb{K}_1 = \mathbb{K}' = \mathbb{K}_2$. Also, the intersection $(H_1 \setminus \mathfrak{L}) \cap (H_2 \setminus \mathfrak{L})$ is an embedding in $\langle W_1 \rangle \cap \langle W_2 \rangle$ isomorphic to an appropriate projection of a full Veronesean of $H_1 \cap H_2$. It follows that $\langle W_1 \rangle \cap \langle W_2 \rangle = \langle W_1 \cap W_2 \rangle$ and $W_i \cap \langle W_1 \cap W_2 \rangle = W_1 \cap W_2$, $i = 1, 2$ (this can also be seen using planes of $\mathcal{S}$ through x meeting $H_1 \cap H_2$ in lines). Hence $(\pi \cup H_1 \cup H_2) \setminus \mathfrak{L}$ is contained in a unique subspace $\mathsf{PG}(d', \mathbb{K}')$, with $d' = \frac{1}{2}n(n+3) - \frac{1}{2}m(m+3) - 1$.

But now every point z of $\mathcal{P} \setminus (H_1 \cup H_2)$ is contained in $\mathsf{PG}(d', \mathbb{K}')$ since we can include it in a plane π' of $\mathcal{S}$ which contains L. Then the argument above leading to the uniqueness of $\mathsf{PG}(5, \mathbb{K}')$ can now be recycled to show that π' is entirely contained in $\mathsf{PG}(d', \mathbb{K}')$.

This concludes the proof of our Main Result.

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