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Semipartial geometries and $(0, \alpha)$-geometries fully embedded in finite projective and affine spaces

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Preface

The work that is presented in this thesis, is situated in the field of finite geometry. One of the major topics in finite geometry is the study of point-line geometries, also called rank 2 geometries. Several interesting classes of rank 2 geometries have been studied in the past half century, including generalized polygons, partial geometries and, more recently, semipartial geometries and $(0,\alpha)$-geometries (see De Clerck and Van Maldeghem [31] for a good overview). Classical problems in this area are the construction of new geometries, and the characterization of the known ones by some of their properties.

Points and lines of a rank 2 geometry are abstract objects. They are points and lines for no other reason but that we call them so. Therefore it is interesting to consider geometries whose points and lines are not just abstract objects, but really points and lines. But what are "real" points and "real" lines? In the context of finite geometry, the most natural choice is to consider points and lines of a projective or affine space over a finite field. A rank 2 geometry whose points and lines are points and lines of a finite projective or affine space R is said to be *laxly embedded* in R. It is said to be *fully embedded* in R if furthermore every point of R which is on a line of the geometry, is itself a point of the geometry. A rank 2 geometry which is fully embedded in R is also called *projective* when R is a projective space, and *affine* when R is an affine space.

For various classes of point-line geometries, the geometries that are fully embedded in a finite projective or affine space, are classified. For example, generalized quadrangles and partial geometries fully embedded in $\mathrm{PG}(n,q)$ and $\mathrm{AG}(n,q)$ are classified (see Buekenhout and Lefèvre [15] for the classification of projective generalized quadrangles, De Clerck and Thas [29] for the classification of projective partial geometries, and Thas [78] for the classification of affine generalized quadrangles and partial geometries). Also, the semipartial geometries and $(0,\alpha)$-geometries, $\alpha > 1$, fully embedded in $\mathrm{PG}(n,q)$, $q > 2$, are classified (see De Clerck and Thas [30] and Thas, Debroey and De Clerck [81]), except the $(0,\alpha)$-geometries fully embedded in

PG$(3, q)$. In this thesis, we investigate $(0, \alpha)$-geometries fully embedded in PG$(3, q)$, and semipartial geometries and $(0, \alpha)$-geometries fully embedded in AG(n, q). For reasons explained in Sections 1.4.3 and 1.4.7, we will always assume that $\alpha > 1$.

In Chapter 1, we introduce the basic concepts and definitions. We give an overview of the known results about projective and affine generalized quadrangles, partial geometries, semipartial geometries and $(0, \alpha)$-geometries. Also, we give the construction of some geometries that will be relevant later on.

Chapter 2 is about $(0, \alpha)$-geometries fully embedded in PG$(3, q)$. The Plücker correspondence is used to transform the line set of such a geometry into a set of points on the Klein quadric $\mathrm{Q}^+(5, q)$. It can be shown that $(0, \alpha)$-geometries fully embedded in PG$(3, q)$ and $(0, \alpha)$-sets on $\mathrm{Q}^+(5, q)$, i. e., sets of points sharing either 0 or α points with every line of $\mathrm{Q}^+(5, q)$, are equivalent. We construct new $(0, \alpha)$-sets on $\mathrm{Q}^+(5, q)$, $q = 2^h$, and hence new $(0, \alpha)$-geometries fully embedded in PG$(3, q)$, $q = 2^h$.

In Chapter 3, we investigate sets of q^2 ovals in PG$(2, q)$, $q = 2^h$, which have a common nucleus and are such that any two ovals have exactly one point in common. These sets are called *planar oval sets*. We show that every planar oval set which satisfies a certain condition, consists of the orbit of an oval under the group of all elations of PG$(2, q)$ with center the nucleus of that oval. This result will be used in Chapter 5 for the classification of $(0, 2)$-geometries fully embedded in AG$(3, q)$, $q = 2^h$.

In Chapters 4, 5 and 6, we obtain a complete classification of semipartial geometries and $(0, \alpha)$-geometries, $\alpha > 1$, fully embedded in AG(n, q), that are not linear representations. De Clerck and Delanote [27] have shown that if an affine semipartial geometry or $(0, \alpha)$-geometry, $\alpha > 1$, is not a linear representation, then $q = 2^h$ and $\alpha = 2$. There is no complete classification yet of linear representations of semipartial geometries, $\alpha > 1$. This problem is solved only in small dimensions: the case AG$(3, q)$ is handled by Debroey and Thas [40], and the case AG$(4, q)$ by De Winter [38].

In Chapter 4, we explain the general method that we will use. It relies on the fact that, roughly speaking, the intersection of an affine $(0, \alpha)$-geometry with an affine subspace is always the disjoint union of affine $(0, \alpha)$-geometries. We give an overview of all the known results about affine semipartial geometries and $(0, \alpha)$-geometries, including our own results (see also [26]). We construct two new infinite classes of affine $(0, 2)$-geometries. We end this chapter with a detailed study of each of the known affine semipartial geometries and $(0, \alpha)$-geometries.

In Chapter 5, we classify all $(0, \alpha)$-geometries, $\alpha > 1$, fully embedded in AG$(3, q)$, that are not linear representations. As we already mentioned, we may assume that $q = 2^h$ and $\alpha = 2$. The main result of Chapter 3 on planar oval sets is used here to complete the classification.

The results of Chapter 5 enable us to classify in Chapter 6 all $(0, \alpha)$-geometries, $\alpha > 1$, fully embedded in $\mathrm{AG}(n, q)$, $n \geq 4$, that are not linear representations. Again we may assume that $q = 2^h$ and $\alpha = 2$.

Chapter 7 gives an overview of the main results of the thesis.

Finally, in Appendix A, we study the $(0, 2)$-geometry $\mathcal{I}(n, q, e)$ fully embedded in $\mathrm{AG}(n, q)$, $q = 2^h$. This geometry was constructed in Chapter 4, where we already deduced some of its properties. However, further study is required for a good understanding of the geometry $\mathcal{I}(n, q, e)$. Firstly, we give an explicit description of the point set of $\mathcal{I}(n, q, e)$, and we describe the intersection with affine lines. Next, we consider a natural partition in two parts of the point and line sets of $\mathcal{I}(n, q, e)$, and we study these two parts. Finally, we obtain some isomorphisms of the geometry $\mathcal{I}(n, q, e)$.

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Chapter 1

Introduction

1.1 Graphs

1.1.1 Definitions

A *graph* Γ is a pair (V, E) consisting of a set of *vertices* V and a set of *edges* E. Edges are unordered pairs of vertices. For our purposes the vertex set V will always be finite, every edge will consist of a pair of distinct vertices, and no two distinct edges consist of the same pair of vertices. Two vertices x and y are called *adjacent* vertices or *neighbors* if the pair $\{x, y\}$ is an edge, and we write $x \sim y$. The *complement* of a graph $\Gamma = (V, E)$ is the graph $\overline{\Gamma} = (V, \overline{E})$ where $\overline{E} = \binom{V}{2} \setminus E$. The graph Γ is called a *complete graph* if any two distinct vertices of Γ are adjacent, and it is called a *void graph* if no two vertices of Γ are adjacent.

A *clique* of a graph Γ is a set of vertices of Γ such that any two of them are adjacent. A *coclique* of a graph Γ is a set of vertices of Γ such that no two of them are adjacent. The *subgraph* of a graph $\Gamma = (V, E)$ *induced on* a subset $V' \subseteq V$ is the graph $\Gamma' = (V', E')$ where E' is the set of all edges $\{x, y\} \in E$ such that $x, y \in V'$.

A *path* in a graph Γ is an ordered set $(x_0, x_1, \ldots, x_n)$ of mutually distinct vertices of Γ such that any two consecutive vertices are adjacent. We say that this path has *length* n, and that it is a path between the vertices x_0 and x_n. A graph $\Gamma = (V, E)$ is said to be *connected* if there is at least one path between any two vertices. If Γ is not connected then it is easily seen that there exists a unique partition $V = V_1 \cup V_2 \cup \ldots \cup V_k$ such that $V_i \neq \emptyset$, $i = 1, \ldots, k$, every edge is contained in some V_i and for every $i = 1, \ldots, k$ the subgraph Γ_i of Γ induced on V_i is connected. The subgraphs Γ_i are called the *connected components* of Γ. If for some $i = 1, \ldots, k$ the set V_i consists of a single vertex x, then x is said to be an *isolated vertex*.

An important notion in graph theory is the *distance* between two vertices x and y of a connected graph Γ. This is the length of the shortest path in Γ between x and y. A vertex is also said to be at distance 0 from itself. Notice that there is not necessarily a unique shortest path between x and y. The *diameter* of a connected graph Γ is the maximal distance between two vertices in Γ. If Γ is a graph of diameter d, then for every $i = 0, \ldots, d$ and for every vertex x of Γ we define $\Gamma_i(x)$ to be the set of vertices of Γ at distance i from x. The set $\Gamma_1(x)$ is also denoted by $x^\perp$.

The *adjacency matrix* of a graph Γ with v vertices is the $v \times v$-matrix A indexed by the vertices of Γ such that the entry $a_{xy} = 1$ if $x \sim y$ and $a_{xy} = 0$ otherwise.

1.1.2 Strongly regular graphs

A graph Γ is *regular* of *degree* or *valency* k if every vertex of Γ has exactly k neighbors. A regular graph with v vertices and valency k is *edge-regular* with parameters (v, k, λ) if any two adjacent vertices have exactly λ common neighbors. An edge-regular graph Γ with parameters (v, k, λ) is *strongly regular* with parameters (v, k, λ, μ) if any two nonadjacent vertices have exactly μ common neighbors. We then say that Γ is an $\mathrm{srg}(v, k, \lambda, \mu)$. To exclude trivial cases we will always assume that $0 < \mu < k$.

In the following theorem some necessary conditions on the parameters of a strongly regular graph are stated. The proofs can be found in [7, 19, 84].

Theorem 1.1.1 *If Γ is an $\mathrm{srg}(v, k, \lambda, \mu)$, then the following holds:*

1. $k(k - \lambda - 1) = \mu(v - k - 1)$.
2. $\overline{\Gamma}$ *is an* $\mathrm{srg}(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$.
3. *If A is the adjacency matrix of Γ, then*

$$AJ = kJ, \qquad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

and A has three eigenvalues k, r, l where

$$r = \frac{\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}, l = \frac{\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}.$$

Note that $r > 0$ and $l < 0$. The multiplicities of the eigenvalues k, r, l are $1, f, g$ respectively, where

$$f = \frac{-k(l + 1)(k - l)}{(k + rl)(r - l)}, g = \frac{k(r + 1)(k - r)}{(k + rl)(r - l)}.$$

Clearly f and g must be integers.

4. *The eigenvalues r and l are both integers, except for one family of graphs, the conference graphs, which are* $\mathrm{srg}(2k+1, k, \frac{k}{2}-1, \frac{k}{2})$. *For a conference graph the number of vertices can be written as a sum of two squares, and the eigenvalues are* $\frac{-1+\sqrt{v}}{2}$ *and* $\frac{-1-\sqrt{v}}{2}$.

5. *The two Krein conditions:*
   - $(r+1)(k+r+2rl) \le (k+r)(l+1)^2$,
   - $(l+1)(k+l+2rl) \le (k+l)(r+1)^2$.

6. *The two absolute bounds:*
   - $v \le \frac{1}{2}f(f+3)$, *and if there is no equality in the first Krein condition then* $v \le \frac{1}{2}f(f+1)$;
   - $v \le \frac{1}{2}g(g+3)$, *and if there is no equality in the second Krein condition then* $v \le \frac{1}{2}g(g+1)$.

7. *The claw bound. If* $\mu \neq l^2$ *and* $\mu \neq l(l+1)$, *then* $2(r+1) \le l(l+1)(\mu+1)$.

8. *The Hoffman bound.*
   - *If C is a clique of Γ, then* $|C| \le 1 - \frac{k}{l}$, *with equality if and only if every vertex $x \notin C$ has the same number of neighbors (namely* $\frac{\mu}{-l}$*) in C.*
   - *If C is a coclique of Γ, then* $|C| \le v(1-\frac{k}{l})^{-1}$, *with equality if and only if every vertex $x \notin C$ has the same number of neighbors (namely $-l$) in C.*

Notwithstanding the severe necessary conditions of Theorem 1.1.1 a lot of examples of strongly regular graphs are known, see for example [8, 55].

1.2 Incidence structures

An *incidence structure* $\mathcal{S}$ is a triple $(\mathcal{P}, \mathcal{B}, \mathrm{I})$ consisting of a set of *points* $\mathcal{P}$, a set of *blocks* $\mathcal{B}$, such that $\mathcal{P} \cup \mathcal{B} \neq \emptyset$, and a symmetric *incidence relation* $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$. For our purposes the sets $\mathcal{P}$ and $\mathcal{B}$ will be finite. If for a point p and a block B we have that $(p, B) \in \mathrm{I}$, then we say that p and B are *incident*, that p *is on* B, that B *passes through* or *contains* p, and we write $p \,\mathrm{I}\, B$ or $p \in B$. A *flag* of $\mathcal{S}$ is just an incident pair $\{p, B\}$ with $p \in \mathcal{P}$, $B \in \mathcal{B}$. A pair $\{p, B\}$ with $p \in \mathcal{P}$, $B \in \mathcal{B}$, and p not incident with B, is called an *anti-flag*. The *dual* of an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is the incidence structure $\mathcal{S}^D = (\mathcal{P}^D, \mathcal{B}^D, \mathrm{I}^D)$ with $\mathcal{P}^D = \mathcal{B}$, $\mathcal{B}^D = \mathcal{P}$, $\mathrm{I}^D = \mathrm{I}$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be an incidence structure and let $X \subseteq \mathcal{P} \cup \mathcal{B}$. Then we define the *sub incidence structure of* $\mathcal{S}$ *induced on* the set X to be $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ where $\mathcal{P}' = \mathcal{P} \cap X$, $\mathcal{B}' = \mathcal{B} \cap X$ and $\mathrm{I}' = \mathrm{I} \cap ((\mathcal{P}' \times \mathcal{B}') \cup (\mathcal{B}' \times \mathcal{P}'))$. The *incidence graph* $\mathcal{I}(\mathcal{S})$ of an incidence structure $\mathcal{S}$ is the graph with vertex set $\mathcal{P} \cup \mathcal{B}$, two vertices being adjacent if they are incident in $\mathcal{S}$. We say that $\mathcal{S}$ is *connected* if the incidence graph $\mathcal{I}(\mathcal{S})$ is connected, and we define the *connected components* of $\mathcal{S}$ to be the sub incidence structures of $\mathcal{S}$ induced on the vertex sets of the connected components of $\mathcal{I}(\mathcal{S})$. If $\mathcal{S}$ has a connected component consisting of a single point, then we call this point an *isolated point*.

An *isomorphism* from an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ to an incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{B}', \mathrm{I}')$ is a bijection $\varphi : \mathcal{P} \cup \mathcal{B} \to \mathcal{P}' \cup \mathcal{B}'$ mapping points to points and blocks to blocks in such a way that incidence is preserved. If there exists an isomorphism from $\mathcal{S}$ to $\mathcal{S}'$ we say that $\mathcal{S}$ and $\mathcal{S}'$ are *isomorphic* and we write $\mathcal{S} \cong \mathcal{S}'$.

A *partial linear space* is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ such that any point is on at least 2 blocks, any block contains at least 2 points, and two distinct blocks intersect in at most one point. In this case we call the elements of $\mathcal{B}$ *lines*. If two (not necessarily distinct) points p, q are on a common line then they are called *collinear* and we write $p \sim q$. The set of points collinear with a given point p is denoted $p^{\perp}$. Note that every point is collinear with itself, whereas in the context of graphs a vertex is not adjacent to itself. However the same notation is used for collinearity and adjacency. If two (not necessarily distinct) lines L, M contain a common point then they are called *concurrent* or *intersecting* lines and we write $L \sim M$. The *collinearity graph* or *point graph* $\Gamma(\mathcal{S})$ of a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is the graph with vertex set $\mathcal{P}$, two vertices being adjacent if they are distinct and collinear. The *line graph* or *block graph* of $\mathcal{S}$ is the graph with vertex set $\mathcal{B}$, two vertices being adjacent if they are distinct and concurrent.

A partial linear space $\mathcal{S}$ is said to have an *order* (s, t) where $s, t \geq 1$ if every line contains precisely $s+1$ points and if every point of $\mathcal{S}$ is on precisely $t+1$ lines. For every anti-flag $\{p, L\}$ of a partial linear space $\mathcal{S}$ the *incidence number* $\alpha(p, L)$ is the number of lines through p which intersect L.

A *linear space* is a partial linear space $\mathcal{S}$ such that any two distinct points are on a unique common line. If every line of the linear space $\mathcal{S}$ contains a constant number k of points, then $\mathcal{S}$ is called a $2-(v, k, 1)$ *design*, where v is the number of points of $\mathcal{S}$.

1.2.1 (α, β)-geometries and partial geometries

An incidence structure $\mathcal{S}$ is an (α, β)-*geometry* if it is a connected partial linear space of order (s, t) such that for every anti-flag of $\mathcal{S}$ the incidence number is either α or β.

An (α, β)-geometry $\mathcal{S}$ with $\alpha = 1$ and $\beta = s + 1$ is also called a *polar space* (for an introduction to polar spaces, see [18]).

An (α, β)-geometry $\mathcal{S}$ with $\alpha = \beta$ is called a *partial geometry* with parameters s, t and α. We say that $\mathcal{S}$ is a $\mathrm{pg}(s, t, \alpha)$. Partial geometries were introduced by Bose [6]. It is easy to see that the point graph $\Gamma(\mathcal{S})$ of a $\mathrm{pg}(s, t, \alpha)$ $\mathcal{S}$ is an

$$\mathrm{srg}((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)).$$

Notice that the definition of a partial geometry is self-dual, hence the dual of a $\mathrm{pg}(s, t, \alpha)$ is a $\mathrm{pg}(t, s, \alpha)$. The partial geometries can be divided into the following four (non disjoint) classes.

1. The partial geometries with $\alpha = 1$ are called *generalized quadrangles*. In section 1.3.3 we give a brief introduction to generalized quadrangles. For more information we refer to the standard work [66].

2. The partial geometries with $\alpha = s + 1$, or dually $\alpha = t + 1$, are the $2 - (v, s + 1, 1)$ designs and their duals. See for example [4, 5].

3. The partial geometries with $\alpha = s$ or dually $\alpha = t$. The partial geometries with $\alpha = t$ are the *(Bruck) nets of order* $s + 1$ *and degree* $t + 1$, which were introduced by Bruck [12].

4. The partial geometries with $1 < \alpha < \min(s, t)$ are called *proper* partial geometries. For an overview of the known proper partial geometries see [31] and [24].

1.2.2 $(0, \alpha)$-geometries

A $(0, \alpha)$-*geometry* ($\alpha \geq 1$) is an (α, β)-geometry with $\beta = 0$. In other words, an incidence structure $\mathcal{S}$ is a $(0, \alpha)$-geometry ($\alpha \geq 1$) if it satisfies the following conditions.

(zag1) $\mathcal{S}$ is a connected partial linear space of order (s, t).

(zag2) Every anti-flag of $\mathcal{S}$ has incidence number 0 or α.

This definition is self-dual, so the dual of a $(0, \alpha)$-geometry will again be a $(0, \alpha)$-geometry. The conditions (zag1) and (zag2) are not entirely independent. Consider the following condition.

(zag1') $\mathcal{S}$ is a connected incidence structure such that any two distinct lines intersect in at most one point, there is a point on at least two lines and a line containing at least two points.

The following lemma shows that if $\alpha > 1$ we may replace (zag1) by the weaker condition (zag1').

Lemma 1.2.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be an incidence structure satisfying* (zag1') *and* (zag2). *If $\alpha > 1$, then $\mathcal{S}$ has an order (s, t) with $s, t \geq 1$, so $\mathcal{S}$ is a $(0, \alpha)$-geometry.*

Proof. Suppose that $\alpha > 1$. Consider two distinct points p_1 and p_2 on a line L. Let $s + 1$ be the number of points on L and let $t_i + 1$ be the number of lines through p_i, $i = 1, 2$. The number of points collinear to both p_1 and p_2 is $s - 1 + t_1(\alpha - 1)$ since by (zag2) there are α points on every line through p_1, different from L, which are collinear to p_2. Since we can interchange p_1 and p_2 in this reasoning we get that $s - 1 + t_1(\alpha - 1) = s - 1 + t_2(\alpha - 1)$. Hence $t_1 = t_2$ since $\alpha > 1$, and by connectedness we have that the number of lines through a point is constant. Dually the number of points on a line is constant, so $\mathcal{S}$ has an order, say (s, t). Then $s, t \geq 1$ since there is a point on at least two lines and a line containing at least two points. $\square$

Lemma 1.2.1 illustrates that $(0, 1)$-geometries are quite different from $(0, \alpha)$-geometries with $\alpha > 1$. Hence it is natural to treat $(0, 1)$-geometries separately from $(0, \alpha)$-geometries with $\alpha > 1$.

1.2.3 Copolar and cotriangular spaces

The $(0, \alpha)$-geometries with $\alpha = s$, $s > 1$, are the *indecomposable* (that is, connected) *copolar spaces* of order at least 2, which do not consist of a single line [48]. If p and r are noncollinear points of the copolar space $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ then we write $p \approx r$ if $p^\perp \setminus \{p\} = r^\perp \setminus \{r\}$. Clearly $\approx$ is an equivalence relation. We call an indecomposable copolar space *reduced* if all its $\approx$-classes have size 1.

Copolar spaces were studied by Hall [48]. In particular the reduced copolar spaces of order at least 2 were classified. The case $s = 2$ was first solved by Seidel [72] and by Shult [73]. A copolar space of order $s = 2$ is also called a *cotriangular space*.

1.2.4 Semipartial geometries

The point graph $\Gamma(\mathcal{S})$ of a $(0,\alpha)$-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is an edge-regular graph with parameters $(v, s(t{+}1), s{-}1{+}t(\alpha{-}1))$, where $v = |\,\mathcal{P}\,|$. By duality the block graph of $\mathcal{S}$ is an edge-regular graph with parameters $(b, t(s+1), t-1+s(\alpha-1))$, where $b = |\,\mathcal{B}\,|$. However the point graph $\Gamma(\mathcal{S})$ of $\mathcal{S}$ is not necessarily a strongly regular graph. If this is the case then we say that $\mathcal{S}$ is a *semipartial geometry*. So an incidence structure $\mathcal{S}$ is a semipartial geometry if it satisfies the following properties.

(spg1) $\mathcal{S}$ is a partial linear space of order (s, t).

(spg2) Every anti-flag of $\mathcal{S}$ has incidence number 0 or α.

(spg3) For every two noncollinear points p_1, p_2 there are precisely $\mu > 0$ points collinear to both p_1 and p_2.

Semipartial geometries were introduced by Debroey and Thas [41]. See [31] and [24] for a list of the known semipartial geometries. We call s, t, α and μ the parameters of the semipartial geometry and we say that $\mathcal{S}$ is an $\mathrm{spg}(s,t,\alpha,\mu)$. The point graph of an $\mathrm{spg}(s,t,\alpha,\mu)$ is an

$$\mathrm{srg}(1 + \frac{(t+1)s(\mu + t(s-\alpha+1))}{\mu}, s(t+1), s-1+t(\alpha-1), \mu).$$

We call an $\mathrm{spg}(s,t,\alpha,\mu)$ a *proper semipartial geometry* if $\mu < (t+1)\alpha$, that is, if it is not a partial geometry. Notice that the definition of a semipartial geometry is not self-dual. It was proven by Debroey [39] that the dual of a semipartial geometry is again a semipartial geometry if and only if either it is a partial geometry or $s = t$.

A semipartial geometry $\mathrm{spg}(s,t,1,\mu)$ is called a *partial quadrangle* and is denoted by $\mathrm{PQ}(s,t,\mu)$. Partial quadrangles were introduced by Cameron [17]. A partial quadrangle is called *proper* if it is not a generalized quadrangle.

1.3 Projective geometry

We assume that the reader is familiar with the basic notions of a projective geometry $\mathrm{PG}(n,q)$ of dimension n over a finite field $\mathrm{GF}(q)$. We will sometimes identify a subspace of $\mathrm{PG}(n,q)$ with the set of points it contains. For example if $\mathcal{K}$ is a set of points of $\mathrm{PG}(n,q)$ and U a subspace, then we will write $\mathcal{K} \cap U$ for the set of points of $\mathcal{K}$ lying in U.

We will often denote a point of $\mathrm{PG}(n,q)$ by a vector which generates this point. If a basis is chosen in $\mathrm{PG}(n,q)$, then the coordinates with respect to this basis will be denoted by $(X_0, \ldots, X_n)$. Let U be an m-dimensional subspace of $\mathrm{PG}(n, q)$, and let A be a matrix whose rows are the coordinate vectors (with respect to the chosen basis in $\mathrm{PG}(n, q)$) of a basis of U. We say that the coordinates $X_{i_0}, \ldots, X_{i_m}$, with $0 \leq i_0 < i_1 < \ldots < i_m \leq n$, coordinatize the subspace U if the submatrix A' of A formed by columns $i_0, \ldots, i_m$ of A is nonsingular. If this is so, then the map

$$\begin{array}{rrcl} \psi: & U & \rightarrow & \mathrm{PG}(m, q) \\ & (x_0, \ldots, x_n) & \mapsto & (x_{i_0}, \ldots, x_{i_m}) \end{array}$$

is a collineation from U to $\mathrm{PG}(m, q)$.

When we discuss the affine space $\mathrm{AG}(n, q)$, then $\mathrm{PG}(n, q)$ denotes the projective completion of $\mathrm{AG}(n, q)$, and Π_∞ denotes the subspace at infinity of $\mathrm{AG}(n, q)$. Sometimes subspaces of $\mathrm{AG}(n, q)$ will be treated as subspaces of $\mathrm{PG}(n, q)$. For example we will say that two parallel lines of $\mathrm{AG}(n, q)$ intersect in a point of Π_∞.

1.3.1 Polarities, quadrics and hermitian varieties

In this section we recall some facts and results about polarities, quadrics and hermitian varieties that we need in the thesis. For more information we refer to the standard works [50, 51, 54].

Polarities

A *polarity* β of $\mathrm{PG}(n, q)$, $n \geq 1$, is an anti-automorphism such that β^2 is the identity. With respect to an arbitrary basis of $\mathrm{PG}(n, q)$, a polarity β can be represented by a pair (A, θ), where θ is an involutory automorphism of $\mathrm{GF}(q)$, and A is a nonsingular $(n+1) \times (n+1)$-matrix over $\mathrm{GF}(q)$ such that $A' = \pm A$ if $\theta = \mathbf{1}$, and $A'^\theta = A$ if $\theta \neq \mathbf{1} = \theta^2$ (here A' stands for the transposed matrix of A); a point p with coordinate vector $\overline{x} = (x_0 \ldots x_n)$ is then mapped by β to the hyperplane with coordinate vector $A\overline{x}'^\theta$. Conversely, every such pair (A, θ) represents a polarity β of $\mathrm{PG}(n, q)$.

A square matrix A is called *skew-symmetric* if $A' = -A$ and all diagonal entries of A are equal to zero. A skew-symmetric $m \times m$-matrix is always singular if m is odd. Therefore, we have the following possibilities for the pair (A, θ), representing a polarity β with respect to a basis of $\mathrm{PG}(n, q)$.

1. n is odd, $\theta = \mathbf{1}$, and A is skew-symmetric. Then β is said to be a *symplectic polarity.*

2. q is odd, $\theta = \mathbf{1}$, and $A' = A$. Then β is said to be an *orthogonal polarity.*

3. q is even, $\theta = \mathbf{1}$, $A' = A$, and A is not skew-symmetric (that is, not all diagonal entries of A are equal to zero). Then β is said to be a *pseudo polarity.*

4. $\theta \neq \mathbf{1} = \theta^2$, and $A'^{\theta} = A$. Then β is said to be a *hermitian polarity.*

One verifies that all four types of polarities do occur, and that the type of a polarity is independent of the chosen basis of $\mathrm{PG}(n, q)$.

A subspace U of $\mathrm{PG}(n, q)$ is called *totally isotropic* with respect to a polarity β if U is contained in U^β. Let β be a polarity of $\mathrm{PG}(n, q)$, represented, with respect to a certain basis, by the pair (A, θ). Let $A = (a_{ij})_{0 \leq i,j \leq n}$.

1. If β is a symplectic polarity, then every point of $\mathrm{PG}(n, q)$ is totally isotropic.

2. If β is an orthogonal polarity, then a point is totally isotropic if and only if its coordinates satisfy $\sum_{i=0}^{n} a_{ii}X_i^2 + 2\sum_{i<j} a_{ij}X_iX_j = 0$.

3. If β is a pseudo polarity, then the totally isotropic points are all the points of the hyperplane $U : a_{00}^{1/2}X_0 + \ldots + a_{nn}^{1/2}X_n = 0$.

4. If β is a hermitian polarity, then a point is totally isotropic if and only if its coordinates satisfy $\sum_{i,j=0}^{n} a_{ij}X_iX_j^{\theta} = 0$.

Choose a basis in $\mathrm{PG}(n, q)$. Then a polarity is determined by a pair (A, θ). One may modify the definition of a polarity so that also singular matrices A are allowed. In this case we say that the polarity is *singular*. The set of points having no image with respect to a singular polarity β of $\mathrm{PG}(n, q)$ forms an r-dimensional subspace called the *vertex* of β. For $n \geq r + 2$, the projection of β from the vertex on an $(n - r - 1)$-dimensional subspace U skew to the vertex is always a nonsingular polarity of U.

Quadrics

A *quadric* $\mathcal{Q}_n$ of $\mathrm{PG}(n, q)$, $n \geq 1$, is the set of points of $\mathrm{PG}(n, q)$ whose coordinates satisfy a homogeneous quadratic equation over $\mathrm{GF}(q)$,

$$\sum_{i \leq j = 0}^{n} a_{ij}X_iX_j = 0,$$

such that not all a_{ij} are zero. A quadric $\mathcal{Q}_2$ in $\mathrm{PG}(2, q)$ is also called a *conic*. The intersection of a quadric $\mathcal{Q}_n$ with a subspace of dimension at least 1 of $\mathrm{PG}(n, q)$ is always a quadric in this subspace. Any line of $\mathrm{PG}(n, q)$ intersects a quadric $\mathcal{Q}_n$ in 0, 1, 2 or $q+1$ points. A line containing 0 or 2 points of $\mathcal{Q}_n$ is called an *external* or a *secant line* respectively; the other lines are called *tangent lines*. A quadric $\mathcal{Q}_n$ is said to be *singular* if there is a point $p \in \mathcal{Q}_n$ such that every line through p is a tangent line; the point p is then called a *singular point*. If $\mathcal{Q}_n$ is singular then its singular points constitute a subspace π_r of dimension r, and for $n \geq r+2$, $\mathcal{Q}_n$ is a cone $\pi_r\mathcal{Q}_{n-r-1}$ with vertex π_r and base a nonsingular quadric $\mathcal{Q}_{n-r-1}$ in an $(n-r-1)$-dimensional subspace skew to π_r.

A *generator* of a quadric $\mathcal{Q}_n$ is a subspace of maximal dimension contained in $\mathcal{Q}_n$. The *projective index* of $\mathcal{Q}_n$ is the dimension of a generator. In $\mathrm{PG}(2n,q)$ any two nonsingular quadrics are projectively equivalent. A nonsingular quadric $\mathcal{Q}_{2n}$ in $\mathrm{PG}(2n,q)$ is called a *nonsingular parabolic quadric* and is also denoted by $\mathrm{Q}(2n,q)$. The projective index of $\mathrm{Q}(2n,q)$ is $n-1$. In $\mathrm{PG}(2n+1,q)$ there are two types of nonsingular quadrics. A quadric of the first type has projective index n, is called a *nonsingular hyperbolic quadric* and is also denoted by $\mathrm{Q}^+(2n+1,q)$. A quadric of the second type has projective index $n-1$, is called a *nonsingular elliptic quadric* and is also denoted by $\mathrm{Q}^-(2n+1,q)$. The numbers of points of nonsingular quadrics are listed below.

$$\begin{aligned}
|\,\mathrm{Q}(2n,q)\,| &= \frac{q^{2n}-1}{q-1},\\
|\,\mathrm{Q}^+(2n+1,q)\,| &= \frac{(q^n+1)(q^{n+1}-1)}{q-1},\\
|\,\mathrm{Q}^-(2n+1,q)\,| &= \frac{(q^n-1)(q^{n+1}+1)}{q-1}.
\end{aligned}$$

Consider the following relation γ on the set of generators of a nonsingular hyperbolic quadric $\mathrm{Q}^+(2n+1,q)$ in $\mathrm{PG}(2n+1,q)$. If U_1 and U_2 are generators of $\mathrm{Q}^+(2n+1,q)$, then $(U_1,U_2) \in \gamma$ if the dimension of $U_1 \cap U_2$ has the same parity as n, the projective index of $\mathrm{Q}^+(2n+1,q)$. The relation γ is an equivalence relation, and it has precisely two equivalence classes.

Let $\mathcal{Q}_n$ be a nonsingular quadric in $\mathrm{PG}(n,q)$, and let p be a point of $\mathcal{Q}_n$. Then there is a unique hyperplane $T_p(\mathcal{Q}_n)$ containing p such that the set of tangent lines through p is exactly the set of lines through p contained in $T_p(\mathcal{Q}_n)$. We call $T_p(\mathcal{Q}_n)$ the *tangent hyperplane at* p. For $n \geq 3$, the intersection of $\mathcal{Q}_n$ with $T_p(\mathcal{Q}_n)$ is a cone $p\mathcal{Q}_{n-2}$, where $\mathcal{Q}_{n-2}$ is a nonsingular quadric of the same type as $\mathcal{Q}_n$.

With a nonsingular quadric $\mathcal{Q}_n$ in $\mathrm{PG}(n,q)$, q odd, one can associate in a natural way an orthogonal polarity β such that $\mathcal{Q}_n$ is the set of totally isotropic points with respect to β. A point $p \in \mathcal{Q}_n$ is mapped by β to the tangent hyperplane $T_p(\mathcal{Q}_n)$.

Let $\mathcal{Q}_{2n}$ be a nonsingular parabolic quadric in $\mathrm{PG}(2n,q)$, q even. Then there is a unique point $p \notin \mathcal{Q}_{2n}$ such that every line through p is a tangent line to $\mathcal{Q}_{2n}$. We call this point the *nucleus* of $\mathcal{Q}_{2n}$. Let p' be a point not on $\mathcal{Q}_{2n}$ and different from the nucleus p, and let p'' be the unique point of $\mathcal{Q}_{2n}$ on the line pp'. Then $p' \in T_{p''}(\mathcal{Q}_{2n})$, and the set of tangent lines through p' is exactly the set of lines through p' contained in $T_{p''}(\mathcal{Q}_{2n})$.

Let $\mathcal{Q}_{2n+1}$ be a nonsingular elliptic or hyperbolic quadric in $\mathrm{PG}(2n+1,q)$, q even, and let p be a point not on $\mathcal{Q}_{2n+1}$. Then there is a unique hyperplane $U_p(\mathcal{Q}_{2n+1})$ containing p such that the set of tangent lines through p is exactly the set of lines through p contained in $U_p(\mathcal{Q}_{2n+1})$. The hyperplane $U_p(\mathcal{Q}_{2n+1})$ intersects $\mathcal{Q}_{2n+1}$ in a nonsingular parabolic quadric $\mathcal{Q}_{2n}$ of which p is the nucleus. With $\mathcal{Q}_{2n+1}$ one can associate in a natural way a symplectic polarity β. A point $p \in \mathcal{Q}_{2n+1}$ is mapped by β to the hyperplane $T_p(\mathcal{Q}_{2n+1})$, and a point $p \notin \mathcal{Q}_{2n+1}$ is mapped by β to the hyperplane $U_p(\mathcal{Q}_{2n+1})$. A line of $\mathrm{PG}(2n+1,q)$ is totally isotropic with respect to β if and only if it is a tangent line of $\mathcal{Q}_{2n+1}$.

Hermitian varieties

A *hermitian variety* $\mathcal{H}_n$ of $\mathrm{PG}(n,q^2)$, $n \geq 1$, is a set of points whose coordinates satisfy an equation over $\mathrm{GF}(q^2)$ of the form

$$\sum_{i,j=0}^{n} a_{ij} X_i X_j^q = 0,$$

such that not all a_{ij} are zero and $a_{ij}^q = a_{ji}$ for all $0 \leq i,j \leq n$. Any line intersects a hermitian variety $\mathcal{H}_n$ in 1, $q+1$ or q^2+1 points. The lines intersecting $\mathcal{H}_n$ in $q+1$ points are called *secant lines*, the other lines are called *tangent lines*.

Hermitian varieties and quadrics have many similar properties. Singularity is defined in the same way as for quadrics. The set of singular points of a hermitian variety $\mathcal{H}_n$ forms a subspace π_r of $\mathrm{PG}(n,q^2)$ of dimension r, and if $\mathcal{H}_n$ is singular and $n \geq r+2$, then $\mathcal{H}_n$ is a cone $\pi_r\mathcal{H}_{n-r-1}$ with vertex π_r and base a nonsingular hermitian variety $\mathcal{H}_{n-r-1}$ in an $(n-r-1)$-dimensional subspace skew to π_r. Any two nonsingular hermitian varieties in $\mathrm{PG}(n,q^2)$ are projectively equivalent. A nonsingular hermitian variety in $\mathrm{PG}(n,q^2)$ is denoted by $\mathrm{H}(n,q^2)$. The projective index (which is defined in the same way as for quadrics) of $\mathrm{H}(2n,q^2)$ is $n-1$, and the projective index of $\mathrm{H}(2n+1,q^2)$ is n. The number of points of a nonsingular hermitian variety is given by the following expressions.

$$
\begin{aligned}
|\,\mathrm{H}(2n,q^2)\,| &= \frac{(q^{2n+1}+1)(q^{2n}-1)}{q^2-1}\\
|\,\mathrm{H}(2n+1,q^2)\,| &= \frac{(q^{2n+2}-1)(q^{2n+1}+1)}{q^2-1}.
\end{aligned}
$$

Like for quadrics, the tangent lines through a point $p \in \mathrm{H}(n,q^2)$ lie in a hyperplane $T_p(\mathrm{H}(n,q^2))$, called the tangent hyperplane at p. With a nonsingular hermitian variety $\mathrm{H}(n,q^2)$ one can associate a hermitian polarity such that the set of totally isotropic points is $\mathrm{H}(n,q^2)$ and such that any point $p \in \mathrm{H}(n,q^2)$ is mapped to $T_p(\mathrm{H}(n,q^2))$.

A nonsingular hermitian variety $\mathrm{H}(2,q^2)$ is also called a *hermitian curve*. It is a set of q^3+1 points in $\mathrm{PG}(2,q^2)$ such that every line intersects it in 1 or $q+1$ points. A set of q^3+1 points of $\mathrm{PG}(2,q^2)$ with this property is called a *unital*. Examples of unitals which are not hermitian curves are the so-called Buekenhout-Metz unitals [14, 62].

1.3.2 The Plücker correspondence

The well-known *Plücker correspondence* establishes a link between the projective geometry $\mathrm{PG}(3,q)$ and the nonsingular hyperbolic quadric $\mathrm{Q}^+(5,q)$ in $\mathrm{PG}(5,q)$, which is commonly referred to as the *Klein quadric*. It goes as follows. Choose a basis in $\mathrm{PG}(3,q)$. A line L of $\mathrm{PG}(3,q)$, which is determined by two of its points $p_1(x_0,x_1,x_2,x_3)$ and $p_2(y_0,y_1,y_2,y_3)$, $p_1 \neq p_2$, is said to have *Plücker coordinates* $L(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12})$, where

$$
l_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \quad , 0 \leq i,j \leq 3.
$$

Let $\mathcal{L}$ be the line set of $\mathrm{PG}(3,q)$. Then

$$
\begin{aligned}
\kappa: \quad \mathcal{L} &\rightarrow \mathrm{PG}(5,q),\\
L(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12}) &\mapsto p(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12}),
\end{aligned}
$$

is an injective map, and the image of κ is the point set of the Klein quadric

$$
\mathrm{Q}^+(5,q): \quad X_0X_3 + X_1X_4 + X_2X_5 = 0.
$$

The map κ is called the *Plücker correspondence*. We list here some of its properties; for more information, see Section 15.4 in [51]. Recall that with the Klein quadric $\mathrm{Q}^+(5,q)$, we can associate in a natural way a nonsingular polarity β of $\mathrm{PG}(5,q)$. If q is even, then β is a symplectic polarity; if q is odd, then β is an orthogonal polarity.

Two distinct lines L_1 and L_2 of $\mathrm{PG}(3,q)$ are concurrent if and only if the line of $\mathrm{PG}(5,q)$ joining the points L_1^κ and L_2^κ is contained in $\mathrm{Q}^+(5,q)$.

The set of lines through a given point p of $\mathrm{PG}(3,q)$ is mapped by κ to the set of points of a generator of $\mathrm{Q}^+(5,q)$. Recall that the generators of $\mathrm{Q}^+(5,q)$ are planes. The set of lines contained in a given plane π of $\mathrm{PG}(3,q)$ is mapped by κ to the set of points of a generator of $\mathrm{Q}^+(5,q)$. In this manner, κ induces a bijection of the set of points and planes of $\mathrm{PG}(3,q)$ onto the set of generators of $\mathrm{Q}^+(5,q)$. Moreover, if Π_1 is the set of generators of $\mathrm{Q}^+(5,q)$ which correspond to points of $\mathrm{PG}(3,q)$, and Π_2 is the set of generators of $\mathrm{Q}^+(5,q)$ which correspond to planes of $\mathrm{PG}(3,q)$, then Π_1 and Π_2 are precisely the two equivalence classes of the equivalence relation γ, defined in Section 1.3.1, on the set of generators of $\mathrm{Q}^+(5,q)$.

From now on, we regard the Plücker correspondence κ as a symmetrized bijection between the set of points, lines and planes of $\mathrm{PG}(3,q)$, and the set of points and generators of $\mathrm{Q}^+(5,q)$. Incidence of points and lines, and of lines and planes of $\mathrm{PG}(3,q)$, is naturally preserved by κ. A point and a plane of $\mathrm{PG}(3,q)$ are incident if and only if the corresponding generators of $\mathrm{Q}^+(5,q)$ intersect in a line.

A *pencil of lines* in $\mathrm{PG}(3,q)$, that is, the set of $q+1$ lines contained in a given plane π and through a given point $p \in \pi$ of $\mathrm{PG}(3,q)$, corresponds via κ to the set of points on a line of $\mathrm{PG}(5,q)$, contained in $\mathrm{Q}^+(5,q)$. The Plücker correspondence induces a bijection from the set of pencils of $\mathrm{PG}(3,q)$ to the set of lines of $\mathrm{PG}(5,q)$, contained in $\mathrm{Q}^+(5,q)$.

A *regulus* in $\mathrm{PG}(3,q)$, that is, the set of $q+1$ lines intersecting three mutually skew lines of $\mathrm{PG}(3,q)$, corresponds via κ to the set of points of a nondegenerate conic on $\mathrm{Q}^+(5,q)$. If the regulus R_1 of $\mathrm{PG}(3,q)$ corresponds to the nondegenerate conic $C_1 = \mathrm{Q}^+(5,q) \cap \pi$, with π a plane of $\mathrm{PG}(5,q)$, then the *opposite regulus* R_2 of R_1, that is, the set of $q+1$ lines intersecting each line of R_1, corresponds to the set of points of the nondegenerate conic $C_2 = \mathrm{Q}^+(5,q) \cap \pi^\beta$.

The set of lines which are totally isotropic with respect to a symplectic polarity of $\mathrm{PG}(3,q)$, corresponds via κ to the set of points of $\mathrm{Q}^+(5,q)$ in a hyperplane of $\mathrm{PG}(5,q)$, not tangent to $\mathrm{Q}^+(5,q)$.

Next, we describe some properties of the Klein quadric $\mathrm{Q}^+(5,q)$, $q = 2^h$. Since q is even, the polarity β of $\mathrm{PG}(5,q)$, associated with $\mathrm{Q}^+(5,q)$, is a symplectic polarity.

Let V be a 3-space of $\mathrm{PG}(5,q)$, and let L be the line V^β. Then L is skew to V if and only if the quadric $\mathrm{Q}^+(5,q) \cap V$ is nonsingular, if and only if L is an external line or a secant to $\mathrm{Q}^+(5,q)$. Moreover, $\mathrm{Q}^+(5,q) \cap V$ is a nonsingular elliptic quadric if and only if L is an external line of $\mathrm{Q}^+(5,q)$, and $\mathrm{Q}^+(5,q) \cap V$ is a nonsingular hyperbolic quadric if and only if L is a secant line of $\mathrm{Q}^+(5,q)$.

Assume that L is skew to V, and that $E = \mathrm{Q}^+(5,q) \cap V$ is a nonsingular elliptic quadric. Then $\mathrm{Q}^+(5,q) \cap L = \emptyset$. Let p be a point of L. Then $p^\beta = \langle p, V \rangle$, and $\mathrm{Q}^+(5,q) \cap p^\beta$ is a nonsingular parabolic quadric with nucleus p. So the set of tangent lines of $\mathrm{Q}^+(5,q)$ through p is precisely the set of lines of $\mathrm{PG}(5,q)$ through p which intersect V.

Let p be a point of V, and let π be the plane $\langle p, L \rangle$. If $p \in E$, then $\mathrm{Q}^+(5,q) \cap \pi$ is the point p. If $p \notin E$, then $\mathrm{Q}^+(5,q) \cap \pi$ is a nondegenerate conic with nucleus p.

Let M be a line of V. Then the 3-space $W = \langle L, M \rangle$ intersects $\mathrm{Q}^+(5,q)$ in a nonsingular elliptic quadric, respectively a nonsingular hyperbolic quadric, or a quadratic cone, if and only if M is a secant line, respectively an external line, or a tangent line, of E.

1.3.3 Generalized quadrangles

As we have seen already in Section 1.2.1, an incidence structure $\mathcal{S}$ is called a *generalized quadrangle* if $\mathcal{S}$ is a partial linear space of order (s,t), such that for every anti-flag $\{p, L\}$ of $\mathcal{S}$, $\alpha(p, L) = 1$. If $s = t$, then $\mathcal{S}$ is said to have order s. For an extensive treatment of finite generalized quadrangles, see [66].

A set of points O of a generalized quadrangle $\mathcal{S}$ is called an *ovoid* of $\mathcal{S}$ if every line of $\mathcal{S}$ contains exactly one point of O. Dually, a set of lines S of a generalized quadrangle $\mathcal{S}$ is called a *spread* of $\mathcal{S}$ if every point of $\mathcal{S}$ is on exactly one line of S.

Of special interest in the context of this thesis are the so-called *classical generalized quadrangles*. These are generalized quadrangles which are associated with a quadric, a hermitian variety or a symplectic polarity of a projective space.

Let $\mathcal{Q}_n$ be a nonsingular quadric of projective index 1 in $\mathrm{PG}(n,q)$. Then $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the set of points of $\mathcal{Q}_n$, $\mathcal{B}$ is the set of generators of $\mathcal{Q}_n$, and I is the natural incidence, is a generalized quadrangle. The nonsingular quadrics of projective index 1 are the following.

1. $n = 3$ and $\mathcal{Q}_3 = \mathrm{Q}^+(3,q)$. Then $\mathcal{S}$ is a generalized quadrangle of order $(q,1)$, and is denoted by $\mathsf{Q}(3,\mathsf{q})$.

2. $n = 4$ and $\mathcal{Q}_4 = \mathrm{Q}(4,q)$. Then $\mathcal{S}$ is a generalized quadrangle of order q, and is denoted by $\mathsf{Q}(4,\mathsf{q})$.

3. $n = 5$ and $\mathcal{Q}_5 = \mathrm{Q}^-(5, q)$. Then $\mathcal{S}$ is a generalized quadrangle of order (q, q^2), and is denoted by $\mathsf{Q(5,q)}$.

Let $\mathcal{H}_n$ be a nonsingular hermitian variety in $\mathrm{PG}(n, q^2)$ of projective index 1. Then $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the set of points of $\mathcal{H}_n$, $\mathcal{B}$ is the set of generators of $\mathcal{H}_n$, and I is the natural incidence, is a generalized quadrangle. The nonsingular hermitian varieties of projective index 1 are the following.

1. $n = 3$ and $\mathcal{H}_3 = \mathrm{H}(3, q)$. Then $\mathcal{S}$ is a generalized quadrangle of order (q^2, q), and is denoted by $\mathsf{H(3,q^2)}$.

2. $n = 4$ and $\mathcal{H}_4 = \mathrm{H}(4, q)$. Then $\mathcal{S}$ is a generalized quadrangle of order (q^2, q^3), and is denoted by $\mathsf{H(4,q^2)}$.

Let β be a symplectic polarity of $\mathrm{PG}(3, q)$. Let $\mathsf{W(q)} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the point set of $\mathrm{PG}(3, q)$, $\mathcal{B}$ is the set of totally isotropic lines with respect to β, and I is the natural incidence. Then $\mathsf{W(q)}$ is a generalized quadrangle of order q.

The isomorphisms between classical generalized quadrangles are the following. For all prime powers q, $\mathsf{W(q)} \cong \mathsf{Q(4,q)}^D$ and $\mathsf{Q(5,q)} \cong \mathsf{H(3,q^2)}^D$. Furthermore, $\mathsf{W(q)} \cong \mathsf{Q(4,q)}$ if and only if q is even.

1.3.4 Arcs and caps

A *k-arc* in $\mathrm{PG}(n, q)$ is a set of k points of $\mathrm{PG}(n, q)$, not contained in a hyperplane of $\mathrm{PG}(n, q)$, with at most n points in every hyperplane of $\mathrm{PG}(n, q)$. The size of the largest k-arc in $\mathrm{PG}(n, q)$ is denoted by $m(n, q)$. For a survey of k-arcs we refer to Chapter 21 of [51] and Chapter 27 of [54].

A *k-cap* in $\mathrm{PG}(n, q)$ is a set of k points of $\mathrm{PG}(n, q)$ such that no three are collinear. A k-cap is called *complete* or *maximal* if it is not contained in a larger cap. A line is called *external*, *tangent* or *secant* with respect to a cap if it contains respectively $0, 1, 2$ points of the cap.

The size of the largest cap in $\mathrm{PG}(n, q)$ is denoted by $m_2(n, q)$. This number $m_2(n, q)$ is known for only a few values of (n, q). It is known that $m_2(2, q) = q + 1$ if q is odd and $m_2(2, q) = q + 2$ if q is even, $m_2(3, q) = q^2 + 1$ if $q > 2$, $m_2(n, 2) = 2^n$ for all $n \geq 2$, $m_2(4, 3) = 20$ and $m_2(5, 3) = 56$.

1.3.5 Ovals and hyperovals

The maximum size $m_2(2, q)$ of a cap in $\mathrm{PG}(2, q)$, equals $q + 1$ if q is odd and $q + 2$ if q is even. In a projective plane $\mathrm{PG}(2, q)$, a cap of size $q + 1$ is called an *oval*. Caps of size $q + 2$ in $\mathrm{PG}(2, q)$ only exist when q is even, and they are called *hyperovals*. For a recent survey of the theory of (hyper)ovals, see [9] or [20, 21].

The classical example of an oval is a nondegenerate conic. Segre [67, 68] proved that in $\mathrm{PG}(2,q)$, q odd, every oval is a conic.

In $\mathrm{PG}(2,q)$, q even, several projectively distinct examples of ovals and hyperovals are known. It is easy to prove that in $\mathrm{PG}(2,q)$, q even, every oval has a unique *nucleus*, this is a point not on the oval such that every line through it is tangent to the oval. Hence every oval in $\mathrm{PG}(2,q)$, q even, is contained in a unique hyperoval. Conversely, every hyperoval contains $q+2$ ovals which may well be projectively distinct. A hyperoval which contains a conic is called a *regular hyperoval*.

Let $\mathcal{H}$ be a hyperoval in $\mathrm{PG}(2,q)$, $q = 2^h$. Then we can coordinatize in such a way that $\mathcal{H}$ contains the points with coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$. Consequently we may write $\mathcal{H}$ as

$$\mathcal{H} = \{(1,t,f(t)) \mid t \in \mathrm{GF}(q)\} \cup \{(0,0,1),(0,1,0)\},$$

where f is a permutation of $\mathrm{GF}(q)$ fixing 0 and 1. The permutation f may be written in a unique way as a polynomial over $\mathrm{GF}(q)$ of degree at most $q-1$. Any polynomial of degree at most $q-1$ that arises as above from a hyperoval is called an *o-polynomial.* Notice that projectively equivalent hyperovals can be represented by different o-polynomials. For example each of $f(t) = t^2$, $f(t) = \sqrt{t}$, $f(t) = t^{q-2}$ is an o-polynomial that correspond to the regular hyperoval. Another o-polynomial is $f(t) = t^{2^i}$, where i is an integer such that $\gcd(i,h) = 1$. The corresponding hyperovals are called *translation hyperovals* and they are not regular provided $i \neq 1, h-1$.

The following result, due to Payne, will be used in Section 5.2.

Theorem 1.3.1 (Payne [65]) *If f is an additive o-polynomial over $\mathrm{GF}(2^h)$, then $f(t) = t^{2^i}$ where i is an integer such that $\gcd(i,h) = 1$.*

Remark

Throughout this thesis all maps are written in exponential notation. However we make an exception when it comes to o-polynomials, since they are traditionally written as functions.

1.3.6 Ovoids

The maximum size $m_2(3,q)$ of a cap in $\mathrm{PG}(3,q)$, $q > 2$, is q^2+1. A cap of size q^2+1 in $\mathrm{PG}(3,q)$, $q > 2$, is called an *ovoid*. For $q = 2$ we have $m_2(3,2) = 8$. An ovoid of $\mathrm{PG}(3,2)$ is a set of five points, no four of which are coplanar.

The classical example of an ovoid in $\mathrm{PG}(3, q)$ is a nonsingular elliptic quadric $\mathrm{Q}^-(3, q)$. For a recent survey of the theory of ovoids in $\mathrm{PG}(3, q)$, see [9].

It can be shown that every plane of $\mathrm{PG}(3, q)$ intersects an ovoid in either a single point or an oval. Barlotti [3] and Panella [63] proved independently that, when q is odd, every ovoid is a nonsingular elliptic quadric.

If q is even, the situation is different. Tits [82] found an example of an ovoid in $\mathrm{PG}(3, q)$, $q = 2^{2e+1}$, $e \geq 1$, which is not an elliptic quadric (for $q = 8$ it was discovered earlier by Segre [69]). As the Suzuki simple group $\mathrm{Sz}(q)$ acts in a natural way on this ovoid, it is called the *Suzuki-Tits ovoid*. The Suzuki-Tits ovoid is the only known ovoid which is not an elliptic quadric. With respect to a basis of $\mathrm{PG}(3, 2^{2e+1})$, it has the following canonical form:

$$\{(1, xy + x^{\sigma+2} + y^{\sigma}, x, y) \,|\, x, y \in \mathrm{GF}(q)\} \cup \{(0, 1, 0, 0)\},$$

where $\sigma : x \mapsto x^{2^{e+1}}$.

Brown [10] proved that, if an ovoid of $\mathrm{PG}(3, q)$, q even, intersects one plane in a conic, then it is a nonsingular elliptic quadric.

Let O be an ovoid of $\mathrm{PG}(3, q)$, q even. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the point set of $\mathrm{PG}(3, q)$, $\mathcal{B}$ is the set of tangent lines to O, and I is the natural incidence. Then $\mathcal{S}$ is the generalized quadrangle $\mathsf{W(q)}$, and O is an ovoid of $\mathcal{S}$. Conversely, Thas [76] proved that every ovoid of $\mathsf{W(q)}$, q even, is an ovoid of $\mathrm{PG}(3, q)$. Hence, when q is even, there is a one-to-one correspondence between ovoids of $\mathrm{PG}(3, q)$ and ovoids of $\mathsf{W(q)}$, and, since $\mathsf{W(q)} \cong \mathsf{Q(4, q)}$, also between ovoids of $\mathrm{PG}(3, q)$ and ovoids of $\mathsf{Q(4, q)}$.

1.3.7 Maximal arcs

A *$\{k; d\}$-arc* in $\mathrm{PG}(2, q)$ is a set of $k > 0$ points such that the maximum number of points of this set on a line is d. A $\{k; 2\}$-arc in $\mathrm{PG}(2, q)$ is then plainly a k-arc in $\mathrm{PG}(2, q)$. Standard counting arguments show that for a $\{k; d\}$-arc in $\mathrm{PG}(2, q)$ we have $k \leq qd - q + d$ and that in case of equality $d = q + 1$ or d divides q. A $\{qd - q + d; d\}$-arc in $\mathrm{PG}(2, q)$ is called a *maximal arc of degree d*. Every line intersects a maximal arc of degree d in either 0 or d points. The set of lines not intersecting a maximal arc of degree d, with $d \neq q + 1$, is a maximal arc of degree q/d in the dual plane of $\mathrm{PG}(2, q)$. A maximal arc is called *nontrivial* if $1 < d < q$. Hyperovals are maximal arcs of degree 2.

Theorem 1.3.2 (Ball, Blokhuis, Mazzocca [2]) *In* $\mathrm{PG}(2, q)$*, q odd, nontrivial maximal arcs do not exist.*

Theorem 1.3.3 (Denniston [44]) *Let x^2+bx+1 be an irreducible quadratic form over* $\mathrm{GF}(q)$*, $q = 2^h$, and let $\mathcal{C}_\lambda$, $\lambda \in \mathrm{GF}(q) \cup \{\infty\}$, be the conic in* $\mathrm{PG}(2,q)$ *with equation $X_0^2 + bX_0X_1 + X_1^2 + \lambda X_2^2 = 0$. Let H be a subgroup of order $d = 2^m$ of the additive group of* $\mathrm{GF}(q)$*. Then the set $\mathcal{K} = \cup_{\lambda \in H} \mathcal{C}_\lambda$ is a maximal arc of degree d in* $\mathrm{PG}(2,q)$*.*

The maximal arcs arising in Theorem 1.3.3 are called *Denniston maximal arcs* or *maximal arcs of Denniston type*. Maximal arcs that are not of Denniston type were constructed by Thas [77, 79], by Mathon [60] and by Hamilton and Mathon [49].

Let $\mathcal{K}$ be the Denniston type maximal arc of degree $d = 2^m$, which arises from a subgroup H of order d of the additive group of $\mathrm{GF}(q)$, $q = 2^h$. Since, for every $d' \mid d$, the group H has a subgroup H' of order d', $\mathcal{K}$ contains a Denniston type maximal arc $\mathcal{K}'$ of degree d' for every divisor d' of d.

1.3.8 Sets of type $(1, m, q+1)$ in $\mathrm{PG}(n,q)$

A *set of type* $(t_1, \ldots, t_m)$ *with respect to a set X of lines of* $\mathrm{PG}(n,q)$, with $n \geq 1$, is a set $\mathcal{K}$ of points of $\mathrm{PG}(n,q)$ such that every line of X intersects $\mathcal{K}$ in t_i points for some $i \in \{1, \ldots, m\}$. A *set of type* $(t_1, \ldots, t_m)$ *in* $\mathrm{PG}(n,q)$ is a set of type $(t_1, \ldots, t_m)$ with respect to the set of all lines of $\mathrm{PG}(n,q)$. If k is the number of points of a set of type $(t_1, \ldots, t_m)$ in $\mathrm{PG}(n,q)$, then this set is also called a k-set of type $(t_1, \ldots, t_m)$ in $\mathrm{PG}(n,q)$.

Of special interest are sets of type $(1, m, q+1)$ in $\mathrm{PG}(n,q)$. A set $\mathcal{K}$ of type $(1, m, q+1)$ is called *singular* if there is a *singular point*, that is, a point p of $\mathcal{K}$ such that every line through p contains either 1 or $q+1$ points of $\mathcal{K}$.

Tallini Scafati [75] characterizes, for $q > 4$, the nonsingular hermitian varieties as the nonsingular sets of type $(1, m, q+1)$ in $\mathrm{PG}(n,q)$. However an arithmetical error invalidates the conclusion if q is even and $m = \frac{1}{2}q + 1$. Hirschfeld and Thas [53, 52] provide a counterexample in this case and give a complete classification of sets of type $(1, m, q+1)$ in $\mathrm{PG}(n,q)$.

The counterexample is the following. Let $\mathcal{Q}_{n+1}$ be a nonsingular quadric in $\mathrm{PG}(n+1,q)$, $n \geq 1$, q even, and let r be a point not on $\mathcal{Q}_{n+1}$ and different from its nucleus if $n+1$ is even. Let $\mathcal{R}_n$ be the projection of $\mathcal{Q}_{n+1}$ from r onto a hyperplane $\mathrm{PG}(n,q)$ of $\mathrm{PG}(n+1,q)$, not containing r. Then $\mathcal{R}_n$ is a nonsingular set of type $(1, \frac{1}{2}q+1, q+1)$ in $\mathrm{PG}(n,q)$. If $n+1$ is odd and $\mathcal{Q}_{n+1} = \mathrm{Q}^+(n+1,q)$ is a hyperbolic quadric, we write $\mathcal{R}_n = \mathcal{R}_n^+$. If $n+1$ is odd and $\mathcal{Q}_{n+1} = \mathrm{Q}^-(n+1,q)$ is an elliptic quadric, we write $\mathcal{R}_n = \mathcal{R}_n^-$.

Notice that every set $\mathcal{R}_n$ contains a hyperplane. Indeed, the set of tangent lines of $\mathcal{Q}_{n+1}$ through r is the set of lines through r which lie in a hyperplane of $\mathrm{PG}(n+1,q)$ containing r. This hyperplane intersects $\mathrm{PG}(n,q)$ in a hyperplane which is hence completely contained in $\mathcal{R}_n$.

The number of points of the set $\mathcal{R}_n$ is as follows.

$$\begin{aligned} |\,\mathcal{R}^+_{2n}\,| &= \frac{1}{2}q^n(q^n+1)+\frac{q^{2n}-1}{q-1}, \\ |\,\mathcal{R}^-_{2n}\,| &= \frac{1}{2}q^n(q^n-1)+\frac{q^{2n}-1}{q-1}, \\ |\,\mathcal{R}_{2n+1}\,| &= \frac{1}{2}q^{2n+1}+\frac{q^{2n+1}-1}{q-1}. \end{aligned}$$

We mention here only the following special cases of the classification of Hirschfeld and Thas.

Theorem 1.3.4 (Hirschfeld, Thas [52, 53], Glynn [46]) *If $\mathcal{K}$ is a non-singular set of type $(1, \frac{1}{2}q+1, q+1)$ in $\mathrm{PG}(n,q)$ with $n \geq 3$ and $q = 2^h$, $h > 2$, then $\mathcal{K} = \mathcal{R}_n$. For $q = 4$ the same conclusion holds if there is no plane intersecting $\mathcal{K}$ in a unital or a Baer subplane.*

The proof of the case $n = 3$ in Theorem 1.3.4 was completed by Glynn [46].

Theorem 1.3.5 (Hirschfeld, Thas [52, 53], Glynn [46]) *If $\mathcal{K}$ is a non-singular set of type $(1, m, q+1)$ in $\mathrm{PG}(3,q)$, $q > 2$, which contains a plane, then one of the following holds.*

1. *$m = 2$ and $\mathcal{K}$ is the union of a plane with a point not on that plane.*
2. *$m = q$ and $\mathcal{K}$ is the complement of a point.*
3. *$q = 2^h$, $m = \frac{1}{2}q+1$, and $\mathcal{K} = \mathcal{R}_3$.*

1.4 Projective and affine incidence structures

Let R be a projective or affine space. An incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is said to be *laxly embedded* in R if $\mathcal{P}$ is a subset of the point set of R, if $\mathcal{B}$ is a subset of the line set of R, if for every pair $(p, L) \in \mathcal{P} \times \mathcal{B}$ we have that $(p, L) \in \mathrm{I}$ if and only if p and L are incident in R, and if $\mathcal{P}$ is not contained in a hyperplane of R. If furthermore for every line L of $\mathcal{S}$ we have that every point of R incident with L is a point of $\mathcal{S}$, then $\mathcal{S}$ is said to be *fully embedded* in R. An incidence structure fully embedded in a projective space will be called *projective*, and an incidence structure fully embedded in an affine space will be called *affine*.

Two incidence structures $\mathcal{S}$ and $\mathcal{S}'$ fully embedded in a projective or affine space R are called *projectively* or *affinely equivalent* if there exists a collineation of R which induces an isomorphism from $\mathcal{S}$ to $\mathcal{S}'$. In this case we write $\mathcal{S} \simeq \mathcal{S}'$. We recall that the notation for isomorphic incidence structures $\mathcal{S}$ and $\mathcal{S}'$ is $\mathcal{S} \cong \mathcal{S}'$. Note that if $\mathcal{S} \simeq \mathcal{S}'$, then $\mathcal{S} \cong \mathcal{S}'$, but not vice versa. Indeed, two projective or affine incidence structures may well be isomorphic without being projectively or affinely equivalent.

An important question in finite geometry is which incidence structures can be fully embedded in projective or affine spaces, and how these embeddings look. This question has been answered for various classes of incidence structures. In the following sections we will give an overview of these results, as well as the constructions of those examples that are relevant for the rest of the thesis. To summarize we can say that the embedding problem is solved for projective and affine generalized quadrangles, for projective and affine partial geometries, for projective semipartial geometries and, with some exceptions, for projective $(0, \alpha)$-geometries.

1.4.1 Projective GQ and (dual) partial quadrangles

Theorem 1.4.1 (Buekenhout, Lefèvre [15]) *If $\mathcal{S}$ is a projective generalized quadrangle, then $\mathcal{S}$ is projectively equivalent to one of the classical generalized quadrangles* $\mathsf{W(q)}$, $\mathsf{Q(3,q)}$, $\mathsf{Q(4,q)}$, $\mathsf{Q(5,q)}$, $\mathsf{H(3,q^2)}$ *or* $\mathsf{H(4,q^2)}$.

About projective partial quadrangles and dual partial quadrangles, not so much is known. No example is known to us of a projective proper partial quadrangle. An example of a dual partial quadrangle fully embedded in $\mathrm{PG}(3, q^2)$ is $\mathsf{H(3,q^2)}^*$ [28] which is obtained by deleting from the classical generalized quadrangle $\mathsf{H(3,q^2)}$ a line L, the points on L and all lines concurrent with L. The geometry $\mathsf{H(3,q^2)}^*$ is the dual of a $\mathrm{PQ}(q-1, q^2, q^2-q)$. De Clerck, Durante and Thas [28] prove that if $\mathcal{S}$ is the dual of a $\mathrm{PQ}(t, q, \mu)$ and fully embedded in $\mathrm{PG}(3, q)$, then $\mu \leq q - \frac{q}{t+1}$ and if equality holds then q is a square and $\mathcal{S} \simeq \mathsf{H(3,q)}^*$.

Recently a new example of a partial quadrangle $\mathrm{PQ}(\frac{1}{2}(q-1), q^2, \frac{1}{2}(q-1)^2)$, with q any odd prime power, was found by Cossidente and Penttila [23]. It arises from a hemisystem of the classical generalized quadrangle $\mathsf{H(3,q^2)}$. A *hemisystem* of $\mathsf{H(3,q^2)}$ is a set of lines of $\mathsf{H(3,q^2)}$ such that any point of $\mathsf{H(3,q^2)}$ is contained in exactly $\frac{1}{2}(q+1)$ lines of that set. Hemisystems were introduced by Segre [71], and it was proven by Thas [80] that the incidence structure with as points the points of $\mathsf{H(3,q^2)}$ and as lines the lines of a hemisystem of $\mathsf{H}(3, \mathsf{q}^2)$ is the dual of a $\mathrm{PQ}(\frac{1}{2}(q-1), q^2, \frac{1}{2}(q-1)^2)$. Now since the generalized quadrangle $\mathsf{H}(3, \mathsf{q}^2)$ is fully embedded in $\mathrm{PG}(3, q^2)$, so is this dual partial quadrangle.

1.4.2 Projective partial geometries

Projective partial geometries were completely classified by De Clerck and Thas [29]. The examples are the following. Notice that none of them is a proper partial geometry.

1. The design of all points and all lines of $\mathrm{PG}(n, q)$, $n \geq 2$, is a $\mathrm{pg}(q, (q^n - q)/(q-1), q+1)$ (or, equivalently, a $2-((q^{n+1}-1)/(q-1), q+1, 1)$ design) fully embedded in $\mathrm{PG}(n, q)$.

2. Let $\mathcal{B}$ be a set of lines in $\mathrm{PG}(2, q)$, q even, which forms a maximal arc of degree $d \geq 2$ in the dual plane of $\mathrm{PG}(2, q)$. Let $\mathcal{P}$ be the set of points of $\mathrm{PG}(2, q)$ on the lines of $\mathcal{B}$ and let I be the natural incidence. Then $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a $\mathrm{pg}(q, d-1, d)$ (or, equivalently, the dual of a $2-(qd-q+d, d, 1)$ design) fully embedded in $\mathrm{PG}(2, q)$. We call $\mathcal{S}$ a *dual maximal arc in* $\mathrm{PG}(2, q)$.

3. The incidence structure H_q^n formed by the points and lines of $\mathrm{PG}(n, q)$, $n \geq 2$, having an empty intersection with a given $(n-2)$-dimensional subspace of $\mathrm{PG}(n, q)$ is a $\mathrm{pg}(q, q^{n-1}-1, q)$ (or, equivalently, the dual of a net of order q^{n-1} and degree $q+1$) fully embedded in $\mathrm{PG}(n, q)$ [29].

Theorem 1.4.2 (De Clerck, Thas [29]) *If $\mathcal{S}$ is a partial geometry fully embedded in* $\mathrm{PG}(n, q)$ *then $\mathcal{S}$ is a generalized quadrangle or $\mathcal{S}$ is one of the examples in the list above.*

1.4.3 Projective (dual) semipartial geometries and $(0, \alpha)$-geometries

The projective (dual) semipartial geometries with $\alpha > 1$ and the projective $(0, \alpha)$-geometries with $\alpha > 1$ were almost completely classified by Debroey, De Clerck and Thas [30, 81]. The cases that are not solved are the semipartial geometries fully embedded in $\mathrm{PG}(n, 2)$ and the $(0, \alpha)$-geometries fully embedded in $\mathrm{PG}(3, q)$ and in $\mathrm{PG}(n, 2)$.

Projective $(0, 1)$-geometries were not studied in [30, 81] because the embedding in $\mathrm{PG}(n, q)$ does not induce much structure on a $(0, 1)$-geometry. For instance if a plane π of $\mathrm{PG}(n, q)$ contains two intersecting lines of a $(0, 1)$-geometry $\mathcal{S}$ fully embedded in $\mathrm{PG}(n, q)$, then π may contain a priori any number $\gamma \in \{2, \ldots, q+1\}$ of lines of $\mathcal{S}$ centered in a point. Thus we obtain far less information than in the case $\alpha > 1$, as we will see in Lemma 1.4.7.

The following list contains the examples of projective $(0, \alpha)$-geometries with $\alpha > 1$ that appear in [81]. In Chapter 2 we construct new examples of $(0, \alpha)$-geometries with $\alpha > 1$ fully embedded in $\mathrm{PG}(3, 2^h)$.

1. Let X be a set with $m \geq 4$ elements, $U_2 = \{T \subset X \| \, |T| = 2\}$, $U_3 = \{T \subset X \| \, |T| = 3\}$ and I symmetric inclusion. Then $U_{2,3}(m) = (U_2, U_3, \mathrm{I})$ is a cotriangular space with $t = m - 3$. Note that $U_{2,3}(m)$ is also an $\mathrm{spg}(2, m-3, 2, 4)$. For some values of m and n the geometry $U_{2,3}(m)$ can be fully embedded in $\mathrm{PG}(n, 2)$. Examples such that $(m, n) \in \{(5, 3), (7, 4), (9, 5)\}$ are described in [81], and examples with $m \in \{n+2, n+3\}$ are described in [57].

2. Let A be a skew-symmetric $(n+1) \times (n+1)$-matrix over $\mathrm{GF}(q)$, $n \geq 2$. Then the rank of A is even; let $\operatorname{rank} A = 2k$, with $k > 0$. Let β be the (possibly singular) symplectic polarity of $\mathrm{PG}(n, q)$ defined by A, and let U be the vertex of β. Let $\mathcal{P}$ be the set of points of $\mathrm{PG}(n, q)$ not in U, let $\mathcal{B}$ be the set of lines of $\mathrm{PG}(n, q)$ which are disjoint from U and which are not totally isotropic with respect to β, and let I be the natural incidence. Then $\overline{W(n, 2k, q)} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a copolar space of order $(q, q^{n-1} - 1)$ (so a $(0, \alpha)$-geometry with $s = \alpha = q$ and $t = q^{n-1} - 1$), fully embedded in $\mathrm{PG}(n, q)$ [30].

   If $k = 1$ then $\overline{W(n, 2, q)} \simeq H_q^n$. If n is odd and $2k = n + 1$ then the symplectic polarity β is nonsingular. In this case the geometry $\overline{W(n, n+1, q)}$ is an $\mathrm{spg}(q, q^{n-1} - 1, q, q^{n-1}(q-1))$ and is shortly denoted by $\overline{W(n, q)}$ [41]. In all other cases $\overline{W(n, 2k, q)}$ is not a semipartial geometry.

3. Let $\mathcal{Q}_n$ be a (possibly singular) quadric in $\mathrm{PG}(n, 2)$. Let $\mathcal{B}$ be the set of external lines of $\mathcal{Q}_n$, let $\mathcal{P}$ be the set of points of $\mathrm{PG}(n, 2)$ on the elements of $\mathcal{B}$, and let I be the natural incidence. Then $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a cotriangular space fully embedded in $\mathrm{PG}(n, 2)$ unless $\mathcal{Q}_n$ consists of one or two hyperplanes, or $n = 3$ and $\mathcal{Q}_3 = \mathrm{Q}^+(3, 2)$, or $n \geq 4$ and $\mathcal{Q}_n$ is a cone with vertex an $(n-4)$-dimensional subspace U and base a quadric $\mathrm{Q}^+(3, 2)$ in a 3-space skew to U (in the last two cases the geometry is not connected) [30].

   If $n = 2d - 1$ and $\mathcal{Q}_{2d-1} = \mathrm{Q}^{\pm}(2d-1, 2)$ then $\mathcal{S}$ is a semipartial geometry $\mathrm{spg}(2, 2^{2d-3} - \varepsilon 2^{d-2} - 1, 2, 2^{2d-3} - \varepsilon 2^{d-1})$, where ε is 1 or -1

according as $\mathcal{Q}_{2d-1}$ is hyperbolic or elliptic. This semipartial geometry is denoted by $\mathrm{NQ}^{\pm}(2d-1,2)$.

If $n = 2d$ and $\mathcal{Q}_{2d} = \mathrm{Q}(2d,2)$ then $\mathcal{S}$ is a semipartial geometry, denoted by $\mathrm{NQ}(2d,2)$, isomorphic to $\overline{W(2d-1,2)}$.

In all other cases $\mathcal{S}$ is not a semipartial geometry.

4. Consider the nonsingular hyperbolic quadric $\mathrm{Q}^+(3,q)$ in $\mathrm{PG}(3,q)$, where $q = 2^h$ and $h > 1$. Then the same construction as in example 3 yields a $(0,\alpha)$-geometry $\mathrm{NQ}^+(3,q)$ fully embedded in $\mathrm{PG}(3,q)$ with $s = q$, $\alpha = \frac{1}{2}q$ and $t = \frac{1}{2}q^2 - \frac{1}{2}q - 1$. This geometry is never a semipartial geometry [30].

Proposition 1.4.3 *If $\mathcal{S}$ is a $(0,\alpha)$-geometry fully embedded in $\mathrm{PG}(2,q)$, then either $\mathcal{S}$ is the design of all points and all lines of $\mathrm{PG}(2,q)$ or $\mathcal{S}$ is a dual maximal arc.*

Proof. Since any two lines of $\mathrm{PG}(2,q)$ intersect, $\mathcal{S}$ is a dual design and hence a partial geometry. Now we can apply Theorem 1.4.2. □

Theorem 1.4.4 (Debroey, De Clerck, Thas [81]) *If $\mathcal{S}$ is a $(0,\alpha)$-geometry with $\alpha > 1$ fully embedded in $\mathrm{PG}(n,q)$, $n \geq 4$, $q > 2$, then we have one of the following cases.*

1. *$\mathcal{S}$ is the design of all points and all lines of $\mathrm{PG}(n,q)$.*
2. *$\mathcal{S} \simeq \overline{W(n,2k,q)}$.*

There is no complete classification of $(0,2)$-geometries fully embedded in $\mathrm{PG}(n,2)$, but partial results are obtained in [81]. There is also no complete classification of $(0,\alpha)$-geometries with $\alpha > 1$ fully embedded in $\mathrm{PG}(3,q)$. Partial results are obtained in [30] and in Section 26.8 of [54], and we will state them below. In the case of semipartial geometries however, there is also a complete classification of full embeddings in $\mathrm{PG}(3,q)$.

Theorem 1.4.5 (Debroey, De Clerck, Thas [81]) *If $\mathcal{S}$ is a proper semipartial geometry with $\alpha > 1$, fully embedded in $\mathrm{PG}(n,q)$, $n \geq 3$, $q > 2$, then n is odd and $\mathcal{S} \simeq \overline{W(n,q)}$.*

Theorem 1.4.6 (De Clerck, Thas [30]) *If $\mathcal{S}$ is a proper dual semipartial geometry with $\alpha > 1$, fully embedded in $\mathrm{PG}(n,q)$, $n \geq 3$, then $n = 3$, $q = 2$ and $\mathcal{S} \simeq \mathrm{NQ}^-(3,2)$.*

Before we come to the results on $(0,\alpha)$-geometries fully embedded in $\mathrm{PG}(3,q)$, we need the following lemma.

Lemma 1.4.7 (De Clerck, Thas [30]) *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in $\mathrm{PG}(n,q)$, $n \geq 3$, and let π be a plane of $\mathrm{PG}(n,q)$. Let X_π be the set of points and lines of $\mathcal{S}$ contained in π and let $\mathcal{S}_\pi$ be the sub incidence structure of $\mathcal{S}$ induced on X_π. Then the plane π is of one of the following three types.*

Type (a). *$\mathcal{S}_\pi$ consists of a $\mathrm{pg}(q,\alpha-1,\alpha)$ (that is, a dual maximal arc of degree α) and possibly some isolated points.*

Type (b). *$\mathcal{S}_\pi$ consists of exactly one line and possibly some isolated points.*

Type (c). *$\mathcal{S}_\pi$ only consists of some isolated points.*

Let $\mathcal{S} = (\mathcal{P},\mathcal{B},\mathrm{I})$ be a $(0,\alpha)$-geometry, $\alpha > 1$, of order (q,t) fully embedded in $\mathrm{PG}(3,q)$, and let π be a plane of type (a) containing m isolated points. Then, since every line of $\mathcal{S}$ is either contained in π or intersects π in a point of $\mathcal{S}_\pi$,

$$|\,\mathcal{B}\,| = (q\alpha - q + \alpha)\frac{q+1}{\alpha}(t+1-\alpha) + q\alpha - q + \alpha + m(t+1).$$

Since this holds for every plane of type (a), it follows that every plane of type (a) contains the same number m of isolated points.

Theorem 1.4.8 (De Clerck, Thas [30]) *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry with $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$. If $m = 0$, then one of the following holds.*

1. *$\mathcal{S}$ is the design of all points and all lines in $\mathrm{PG}(3,q)$.*
2. *$\mathcal{S} \simeq H_q^3$.*
3. *$q = 2$ and $\mathcal{S} \simeq \mathrm{NQ}^-(3,2)$.*

Theorem 1.4.9 (De Clerck, Thas [30]) *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry with $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$. If $m \neq 0$ then there is no plane of type* (b).

Theorem 1.4.10 (Hirschfeld, Thas [54], Theorem 26.8.6) *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry with $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$. If $m = 1$, then one of the following holds.*

1. *$\alpha = q$ and $\mathcal{S} \simeq \overline{W(3,q)}$.*
2. *$q = 2^h$, $\alpha = q/2$ and $\mathcal{S} \simeq \mathrm{NQ}^+(3,q)$.*

Theorem 1.4.11 (Hirschfeld, Thas [54], Theorem 26.8.7) *If $\mathcal{S}$ is a $(0,\alpha)$-geometry with $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$, then $m \neq 2$.*

1.4.4 Linear representations

An important class of affine incidence structures is formed by the *linear representations*. Consider the affine space $\mathrm{AG}(n,q)$. We recall that Π_∞ denotes the space at infinity of $\mathrm{AG}(n,q)$. Let $\mathcal{K}_\infty$ be a set of points of Π_∞. Then the linear representation of the set $\mathcal{K}_\infty$ is the incidence structure $T^*_{n-1}(\mathcal{K}_\infty) = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ where $\mathcal{P}$ is the set of all points of $\mathrm{AG}(n,q)$, $\mathcal{B}$ is the set of all affine lines which intersect Π_∞ in a point of $\mathcal{K}_\infty$, and I is the natural incidence. One can prove the following proposition (see, for example, [64], Section 2.3).

Proposition 1.4.12 *A linear representation $T^*_{n-1}(\mathcal{K}_\infty)$ is connected if and only if the set $\mathcal{K}_\infty$ spans Π_∞.*

It is easy to see that for an anti-flag $\{p, L\}$ of $T^*_{n-1}(\mathcal{K}_\infty)$ the incidence number $\alpha(p, L) = k - 1$, where k is the number of points of $\mathcal{K}_\infty$ on the line at infinity of the plane $\langle p, L\rangle$.

Theorem 1.4.13 (Delsarte [43]) *The point graph of a linear representation $T^*_{n-1}(\mathcal{K}_\infty)$ is strongly regular if and only if the set $\mathcal{K}_\infty$ has two intersection numbers with respect to hyperplanes of Π_∞.*

1.4.5 Affine GQ and (dual) partial quadrangles

The affine generalized quadrangles were completely classified by Thas [78]. Apart from some trivial examples, there are three infinite classes of generalized quadrangles fully embedded in $\mathrm{AG}(3,q)$ and five sporadic examples of affine generalized quadrangles.

There is no complete classification of affine partial quadrangles or affine dual partial quadrangles. Debroey and Thas [40] proved that no nontrivial proper partial quadrangles can be fully embedded in $\mathrm{AG}(2,q)$ or $\mathrm{AG}(3,q)$. The affine partial quadrangles that are linear representations are almost completely classified by Calderbank [16] (see also [31] for a good overview of this and related results). The only infinite class of linear representations that are partial quadrangles is $T^*_3(\mathcal{O}_\infty)$ with $\mathcal{O}_\infty$ an ovoid.

The following is an example of an affine partial quadrangle that is not a linear representation. Consider the affine space $\mathrm{AG}(5,q)$ and its projective completion $\mathrm{PG}(5,q)$. Consider the classical generalized quadrangle $\mathsf{Q}(5,\mathsf{q})$ fully embedded in $\mathrm{PG}(5,q)$, in such a way that Π_∞ is a tangent hyperplane to the nonsingular elliptic quadric $\mathrm{Q}^-(5,q)$. Then the incidence structure of points and lines of $\mathsf{Q}(5,\mathsf{q})$, not contained in Π_∞, is a partial quadrangle $\mathrm{PQ}(q-1, q^2, q^2-q)$ fully embedded in $\mathrm{AG}(5,q)$.

De Clerck and Delanote [27] proved that an affine dual partial quadrangle can't be a linear representation.

1.4.6 Affine partial geometries

The affine partial geometries were completely classified by Thas [78]. Examples are the following.

1. The design of all points and all lines of $\mathrm{AG}(n,q)$ is a $\mathrm{pg}(q-1, (q^n-q)/(q-1), q)$ (or, equivalently, a $2-(q^n,q,1)$-design) fully embedded in $\mathrm{AG}(n,q)$.

2. Let $\mathcal{P}$ be a set of $t+2$ points of $\mathrm{AG}(n,2)$ not in a hyperplane (so $t \in \{n-1,\dots,2^n-2\}$), and let $\mathcal{B}$ be the set of all pairs of points of $\mathcal{P}$. Then $(\mathcal{P},\mathcal{B},\in)$ is a $\mathrm{pg}(1,t,2)$ (or, equivalently, a $2-(t+2,2,1)$ design) fully embedded in $\mathrm{AG}(n,2)$.

3. If $\mathcal{P}$ is the set of all points of $\mathrm{AG}(2,q)$, $\mathcal{B}$ is the union of $\alpha+1 \geq 2$ parallel classes of lines of $\mathrm{AG}(2,q)$ and I is the natural incidence, then $\mathcal{S} = (\mathcal{P},\mathcal{B},\mathrm{I})$ is a $\mathrm{pg}(q-1,\alpha,\alpha)$ (or, equivalently, a net of order q and degree $\alpha+1$) fully embedded in $\mathrm{AG}(2,q)$. We say that $\mathcal{S}$ is a *planar net*.

4. Let $\mathcal{B}$ be a set of 2^h+1 lines in $\mathrm{AG}(2,2^h)$ such that $\mathcal{B} \cup \{\Pi_\infty\}$ forms a hyperoval in the dual of the plane $\mathrm{PG}(2,2^h)$. Then $\mathcal{S} = (\mathcal{P},\mathcal{B},\mathrm{I})$ where $\mathcal{P}$ is the set of points of $\mathrm{AG}(2,2^h)$ on the lines of $\mathcal{B}$ and I is the natural incidence, is a $\mathrm{pg}(2^h-1,1,2)$ (or, equivalently, the dual of a $2-(q+1,2,1)$ design) fully embedded in $\mathrm{AG}(2,2^h)$. We call $\mathcal{S}$ a *dual oval.*

5. Consider $\mathrm{AG}(n,q)$. Let $\mathcal{K}_\infty$ be the complement of a hyperplane in Π_∞. Then $T^*_{n-1}(\mathcal{K}_\infty)$ is a $\mathrm{pg}(q-1,q^{n-1}-1,q-1)$ (or, equivalently, the dual of a net of order q^{n-1} and degree q) fully embedded in $\mathrm{AG}(n,q)$.

6. Consider $\mathrm{AG}(3,q)$. Let $\mathcal{M}_\infty$ be a maximal arc of degree $d \geq 2$ in Π_∞. Then $T^*_2(\mathcal{M}_\infty)$ is a $\mathrm{pg}(q-1,(q+1)(d-1),d-1)$ fully embedded in $\mathrm{AG}(3,q)$ [77]. It is a proper partial geometry when $\mathcal{M}_\infty$ is a nontrivial maximal arc, so in this case $q=2^h$.

Theorem 1.4.14 (Thas [78]) *If $\mathcal{S}$ is a partial geometry fully embedded in $\mathrm{AG}(n,q)$, then it is a generalized quadrangle or it occurs in the list above. In particular if $n=2$ and $q>2$ we have one of the following cases.*

1. *$\mathcal{S}$ is the design of all points and all lines of* $\mathrm{AG}(2, q)$.

2. *$\mathcal{S}$ is a planar net.*

3. *$q = 2^h$ and $\mathcal{S}$ is a dual oval.*

1.4.7 Affine (dual) semipartial geometries and $(0, \alpha)$-geometries

In this section we mention some results about affine (dual) semipartial geometries and $(0, \alpha)$-geometries with $\alpha > 1$. However the most recent results in this area are discussed in Chapter 4, where we also summarize our own results in this area.

Affine $(0, 1)$-geometries are not studied for the same reason why projective $(0, 1)$-geometries are not studied: because the embedding in $\mathrm{AG}(n, q)$ does not induce much structure on a $(0, 1)$-geometry. For instance, if a plane π of $\mathrm{AG}(n, q)$ contains two intersecting lines of a $(0, 1)$-geometry $\mathcal{S}$ fully embedded in $\mathrm{AG}(n, q)$, then π may contain a priori any number $\gamma \in \{2, \ldots, q + 1\}$ of lines of $\mathcal{S}$ centered in a point. Thus we obtain far less information than in the case $\alpha > 1$, as we will see in Lemma 4.1.3.

For the sake of completeness, we give the explicit conditions for a linear representation to be a $(0, \alpha)$-geometry or a semipartial geometry.

Proposition 1.4.15 *Consider* $\mathrm{AG}(n, q)$*, and let $\mathcal{K}_\infty$ be a set of points in Π_∞. Then $T^*_{n-1}(\mathcal{K}_\infty)$ is a $(0, \alpha)$-geometry if and only if $\mathcal{K}_\infty$ spans Π_∞ and $\mathcal{K}_\infty$ is a set of type $(0, 1, \alpha + 1)$ in Π_∞. If $T^*_{n-1}(\mathcal{K}_\infty)$ is a $(0, \alpha)$-geometry, then the following are equivalent.*

1. *$T^*_{n-1}(\mathcal{K}_\infty)$ is a semipartial geometry* $\mathrm{spg}(q - 1, |\mathcal{K}_\infty| - 1, \alpha, \mu)$.

2. *Every point of Π_∞, not in $\mathcal{K}_\infty$, is on precisely $\mu/(\alpha(\alpha + 1))$ lines which intersect $\mathcal{K}_\infty$ in $\alpha + 1$ points.*

3. *The set $\mathcal{K}_\infty$ has two intersection numbers with respect to hyperplanes of Π_∞.*

Proof. By Theorem 1.4.13, the point graph of $T^*_{n-1}(\mathcal{K}_\infty)$ is strongly regular if and only the set $\mathcal{K}_\infty$ has two intersection numbers with respect to hyperplanes of Π_∞. The rest of the proof is straightforward. □

The following list contains all the known examples of proper semipartial geometries with $\alpha > 1$ fully embedded in $\mathrm{AG}(n, q)$.

1. Consider $\mathrm{AG}(3,q^2)$, and let $\mathcal{U}_\infty$ be a unital in Π_∞. Then $T_2^*(\mathcal{U}_\infty)$ is an $\mathrm{spg}(q^2-1, q^3, q, q^2(q^2-1))$ fully embedded in $\mathrm{AG}(3,q^2)$ [41].

2. Consider $\mathrm{AG}(n,q^2)$, and let $\mathcal{B}_\infty$ be the point set of a Baer subspace of Π_∞. Then $T_{n-1}^*(\mathcal{B}_\infty)$ is an $\mathrm{spg}(q^2-1, (q^n-q)/(q-1), q, q(q+1))$ fully embedded in $\mathrm{AG}(n,q^2)$ [41, 31].

3. Consider $\mathrm{AG}(4,2^h)$ and consider the set $\mathcal{R}_4^-$ in $\mathrm{PG}(4,2^h)$ (see section 1.3.8) in such a way that Π_∞ is the unique hyperplane of $\mathrm{PG}(4,2^h)$ completely contained in $\mathcal{R}_4^-$. Let $\mathcal{P}$ be the set of affine points of $\mathcal{R}_4^-$, let $\mathcal{B}$ be the set of affine lines completely contained in $\mathcal{R}_4^-$, and let I be the natural incidence. Then $\mathrm{TQ}(4,2^h) = (\mathcal{P},\mathcal{B},\mathrm{I})$ is an $\mathrm{spg}(2^h-1, 2^{2h}, 2, 2^{h+1}(2^h-1))$ fully embedded in $\mathrm{AG}(4,2^h)$ [53].

Debroey and Thas [40] classified the semipartial geometries fully embedded in $\mathrm{AG}(2,q)$ and $\mathrm{AG}(3,q)$.

Theorem 1.4.16 (Debroey, Thas [40]) *If $\mathcal{S}$ is a proper semipartial geometry with $\alpha > 1$ fully embedded in $\mathrm{AG}(n,q)$ where $n \leq 3$, then we have one of the following cases.*

1. *$n = 3$, q is a square and $\mathcal{S} \simeq T_2^*(\mathcal{U}_\infty)$ with $\mathcal{U}_\infty$ a unital of Π_∞.*

2. *$n = 3$, q is a square and $\mathcal{S} \simeq T_2^*(\mathcal{B}_\infty)$ with $\mathcal{B}_\infty$ a Baer subplane of Π_∞.*

The following theorem characterizes $\mathrm{TQ}(4,q)$ by its parameters and its full embedding in $\mathrm{AG}(4,q)$.

Theorem 1.4.17 (Brown, De Clerck, Delanote [11]) *If $\mathcal{S}$ is a semipartial geometry $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ fully embedded in $\mathrm{AG}(4,q)$, then $q = 2^h$ and $\mathcal{S} \simeq \mathrm{TQ}(4,q)$.*

The full embedding in $\mathrm{AG}(n,q)$ of dual semipartial geometries with $\alpha > 1$ is solved by the following theorem.

Theorem 1.4.18 (De Clerck, Delanote [27]) *If $\mathcal{S}$ is a dual semipartial geometry with $\alpha > 1$ fully embedded in $\mathrm{AG}(n,q)$, then $\mathcal{S}$ is a partial geometry.*

We conclude this section with some examples of affine $(0,\alpha)$-geometries.

1. Consider $\mathrm{AG}(n,q)$ and a set $\mathcal{K}_\infty$ of type $(0,1,k)$ in Π_∞, with $|\mathcal{K}_\infty| > 1$ and $k > 1$, which spans Π_∞. Then $T_{n-1}^*(\mathcal{K}_\infty)$ is a $(0,\alpha)$-geometry with $s = q-1$, $t = |\mathcal{K}_\infty| - 1$ and $\alpha = k-1$, fully embedded in $\mathrm{AG}(n,q)$.

An example of a set of type $(0,1,k)$ in $\mathrm{PG}(n,q)$ is the point set of a projective subspace $\mathrm{PG}(n,q')$ of $\mathrm{PG}(n,q)$, where $\mathrm{GF}(q')$ is a subfield of $\mathrm{GF}(q)$. Here $k = q' + 1$. One can also construct sets of type $(0,1,k)$ in the following ways. If $\mathcal{K}$ is a set of type $(0,1,k)$ in $\mathrm{PG}(n,q)$ and $\mathrm{PG}(n,q)$ is embedded in $\mathrm{PG}(n,q^m)$, then $\mathcal{K}$ is a set of type $(0,1,k)$ in $\mathrm{PG}(n,q^m)$. If $\mathcal{K}$ is a set of type $(0,1,k)$ in $\mathrm{PG}(n,q)$, $n \geq 2$, and $p \notin \mathcal{K}$ is a point such that every plane through p contains at most k points of $\mathcal{K}$, then the projection of $\mathcal{K}$ from p onto a hyperplane $\mathrm{PG}(n-1,q)$ of $\mathrm{PG}(n,q)$, not containing p, yields a set of type $(0,1,k)$ in $\mathrm{PG}(n-1,q)$.

2. Consider $\mathrm{AG}(3,2^h)$ and consider the set $\mathcal{R}_3$ in $\mathrm{PG}(3,2^h)$ (see Section 1.3.8) in such a way that Π_∞ is the unique hyperplane of $\mathrm{PG}(3,2^h)$ completely contained in $\mathcal{R}_3$. Let $\mathcal{P}$ be the set of affine points of $\mathcal{R}_3$, let $\mathcal{B}$ be the set of affine lines completely contained in $\mathcal{R}_3$, and let I be the natural incidence. Then $\mathrm{HT} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a $(0,\alpha)$-geometry with $s = q-1$, $t = q$ and $\alpha = 2$, fully embedded in $\mathrm{AG}(3,2^h)$ [53]. This geometry is not a semipartial geometry.

Chapter 2

On $(0, \alpha)$-geometries fully embedded in PG$(3, q)$ and $(0, \alpha)$-sets on the Klein quadric

Theorem 1.4.4 classifies all $(0, \alpha)$-geometries, $\alpha > 1$, fully embedded in PG(n, q), $n \geq 4$, $q > 2$. In this chapter, we consider $(0, \alpha)$-geometries, $\alpha > 1$, fully embedded in PG$(3, q)$. In Section 2.2, the Plücker correspondence is used to transform the line set of a $(0, \alpha)$-geometry, $\alpha > 1$, fully embedded in PG$(3, q)$, $q > 2$, into a set of points on the Klein quadric, a so-called $(0, \alpha)$-set. In Section 2.3, we discuss the link between caps and $(0, 2)$-sets on the Klein quadric. Next, in Section 2.4, we give the explicit construction of some caps on the Klein quadric $\mathrm{Q}^+(5, q)$, q even, which are due to Ebert, Metsch and Szőnyi [45]. De Clerck and Durante (private communication) observed that these caps yield previously unknown $(0, 2)$-geometries fully embedded in PG$(3, q)$, q even. In Section 2.5, we investigate the structure of the caps of Ebert, Metsch and Szőnyi. Finally, in Section 2.6, we show that, by slightly modifying the construction, sets of points on the Klein quadric may be found which correspond to new $(0, \alpha)$-geometries, $\alpha > 1$, fully embedded in PG$(3, q)$, q even.

The results of this chapter are joint work with De Clerck and Durante, and are published in [25].

2.1 Preliminaries

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a $(0, \alpha)$-geometry, $\alpha > 1$, fully embedded in PG$(3, q)$, and let π be a plane of PG$(3, q)$. By Lemma 1.4.7, π is of one of the following types.

Type (a). $\mathcal{S}_\pi$ consists of a $\mathrm{pg}(q,\alpha-1,\alpha)$ (that is, a dual maximal arc of degree α) and possibly some isolated points.

Type (b). $\mathcal{S}_\pi$ consists of exactly one line and possibly some isolated points.

Type (c). $\mathcal{S}_\pi$ only consists of some isolated points.

In Section 1.4.3, it was shown that every plane of type (a) contains a constant number m of isolated points.

Proposition 2.1.1 (De Clerck, Thas [30]) *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in $\mathrm{PG}(3,2)$. Then either $\alpha = 3$ and $\mathcal{S}$ is the design of all points and all lines of $\mathrm{PG}(3,2)$, or $\alpha = 2$ and $\mathcal{S} \simeq \overline{W(3,2)}$, $\mathcal{S} \simeq H_2^3$ or $\mathcal{S} \simeq \mathrm{NQ}^-(3,2)$.*

By Proposition 2.1.1, we may assume from now on that $q > 2$. The following result is an immediate consequence of Theorems 1.4.8 and 1.4.9, due to De Clerck and Thas [30].

Theorem 2.1.2 *Let $\mathcal{B}$ be a nonempty set of lines of lines of $\mathrm{PG}(3,q)$, $q > 2$. Let $\mathcal{S} = (\mathcal{P},\mathcal{B},\mathrm{I})$, where $\mathcal{P}$ is the set of all points of $\mathrm{PG}(3,q)$ on the lines of $\mathcal{B}$, and I is the natural incidence. Then $\mathcal{S}$ is a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$ if and only if every pencil of $\mathrm{PG}(3,q)$ contains either 0 or α lines of $\mathcal{B}$. Hence there are no planes of type* (b).

Proof. Suppose that $\mathcal{S}$ is a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$. Then either $m \neq 0$, in which case Theorem 1.4.9 states that there are no planes of type (b), or $m = 0$, in which case Theorem 1.4.8 applies and either $\mathcal{S}$ is the design of all points and lines of $\mathrm{PG}(3,q)$, or $\mathcal{S} \simeq H_q^3$. So, in any case, there are no planes of type (b). It follows that every plane is of type (a) or (c), hence every pencil of $\mathrm{PG}(3,q)$ contains either 0 or α lines of $\mathcal{B}$.

Suppose that every pencil of $\mathrm{PG}(3,q)$ contains either 0 or $\alpha > 1$ lines of $\mathcal{B}$. Let $\{p,L\}$ be an anti-flag of $\mathcal{S}$. Let π be the plane $\langle p,L\rangle$. Since the pencil of lines of $\mathrm{PG}(3,q)$ through p, contained in π, contains either 0 or α lines of $\mathcal{B}$, $\alpha(p,L)$ is either 0 or α. So $\mathcal{S}$ satisfies property (zag2) (see Section 1.2.2).

Let p_1 and p_2 be noncollinear points of $\mathcal{S}$. Let $L = \langle p_1,p_2\rangle$, let π be a plane not containing p_1 or p_2, and let $p = L\cap\pi$. Every pencil of $\mathrm{PG}(3,q)$ contains either 0 or α lines of $\mathcal{B}$, so the lines of $\mathcal{S}$ through p_i intersect π in a maximal arc $\mathcal{M}_i$ of degree α, $i = 1,2$. Since $p \notin \mathcal{M}_i$, there are $q+1-q/\alpha$ lines of π through p which intersect $\mathcal{M}_i$ in α points, $i = 1,2$. Since $\alpha > 1$, there is a line $L' \subseteq \pi$ through p which intersects $\mathcal{M}_1$ and $\mathcal{M}_2$ in α points.

Hence the plane $\pi' = \langle L, L' \rangle$ contains α lines of $\mathcal{S}$ through p_i, $i = 1, 2$. So there is a point of $\mathcal{S}$ collinear to both p_1 and p_2. Hence $\mathcal{S}$ is connected, so $\mathcal{S}$ satisfies property (zag1'). By Lemma 1.2.1, $\mathcal{S}$ is a $(0, \alpha)$-geometry. Clearly, $\mathcal{S}$ is fully embedded in $\mathrm{PG}(3, q)$. $\square$

Let $\mathcal{S}$ be a $(0, \alpha)$-geometry, $\alpha > 1$, fully embedded in $\mathrm{PG}(3, q)$, $q > 2$. Since every plane π of type (a) contains a constant number m of isolated points, the number d of points of π which do not belong to $\mathcal{S}$, is also a constant. In fact $d = q(q + 1 - \alpha)/\alpha - m$. The number d is called the *deficiency* of the $(0, \alpha)$-geometry $\mathcal{S}$.

Proposition 2.1.3 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a* $(0, \alpha)$*-geometry,* $\alpha > 1$*, fully embedded in* $\mathrm{PG}(3, q)$*,* $q > 2$*. Then the order of* $\mathcal{S}$ *is* $(q, (q + 1)(\alpha - 1))$*,* $|\,\mathcal{P}\,| = (q+1)(q^2+1-d)$*,* $|\,\mathcal{B}\,| = (q\alpha - q + \alpha)(q^2+1-d)$*, and the number of planes of type* (a) *is* $(q+1)(q^2+1-d)$*.*

Proof. By Theorem 2.1.2, every pencil of $\mathrm{PG}(3, q)$ contains either 0 or α lines of $\mathcal{S}$. Hence the lines of $\mathcal{S}$ through a point p of $\mathcal{S}$ intersect a plane, not containing p, in a maximal arc of degree α. Hence the order of $\mathcal{S}$ is $(q, (q + 1)(\alpha - 1))$.

Consider a line L of $\mathcal{S}$. Every plane through L is of type (a) and contains $m = q(q + 1 - \alpha)/\alpha - d$ isolated points. Hence $|\,\mathcal{P}\,| = (q + 1)(q^2 + 1 - d)$. Counting the flags of $\mathcal{S}$ yields $|\,\mathcal{B}\,| = (q\alpha - q + \alpha)(q^2 + 1 - d)$. Counting the number of pairs (L, π), where π is a plane of type (a) and L a line of $\mathcal{S}$ in π, yields the number of planes of type (a). $\square$

2.2 The Plücker correspondence

Let $\mathcal{B}$ be a set of lines of $\mathrm{PG}(3, q)$, and let $\mathcal{K}$ be the image of $\mathcal{B}$ under the Plücker correspondence κ. The set of pencils of $\mathrm{PG}(3, q)$ corresponds via κ to the set of lines of $\mathrm{PG}(5, q)$ which are contained in the Klein quadric $\mathrm{Q}^+(5, q)$. Hence every pencil of $\mathrm{PG}(3, q)$ contains either 0 or α ($1 \le \alpha \le q + 1$) lines of $\mathcal{B}$ if and only if $\mathcal{K}$ is a set of type $(0, \alpha)$ with respect to the lines of $\mathrm{PG}(5, q)$ which are contained in $\mathrm{Q}^+(5, q)$. A set of points on the Klein quadric $\mathrm{Q}^+(5, q)$ which has the latter property is called a $(0, \alpha)$*-set on* $\mathrm{Q}^+(5, q)$. By Theorem 2.1.2, the following objects are equivalent whenever $q > 2$ and $\alpha > 1$.

1. A $(0, \alpha)$-geometry fully embedded in $\mathrm{PG}(3, q)$.

2. A set $\mathcal{B}$ of lines of $\mathrm{PG}(3, q)$ such that every pencil of $\mathrm{PG}(3, q)$ contains either 0 or α lines of $\mathcal{B}$.

3. A $(0,\alpha)$-set on the Klein quadric $\mathrm{Q}^+(5,q)$.

Let $\mathcal{K}$ be a $(0,\alpha)$-set, $\alpha > 1$, on the Klein quadric $\mathrm{Q}^+(5,q)$, $q > 2$, and let $\mathcal{S}$ be the corresponding $(0,\alpha)$-geometry fully embedded in $\mathrm{PG}(3,q)$. Then the deficiency d of $\mathcal{S}$ is also called the *deficiency of the* $(0,\alpha)$*-set* $\mathcal{K}$. By Proposition 2.1.3, $|\mathcal{K}| = (q\alpha - q + \alpha)(q^2 + 1 - d)$.

The following list contains the examples of $(0,\alpha)$-geometries $\mathcal{S}$, $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$, $q > 2$, that appear in [81]. For every example, we describe the corresponding $(0,\alpha)$-set $\mathcal{K}$ on the Klein quadric.

1. $\mathcal{S}$ is the design of all points and all lines of $\mathrm{PG}(3,q)$. Here $\alpha = q+1$, $d = 0$, and $\mathcal{K}$ is the set of all points of $\mathrm{Q}^+(5,q)$.

2. $\mathcal{S} = \overline{W(3,q)}$. Here $\alpha = q$, and $d = 0$. The line set of $\mathcal{S}$ consists of the lines which are not totally isotropic with respect to a symplectic polarity φ of $\mathrm{PG}(3,q)$. The set of totally isotropic lines with respect to φ corresponds to the set of points of $\mathrm{Q}^+(5,q)$ in a hyperplane U of $\mathrm{PG}(5,q)$, which is not tangent to $\mathrm{Q}^+(5,q)$. Hence $\mathcal{K}$ is the set of points of $\mathrm{Q}^+(5,q)$, not in U.

3. $\mathcal{S} = H_q^3$. Here $\alpha = q$, and $d = 1$. The line set of $\mathcal{S}$ consists of the lines which are skew to a line L of $\mathrm{PG}(3,q)$. The set of lines of $\mathrm{PG}(3,q)$ which are not skew to L, corresponds via κ to the set of points of $\mathrm{Q}^+(5,q)$ which are collinear to the point $p = L^\kappa$. A point of $\mathrm{Q}^+(5,q)$ is collinear to p if and only if it is in the tangent hyperplane U to $\mathrm{Q}^+(5,q)$ at p. Hence $\mathcal{K}$ is the set of points of $\mathrm{Q}^+(5,q)$, not in U.

4. $\mathcal{S} = \mathrm{NQ}^+(3,2^h)$. Here $q = 2^h$, $\alpha = q/2$, and $d = q+1$. The line set of $\mathcal{S}$ consists of the lines which are skew to a nonsingular hyperbolic quadric $\mathrm{Q}^+(3,q)$ in $\mathrm{PG}(3,q)$. Let R_1 and R_2 be the reguli of $\mathrm{Q}^+(3,q)$. Then R_i corresponds via κ to a nondegenerate conic $C_i = \mathrm{Q}^+(5,q) \cap \pi_i$, π_i a plane of $\mathrm{PG}(5,q)$, $i = 1,2$. A line of $\mathrm{PG}(3,q)$ belongs to $\mathcal{S}$ if and only if it is skew to all lines of R_1. So $\mathcal{K}$ is the set of points of $\mathrm{Q}^+(5,q)$ which are not collinear to any point of C_1.

It was conjectured in [30] that the design of all points and all lines of $\mathrm{PG}(3,q)$, H_q^3, $\overline{W(3,q)}$ and $\mathrm{NQ}^+(3,q)$ are the only $(0,\alpha)$-geometries, $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$, $q > 2$. This conjecture is false as will be clear from Sections 2.4 and 2.6. However, when one assumes additionally that $m \leq 2$, then Theorems 1.4.8, 1.4.10 and 1.4.11 prove the conjecture. Also, the conjecture is valid if we restrict ourselves to the q odd case. We state this result in terms of $(0,\alpha)$-sets on the Klein quadric.

Theorem 2.2.1 *Let $\mathcal{K}$ be a $(0,\alpha)$-set, $\alpha > 1$, on the Klein quadric $\mathrm{Q}^+(5,q)$, q odd. Then one of the following possibilities occurs.*

1. *$\alpha = q+1$ and $\mathcal{K}$ is the set of all points of $\mathrm{Q}^+(5,q)$.*

2. *$\alpha = q$ and $\mathcal{K}$ is the set of points of $\mathrm{Q}^+(5,q)$, not in a given hyperplane of $\mathrm{PG}(5,q)$.*

Proof. Let π be a generator of $\mathrm{Q}^+(5,q)$ such that $\mathcal{M} = \mathcal{K} \cap \pi$ is not empty. Since every line of π intersects $\mathcal{M}$ in either 0 or α points, $\mathcal{M}$ is a maximal arc of degree α in the plane π. By Theorem 1.3.2, α is either q or $q+1$.

If $\alpha = q+1$, then clearly $\mathcal{K}$ is the set of all points of $\mathrm{Q}^+(5,q)$. Assume that $\alpha = q$, and let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be the $(0,q)$-geometry fully embedded in $\mathrm{PG}(3,q)$ which corresponds to $\mathcal{K}$. Then $m+d=1$, so either $m=0$ and $d=1$, or $m=1$ and $d=0$. If $m=0$, then, by Theorem 1.4.8, $\mathcal{S} \simeq H_q^3$. If $m=1$, then, by Theorem 1.4.10, $\mathcal{S} \simeq \overline{W(3,q)}$. $\square$

Note that when $\alpha = q$ in Theorem 2.2.1, though we did not use it, the classification of copolar spaces by Hall [48] applies.

The second part of the proof of Theorem 2.2.1 is also valid for q even. So the conclusion of Theorem 2.2.1 holds, when, instead of assuming that q is odd, one assumes that $\alpha \geq q$.

2.3 Caps and $(0,2)$-sets on the Klein quadric

Since it is very hard to find in general the exact value of $m_2(n,q)$, the size of the largest cap in $\mathrm{PG}(n,q)$, and to construct caps of this size, it is interesting to investigate and construct caps that are contained in certain varieties, such as quadrics. For example, caps embedded in the Klein quadric $\mathrm{Q}^+(5,q)$ have been studied by various authors.

Since the intersection of a cap $\mathcal{K}$ in $\mathrm{Q}^+(5,q)$ with a generator π of $\mathrm{Q}^+(5,q)$, is a cap in the projective plane π, $|\,\mathcal{K} \cap \pi\,| \leq m_2(2,q)$. An easy counting argument shows that $|\,\mathcal{K}\,| \leq m_2(2,q)(q^2+1)$. So, if q is odd, the (theoretically) maximum size of $\mathcal{K}$ is $(q+1)(q^2+1)$, and if q is even, it is $(q+2)(q^2+1)$.

In the q odd case, Glynn [47] constructs an example of a cap in $\mathrm{Q}^+(5,q)$ which has the maximum size $(q+1)(q^2+1)$. It is the intersection of $\mathrm{Q}^+(5,q)$ with a singular quadric with vertex a line L and base a nonsingular elliptic quadric in a 3-space skew to L. In [74], Storme proves that every $(q+1)(q^2+1)$-cap in $\mathrm{Q}^+(5,q)$, with q odd and sufficiently large, is the intersection of $\mathrm{Q}^+(5,q)$ with another quadric. Metsch [61] shows that every cap of size $k > q^3+q^2+2$ in $\mathrm{Q}^+(5,q)$, with q odd and sufficiently large, can be extended to a $(q+1)(q^2+1)$-cap in $\mathrm{Q}^+(5,q)$.

When q is even, a cap in $\mathrm{Q}^+(5,q)$ of the maximum size $(q+2)(q^2+1)$ is only known to exist for $q=2$ (it is the complement in $\mathrm{Q}^+(5,2)$ of a hyperplane which is not tangent to $\mathrm{Q}^+(5,2)$).

Proposition 2.3.1 *Let* $\mathcal{K}$ *be a set of points on* $\mathrm{Q}^+(5,q)$, $q=2^h$. *Then* $\mathcal{K}$ *is a cap of size* $(q+2)(q^2+1)$ *if and only if* $\mathcal{K}$ *is a* $(0,2)$*-set on* $\mathrm{Q}^+(5,q)$ *of deficiency* 0.

Proof. Suppose that $\mathcal{K}$ is a cap of size $(q+2)(q^2+1)$. Since every generator of $\mathrm{Q}^+(5,q)$ contains at most $m_2(2,q)=q+2$ points of $\mathcal{K}$, a simple counting argument shows that every generator of $\mathrm{Q}^+(5,q)$ contains $q+2$ points of $\mathcal{K}$. In other words, for every generator π of $\mathrm{Q}^+(5,q)$, $\mathrm{Q}^+(5,q)\cap\pi$ is a hyperoval. So, every line on $\mathrm{Q}^+(5,q)$ intersects $\mathcal{K}$ in either 0 or 2 points. Since $|\,\mathcal{K}\,|=(q+2)(q^2+1)$, $d=0$.

Clearly, if $\mathcal{K}$ is a $(0,2)$-set on $\mathrm{Q}^+(5,q)$ of deficiency 0, then $\mathcal{K}$ is a cap of size $(q+2)(q^2+1)$. $\square$

The largest known cap in $\mathrm{Q}^+(5,q)$, q even and $q>2$, is a cap of size $q^3+2q^2+1=(q+2)(q^2+1)-q-1$, constructed by Ebert, Metsch and Szőnyi [45]. We give the construction in Section 2.4. This cap is maximal in $\mathrm{Q}^+(5,q)$, that is, it cannot be extended by adding a point of $\mathrm{Q}^+(5,q)$.

In [45], it is shown that a cap of size $k>(q+2)(q^2+1)-q-1$ in $\mathrm{Q}^+(5,q)$, q even, can always be extended to a cap of size $(q+2)(q^2+1)$ in $\mathrm{Q}^+(5,q)$. Furthermore, the structure of maximal caps of size $(q+2)(q^2+1)-q-1$ on $\mathrm{Q}^+(5,q)$, q even, is investigated in [45]. The main result may be translated as follows in terms of $(0,2)$-sets on $\mathrm{Q}^+(5,q)$.

Theorem 2.3.2 (Ebert, Metsch, Szőnyi [45]) *Let* $\mathcal{K}$ *be a maximal cap of size* $(q+2)(q^2+1)-q-1$ *on* $\mathrm{Q}^+(5,q)$, q *even. Then there is a point* $p\in\mathcal{K}$ *which is not collinear to any other point of* $\mathcal{K}$. *The set* $\mathcal{K}\setminus\{p\}$ *is a* $(0,2)$*-set of deficiency* 1 *on* $\mathrm{Q}^+(5,q)$.

The converse to Theorem 2.3.2 is much easier to prove.

Theorem 2.3.3 *Let* $\mathcal{K}$ *be a* $(0,2)$*-set of deficiency* 1 *on* $\mathrm{Q}^+(5,q)$, $q=2^h$, $h>1$. *Then there is a point* $p\in\mathcal{K}$ *such that* $\mathcal{K}\cup\{p\}$ *is a maximal cap of size* $(q+2)(q^2+1)-q-1$ *on* $\mathrm{Q}^+(5,q)$.

Proof. Let $\mathcal{S}$ be the $(0,2)$-geometry fully embedded in $\mathrm{PG}(3,q)$, corresponding to $\mathcal{K}$. By Proposition 2.1.3, there are precisely $q+1$ points of

PG$(3,q)$ which do not belong to $\mathcal{S}$, and precisely $q+1$ planes of PG$(3,q)$ which are of type (c). Since $d=1$, any plane which contains at least 2 points not belonging to $\mathcal{S}$, is a plane of type (c). It follows that the $q+1$ points not belonging to $\mathcal{S}$ lie on a line L of PG$(3,q)$ and that the $q+1$ planes of type (c) are exactly the planes containing L.

Clearly no line of $\mathcal{S}$ is concurrent with L. Hence $\mathcal{K}\cup\{p\}$, where p is the point of $\mathrm{Q}^+(5,q)$ which corresponds to L, is a cap of size

$$q^2(q+2)+1=(q+2)(q^2+1)-q-1$$

on $\mathrm{Q}^+(5,q)$.

Suppose that there is a point $p'\in\mathrm{Q}^+(5,q)\setminus(\mathcal{K}\cup\{p\})$ such that $\mathcal{K}\cup\{p,p'\}$ is a cap. Let L' be the line of PG$(3,q)$ which corresponds to p'. Let r be a point of $\mathcal{S}$ on L'. Then the lines of $\mathcal{S}$ through r intersect a plane π not containing r in a hyperoval H. Since L' is not a line of $\mathcal{S}$, $r'=L'\cap\pi\notin H$. Let M be a line of π through r' which intersects H in 2 points. Then the pencil of lines through r in the plane $\langle L',M\rangle$ contains L' and 2 lines of $\mathcal{S}$. But this contradicts the fact that $\mathcal{K}\cup\{p,p'\}$ is a cap. So $\mathcal{K}\cup\{p\}$ is a maximal cap of $\mathrm{Q}^+(5,q)$. $\square$

2.4 A construction of caps on the Klein quadric

In this section, we give a construction of caps on the Klein quadric, due to Ebert, Metsch and Szőnyi [45].

Consider the Klein quadric $\mathrm{Q}^+(5,q)$ in PG$(5,q)$, $q=2^h$. Since q is even, the polarity β which is associated with $\mathrm{Q}^+(5,q)$, is a symplectic polarity. Let V be a 3-space of PG$(5,q)$, such that $E=\mathrm{Q}^+(5,q)\cap V$ is a nonsingular elliptic quadric. Let L be the line V^β. Then L is an external line of $\mathrm{Q}^+(5,q)$.

Let O be an ovoid of V which has the same set of tangent lines as the elliptic quadric E. For every point $p\in O\setminus E$, the plane $\pi=\langle p,L\rangle$ intersects $\mathrm{Q}^+(5,q)$ in a nondegenerate conic with nucleus p, whereas, for every point $p\in O\cap E$, the plane π intersects $\mathrm{Q}^+(5,q)$ in the point p only. Let

$$\mathcal{K}=\bigcup_{p\in O\setminus E}(\mathrm{Q}^+(5,q)\cap\pi)\cup(E\setminus O),$$

and let $\mathcal{K}'=\mathcal{K}\cup(E\cap O)$. Then $\mathcal{K}'$ is the intersection of $\mathrm{Q}^+(5,q)$ with the cone with vertex L and base $E\cup O$, and $\mathcal{K}$ is the intersection of $\mathrm{Q}^+(5,q)$ with the cone with vertex L and base the symmetric difference $E\bigtriangleup O$.

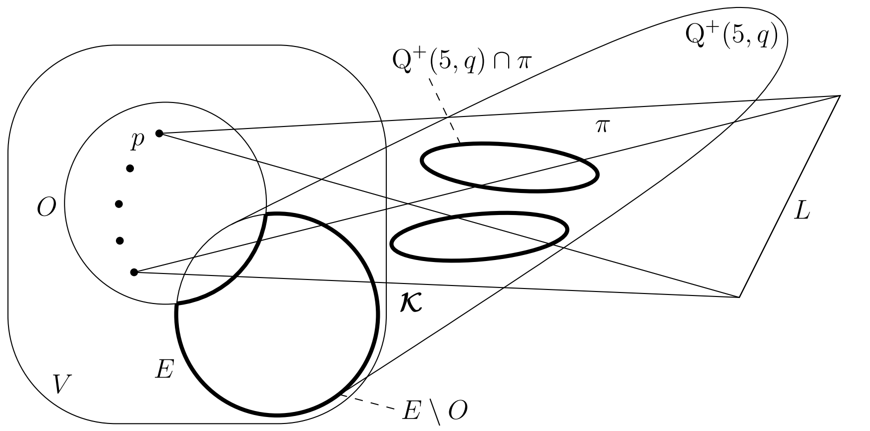

Figure 2.1: Construction of the $(0,2)$-sets $\mathcal{E}_d$ and $\mathcal{T}_{q\pm\sqrt{2q}+1}$.

It is shown in [45] that $\mathcal{K}'$ is a maximal cap of size $q^2+1+(q+1)\,|\,O\setminus E\,|$ on the Klein quadric. The fact that $\mathcal{K}$ is a $(0,2)$-set on the Klein quadric, was observed by De Clerck and Durante (private communication).

Theorem 2.4.1 *The set $\mathcal{K}$ is a $(0,2)$-set of deficiency $d = |\,E\cap O\,|$ on the Klein quadric.*

Proof. Let M be a line of $\mathrm{PG}(5,q)$ which is contained in the Klein quadric $\mathrm{Q}^+(5,q)$. Since L is an external line to $\mathrm{Q}^+(5,q)$, L and M are skew. Since $E = \mathrm{Q}^+(5,q)\cap V$ is an elliptic quadric, there are two possibilities: either M is skew to V, or M intersects V in a point of E.

Assume that M and V are skew. Let p be a point of M, let $\pi = \langle p, L\rangle$, and let $p' = \pi \cap V$. Suppose that $p' \in E$. Then π intersects $\mathrm{Q}^+(5,q)$ only in p', a contradiction since $p \in \mathrm{Q}^+(5,q)\cap\pi$. So $p' \notin E$, and $\mathrm{Q}^+(5,q)\cap\pi$ is a nondegenerate conic with nucleus p'. From the definition of $\mathcal{K}$, it follows that $p\in\mathcal{K}$ if and only if $p' \in O\setminus E$.

Let M' be the projection of the line M from L onto V. Then M' is an external line to E, and the number of points of $\mathcal{K}$ on M equals the number of points of $O\setminus E$ on M'. Since M' is not a tangent line to E, it is not a tangent line to O. So $|\,(O\setminus E)\cap M'\,| = |\,O\cap M'\,| \in \{0,2\}$. Hence M contains either 0 or 2 points of $\mathcal{K}$.

Assume that M intersects V in the point r. As in the previous case, for every point $p\neq r$ of M, the projection p' of p from L onto V is a point, not on E, and $p\in\mathcal{K}$ if and only if $p'\in O\setminus E$. Again, let M' be the projection of the line M from L onto V. Then M' is a tangent line to E at the point r; the number of points of $M \setminus \{r\}$, which belong to $\mathcal{K}$, equals the number of points of $M' \setminus \{r\}$, which are on $O \setminus E$.

Since the line M' is tangent to E, it is tangent to O. Let r' be the point $O \cap M'$. Assume that $r \in E \cap O$. Then $r \notin \mathcal{K}$. Also, $r' = r \notin O \setminus E$, so $(O \setminus E) \cap (M' \setminus \{r\}) = \emptyset$. Hence M does not contain any points of $\mathcal{K}$.

Assume that $r \in E \setminus O$. Then $r \in \mathcal{K}$. Also, $r \neq r'$ and $r' \in O \setminus E$, so $(O \setminus E) \cap (M' \setminus \{r\}) = \{r'\}$. Hence M contains exactly 2 points of $\mathcal{K}$.

We conclude that $\mathcal{K}$ is a $(0, 2)$-set on the Klein quadric. Since

$$\begin{aligned} |\,\mathcal{K}\,| &= |\,E \setminus O\,| + (q+1)\,|\,O \setminus E\,| \\ &= |\,O \setminus E\,| + (q+1)\,|\,O \setminus E\,| \\ &= (q+2)(q^2 + 1 - |\,E \cap O\,|), \end{aligned}$$

$\mathcal{K}$ has deficiency $d = |\,E \cap O\,|$. □

The question remains which deficiencies can occur. Suppose that O is a nonsingular elliptic quadric in V which has the same set of tangent lines as E. Then, by Bruen and Hirschfeld [13], either E and O intersect in a unique point p, and they have a common tangent plane at p, or E and O intersect in $q + 1$ points which form a nondegenerate conic in a plane of V (Types 1(i) and 3(g)(ii) in Table 2 of [13]). We will denote the corresponding $(0, 2)$-set $\mathcal{K}$ by $\mathcal{E}_1$ if $|\,E \cap O\,| = 1$ and by $\mathcal{E}_{q+1}$ if $|\,E \cap O\,| = q + 1$.

Suppose that $q = 2^{2e+1}$, and that O is a Suzuki-Tits ovoid in V, which has the same set of tangent lines as E. Then, by Bagchi and Sastry [1], E and O intersect in $q \pm \sqrt{2q} + 1$ points, and both intersection sizes do occur. (For an alternative proof, see De Smet and Van Maldeghem [37].) We will denote the corresponding $(0, 2)$-set $\mathcal{K}$ by $\mathcal{T}_{q-\sqrt{2q}+1}$ if $|\,E \cap O\,| = q - \sqrt{2q} + 1$ and by $\mathcal{T}_{q+\sqrt{2q}+1}$ if $|\,E \cap O\,| = q + \sqrt{2q} + 1$.

2.5 Unions of elliptic quadrics

Consider a $(0, 2)$-set $\mathcal{K} \in \{\mathcal{E}_1, \mathcal{E}_{q+1}\}$ in $\mathrm{Q}^+(5, q)$, $q = 2^h$. Let U be a hyperplane containing V, and let $p = U \cap L$, where $L = V^\beta$. Then U intersects $\mathrm{Q}^+(5, q)$ in a nonsingular parabolic quadric $\mathrm{Q}(4, q)$ with nucleus p. Since $\mathcal{K}$ is the intersection of $\mathrm{Q}^+(5, q)$ with the cone with vertex L and base the symmetric difference $E \triangle O$, $\mathcal{K} \cap U$ is the intersection of $\mathrm{Q}(4, q)$ with the cone with vertex p and base $E \triangle O$.

The projection of $\mathrm{Q}(4, q)$ from p onto V yields an isomorphism from the classical generalized quadrangle $\mathsf{Q}(4, q)$ to the classical generalized quadrangle $\mathsf{W}(q)$, which consists of the points of V and the lines of V that are tangent to E. This isomorphism induces a bijection from the set of ovoids of $\mathsf{Q}(4,\mathsf{q})$ to the set of ovoids of $\mathsf{W}(\mathsf{q})$. Since the ovoid O has the same set of tangent lines as E, it is an ovoid of the generalized quadrangle $\mathsf{W}(\mathsf{q})$. Hence O is the projection from p onto V of an ovoid $\overline{O}$ of $\mathsf{Q}(4,\mathsf{q})$. So $\mathcal{K}\cap U$ is the symmetric difference $E \triangle \overline{O}$. Since O is a nonsingular elliptic quadric in V, $\overline{O}$ is a nonsingular elliptic quadric in a 3-space $\overline{V} \subseteq U$, that is, $\overline{O} = \mathrm{Q}^+(5,q)\cap\overline{V}$. Let $\overline{\pi}$ be the plane $V\cap\overline{V}$. Then we may also write $\mathcal{K}\cap U = \mathrm{Q}(4,q)\cap(V\cup\overline{V})\setminus\overline{\pi}$.

From the definition of $\mathcal{E}_1$ and $\mathcal{E}_{q+1}$, it follows that there is exactly one plane $\pi \subseteq V$ such that $\pi\cap\mathrm{Q}(4,q) = E\cap O$. Indeed, if $\mathcal{K} = \mathcal{E}_1$, then E and O intersect in exactly one point and π is the unique tangent plane in V to E at this point. If $\mathcal{K} = \mathcal{E}_{q+1}$, then E and O intersect in a nondegenerate conic and π is the ambient plane of this conic. We prove that $\overline{\pi} = \pi$. Since O is the projection of $\overline{O}$ from p on V, $E\cap\overline{O} = E\cap O$. Since $\overline{O} = \overline{V}\cap\mathrm{Q}(4,q)$, $\overline{\pi}\cap\mathrm{Q}(4,q) = V\cap\overline{V}\cap\mathrm{Q}(4,q) = V\cap\overline{O} = E\cap\overline{O} = E\cap O$. So $\overline{\pi}$ is a plane in V such that $\overline{\pi}\cap\mathrm{Q}(4,q) = E\cap O$. This means that $\overline{\pi} = \pi$.

So $\mathcal{K}\cap U$ is the symmetric difference of elliptic quadrics E and $\overline{O}$ on $\mathrm{Q}(4,q)$, with ambient 3-spaces V and $\overline{V}$, intersecting in the plane π. Since this holds for all hyperplanes U containing V, we conclude that there exist 3-spaces $V_0 = V, V_1, \ldots, V_{q+1}$, mutually intersecting in the plane π, such that each intersects $\mathrm{Q}^+(5,q)$ in an elliptic quadric and such that

$$\mathcal{K} = \mathrm{Q}^+(5,q)\cap(V_0\cup V_1\cup\ldots\cup V_{q+1})\setminus\pi.$$

What remains to be verified is the position of the 3-spaces V_i. Consider a plane π' spanned by L and a point $r\in O\setminus E$. One verifies in the respective cases $\mathcal{K} = \mathcal{E}_1$ and $\mathcal{K} = \mathcal{E}_{q+1}$ that $\pi\cap O = \pi\cap E = E\cap O$, so $r\notin\pi$. Hence π' is skew to π. We determine the points of intersection of V_i, $i = 0,\ldots,q+1$, with π'. Clearly $V_0\cap\pi' = V\cap\pi' = r$. Let $i\in\{1,\ldots,q+1\}$ and let $p_i\in L$ be such that $V_i\subseteq\langle p_i, V\rangle$. Let r_i be the unique point of $\mathrm{Q}^+(5,q)$ on the line $\langle p_i, r\rangle$. Since $r\in O\setminus E$, r_i is a point of $\mathcal{K}$, and hence of V_i. But also $r_i\in\pi'$, so $V_i\cap\pi' = r_i$. Repeating this reasoning for all points p_i on L, we see that the 3-spaces V_i, $i = 1,\ldots,q+1$, intersect π' in the points of the nondegenerate conic $C' = \pi'\cap\mathrm{Q}^+(5,q)$, and that V intersects π' in the point r, which is the nucleus of the conic C'. We have now proven the following theorem which completely determines the structure of the $(0,2)$-sets $\mathcal{E}_1$ and $\mathcal{E}_{q+1}$.

Theorem 2.5.1 *Let $\mathcal{K}\in\{\mathcal{E}_1,\mathcal{E}_{q+1}\}$ and let π be the unique plane in V such that $\pi\cap\mathrm{Q}^+(5,q) = E\cap O$. Then*

$$\mathcal{K} = (E\cup O_1\cup\ldots\cup O_{q+1})\setminus\pi,$$

where O_i, $1\le i\le q+1$, is a nonsingular elliptic quadric on $\mathrm{Q}^+(5,q)$ such that its ambient 3-space V_i intersects V in the plane π. In particular the

3-spaces $V_1, \ldots, V_{q+1}$ intersect each plane $\pi' = \langle r, L\rangle$, with $r \in O \setminus E$, in the points of the nondegenerate conic $C' = \pi' \cap \mathrm{Q}^+(5,q)$, while V intersects π' in the nucleus r of the conic C'.

Remark

We can apply the same reasoning to the $(0,2)$-sets $\mathcal{T}_{q\pm\sqrt{2q}+1}$. Let $p_1, \ldots, p_{q+1}$ be the points of L. For every point $p_i \in L$, let $\mathcal{S}_i \cong \mathsf{Q}(4,\mathsf{q})$ be the classical generalized quadrangle formed by the points and lines of the quadric $\mathrm{Q}^+(5,q) \cap \langle p_i, V\rangle$. Then $\mathcal{T}_{q\pm\sqrt{2q}+1}$ can be written as

$$(E \cup O_1 \cup \ldots O_{q+1}) \setminus (E \cap O),$$

where O_i is a Suzuki-Tits ovoid of $\mathcal{S}_i$, $i = 1, \ldots, q+1$, such that O is the projection of O_i from p_i on V. However, this was already shown by Cossidente [22].

2.6 A new construction

The following construction is inspired by Theorem 2.5.1. Let π be a plane of $\mathrm{PG}(5,q)$, $q = 2^h$, which does not contain any line contained in $\mathrm{Q}^+(5,q)$, and let π' be a plane skew to π. Let $\mathcal{D}$ denote the set of points $p \in \pi'$ such that the 3-space $V = \langle p, \pi\rangle$ intersects $\mathrm{Q}^+(5,q)$ in a nonsingular elliptic quadric. Assume that A is a maximal arc of degree $\alpha > 1$ in π' such that $A \subseteq \mathcal{D}$. Then we define

$$\mathcal{M}^\alpha(A) = \bigcup_{p\in A} (\mathrm{Q}^+(5,q) \cap V) \setminus \pi.$$

In other words, $\mathcal{M}^\alpha(A)$ is the intersection of $\mathrm{Q}^+(5,q)$ with the cone with vertex π and base A, minus the points of $\mathrm{Q}^+(5,q)$ in π.

Theorem 2.6.1 *The set $\mathcal{M}^\alpha(A)$ is a $(0,\alpha)$-set on $\mathrm{Q}^+(5,q)$ of deficiency $d = |\,\mathrm{Q}^+(5,q) \cap \pi\,|$.*

Proof. Let L be a line of $\mathrm{PG}(5,q)$ which is contained in $\mathrm{Q}^+(5,q)$. Since π does not contain any lines of $\mathrm{Q}^+(5,q)$, there are two possibilities.

Firstly, suppose that L intersects the plane π in a point. Then the 3-space $V = \langle L, \pi\rangle$ contains a line of $\mathrm{Q}^+(5,q)$. Hence $\mathrm{Q}^+(5,q) \cap V$ is not a nonsingular elliptic quadric. So the point $p = V \cap \pi'$ is not in $\mathcal{D}$, and hence not in A. This means that there are no points of $\mathcal{M}^\alpha(A)$ in V and hence also none on L.

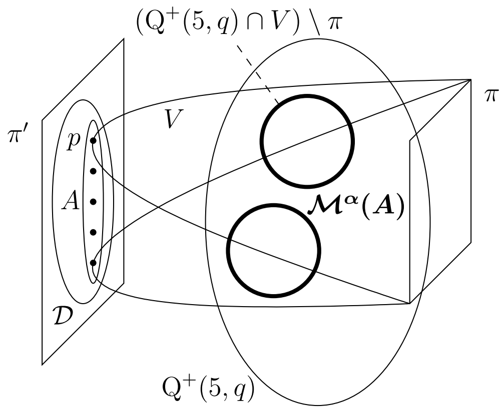

Figure 2.2: Construction of the $(0,\alpha)$-set $\mathcal{M}^\alpha(A)$.

Secondly, suppose that L is skew to π. A point $p \in L$ is in $\mathcal{M}^\alpha(A)$ if and only if $V = \langle p, \pi \rangle$ intersects π' in a point of A, if and only if the projection of p from π onto π' is a point of A. So, if L' is the projection of L from π onto π', then $|\,\mathcal{M}^\alpha(A) \cap L\,| = |\,A \cap L'\,| \in \{0, \alpha\}$. So every line on $\mathrm{Q}^+(5,q)$ intersects $\mathcal{M}^\alpha(A)$ in 0 or α points.

From the construction, it follows that

$$\begin{aligned} |\,\mathcal{M}^\alpha(A)\,| &= |\,A\,|\,(q^2 + 1 - |\,\mathrm{Q}^+(5,q) \cap \pi\,|\,) \\ &= (q\alpha + q - \alpha)(q^2 + 1 - |\,\mathrm{Q}^+(5,q) \cap \pi\,|\,). \end{aligned}$$

Hence $\mathcal{M}^\alpha(A)$ has deficiency $d = |\,\mathrm{Q}^+(5,q) \cap \pi\,|$. □

Since the plane π does not contain any line of $\mathrm{Q}^+(5,q)$, there are two possibilities: either $\mathrm{Q}^+(5,q) \cap \pi$ is a single point or it is a nondegenerate conic. In the former case the $(0,\alpha)$-set has deficiency 1 and it is denoted by $\mathcal{M}_1^\alpha(A)$. In the latter case the $(0,\alpha)$-set has deficiency $q+1$ and it is denoted by $\mathcal{M}_{q+1}^\alpha(A)$.

Assume that the plane π intersects the Klein quadric $\mathrm{Q}^+(5,q)$ in a single point p. Let V be a 3-space containing π, such that $\mathrm{Q}^+(5,q) \cap V$ is not a nonsingular elliptic quadric. Then $\mathrm{Q}^+(5,q) \cap V$ is a nonsingular hyperbolic quadric, a quadratic cone or the union of two distinct planes. Since π intersects $\mathrm{Q}^+(5,q)$ in the point p only, it is immediately clear that $\mathrm{Q}^+(5,q) \cap V$ is a quadratic cone with vertex p. Hence V is contained in the tangent hyperplane p^β to $\mathrm{Q}^+(5,q)$ at p. Conversely, if $V \supseteq \pi$ is a 3-space contained in p^β, then $\mathrm{Q}^+(5,q) \cap V$ is a quadratic cone with vertex p. We conclude that the set

$\mathcal{D}$ of points $p' \in \pi'$ such that $V = \langle p', \pi \rangle$ intersects $\mathrm{Q}^+(5,q)$ in a nonsingular elliptic quadric, is the set of points of π' which are not on the line $\pi' \cap p^\beta$. Clearly, in this case, the set $\mathcal{D}$ contains a maximal arc A of degree α for every $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$.

Assume that the plane π intersects the Klein quadric $\mathrm{Q}^+(5,q)$ in a nondegenerate conic C. Then the plane π^β also intersects $\mathrm{Q}^+(5,q)$ in a nondegenerate conic C'. Furthermore, β induces an anti-automorphism between the projective plane π^β and the projective plane having as points the 3-spaces through π and as lines the hyperplanes through π. This anti-automorphism is such that a 3-space containing π intersects $\mathrm{Q}^+(5,q)$ in a nonsingular elliptic quadric if and only if the corresponding line of π^β is external to the conic C'. Hence the set $\mathcal{D}$ in the plane π' is the dual of the set of external lines to a nondegenerate conic. It follows that $\mathcal{D}$ is a Denniston type maximal arc of degree $q/2$, and hence that $\mathcal{D}$ contains a maximal arc A of degree α for every $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$. We have proven the following theorem.

Theorem 2.6.2 *There exists a $(0,\alpha)$-set $\mathcal{M}_d^\alpha(A)$ on $\mathrm{Q}^+(5,q)$, $q = 2^h$, of deficiency $d \in \{1, q+1\}$ for all $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$.*

Corollary 2.6.3 *There exist $(0,\alpha)$-geometries fully embedded in $\mathrm{PG}(3,q)$, $q = 2^h$, of deficiency $d \in \{1, q+1\}$ for all $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$.*

By Theorem 2.5.1, the $(0,2)$-set $\mathcal{E}_d$, $d = 1, q+1$, is of the form $\mathcal{M}_d^2(H)$, with H a regular hyperoval.

Let $\mathcal{K}$ be the $(0,q/2)$-set corresponding to the $(0,q/2)$-geometry $\mathrm{NQ}^+(3,q)$, q even. As we have shown in Section 2.2, there is a plane π of $\mathrm{PG}(5,q)$ which intersects $\mathrm{Q}^+(5,q)$ in a nondegenerate conic C, such that $\mathcal{K}$ is the set of points of $\mathrm{Q}^+(5,q)$ which are collinear to none of the points on C. So a point p of $\mathrm{Q}^+(5,q)$ is in $\mathcal{K}$ if and only if $p \notin \pi$ and the 3-space $V = \langle p, \pi \rangle$ intersects $\mathrm{Q}^+(5,q)$ in a nondegenerate elliptic quadric. Hence $\mathcal{K} = \mathcal{M}_{q+1}^{q/2}(A)$, with $A = \mathcal{D}$.

We conclude this chapter with a list of all the known distinct examples of $(0,\alpha)$-sets $\mathcal{K}$ in $\mathrm{Q}^+(5,q)$, $\alpha > 1$, $q > 2$. In this list d denotes the deficiency of the $(0,\alpha)$-set $\mathcal{K}$, and $\mathcal{S}$ denotes the corresponding $(0,\alpha)$-geometry fully embedded in $\mathrm{PG}(3,q)$.

1. $\alpha = q+1$, $d = 0$, $\mathcal{K}$ is the set of all points of $\mathrm{Q}^+(5,q)$, and $\mathcal{S}$ is the design of all points and all lines of $\mathrm{PG}(3,q)$.

2. $\alpha = q$, $d = 0$, $\mathcal{K}$ is the complement in $\mathrm{Q}^+(5,q)$ of a hyperplane which is not tangent to $\mathrm{Q}^+(5,q)$, and $\mathcal{S} = \overline{W(3,q)}$.

3. $\alpha = q$, $d = 1$, $\mathcal{K}$ is the complement in $\mathrm{Q}^+(5,q)$ of a hyperplane which is tangent to $\mathrm{Q}^+(5,q)$, and $\mathcal{S} = H_q^3$.
4. $q = 2^h$, $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$, $d \in \{1, q+1\}$ and $\mathcal{K} = \mathcal{M}_d^\alpha(A)$.
5. $q = 2^{2e+1}$, $\alpha = 2$, $d = q \pm \sqrt{2q} + 1$, and $\mathcal{K} = \mathcal{T}_{q \pm \sqrt{2q}+1}$.

Chapter 3

Planar oval sets in Desarguesian planes of even order

In this chapter, we provide a characterization of certain sets of ovals in $\mathrm{PG}(2,q)$, called *planar oval sets*. We will need this result in our study of affine semipartial geometries and $(0,\alpha)$-geometries, in Chapters 4–6. In particular, planar oval sets play a crucial role in the classification of $(0,2)$-geometries of order $(q-1,q)$ fully embedded in $\mathrm{AG}(3,q)$, $q=2^h$, such that there are no planar nets (see Section 5.3.3).

A *planar oval set* in $\mathrm{PG}(2,q)$, $q=2^h$, is a set Ω of q^2 ovals in $\mathrm{PG}(2,q)$ with common nucleus n, such that the incidence structure $\pi(\Omega)$ having as points the points of $\mathrm{PG}(2,q)$, as lines the elements of Ω and the lines of $\mathrm{PG}(2,q)$ through n, and as incidence the natural one, is a projective plane of order q. Equivalently, a set Ω of q^2 ovals in $\mathrm{PG}(2,q)$ with common nucleus n is a planar oval set if and only if any two elements of Ω intersect in exactly one point. We say that n is the nucleus of the planar oval set Ω.

A planar oval set Ω in $\mathrm{PG}(2,q)$ is called *Desarguesian* if $\pi(\Omega)$ is a Desarguesian projective plane, and it is called a *regular Desarguesian planar oval set* if furthermore there exists a collineation from $\mathrm{PG}(2,q)$ to $\pi(\Omega)$ which fixes every line through the nucleus of Ω.

In Section 3.1, we will construct a regular Desarguesian planar oval set $\Omega(O)$ in $\mathrm{PG}(2,q)$ starting from an arbitrary oval O of $\mathrm{PG}(2,q)$. In Section 3.4, we will show that every regular Desarguesian planar oval set is of type $\Omega(O)$.

The results of this chapter are published in [35]. We remark that throughout this chapter $q=2^h$, unless it is explicitly mentioned otherwise.

3.1 The planar oval set $\Omega(O)$

For any point p of $\mathrm{PG}(2,q)$ let $\mathrm{Persp}(p)$ be the group of perspectivities of $\mathrm{PG}(2,q)$ with center p, and by $\mathrm{El}(p)$ the group of elations of $\mathrm{PG}(2,q)$ with center p. The group $\mathrm{El}(p)$ is an elementary abelian group of order q^2 acting sharply transitively on the set of all lines of $\mathrm{PG}(2,q)$ missing p.

Let O be an oval of $\mathrm{PG}(2,q)$ with nucleus n, and let

$$\Omega(O) = \{O^e \mid e \in \mathrm{El}(n)\}.$$

Since the nucleus n of the oval O is fixed by every elation $e \in \mathrm{El}(n)$, n is the nucleus of every oval $O^e \in \Omega(O)$. So a line through n intersects every element of $\Omega(O)$ in exactly one point. Let O_1 and O_2 be two distinct elements of $\Omega(O)$. Then there exists a nontrivial elation $e \in \mathrm{El}(n)$ such that $O_1^e = O_2$. Let L be the axis of e. Then L intersects O_j in exactly one point p_j, $j = 1, 2$. Since e fixes every point of L and maps O_1 to O_2, $p_1 = p_2$. Suppose that there is a point $p \in O_1 \cap O_2$ which is not on the axis L. Let $M = \langle n, p\rangle$. Then p is the unique point of $O_1 \cap M$ and of $O_2 \cap M$. Since e fixes M and maps O_1 to O_2, $p^e = p$. But then e is the identity, a contradiction. So any two elements of $\Omega(O)$ intersect in exactly one point. Hence $\Omega(O)$ is a planar oval set with nucleus n.

We define a map ξ, which maps points of $\mathrm{PG}(2,q)$ to points of $\pi(\Omega(O))$, and lines of $\mathrm{PG}(2,q)$ to lines of $\pi(\Omega(O))$. Let L be a line of $\mathrm{PG}(2,q)$ which does not contain n. Then for every line M of $\mathrm{PG}(2,q)$ not through n, there is exactly one element e of $\mathrm{El}(n)$, such that $M^e = L$. Let $M^\xi = O^e$. For every line N of $\mathrm{PG}(2,q)$ through n, let $N^\xi = N$. Let $n^\xi = n$. For every point p of $\mathrm{PG}(2,q)$ different from n, let $p^\xi = M^\xi \cap \langle n, p\rangle$, where M is a line of $\mathrm{PG}(2,q)$ which contains p but not n. This definition does not depend on the chosen line M. Indeed, let M_1 and M_2 be distinct lines of $\mathrm{PG}(2,q)$ which contain p but not n, and let e_j be the unique element of $\mathrm{El}(n)$ such that $M_j^{e_j} = L$, $j = 1, 2$. Let $p' = L \cap \langle n, p\rangle$ and let $p'' = O \cap \langle n, p\rangle$. Then $p^{e_1} = p^{e_2} = p'$, so $e_1 e_2^{-1} \in \mathrm{El}(n)$ is such that $p^{e_1 e_2^{-1}} = p$. Hence the line $\langle n, p\rangle$ is the axis of $e_1 e_2^{-1}$, and so e_1 and e_2 have the same action on the line $\langle n, p\rangle$. This means that $p''^{e_1} = p''^{e_2}$. But since p'' is the unique point of O on the line $\langle n, p\rangle$, p''^{e_j} is the unique point of M_j^ξ on the line $\langle n, p\rangle$, $j = 1, 2$. Hence $M_1^\xi \cap \langle n, p\rangle = M_2^\xi \cap \langle n, p\rangle$. So ξ is well-defined.

We prove that ξ is a collineation from $\mathrm{PG}(2,q)$ to $\pi(\Omega(O))$. From the definition of ξ it follows that if $\{p, M\}$ is a flag of $\mathrm{PG}(2,q)$ then $\{p^\xi, M^\xi\}$ is a flag of $\pi(\Omega(O))$. Clearly ξ induces a bijection from the line set of $\mathrm{PG}(2,q)$ to the line set of $\pi(\Omega(O))$. We show that ξ induces a bijection from the point set of $\mathrm{PG}(2,q)$ to the point set of $\pi(\Omega(O))$. Clearly a point $p \neq n$ is mapped by ξ to a point on the line $\langle n, p\rangle$. Suppose that two distinct points p_1 and p_2 such that $n \neq p_1, p_2$ and $n \in \langle p_1, p_2\rangle$, have the same image under ξ. Let M_j be a line of $\mathrm{PG}(2,q)$ which contains p_j but not n, $j = 1, 2$. Then $M_1 \neq M_2$, so $M_1 \cap M_2$ is a point p. Now $p, p_j \in M_j$, so $p^\xi, p_j^\xi \in M_j^\xi$, $j = 1, 2$. Since p, p_j and n are not collinear, $p^\xi \neq p_j^\xi$, $j = 1, 2$. So M_1^ξ and M_2^ξ share two distinct points, namely p^ξ and $p_1^\xi = p_2^\xi$. Since any two distinct elements of $\Omega(O)$ intersect in exactly one point, $M_1^\xi = M_2^\xi$. This implies that $M_1 = M_2$, a contradiction. So ξ induces a bijection from the point set of $\mathrm{PG}(2,q)$ to the point set of $\pi(\Omega(O))$. Hence ξ is a collineation from $\mathrm{PG}(2,q)$ to $\pi(\Omega(O))$ which fixes every line of $\mathrm{PG}(2,q)$ through n. It follows that $\Omega(O)$ is a regular Desarguesian planar oval set.

3.2 Regular Desarguesian planar oval sets

Let Ω be a planar oval set in $\mathrm{PG}(2,q)$ with nucleus n. Let V denote the set of collineations from $\mathrm{PG}(2,q)$ to $\pi(\Omega)$ which fix every line through n. By definition $V \neq \emptyset$ if and only if Ω is a regular Desarguesian planar oval set. Since the point set of $\mathrm{PG}(2,q)$ coincides with the point set of $\pi(\Omega)$, every element of V induces a permutation on this set. For every $\xi \in V$, let $\mathrm{Fix}(\xi)$ denote the set of points of $\mathrm{PG}(2,q)$, different from n, which are fixed by ξ.

Lemma 3.2.1 *Let Ω be a regular Desarguesian planar oval set in* $\mathrm{PG}(2,q)$ *with nucleus n. Then $|\,V\,| = q^2(q-1)$ and for any $\xi \in V$, $V = \mathrm{Persp}(n)\,\xi$.*

Proof. Let $\xi \in V$. A collineation ξ' from $\mathrm{PG}(2,q)$ to $\pi(\Omega)$ is in V if and only if $\xi'\xi^{-1}$ is a collineation of $\mathrm{PG}(2,q)$ fixing every line through n. Hence $V = \mathrm{Persp}(n)\,\xi$. It follows that $|\,V\,| = |\,\mathrm{Persp}(n)\,| = q^2(q-1)$. □

Lemma 3.2.2 *Let Ω be a regular Desarguesian planar oval set in* $\mathrm{PG}(2,q)$ *with nucleus n. For any three points p_1, p_2, p_3 of* $\mathrm{PG}(2,q)$ *such that no three of n, p_1, p_2, p_3 are collinear in* $\mathrm{PG}(2,q)$ *or in $\pi(\Omega)$, there is exactly one $\xi \in V$ such that $p_1, p_2, p_3 \in \mathrm{Fix}(\xi)$.*

Proof. Let p_1, p_2, p_3 be points of $\mathrm{PG}(2,q)$ such that no three of n, p_1, p_2, p_3 are collinear in $\mathrm{PG}(2,q)$ or in $\pi(\Omega)$. Let $\xi_1 \in V$. Since n, p_1 and p_2 are not collinear, there exists an elation $e \in \mathrm{El}(n)$ such that $p_j^e = p_j^{\xi_1^{-1}}$, $j = 1, 2$. From Lemma 3.2.1 we know that $V = \mathrm{Persp}(n)\,\xi_1$, so $\xi_2 = e\xi_1 \in V$ and ξ_2 fixes p_1 and p_2.

Let L be the line of $\mathrm{PG}(2,q)$ through p_1 and p_2. Then by assumption L does not contain p_3. Suppose that $p_3^{\xi_2^{-1}} \in L$. Then p_1, p_2 and p_3 are contained in L^{ξ_2}, a contradiction since we assumed that p_1, p_2 and p_3 are not collinear in $\pi(\Omega)$. Since $p_3, p_3^{\xi_2^{-1}} \notin L$, there is a homology $h \in \mathrm{Persp}(n)$ with axis L which maps p_3 to $p_3^{\xi_2^{-1}}$. Let $\xi = h\xi_2$. Then $\xi \in \mathrm{Persp}(n)\,\xi_2 = V$ and ξ fixes p_1, p_2 and p_3.

Suppose that $\xi' \in V$ fixes the points p_1, p_2 and p_3. By Lemma 3.2.1, $V = \mathrm{Persp}(n)\,\xi$, so $\xi'\xi^{-1}$ is a perspectivity of $\mathrm{PG}(2,q)$ with center n which fixes p_1, p_2 and p_3. Since no three of n, p_1, p_2 and p_3 are collinear, it follows that $\xi'\xi^{-1}$ is the identity. So there is exactly one $\xi \in V$ which fixes p_1, p_2 and p_3. □

Lemma 3.2.3 *Let Ω be a regular Desarguesian planar oval set in* $\mathrm{PG}(2,q)$ *with nucleus n. Let L be a line of* $\mathrm{PG}(2,q)$ *through n, and let $\xi_1, \xi_2 \in V$, $\xi_1 \neq \xi_2$. If $\xi_1\xi_2^{-1} \in \mathrm{El}(n)$, then the sets* $\mathrm{Fix}(\xi_1) \cap L$ *and* $\mathrm{Fix}(\xi_2) \cap L$ *are either disjoint or equal. If $\xi_1\xi_2^{-1} \in \mathrm{Persp}(n) \setminus \mathrm{El}(n)$, then the sets* $\mathrm{Fix}(\xi_1) \cap L$ *and* $\mathrm{Fix}(\xi_2) \cap L$ *have at most one point in common.*

Proof. By Lemma 3.2.1, $V = \mathrm{Persp}(n)\,\xi_2$, so $g = \xi_1\xi_2^{-1} \in \mathrm{Persp}(n)$. Assume that $g \in \mathrm{El}(n)$ and suppose that $\mathrm{Fix}(\xi_1) \cap L$ and $\mathrm{Fix}(\xi_2) \cap L$ have a point p in common. Then g is an elation of $\mathrm{PG}(2,q)$ with center n which fixes p, so the line L is the axis of g. This implies that ξ_1 and ξ_2 have the same action on L, so $\mathrm{Fix}(\xi_1) \cap L = \mathrm{Fix}(\xi_2) \cap L$.

Now assume that $g \in \mathrm{Persp}(n) \setminus \mathrm{El}(n)$, and suppose that $\mathrm{Fix}(\xi_1) \cap L$ and $\mathrm{Fix}(\xi_2) \cap L$ have two points p_1 and p_2 in common. Then g is a perspectivity with center n which fixes $p_1 \neq n$ and $p_2 \neq n$, so the line L through p_1 and p_2 is the axis of g. But L contains n, a contradiction since we assumed that g was not an elation. So $\mathrm{Fix}(\xi_1) \cap L$ and $\mathrm{Fix}(\xi_2) \cap L$ have at most one point in common. □

3.3 $(q+i)$-sets of type $(0,2,i)$ in $\mathrm{PG}(2,q)$

Korchmáros and Mazzocca [56] studied $(q+i)$-sets of type $(0,2,i)$ in $\mathrm{PG}(2,q)$, $i \neq 0, 2$. Note that a $(q+i)$-set of type $(0,2,i)$ is an oval when $i = 1$, a hyperoval when $i = 2$, and the symmetric difference of two lines when $i = q$. Korchmáros and Mazzocca showed that, if $2 < i < q$, then such a set $\mathcal{K}$ can exist only when $q = 2^h$ and i divides q. Furthermore, it was shown that through every point of $\mathcal{K}$ there is exactly one i-secant, and that in general all

i-secants are concurrent at a point called the i-nucleus of $\mathcal{K}$. Exceptions can only occur when $i = 2^m$ and there exist integers b and $c \geq 3$ such that both $h = (b+1)c$ and $m = bc+1$. Examples of $(2^h + 2^m)$-sets of type $(0,2,2^m)$ in $\mathrm{PG}(2,2^h)$ were constructed by Korchmáros and Mazzocca [56] when m is a proper divisor of h.

Lemma 3.3.1 *Let Ω be a regular Desarguesian planar oval set in* $\mathrm{PG}(2,q)$ *with nucleus n, and let $\xi \in V$. Then the set* $\mathrm{Fix}(\xi)$ *is either empty or it is a $(q+i)$-set of type $(0,2,i)$ with i-nucleus n, for some $i \,|\, q$.*

Proof. Let L be a line of $\mathrm{PG}(2,q)$ not through n and let p be a point of L. Since ξ fixes the line $\langle n,p\rangle$, the point p is mapped by ξ to the point $\langle n,p\rangle \cap L^\xi$. Hence $\mathrm{Fix}(\xi) \cap L = L \cap L^\xi$. Since L^ξ is an oval with nucleus n and since $n \notin L$, $\mathrm{Fix}(\xi) \cap L$ consists of either 0 or 2 points. So any line of $\mathrm{PG}(2,q)$ not through n contains either 0 or 2 points of $\mathrm{Fix}(\xi)$.

Suppose that $\mathrm{Fix}(\xi) \neq \emptyset$, and let $i \in \mathbb{Z}$ be such that $|\,\mathrm{Fix}(\xi)\,| = q+i$. Let L be a line of $\mathrm{PG}(2,q)$ through n which contains at least one point p of $\mathrm{Fix}(\xi)$. Then by the preceding paragraph every line of $\mathrm{PG}(2,q)$ through p but not through n contains exactly two points of $\mathrm{Fix}(\xi)$. Hence there are exactly q points of $\mathrm{Fix}(\xi)$ not on L. So there are exactly i points of $\mathrm{Fix}(\xi)$ on L. It follows that every line of $\mathrm{PG}(2,q)$ through n contains either 0 or i points of $\mathrm{Fix}(\xi)$. So $\mathrm{Fix}(\xi)$ is a $(q+i)$-set of type $(0,2,i)$ with i-nucleus n, and $i \,|\, q$. $\square$

Lemma 3.3.2 *Let Ω be a regular Desarguesian planar oval set in* $\mathrm{PG}(2,q)$ *with nucleus n, and let $\xi \in V$. Then there exists an $i \,|\, q$ such that for every $\xi' \in \mathrm{El}(n)\,\xi$, the set* $\mathrm{Fix}(\xi')$ *is either empty or is a $(q+i)$-set of type $(0,2,i)$ with i-nucleus n. If $i \neq 1$, then $i \geq q^{2/3}$. The number of elements $\xi' \in \mathrm{El}(n)\,\xi$ such that* $\mathrm{Fix}(\xi') \neq \emptyset$, *equals $q^2(q+1)/(q+i)$.*

Proof. Let $\xi_1, \xi_2 \in \mathrm{El}(n)\,\xi$ and suppose that $\mathrm{Fix}(\xi_1)$ and $\mathrm{Fix}(\xi_2)$ are not empty. By Lemma 3.3.1, $\mathrm{Fix}(\xi_j)$ is a $(q+i_j)$-set of type $(0,2,i_j)$ with i_j-nucleus n for some $i_j \,|\, q$, $j = 1,2$. Let $p_1 \in \mathrm{Fix}(\xi_1)$ and $p_2 \in \mathrm{Fix}(\xi_2)$ be such that $L_1 = \langle n,p_1\rangle$ is different from $L_2 = \langle n,p_2\rangle$, and let e be the elation with center n and axis L_1 which maps p_2 to $p_2^{\xi_1^{-1}}$. Then

$$\xi_3 = e\xi_1 \in \mathrm{El}(n)\,\xi_1 = \mathrm{El}(n)\,\xi = \mathrm{El}(n)\,\xi_2$$

and ξ_3 fixes p_1 and p_2. By Lemma 3.3.1, $\mathrm{Fix}(\xi_3)$ is a $(q+i)$-set of type $(0,2,i)$ with i-nucleus n for some $i \,|\, q$. Since $\mathrm{Fix}(\xi_j) \cap L_j$ is not disjoint from $\mathrm{Fix}(\xi_3) \cap L_j$, Lemma 3.2.3 implies that $\mathrm{Fix}(\xi_j) \cap L_j = \mathrm{Fix}(\xi_3) \cap L_j$, $j = 1,2$.

Hence $i_1 = i_2 = i$. So there exists an $i \mid q$ such that for every $\xi' \in \mathrm{El}(n)\,\xi$ the set $\mathrm{Fix}(\xi')$ is either empty or is a $(q+i)$-set of type $(0,2,i)$ with i-nucleus n.

For every line L through n, we define the following set S_L.

$$S_L = \{A \neq \emptyset \mid \exists\, \xi' \in \mathrm{El}(n)\,\xi : \mathrm{Fix}(\xi') \cap L = A\}.$$

From the preceding paragraph it follows that every $A \in S_L$ has size i. By Lemma 3.2.3, if $A, B \in S_L$ and $A \neq B$, then A is disjoint from B. Let p be a point of L different from n, and let $e \in \mathrm{El}(n)$ be such that $p^e = p^{\xi^{-1}}$. Then $\xi' = e\xi \in \mathrm{El}(n)\,\xi$ and ξ' fixes p, so there is an element $A = \mathrm{Fix}(\xi') \cap L \in S_L$ which contains p. Now it is clear that S_L forms a partition of the set $L \setminus \{n\}$ into sets of size i.

Let $S = \bigcup S_L$, for all lines L through n. We count the number of ordered pairs (A, ξ') such that $A \in S$, $\xi' \in \mathrm{El}(n)\,\xi$ and $A \subseteq \mathrm{Fix}(\xi')$. Let x denote the number of elements $\xi' \in \mathrm{El}(n)\,\xi$ such that $\mathrm{Fix}(\xi')$ is not empty. If $\mathrm{Fix}(\xi') \neq \emptyset$, then $\mathrm{Fix}(\xi')$ is a $(q+i)$-set of type $(0,2,i)$ with i-nucleus n, so the number of i-secants of $\mathrm{Fix}(\xi')$ is $(q/i)+1$. Hence the number of ordered pairs (A,ξ') equals $x((q/i)+1)$. On the other hand, since for every line L through n, S_L forms a partition of $L \setminus \{n\}$, the number of elements A of S is $q(q+1)/i$. Let $A \in S$, and let L be the line through n which contains the points of A. Then by definition of S_L, there is an element $\xi' \in \mathrm{El}(n)\,\xi$ such that $\mathrm{Fix}(\xi') \cap L = A$. Let $\xi'' \in \mathrm{El}(n)\,\xi$. Then $A \subseteq \mathrm{Fix}(\xi'')$ if and only if $\xi''\xi'^{-1}$ fixes every point of A. Since $\mathrm{Fix}(\xi'')$ is either the empty set or a $(q+i)$-set of type $(0,2,i)$, $A \subseteq \mathrm{Fix}(\xi'')$ if and only if $\mathrm{Fix}(\xi'') \cap L = A$. Since $\xi''\xi'^{-1} \in \mathrm{El}(n)$, $\xi''\xi'^{-1}$ fixes every point of A if and only if L is the axis of $\xi''\xi'^{-1}$. We conclude that the number of elements $\xi'' \in \mathrm{El}(n)\,\xi$ such that $\mathrm{Fix}(\xi'') \cap L = A$ equals the number of elations of $\mathrm{PG}(2,q)$ with center n and axis L, namely q. Hence $x((q/i)+1) = q^2(q+1)/i$. So the number x of elements $\xi' \in \mathrm{El}(n)\,\xi$ such that $\mathrm{Fix}(\xi') \neq \emptyset$ equals $q^2(q+1)/(q+i)$.

Since x is an integer, $q + i \mid q^2(q+1)$. Let $q = 2^h$ and $i = 2^{h-m}$. Then $2^h + 2^{h-m} \mid 2^{2h}(2^h+1)$, so $2^m+1 \mid 2^{h+m}(2^h+1)$. If $i \neq q$, then 2^m+1 is odd, so $2^m+1 \mid 2^h+1$. Let $l \in \mathbb{N}$ and $R \in \{0,1,\ldots,m-1\}$ be such that $h = ml+R$. Then since 2^m+1 divides $2^h+1 = (2^m+1-1)^l 2^R + 1$, $2^m+1 \mid (-1)^l 2^R + 1$. As $0 \leq R < m$ this is impossible if l is even. Hence l is odd and $R = 0$, so $h = ml$ with l odd. In particular, if $i \neq 1$, or, equivalently, $h \neq m$, then $m \leq h/3$, so $i \geq q^{2/3}$. $\square$

We introduce some new notation. Let Ω be a regular Desarguesian planar oval set in $\mathrm{PG}(2,q)$ with nucleus n. By Lemma 3.2.1, $V = \mathrm{Persp}(n)\,\xi$ for any $\xi \in V$. So V can be partitioned into $q-1$ sets $V_1, V_2, \ldots, V_{q-1}$ of size q^2 such that for any $j \in \{1,2,\ldots,q-1\}$ and for any $\xi_1, \xi_2 \in V_j$, $\xi_1\xi_2^{-1} \in \mathrm{El}(n)$.

Lemma 3.3.2 states that for every $j \in \{1, 2, \ldots, q-1\}$, there is an $i_j \mid q$ such that for every $\xi \in V_j$ the set $\mathrm{Fix}(\xi)$ is either the empty set or a $(q+i_j)$-set of type $(0, 2, i_j)$ with i_j-nucleus n. For every $j \in \{1, 2, \ldots, q-1\}$ and for every line L through n we define a set S^j_L as follows.

$$S^j_L = \{A \neq \emptyset \mid \exists\, \xi \in V_j : \mathrm{Fix}(\xi) \cap L = A\}.$$

In the proof of Lemma 3.3.2 we showed that S^j_L forms a partition of $L \setminus \{n\}$ into sets of size i_j. For any $j \in \{1, 2, \ldots, q-1\}$ let S_j denote the union of all sets S^j_L, L a line through n.

Lemma 3.3.3 *Let Ω be a regular Desarguesian planar oval set in* $\mathrm{PG}(2,q)$ *with nucleus n. Then*

$$\sum_{j=1}^{q-1} i_j = 2q - 2.$$

Proof. Let X be the set of ordered sets (p_1, p_2, p_3, ξ) where p_1, p_2 and p_3 are points of $\mathrm{PG}(2,q)$ such that no three of n, p_1, p_2, p_3 are collinear in $\mathrm{PG}(2,q)$ or in $\pi(\Omega)$, and where $\xi \in V$ fixes the points p_1, p_2 and p_3. Lemma 3.2.2 states that, given three such points p_1, p_2 and p_3, there is exactly one $\xi \in V$ which fixes these three points. Hence $|\,X\,| = q^3(q^2-1)(q-2)$.

Let $j \in \{1, \ldots, q-1\}$ and let $\xi \in V_j$ be such that $\mathrm{Fix}(\xi) \neq \emptyset$. We show that any three points $p_1, p_2, p_3 \in \mathrm{Fix}(\xi)$ such that the lines $\langle n, p_1\rangle$, $\langle n, p_2\rangle$, $\langle n, p_3\rangle$ are distinct, are not collinear in $\mathrm{PG}(2,q)$ or in $\pi(\Omega)$. Let p_1, p_2, p_3 be such points. Clearly, since $\mathrm{Fix}(\xi)$ is a $(q+i_j)$-set of type $(0, 2, i_j)$ with i_j-nucleus n, the points p_1, p_2 and p_3 are not collinear in $\mathrm{PG}(2,q)$. Let $\mathcal{L}$ denote the set of lines of $\mathrm{PG}(2,q)$ not through n and let $V^{-1} = \{\xi'^{-1} \mid \xi' \in V\}$. Then $\mathcal{L}$ is a regular Desarguesian planar oval set in $\pi(\Omega)$ and V^{-1} is the set of all collineations from $\pi(\Omega)$ to $\mathrm{PG}(2,q)$ which fix every line through n. Since $\xi^{-1} \in V^{-1}$, Lemma 3.3.1 implies that the set $\mathrm{Fix}(\xi^{-1})$, which contains p_1, p_2 and p_3, is a $(q+i)$-set of type $(0,2,i)$ of the plane $\pi(\Omega)$, with i-nucleus n, for some $i \mid q$. So p_1, p_2 and p_3 are not collinear in $\pi(\Omega)$. It follows that, given a $\xi \in V_j$ such that $\mathrm{Fix}(\xi) \neq \emptyset$, there are exactly $q(q^2 - i_j^2)$ sets $(p_1, p_2, p_3, \xi) \in X$. By Lemma 3.3.2, there are $q^2(q+1)/(q+i_j)$ elements $\xi \in V_j$ such that $\mathrm{Fix}(\xi) \neq \emptyset$. Hence

$$\sum_{j=1}^{q-1} q^3(q+1)(q-i_j) = q^3(q^2-1)(q-2).$$

The lemma follows. □

Theorem 3.3.4 *Let Ω be a regular Desarguesian planar oval set in* $\mathrm{PG}(2,q)$ *with nucleus n. Then there is an $l \in \{1,2,\ldots,q-1\}$ such that $i_l = q$ and $i_j = 1$ for every $j \in \{1,2,\ldots,q-1\} \setminus \{l\}$.*

Proof. By Lemma 3.3.3, $\sum_{j=1}^{q-1} = 2q-2$, so there is at least one $l \in \{1,2,\ldots,q-1\}$ such that $i_l > 1$. By Lemma 3.3.2, $i_l \geq q^{2/3}$. Let $j \in \{1,2,\ldots,q-1\} \setminus \{l\}$, let L be a line of $\mathrm{PG}(2,q)$ through n and let $A \in S_L^j$ and $B \in S_L^l$. Then there exist elements $\xi \in V_j$ and $\xi' \in V_l$ such that $A = \mathrm{Fix}(\xi) \cap L$ and $B = \mathrm{Fix}(\xi') \cap L$. Since $j \neq l$, $\xi\xi'^{-1} \in \mathrm{Persp}(n) \setminus \mathrm{El}(n)$ so by Lemma 3.2.3, A and B have at most one point in common.

Let x be the number of ordered triples (p, A, B) such that $A \in S_L^j$, $B \in S_L^l$ and $p \in A \cap B$. Since both S_L^j and S_L^l partition the point set $L \setminus \{n\}$, for a given $p \in L \setminus \{n\}$, there is exactly one $A \in S_L^j$, respectively $B \in S_L^l$, such that $p \in A$, respectively $p \in B$. So $x = q$. On the other hand, any two sets $A \in S_L^j$ and $B \in S_L^l$ intersect in at most one point p, so $x \leq (q/i_j)(q/i_l)$. Hence $i_j \leq q/i_l \leq q^{1/3}$. Now Lemma 3.3.2 implies that $i_j = 1$. So for every $j \in \{1,2,\ldots,q-1\} \setminus \{l\}$, $i_j = 1$. By Lemma 3.3.3, $i_l = q$. □

Remark

A *Laguerre plane* is a quadruple $(\mathcal{P}, \mathcal{L}, \mathcal{C}, \mathrm{I})$, where $\mathcal{P}$ is a set whose elements are called *points*, $\mathcal{L}$ is a set whose elements are called *lines*, $\mathcal{C}$ is a set whose elements are called *circles*, $\mathrm{I} \subseteq (\mathcal{P} \times (\mathcal{L} \cup \mathcal{C})) \cup ((\mathcal{L} \cup \mathcal{C}) \times \mathcal{P})$ is a symmetric incidence relation, such that the following axioms are satisfied.

(lp1) Any point is on exactly one line.

(lp2) Any three points, no two of which are on a line, are on a unique circle.

(lp3) Each circle intersects each line in exactly one point.

(lp4) There are at least two circles and each circle contains at least 3 points.

(lp5) If C is a circle, and p and r are points not on a common line, such that $p \in C$ and $r \notin C$, then there is a unique circle D through r which is tangent to C at p.

Laguerre planes and the related inversive planes and Minkowski planes are treated in [42]. An example of a Laguerre plane is the following. Consider in $\mathrm{PG}(3,q)$, q not necessarily even, a cone Q with vertex a point p and base an oval O in a plane not containing p. Let $\mathcal{P}$ be the set of points on the cone Q different from the vertex p, let $\mathcal{L}$ be the set of lines on Q, let $\mathcal{C}$ be the set of ovals obtained by intersecting Q with a plane of $\mathrm{PG}(3,q)$ not containing p, and let I be the natural incidence. Then $\mathbf{L}(Q) = (\mathcal{P}, \mathcal{L}, \mathcal{C}, \mathrm{I})$ is a Laguerre plane. Any Laguerre plane which is isomorphic to this example is called *embeddable*.

Let Ω be a regular Desarguesian planar oval set in $\mathrm{PG}(2,q)$ with nucleus n. Assume, as we may by Theorem 3.3.4, that $i_1 = q$. Let $\mathcal{P}$ denote the set of points of $\mathrm{PG}(2,q)$ different from n, let $\mathcal{L}$ denote the set of lines of $\mathrm{PG}(2,q)$ through n, let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, where $\mathcal{C}_1$ is the set of lines of $\mathrm{PG}(2,q)$ not through n, $\mathcal{C}_2 = \Omega$ and

$$\mathcal{C}_3 = \{\mathrm{Fix}(\xi) \mid \xi \in V_j, 2 \leq j \leq q-1\},$$

and let I be the natural incidence. Then Lemma 3.2.2 and Theorem 3.3.4 imply that $\mathbf{L}(\Omega) = (\mathcal{P}, \mathcal{L}, \mathcal{C}, \mathrm{I})$ is a Laguerre plane.

3.4 Classification of regular Desarguesian planar oval sets

Theorem 3.4.1 *Let Ω be a regular Desarguesian planar oval set in* $\mathrm{PG}(2,q)$ *with nucleus n, and let $O \in \Omega$. Then $\Omega = \Omega(O)$.*

Proof. Let $\Omega^* = \Omega(O)$. Since n is the nucleus of the oval O, n is the nucleus of the planar oval set Ω^*. We adopt here the notation of Sections 3.2 and 3.3 for Ω, and also for Ω^*. If confusion is possible, we add a star to the notation related to Ω^*. For example, V is the set of all collineations from $\mathrm{PG}(2,q)$ to $\pi(\Omega)$ which fix every line through n, and V^* is the set of all collineations from $\mathrm{PG}(2,q)$ to $\pi(\Omega^*)$ which fix every line through n.

As a consequence of Theorem 3.3.4, we may assume without loss of generality that $i_1 = i_1^* = q$. So for every $\xi \in V_1$, respectively $\xi^* \in V_1^*$, $\mathrm{Fix}(\xi)$, respectively $\mathrm{Fix}(\xi^*)$, is either the empty set or a $(2q)$-set of type $(0,2,q)$ with q-nucleus n, that is, the symmetric difference of the point sets of two lines of $\mathrm{PG}(2,q)$ which intersect in the point n. We prove that for any two distinct lines L and M of $\mathrm{PG}(2,q)$ which intersect in n, there is an element $\xi \in V_1$, respectively $\xi^* \in V_1^*$, such that $\mathrm{Fix}(\xi)$, respectively $\mathrm{Fix}(\xi^*)$, is the symmetric difference $L \triangle M$. We give the proof for Ω, the proof for Ω^* is analogous. Let L and M be lines of $\mathrm{PG}(2,q)$ such that $L \cap M = n$, let $p \in L \setminus \{n\}$ and let $r \in M \setminus \{n\}$. Let $\xi' \in V_1$. Then there exists an elation $e \in \mathrm{El}(n)$ such that $p^e = p^{\xi'^{-1}}$ and $r^e = r^{\xi'^{-1}}$. Now since $\xi = e\xi' \in V_1$, $\mathrm{Fix}(\xi)$ is either the empty set or a $(2q)$-set of type $(0,2,q)$. Since ξ fixes the points p and r, $\mathrm{Fix}(\xi) = L \triangle M$.

Let L and L' be two lines of $\mathrm{PG}(2,q)$ which intersect in the point n, and let $\xi \in V_1$, respectively $\xi^* \in V_1^*$, be such that $\mathrm{Fix}(\xi) = L \triangle L'$, respectively $\mathrm{Fix}(\xi^*) = L \triangle L'$. Let $p = O \cap L$, let $p' = O \cap L'$ and let $M = \langle p, p' \rangle$. Since ξ, respectively ξ^*, fixes p and p', M^ξ, respectively M^{ξ^*}, is an element of Ω, respectively Ω^*, which contains the points p and p'. Hence $M^\xi = M^{\xi^*} = O$. Let L'' be a line of $\mathrm{PG}(2,q)$ through n, distinct from L and L', and let $p'' = O \cap L''$ and $r = M \cap L''$. Then since $M^\xi = M^{\xi^*} = O$ and since ξ and ξ^* fix the line L'', $r^\xi = r^{\xi^*} = p''$. Let $e \in \mathrm{El}(n)$ be the elation with axis L which maps p'' to r. Then $\xi' = e\xi \in V_1$, respectively $\xi'^* = e\xi^* \in V_1^*$, fixes p'' and every point of L, hence $\mathrm{Fix}(\xi') = \mathrm{Fix}(\xi'^*) = L \triangle L''$. So ξ' and ξ'^* have the same action on L'', and hence so do $\xi = e^{-1}\xi'$ and $\xi^* = e^{-1}\xi'^*$. But this holds for any line L'' of $\mathrm{PG}(2,q)$ through n, distinct from L and L'. Hence $\xi = \xi^*$. Since ξ is a collineation from $\mathrm{PG}(2,q)$ to $\pi(\Omega)$, and since $\xi^* = \xi$ is a collineation from $\mathrm{PG}(2,q)$ to $\pi(\Omega^*)$, $\Omega = \Omega^* = \Omega(O)$. $\square$

Remark

Let O be an oval of $\mathrm{PG}(2,q)$ with nucleus n, and let $\Omega = \Omega(O)$. We prove that the Laguerre plane $\mathbf{L}(\Omega(O))$, constructed in Section 3.3, is embeddable.

We coordinatize the plane $\mathrm{PG}(2,q)$ in such a way that $n(1,0,0)$ and that O contains the points $(0,1,0), (0,0,1)$ and $(1,1,1)$. Let $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ be such that $O = \{(t, f(t), 1) \,|\, t \in \mathrm{GF}(q)\} \cup \{(0,1,0)\}$. Then $f(0) = 0$, $f(1) = 1$, and f is an o-polynomial.

Let $L : X_2 = 0$, let $L' : X_1 = 0$ and let $L'' : X_1 = yX_2$ for some $y \in \mathrm{GF}(q) \setminus \{0\}$. Define ξ, p, p', p'', r, M, e in the same way as in the proof of Theorem 3.4.1. Then $p(0,1,0)$, $p'(0,0,1)$, $p''(f^{-1}(y), y, 1)$, $r(0,y,1)$, $M : X_0 = 0$, and the matrix of the elation e with respect to the chosen basis is

$$\begin{pmatrix} 1 & 0 & f^{-1}(y) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows from the proof of Theorem 3.4.1 that ξ and e^{-1} have the same action on L''. Since q is even, $e^{-1} = e$. So for every point $p(x,y,1) \in L''$, $p^\xi(x + f^{-1}(y), y, 1)$. This holds for every $x \in \mathrm{GF}(q)$ and, since L'' is an arbitrary line through n, distinct from L and L', it also holds for every $y \in \mathrm{GF}(q) \setminus \{0\}$. Every point on L and L' is fixed by ξ, so we have determined explicitly the action of ξ on every point of $\mathrm{PG}(2,q)$, with respect to the chosen basis.

Consider the Laguerre plane $\mathbf{L}(\Omega(O)) = (\mathcal{P}, \mathcal{L}, \mathcal{C}, \mathrm{I})$, where $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, and let $C \in \mathcal{C}_3$. Then, by definition of $\mathcal{C}_3$, there is an element $\xi' \in V \setminus V_1$ such that $\mathrm{Fix}(\xi') = C$. By Lemma 3.2.1, $V = \mathrm{Persp}(n)\,\xi$, so since $V_1 = \mathrm{El}(n)\,\xi$ and $\xi' \in V \setminus V_1$, there is a nontrivial homology h of $\mathrm{PG}(2,q)$ with center n such that $\xi' = h\xi$. The matrix of h with respect to the chosen basis is

$$\begin{pmatrix} 1 & a & b \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix},$$

for some $a, b, c \in \mathrm{GF}(q)$ with $c \neq 0, 1$. One verifies that

$$C = \{(b + ay + cf^{-1}(y), (1+c)y, 1+c) \mid y \in \mathrm{GF}(q)\} \cup \{(a, 1+c, 0)\}.$$

Clearly C is the image of the oval O under the homology $h' \in \mathrm{Persp}(n)$ which has the following matrix with respect to the chosen basis.

$$\begin{pmatrix} c & a & b \\ 0 & 1+c & 0 \\ 0 & 0 & 1+c \end{pmatrix}.$$

We conclude that $\mathcal{C}_3$ is the set of all ovals of $\mathrm{PG}(2,q)$ which are the image of O under a nontrivial homology of $\mathrm{PG}(2,q)$ with center n. So $\mathcal{C}$ is the union of the set of lines of $\mathrm{PG}(2,q)$ not through n, with the set of ovals of $\mathrm{PG}(2,q)$ which are the images of O under all perspectivities of $\mathrm{PG}(2,q)$ with center n.

Embed $\mathrm{PG}(2,q)$ as a plane in $\mathrm{PG}(3,q)$ and consider the cone Q with vertex a point p not in $\mathrm{PG}(2,q)$ and base the oval O. Let r be a point of $\langle p, n\rangle \setminus \{p, n\}$. Then the projection of the set of circles of the Laguerre plane $\mathbf{L}(Q)$ from r on $\mathrm{PG}(2,q)$ is exactly the union of the set of lines of $\mathrm{PG}(2,q)$ not through n with the set of all ovals of $\mathrm{PG}(2,q)$ which are the image of O under a perspectivity of $\mathrm{PG}(2,q)$ with center n. So this projection induces an isomorphism from the Laguerre plane $\mathbf{L}(Q)$ to the Laguerre plane $\mathbf{L}(\Omega(O))$. Hence $\mathbf{L}(\Omega(O))$ is embeddable.

Chapter 4

Affine semipartial geometries and $(0,\alpha)$-geometries

In this chapter we start with the study of affine semipartial geometries and $(0,\alpha)$-geometries. Firstly we discuss the methods that we will use and we explain why it is necessary to study affine $(0,\alpha)$-geometries in order to get results about affine semipartial geometries. Next we construct some new geometries and prove that they are affine $(0,\alpha)$-geometries which are not linear representations. A survey of recent results follows, including a summary of the results that we obtain in Chapters 5 and 6. Our main result in this area is the classification of affine semipartial geometries and $(0,\alpha)$-geometries with $\alpha > 1$ which are not linear representations. We end this chapter with a detailed study of the different affine $(0,\alpha)$-geometries.

The construction of the geometry $\mathcal{A}(O_\infty)$ is published in [36], and the construction of the geometry $\mathcal{I}(n,q,e)$ is published in [33].

4.1 General method

One of the main goals of this thesis is to improve the results in Section 1.4.7 on affine semipartial geometries. Affine semipartial geometries are only classified when they are fully embedded in $\mathrm{AG}(2,q)$ or $\mathrm{AG}(3,q)$ (see Theorem 1.4.16). We would like to obtain results for semipartial geometries fully embedded in an affine space $\mathrm{AG}(n,q)$ of arbitrary dimension. A possible method to prove such results is to use induction on the dimension of the affine space. In this context the following lemma about affine $(0,\alpha)$-geometries with $\alpha > 1$ is very interesting.

If $\mathcal{S}$ is a $(0,\alpha)$-geometry fully embedded in $\mathrm{AG}(n,q)$, and if U is a subspace of $\mathrm{AG}(n,q)$ then we denote by X_U the set of points and lines of $\mathcal{S}$ contained in U and by $\mathcal{S}_U$ the sub incidence structure of $\mathcal{S}$ induced on X_U.

Lemma 4.1.1 *Let $\mathcal{S}$ be a $(0, \alpha)$-geometry with $\alpha > 1$ fully embedded in* $\mathrm{AG}(n, q)$*, $n \geq 3$, and let U be a proper subspace of* $\mathrm{AG}(n, q)$ *of dimension at least* 2*. Then every connected component $\mathcal{S}'$ of $\mathcal{S}_U$ that contains two intersecting lines is a $(0, \alpha)$-geometry fully embedded in a subspace of U.*

Proof. Let $\{p, L\}$ be an anti-flag of $\mathcal{S}'$. Then p and L lie in U, so the plane $\langle p, L\rangle \subseteq U$. Hence every line of $\mathcal{S}$ through p intersecting L lies in U and so is a line of $\mathcal{S}'$. It follows that $\mathcal{S}'$ satisfies (zag2) since $\mathcal{S}$ satisfies (zag2) (see Section 1.2.2). Also $\mathcal{S}'$ satisfies (zag1'). By Lemma 1.2.1 $\mathcal{S}'$ is a $(0, \alpha)$-geometry. Since $\mathcal{S}$ is fully embedded in $\mathrm{AG}(n, q)$ every point of $\mathrm{AG}(n, q)$ on a line L of $\mathcal{S}'$ is a point of $\mathcal{S}$ and since $L \subseteq U$ also of $\mathcal{S}'$. So $\mathcal{S}'$ is fully embedded in a subspace of U. $\square$

Lemma 4.1.1 provides a powerful tool for investigating affine $(0, \alpha)$-geometries with $\alpha > 1$. To illustrate this, let n be a positive integer and assume that the $(0, \alpha)$-geometries with $\alpha > 1$ fully embedded in $\mathrm{AG}(m, q)$ are classified for all $m < n$. Then if $\mathcal{S}$ is a $(0, \alpha)$-geometry with $\alpha > 1$ fully embedded in $\mathrm{AG}(n, q)$, we can take any proper subspace U of $\mathrm{AG}(n, q)$ and consider the connected components of $\mathcal{S}_U$. By Lemma 4.1.1 every connected component of $\mathcal{S}_U$ which contains two intersecting lines is a $(0, \alpha)$-geometry with $\alpha > 1$, fully embedded in some $\mathrm{AG}(m, q)$ with $m < n$. But by assumption these $(0, \alpha)$-geometries are classified! Clearly in this way we get a lot of information about the $(0, \alpha)$-geometry $\mathcal{S}$.

So we will investigate affine $(0, \alpha)$-geometries with $\alpha > 1$ by looking at the full embeddings in $\mathrm{AG}(2, q)$ first. When we are done with these, we will look at full embeddings in $\mathrm{AG}(3, q)$. After that we will look at $\mathrm{AG}(4, q)$, and so on. At a certain point it will be possible to formulate proofs for general affine spaces $\mathrm{AG}(n, q)$ by means of an induction argument on the dimension n of the affine space.

Note that a similar strategy for investigating affine semipartial geometries with $\alpha > 1$ will not work because the analogue of Lemma 4.1.1 for semipartial geometries does not hold. The problem here is that a connected component of $\mathcal{S}_U$ does not necessarily satisfy (spg3) (see Section 1.2.4). So we cannot study affine semipartial geometries as such by using induction on the dimension of the affine space. Instead we must study affine $(0, \alpha)$-geometries, and as a consequence we will get results about affine semipartial geometries.

The first step, namely the classification of $(0, \alpha)$-geometries fully embedded in $\mathrm{AG}(2, q)$, is easy.

Proposition 4.1.2 *If $\mathcal{S}$ is a $(0,\alpha)$-geometry fully embedded in* $\mathrm{AG}(2,q)$ *then $\mathcal{S}$ is a partial geometry. In particular $\mathcal{S}$ is a planar net or $\mathcal{S}$ is a dual oval and $q = 2^h$.*

Proof. Let $\{p, L\}$ be an anti-flag of $\mathcal{S}$ and suppose that $\alpha(p, L) = 0$. Then at most one line of $\mathcal{S}$ passes through p, namely the line parallel to L. But p must be on at least two lines of $\mathcal{S}$, contradiction. So $\alpha(p, L) = \alpha$ for every anti-flag of $\mathcal{S}$, and $\mathcal{S}$ is a partial geometry. We can now apply Theorem 1.4.14. $\square$

Now we can apply the classification for $\mathrm{AG}(2,q)$ to $(0,\alpha)$-geometries fully embedded in $\mathrm{AG}(n,q)$, $n \geq 3$.

Lemma 4.1.3 *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry with $\alpha > 1$ fully embedded in* $\mathrm{AG}(n,q)$*, $n \geq 3$, and let π be a plane of* $\mathrm{AG}(n,q)$*. Then π is of one of the following four types.*

Type I. *π does not contain any line of $\mathcal{S}$.*

Type II. *π contains a number of parallel lines of $\mathcal{S}$ and possibly some isolated points.*

Type III. *$\mathcal{S}_\pi$ is a planar net of order q and degree $\alpha + 1$.*

Type IV. *$\mathcal{S}_\pi$ consists of a* $\mathrm{pg}(q-1,1,2)$ *(that is, a dual oval with nucleus the line at infinity; here necessarily $q = 2^h$ and $\alpha = 2$) and possibly some isolated points.*

Proof. This is an immediate corollary to Lemma 4.1.1 and Proposition 4.1.2. $\square$

Notice that if π is a plane of type IV, then $\mathcal{S}_\pi$ has exactly one line in every parallel class of lines and through a point of $\mathcal{S}_\pi$ there are either 0 or 2 lines of $\mathcal{S}_\pi$. If π is a plane of type III then there are exactly $\alpha + 1$ lines of $\mathcal{S}_\pi$ through every point of $\mathcal{S}_\pi$.

Let $\mathcal{S}$ be an affine incidence structure. Then for every point p of $\mathcal{S}$ let θ_p denote the set of points at infinity of the lines of $\mathcal{S}$ containing p. The following lemma is a consequence of Lemma 4.1.1.

Lemma 4.1.4 *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry with $\alpha > 1$, fully embedded in* $\mathrm{AG}(n,q)$*. Then for every point p of $\mathcal{S}$, the set θ_p spans Π_∞.*

Proof. Let $(q-1,t)$ denote the order of $\mathcal{S}$. Suppose that there is a point p of $\mathcal{S}$ such that θ_p does not span Π_∞. Then there is a proper subspace U_∞ of Π_∞ which contains θ_p. Let $U = \langle p, U_\infty \rangle$, and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ containing p. By Lemma 4.1.1 $\mathcal{S}'$ is a $(0,\alpha)$-geometry. Since all the lines of $\mathcal{S}$ through p are lines of $\mathcal{S}'$, the order of $\mathcal{S}'$ is $(q-1,t)$. But this implies that for every point p' of $\mathcal{S}'$, every line of $\mathcal{S}$ through p' is in $\mathcal{S}'$. Hence, since $\mathcal{S}$ is connected, $\mathcal{S} = \mathcal{S}'$. But this contradicts the fact that $\mathcal{S}$ is not contained in a proper subspace of $\mathrm{AG}(n,q)$. □

4.2 Some new examples of affine $(0,\alpha)$-geometries

4.2.1 The $(0,2)$-geometry $\mathcal{A}(O_\infty)$

Consider $\mathrm{AG}(3,2^h)$. As usual Π_∞ denotes the plane at infinity. Let O_∞ be an oval of Π_∞ with nucleus n_∞. Choose a basis such that $\Pi_\infty : X_3 = 0, n_\infty(1,0,0,0)$ and $(0,1,0,0)$, $(0,0,1,0)$, $(1,1,1,0) \in O_\infty$. Let f be the o-polynomial such that

$$O_\infty = \{(\rho, f(\rho), 1, 0) \,|\, \rho \in \mathrm{GF}(2^h)\} \cup \{(0,1,0,0)\}.$$

For every affine point $p(x,y,z,1)$, consider the oval

$$O_\infty^p = \{(y + zf(\rho) + \rho, f(\rho), 1, 0) \,|\, \rho \in \mathrm{GF}(2^h)\} \cup \{(z,1,0,0)\}.$$

Note that if p and r are affine points then $O_\infty^p = O_\infty^r$ if and only if n_∞, p and r are collinear. Hence the set Ω_∞ of all ovals O_∞^p contains exactly 2^{2h} elements. Note also that the set Ω_∞ is the orbit of the oval O_∞ under the group of elations of Π_∞ with center n_∞. Hence every element of Ω_∞ is an oval with nucleus n_∞, and any two elements of Ω_∞ intersect in exactly one point.

For an affine point p let $\mathcal{L}_p$ be the set of lines through p and the points of O_∞^p. Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ where $\mathcal{P}$ is the point set of $\mathrm{AG}(3,2^h)$, $\mathcal{B} = \bigcup_{p \in \mathcal{P}} \mathcal{L}_p$, and I is the natural incidence.

Theorem 4.2.1 *Every connected component of $\mathcal{S}$ is a $(0,\alpha)$-geometry with $s = 2^h - 1$, $t = 2^h$ and $\alpha = 2$, fully embedded in $\mathrm{AG}(3,2^h)$.*

Proof. Let $p(x,y,z,1)$ be a point of $\mathcal{S}$. Then the elements of $\mathcal{L}_p$ are lines of $\mathcal{S}$ through p. We prove that every line of $\mathcal{S}$ through p is in the set $\mathcal{L}_p$.

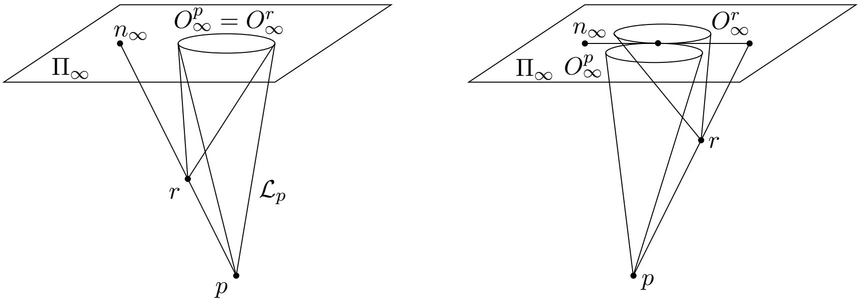

Figure 4.1: Construction of the $(0,2)$-geometry $\mathcal{A}(O_\infty)$.

Suppose that L is a line of $\mathcal{S}$ through p. Then there is a point $r(x',y',z',1)$ on L such that $L \in \mathcal{L}_r$. We may assume that $r \neq p$. Let $p_\infty = L \cap \Pi_\infty$; then $p_\infty(x+x', y+y', z+z', 0)$. Now $L \in \mathcal{L}_r$ implies $p_\infty \in O^r_\infty$. If $z' = z$ then $p_\infty(x+x', y+y', 0, 0)$ and so p_∞ must be the point with coordinates $(z',1,0,0)$. But since $z'=z$, $p_\infty(z',1,0,0) \in O^p_\infty$. So $L \in \mathcal{L}_p$. If $z' \neq z$ then $p_\infty \in O^r_\infty$ implies

$$p_\infty(x+x', y+y', z+z', 0) = (y' + z'f(\rho) + \rho, f(\rho), 1, 0)$$

for some $\rho \in \mathrm{GF}(2^h)$. Hence $f(\rho) = (y+y')/(z+z')$ and

$$(x+x')/(z+z') = y' + z'(y+y')/(z+z') + f^{-1}((y+y')/(z+z')).$$

But since

$$y' + z'(y+y')/(z+z') = y + z(y+y')/(z+z')$$

it follows that

$$p_\infty(y+z(y+y')/(z+z')+f^{-1}((y+y')/(z+z')), (y+y')/(z+z'), 1, 0) \in O^p_\infty.$$

So $L \in \mathcal{L}_p$.

It follows that through every point of $\mathcal{S}$ there are 2^h+1 lines of $\mathcal{S}$. So $\mathcal{S}$ is a partial linear space of order $(2^h-1, 2^h)$ fully embedded in $\mathrm{AG}(3,2^h)$. Hence every connected component of $\mathcal{S}$ satisfies (zag1) (see Section 1.2.2).

We prove that an affine plane containing the point n_∞, contains no two intersecting lines of $\mathcal{S}$, and that an affine plane not containing n_∞, contains no two parallel lines of $\mathcal{S}$. Suppose that π is a plane containing n_∞ and two lines L and M of $\mathcal{S}$ which intersect in an affine point p. By the preceding paragraphs $L, M \in \mathcal{L}_p$. Hence the line $L_\infty = \pi \cap \Pi_\infty$ contains two points of O^p_∞. But L_∞ contains n_∞, the nucleus of the oval O^p_∞, a contradiction.

Now suppose that π is a plane not through the point n_∞ which contains two parallel lines L and M of $\mathcal{S}$. Let $p(x, y, z, 1)$, respectively $p'(x', y', z', 1)$, be a point of $\mathcal{S}$ on L, respectively M. Since $n_\infty \notin \pi$, n_∞, p and p' are not collinear. Hence the ovals O^p_∞ and $O^{p'}_\infty$ intersect in a unique point p_∞. Looking at the explicit forms of the ovals O^p_∞ and $O^{p'}_\infty$ we see that p_∞ has coordinates $(z, 1, 0, 0)$ if $z = z'$ and

$$(y + z(y + y')/(z + z') + f^{-1}((y + y')/(z + z')), (y + y')/(z + z'), 1, 0)$$

if $z \neq z'$. In both cases one verifies that n_∞, p, p' and p_∞ are coplanar.

Let r_∞ be the point at infinity of the parallel lines L and M. Since L is a line of $\mathcal{S}$ and $p \in L$, $r_\infty \in O^p_\infty$. Since M is a line of $\mathcal{S}$ and $p' \in M$, $r_\infty \in O^{p'}_\infty$. So $r_\infty \in O^p_\infty \cap O^{p'}_\infty$. But p_∞ is the unique point of $O^p_\infty \cap O^{p'}_\infty$, so $r_\infty = p_\infty$. So n_∞, p and $M = \langle p', r_\infty \rangle = \langle p', p_\infty \rangle$ are coplanar. But then $n_\infty \in \pi$, a contradiction.

Let $\{p, L\}$ be an anti-flag of $\mathcal{S}$. There are two possibilities. The first is that the plane $\pi = \langle p, L \rangle$ contains the point n_∞. Then no two lines of $\mathcal{S}$ in π intersect, and the incidence number $\alpha(p, L) = 0$.

The second possibility is that $n_\infty \notin \pi$. Then the line $L_\infty = \pi \cap \Pi_\infty$ does not contain n_∞, so L_∞ contains 0 or 2 points of O^p_∞. Hence there are 0 or 2 lines of $\mathcal{L}_p$ in π. Since no two lines of $\mathcal{S}$ in π are parallel, $\alpha(p, L)$ equals either 0 or 2. We conclude that $\mathcal{S}$ satisfies (zag2) with $\alpha = 2$ (see Section 1.2.2). □

Theorem 4.2.2 *If O_∞ is a conic then $\mathcal{S}$ consists of two connected components $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_1 \simeq \mathcal{S}_2 \simeq$ HT. If O_∞ is not a conic then $\mathcal{S}$ is connected.*

Proof. Suppose that $\mathcal{S}$ is disconnected. We prove that $\mathcal{S}$ has exactly two connected components $\mathcal{S}_1$ and $\mathcal{S}_2$, that $\mathcal{S}_1 \simeq \mathcal{S}_2 \simeq$ HT and that O_∞ is a conic. Let $\mathcal{S}'$ be a connected component of $\mathcal{S}$, and let L be a line of $\mathcal{S}'$. From Theorem 4.2.1 it follows that $\mathcal{S}'$ is a $(0, 2)$-geometry of order $(2^h - 1, 2^h)$. Counting the flags $\{p, M\}$ of $\mathcal{S}'$ such that $p \notin L$ and such that M intersects L in an affine point we get that there are at least $2^{2h-1}(2^h - 1)$ points of $\mathcal{S}'$ not on L. Since $\mathcal{S}$ has 2^{3h} points and every connected component of $\mathcal{S}$ has at least $2^{2h-1}(2^h - 1) + 2^h$ points, it follows that $\mathcal{S}$ has exactly two connected components $\mathcal{S}_1$ and $\mathcal{S}_2$.

We prove that the point sets of $\mathcal{S}_1$ and $\mathcal{S}_2$ are sets of type $(0, 2^{h-1}, 2^h)$ with respect to affine lines. If L is a line of $\mathcal{S}$ then L contains either 0 or 2^h points of $\mathcal{S}_i$, $i = 1, 2$. Let L be an affine line intersecting Π_∞ in n_∞. For any two affine points $p, p' \in L$ we have $O^p_\infty = O^{p'}_\infty$. So the sets $\overline{\mathcal{L}_p}$ and $\overline{\mathcal{L}_{p'}}$ of affine points on the lines of $\mathcal{L}_p$ and $\mathcal{L}_{p'}$ are disjoint for any two affine points $p, p' \in L$. Since each of the sets $\overline{\mathcal{L}_p}$, $p \in L$, contains 2^{2h} points, these sets form a partition of the point set of $\mathcal{S}$. So the point set of $\mathcal{S}_i$, $i = 1, 2$, is a union of sets $\overline{\mathcal{L}_p}$, $p \in L$. Hence the number of points of $\mathcal{S}_i$, $i = 1, 2$, is a multiple of 2^{2h}. Since each $\mathcal{S}_i$, $i = 1, 2$, has at least $2^{2h-1}(2^h - 1) + 2^h$ points, we conclude that $\mathcal{S}_i$, $i = 1, 2$, has exactly 2^{3h-1} points and that L contains 2^{h-1} points of each $\mathcal{S}_i$, $i = 1, 2$.

Let L be an affine line which is not a line of $\mathcal{S}$ and which does not intersect Π_∞ in n_∞. Let M be an affine line through n_∞ which intersects L in the affine point p, let $\pi = \langle L, M\rangle$ and let $L_\infty = \pi \cap \Pi_\infty$. Then $n_\infty \in L_\infty$, so $L_\infty \cap O^p_\infty$ consists of a single point p_∞. Now for every affine point $p' \in M$ we have $O^{p'}_\infty = O^p_\infty$, so $L_\infty \cap O^{p'}_\infty = p_\infty$. Hence $\mathcal{S}_\pi$ consists of 2^h parallel lines intersecting Π_∞ in p_∞. These lines intersect L and M in affine points since $L, M \notin \mathcal{B}$. Hence from the fact that M contains 2^{h-1} points of each $\mathcal{S}_i$, $i = 1, 2$, it follows that L contains 2^{h-1} points of each $\mathcal{S}_i$, $i = 1, 2$.

So the point sets of $\mathcal{S}_i$, $i = 1, 2$, are sets of type $(0, 2^{h-1}, 2^h)$ with respect to affine lines. Hence the set $\mathcal{T}_i$ which is the union of the point set of $\mathcal{S}_i$ with Π_∞ is a set of type $(1, 2^{h-1}+1, 2^h+1)$ in $\mathrm{PG}(3, 2^h)$, $i = 1, 2$. Since every affine line that is not a line of $\mathcal{S}$ intersects $\mathcal{T}_i$ in $2^{h-1} + 1$ points, $\mathcal{T}_i$ is nonsingular, $i = 1, 2$. Now from Theorem 1.3.4 it follows that $\mathcal{T}_i$ is projectively equivalent to $\mathcal{R}_3$, $i = 1, 2$. Hence $\mathcal{S}_1 \simeq \mathcal{S}_2 \simeq \mathrm{HT}$. It is easy to see that in HT the set of lines through a given point forms a quadratic cone. Hence O_∞ is a conic.

We conclude that if $\mathcal{S}$ is disconnected then O_∞ is a conic, and $\mathcal{S}$ has exactly two connected components $\mathcal{S}_1$ and $\mathcal{S}_2$ which are both projectively equivalent to HT. It remains to be proven that if O_∞ is a conic then $\mathcal{S}$ is indeed disconnected.

Let O_∞ be a conic:

$$O_\infty = \{(\rho, \rho^2, 1, 0) \mid \rho \in \mathrm{GF}(2^h)\} \cup \{(0, 1, 0, 0)\}.$$

Let $p(x, y, z, 1)$ and $p'(x', y', z', 1)$ be distinct collinear points of $\mathcal{S}$, and let $p_\infty = \langle p, p'\rangle \cap \Pi_\infty$. Then $p_\infty(x + x', y + y', z + z', 0)$ is on the conic

$$O^p_\infty = \{(y + z\rho^2 + \rho, \rho^2, 1, 0) \mid \rho \in \mathrm{GF}(2^h)\} \cup \{(z, 1, 0, 0)\}.$$

If $z \neq z'$ then

$$p_\infty(x + x', y + y', z + z', 0) = (y + z\rho^2 + \rho, \rho^2, 1, 0)$$

for some $\rho \in \mathrm{GF}(2^h)$. So $\rho^2 = (y + y')/(z + z')$ and

$$\frac{x + x'}{z + z'} = y + z\,\frac{y + y'}{z + z'} + \left(\frac{y + y'}{z + z'}\right)^{1/2}.$$

This is equivalent to

$$x + yz + x' + y'z' = (y + y')(z + z') + ((y + y')(z + z'))^{1/2}.$$

Hence $\mathrm{Tr}(x+yz) = \mathrm{Tr}(x'+y'z')$. If $z = z'$ then $y \neq y'$ and $(x+x')/(y+y') = z$, so $x + yz = x' + y'z = x' + y'z'$. So here also $\mathrm{Tr}(x + yz) = \mathrm{Tr}(x' + y'z')$. We conclude that if $p(x, y, z, 1)$ and $p'(x', y', z', 1)$ are distinct affine points such that $\mathrm{Tr}(x + yz) \neq \mathrm{Tr}(x' + y'z')$, then there is no path from p to p' in the point graph of $\mathcal{S}$. Hence $\mathcal{S}$ is disconnected. $\square$

We let $\mathcal{A}(O_\infty)$ denote any connected component of $\mathcal{S}$. There is no confusion possible since when O_∞ is a conic the two connected components of $\mathcal{S}$ are affinely equivalent. So for any oval O_∞ the geometry $\mathcal{A}(O_\infty)$ is a $(0, 2)$-geometry with $s = 2^h - 1$ and $t = 2^h$ fully embedded in $\mathrm{AG}(3, 2^h)$.

4.2.2 The $(0, 2)$-geometry $\mathcal{I}(n, q, e)$

Let U be a hyperplane of $\mathrm{AG}(n, 2^h)$, $n \geq 3$. Choose a basis such that $\Pi_\infty : X_n = 0$ and $U : X_{n-1} = 0$. Let $e \in \{1, 2, \ldots, h - 1\}$ be such that $\gcd(e, h) = 1$, and let φ be the collineation of $\mathrm{PG}(n, 2^h)$ such that

$$\varphi : p(x_0, x_1, \ldots, x_{n-1}, x_n) \mapsto p^\varphi(x_0^{2^e}, x_1^{2^e}, \ldots, x_n^{2^e}, x_{n-1}^{2^e}).$$

Put $U_\infty = U \cap \Pi_\infty$ and let $\mathcal{K}_\infty$ be the set of points of U_∞ fixed by φ. Then

$$\mathcal{K}_\infty = \{(\varepsilon_0, \ldots, \varepsilon_{n-2}, 0, 0) \neq (0, \ldots, 0) \,|\, \varepsilon_i \in \mathrm{GF}(2), 0 \leq i \leq n - 2\},$$

so $\mathcal{K}_\infty$ is the point set of a projective geometry $\mathrm{PG}(n - 2, 2) \subseteq U_\infty$. Let

$$\mathcal{B}_1 = \{L \subseteq U \| L \not\subseteq \Pi_\infty, L \cap \Pi_\infty \in \mathcal{K}_\infty\},$$

and let

$$\mathcal{B}_2 = \{\langle p, p^\varphi \rangle \| p \in U \setminus \Pi_\infty\}.$$

Let $\mathcal{I}(n, 2^h, e) = (\mathcal{P}, \mathcal{B}_1 \cup \mathcal{B}_2, \mathrm{I})$, where $\mathcal{P}$ is the set of affine points on the lines of $\mathcal{B}_1 \cup \mathcal{B}_2$, and I is the natural incidence.

Theorem 4.2.3 *The geometry $\mathcal{I}(n, 2^h, e)$ is a $(0, \alpha)$-geometry with parameters $s = 2^h - 1$, $t = 2^{n-1} - 1$ and $\alpha = 2$, fully embedded in $\mathrm{AG}(n, 2^h)$.*

Proof. Put $\mathcal{S} = \mathcal{I}(n, 2^h, e)$. Since $\mathcal{K}_\infty$ spans U_∞, the geometry $\mathcal{S}_U$, which is the linear representation in the affine space U of the set $\mathcal{K}_\infty$, is connected. Since every point of $\mathcal{S}$ is collinear with at least one point of $\mathcal{S}_U$, $\mathcal{S}$ is connected. So $\mathcal{S}$ satisfies (zag1') (see Section 1.2.2).

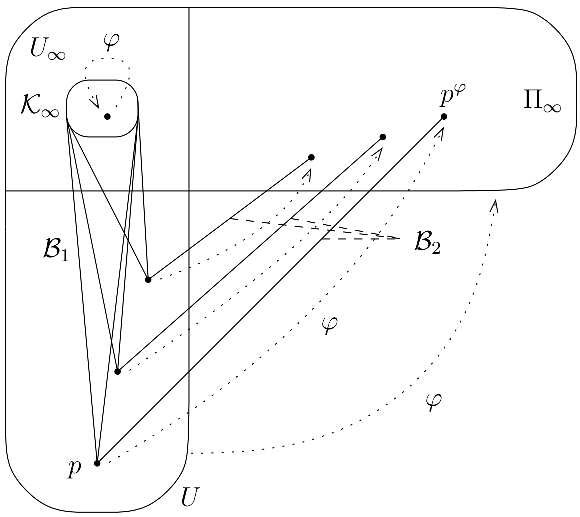

Figure 4.2: Construction of the $(0,2)$-geometry $\mathcal{I}(n,q,e)$.

Let π be a plane containing two lines L and M of $\mathcal{S}$ intersecting in an affine point p. We prove that if $\pi \subseteq U$ then π is a plane of type III containing 3 parallel classes of lines, and that if $\pi \not\subseteq U$ then π is a plane of type IV.

Let $L_\infty = \pi \cap \Pi_\infty$. If $\pi \subseteq U$, then the points at infinity of L and M are in $\mathcal{K}_\infty$, so L_∞ contains 3 points of $\mathcal{K}_\infty$. It follows that π is a plane of type III containing 3 parallel classes of lines.

Now suppose that $\pi \not\subseteq U$. Then at least one of the lines L and M is in $\mathcal{B}_2$, so π intersects U in an affine line N. Let $L_\infty = \pi \cap \Pi_\infty$ and $p_\infty = N \cap L_\infty = \pi \cap U_\infty$. First we prove that $N^\varphi = L_\infty$.

If $N \neq L, M$, then $L, M \in \mathcal{B}_2$, so L, respectively M, intersects U in an affine point r, respectively r', and Π_∞ in the point r^φ, respectively r'^φ. Since $N = \langle r, r' \rangle$ and $L_\infty = \langle r^\varphi, r'^\varphi \rangle$, we have $N^\varphi = L_\infty$. If $N \in \{L, M\}$, say $N = L$, then $L \in \mathcal{B}_1$ so $p_\infty \in \mathcal{K}_\infty$ and hence $p_\infty^\varphi = p_\infty$. Also $M \in \mathcal{B}_2$, so since $M \cap U = p$, $M \cap \Pi_\infty = p^\varphi$. It follows that

$$N^\varphi = L^\varphi = \langle p, p_\infty \rangle^\varphi = \langle p^\varphi, p_\infty \rangle = L_\infty.$$

So in any case $N^\varphi = L_\infty$. Now $p_\infty^\varphi = (N \cap U_\infty)^\varphi = L_\infty \cap U_\infty = p_\infty$, implying that $p_\infty \in \mathcal{K}_\infty$. So $N \in \mathcal{B}_1$. For every affine point $r \in N$, $r^\varphi \in L_\infty$ and hence $\langle r, r^\varphi \rangle \subseteq \pi$ is in $\mathcal{B}_2$. So π contains $2^h + 1$ lines of $\mathcal{S}$. We prove that these lines form a dual oval with nucleus L_∞.

Since $p_\infty \in \mathcal{K}_\infty$, p_∞ has coordinates $(\varepsilon_0, \ldots, \varepsilon_{n-2}, 0, 0)$, with $\varepsilon_i \in \mathrm{GF}(2)$, $0 \le i \le n-2$. Let $i \in \{0, \ldots, n-2\}$ be such that $\varepsilon_i = 1$. Then the point of intersection r of the line N with the hyperplane with equation $X_i = 0$ is an affine point. It has coordinates

$$r(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{n-2}, 0, 1).$$

Then

$$r^\varphi(x_0^{2^e}, \ldots, x_{i-1}^{2^e}, 0, x_{i+1}^{2^e}, \ldots, x_{n-2}^{2^e}, 0, 1).$$

We can coordinatize the plane π by the coordinates X_i, X_{n-1}, X_n. In this coordinate system $p_\infty(1,0,0)$, $r(0,0,1)$, $r^\varphi(0,1,0)$, $N[0,1,0]$ and $L_\infty[0,0,1]$. The lines of $\mathcal{S}$ in the plane π are the line N and the lines $\langle r', r'^\varphi\rangle$, for all affine points $r' \in N$. The affine points $r' \in N$ have coordinates

$$(x_0 + x\varepsilon_0, \ldots, x_{i-1} + x\varepsilon_{i-1}, x, x_{i+1} + x\varepsilon_{i+1}, \ldots, x_{n-2} + x\varepsilon_{n-2}, 0, 1)$$

with $x \in \mathrm{GF}(2^h)$. In the new coordinate system, $r'(x,0,1)$, $r'^\varphi(x^{2^e},1,0)$ and $\langle r', r'^\varphi\rangle = [1, x^{2^e}, x]$. So the lines of $\mathcal{S}$ in π have coordinates $[0,1,0]$ and $[1, x^{2^e}, x]$, $x \in \mathrm{GF}(2^h)$. Clearly they form a translation oval in the dual plane of π, with nucleus $L_\infty[0,0,1]$. So π is a plane of type IV.

Since every plane containing two intersecting lines of $\mathcal{S}$ is a plane of type III with 3 parallel classes of lines or a plane of type IV, we have property (zag2) (see Section 1.2.2) with $\alpha = 2$. By Lemma 1.2.1 $\mathcal{S}$ is a $(0,2)$-geometry. The order of $\mathcal{S}$ is $(2^h - 1, 2^{n-1} - 1)$ since through an affine point of U there are $|\,\mathcal{K}_\infty\,| = 2^{n-1} - 1$ lines of $\mathcal{B}_1$ and one line of $\mathcal{B}_2$. $\square$

4.3 Survey of recent results

A first and important result on affine $(0,\alpha)$-geometries with $\alpha > 1$ is the following theorem.

Theorem 4.3.1 (De Clerck, Delanote [27]) *If $\mathcal{S}$ is a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in* $\mathrm{AG}(n,q)$ *and if there are no planes of type* IV, *then $\mathcal{S} \simeq T^*_{n-1}(\mathcal{K}_\infty)$ is a linear representation of a set $\mathcal{K}_\infty$ in Π_∞. Also, if q is odd or $\alpha > 2$, then the same conclusions hold without restriction on the types of planes.*

As a consequence of Theorem 4.3.1 we are now left with two distinct problems.

Problem 1. Classify all linear representations of $(0, \alpha)$-geometries, $\alpha > 1$. Equivalently, classify all sets of type $(0, 1, k)$ with $k > 2$ in $\mathrm{PG}(n, q)$.

Problem 2. Classify all $(0, 2)$-geometries fully embedded in $\mathrm{AG}(n, 2^h)$ which have at least one plane of type IV.

There is no hope of solving Problem 1 in its full extent. However the following partial solution is known.

Theorem 4.3.2 (Ueberberg [83]) *Let $\mathcal{K}$ be a set of type $(0, 1, k)$ in $\mathrm{PG}(n, q)$, $n \geq 2$, not contained in a hyperplane. If $k \geq \sqrt{q} + 1$ then one of the following possibilities occurs.*

1. *$n = 2$ and $\mathcal{K}$ is a maximal arc.*
2. *$n = 2$, q is a square and $\mathcal{K}$ is a unital.*
3. *q is a square and $\mathcal{K}$ is the point set of a Baer subspace.*
4. *$\mathcal{K}$ is the complement of a hyperplane of $\mathrm{PG}(n, q)$.*
5. *$\mathcal{K}$ is the point set of $\mathrm{PG}(n, q)$.*

Instead of restricting the value of k in Problem 1, one can also consider the restriction of Problem 1 to semipartial geometries.

Problem 1'. Classify all linear representations of semipartial geometries with $\alpha > 1$. Equivalently, classify all sets of type $(0, 1, k)$ with $k > 2$ in $\mathrm{PG}(n, q)$ which have two sizes of intersection with respect to hyperplanes of $\mathrm{PG}(n, q)$.

Problem 1' has only been solved for small dimensions. In the case of linear representations of semipartial geometries in $\mathrm{AG}(2, q)$ and $\mathrm{AG}(3, q)$ the solution is given by Theorem 1.4.16. The case of $\mathrm{AG}(4, q)$ was solved recently by De Winter.

Theorem 4.3.3 (De Winter [38]) *If a linear representation $T_3^*(\mathcal{K}_\infty)$ in $\mathrm{AG}(4, q)$ is a proper semipartial geometry with $\alpha > 1$, then q is a square and $\mathcal{K}_\infty$ is the point set of a Baer subspace of Π_∞.*

Problem 2 is the subject of Chapters 5 and 6 of this thesis. Note that most of the known $(0, \alpha)$-geometries with $\alpha > 1$, which are fully embedded in $\mathrm{AG}(n, q)$ have $\alpha = 2$ and have planes of type IV. Using the method of induction on the dimension of the affine space described in Section 4.1, we solve Problem 2 completely.

Theorem 4.3.4 *If $\mathcal{S}$ is a $(0,2)$-geometry fully embedded in* $\mathrm{AG}(n,q)$*,* $q=2^h$*, such that there is at least one plane of type* IV*, then one of the following cases holds.*

1. $q=2$ *and $\mathcal{S}$ is a* $2-(t+2,2,1)$*-design.*
2. $n=2$ *and $\mathcal{S}$ is a dual oval.*
3. $n=3$ *and* $\mathcal{S}\simeq\mathcal{A}(O_\infty)$*.*
4. $n=4$ *and* $\mathcal{S}\simeq\mathrm{TQ}(4,q)$*.*
5. $n\geq 3$ *and* $\mathcal{S}\simeq\mathcal{I}(n,q,e)$*.*

The case $q=2$ in Theorem 4.3.4 is trivial.

Proposition 4.3.5 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in* $\mathrm{AG}(n,2)$*. Then $\mathcal{S}$ is a* $2-(t+2,2,1)$ *design.*

Proof. The order of $\mathcal{S}$ is $(1,t)$, and by connectedness any two points of $\mathcal{S}$ are forced to be collinear. So $\mathcal{S}$ is a $2-(t+2,2,1)$ design. □

The most difficult part of the proof of Theorem 4.3.4 is when $n=3$. This case is solved in Chapter 5. The general case $n\geq 4$ of Theorem 4.3.4 is solved in Chapter 6.

Theorem 4.3.4 has some interesting corollaries.

Corollary 4.3.6 *If $\mathcal{S}$ is a $(0,\alpha)$-geometry with $\alpha>1$, fully embedded in* $\mathrm{AG}(n,q)$*, then one of the following possibilities holds.*

1. $q=2$ *and $\mathcal{S}$ is a* $2-(t+2,2,1)$*-design.*
2. $n=2$*,* $q=2^h$ *and $\mathcal{S}$ is a dual oval.*
3. $n=3$*,* $q=2^h$ *and* $\mathcal{S}\simeq\mathcal{A}(O_\infty)$*.*
4. $n=4$*,* $q=2^h$ *and* $\mathcal{S}\simeq\mathrm{TQ}(4,q)$*.*
5. $n\geq 3$*,* $q=2^h$ *and* $\mathcal{S}\simeq\mathcal{I}(n,q,e)$*.*
6. $n\geq 2$ *and* $\mathcal{S}\simeq T^*_{n-1}(\mathcal{K}_\infty)$*, with $\mathcal{K}_\infty$ a set of type* $(0,1,\alpha+1)$ *in* Π_∞ *which spans* Π_∞*.*

Proof. This follows immediately from Theorems 4.3.1 and 4.3.4. □

Corollary 4.3.7 *If $\mathcal{S}$ is a semipartial geometry with $\alpha > 1$, fully embedded in* $\mathrm{AG}(n, q)$*, then one of the following possibilities occurs.*

1. $q = 2$ *and* $\mathcal{S}$ *is a* $2-(t+2, 2, 1)$*-design.*
2. $n = 2$, $q = 2^h$ *and* $\mathcal{S}$ *is a dual oval.*
3. $n = 4$, $q = 2^h$ *and* $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.
4. $n \geq 2$ *and* $\mathcal{S} \simeq T^*_{n-1}(\mathcal{K}_\infty)$*, with* $\mathcal{K}_\infty$ *a set of type* $(0, 1, \alpha + 1)$ *in* Π_∞ *which spans* Π_∞ *and has two intersection numbers with respect to hyperplanes of* Π_∞.

Proof. This follows immediately from Theorem 1.4.13 and Corollary 4.3.6. $\square$

Corollary 4.3.8 *If $\mathcal{S}$ is a proper semipartial geometry with $\alpha > 1$, fully embedded in* $\mathrm{AG}(n, q)$, $n \leq 4$*, then one of the following possibilities occurs.*

1. $n = 3$, q *is a square and* $\mathcal{S} \simeq T^*_2(\mathcal{U}_\infty)$*, with* $\mathcal{U}_\infty$ *a unital of* Π_∞.
2. $n = 4$, $q = 2^h$ *and* $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.
3. $n \in \{3, 4\}$, q *is a square and* $\mathcal{S} \simeq T^*_{n-1}(\mathcal{B}_\infty)$*, with* $\mathcal{B}_\infty$ *a Baer subspace of* Π_∞.

Proof. This follows from Theorems 1.4.16, 4.3.3 and Corollary 4.3.7. $\square$

Due to the recent progress regarding Problem 1', made by De Winter [38], we conjecture that Corollary 4.3.8 is true for all values of n.

4.4 A combinatorial study of the affine $(0, \alpha)$-geometries

The purpose of this section is to gain insight in the structure of the known affine $(0, \alpha)$-geometries. Also we deduce some properties that we will need in later chapters. Firstly we introduce some new terminology.

Let $\mathcal{S}$ be a $(0, \alpha)$-geometry with $\alpha > 1$, fully embedded in $\mathrm{AG}(n, q)$, $n \geq 4$, and let U be an m-space of $\mathrm{AG}(n, q)$ with $3 \leq m \leq n - 1$. Then U can be of the following four types.

Type A. Only defined for $m = 3$ or 4. If $m = 3$ then $\mathcal{S}_U$ contains a connected component $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$. If $m = 4$ then $\mathcal{S}_U$ contains a connected component $\mathcal{S}' \simeq \mathrm{TQ}(4, q)$.

Type B. $\mathcal{S}_U$ contains a connected component $\mathcal{S}' \simeq \mathcal{I}(m, q, e)$.

Type C. $\mathcal{S}_U$ is a connected linear representation.

Type D. Every connected component of $\mathcal{S}_U$ is contained in a proper subspace of U.

Note that at this stage we cannot say that every m-space U with $m \geq 3$ is of type **A**, **B**, **C** or **D**. However we can prove that if U is of one type, then U is not of any other type. Indeed, if U is of type **C** or **D** then obviously U can not be of any other type. Suppose that U is at the same time of type **A** and type **B**. Then m equals 3 or 4 and $\mathcal{S}_U$ contains two connected components $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_2 \simeq \mathcal{I}(m, q, e)$ and $\mathcal{S}_1 \simeq \mathcal{A}(O_\infty)$ if $m = 3$ or $\mathcal{S}_1 \simeq \mathrm{TQ}(4, q)$ if $m = 4$. From the construction of $\mathcal{I}(n, q, e)$ it follows that there is an $(m-1)$-space $V \subseteq U$ completely contained in the point set of $\mathcal{S}_2$. Now since $\mathcal{S}_1$ and $\mathcal{S}_2$ are two distinct connected components, their point sets must be disjoint, so $\mathcal{S}_1$ does not contain any point of V. But V is a hyperplane of the affine space U. It follows from the construction of $\mathcal{A}(O_\infty)$ and $\mathrm{TQ}(4, q)$ that this is impossible. So if U is of type **A**, **B**, **C** or **D** then it is not of any other type.

4.4.1 The $(0, 2)$-geometry HT

Consider the $(0, 2)$-geometry HT fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, constructed in Section 1.4.7. We proved in Theorem 4.2.2 that $\mathrm{HT} \simeq \mathcal{A}(O_\infty)$ with O_∞ a conic. So we refer to section 4.4.2 for some properties of HT.

4.4.2 The $(0, 2)$-geometry $\mathcal{A}(O_\infty)$

Consider the geometry $\mathcal{A}(O_\infty)$ fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, constructed in Section 4.2.1. We call the point n_∞ the *hole* of the geometry $\mathcal{A}(O_\infty)$. Notice that from the construction of $\mathcal{A}(O_\infty)$ it follows that none of the affine lines through the hole n_∞ is a line of $\mathcal{A}(O_\infty)$.

Lemma 4.4.1 *If O_∞ is a conic then the point set of the geometry $\mathcal{A}(O_\infty)$ is a set of type $(0, \frac{1}{2}q, q)$ of size $\frac{1}{2}q^3$. If O_∞ is an oval which is not a conic then the point set of $\mathcal{A}(O_\infty)$ is the set of all affine points.*

Proof. This follows immediately from the construction of $\mathcal{A}(O_\infty)$ and Theorem 4.2.2. □

Lemma 4.4.2 *The geometry $\mathcal{A}(O_\infty)$ has no planar nets.*

Proof. In the proof of Theorem 4.2.1 it was shown that a plane of $\mathrm{AG}(3,q)$ containing the hole n_∞ does not contain two intersecting lines of $\mathcal{A}(O_\infty)$, and that a plane of $\mathrm{AG}(3,q)$ not containing the hole does not contain two parallel lines of $\mathcal{S}$. Hence there are no planes of type III. □

Lemma 4.4.3 *Every line L of $\mathcal{A}(O_\infty)$ is contained in one plane of type* II *and q planes of type* IV.

Proof. Let $\mathcal{S} = \mathcal{A}(O_\infty)$. From Lemmas 4.1.3 and 4.4.2 it follows that every plane containing two intersecting lines of $\mathcal{S}$ is of type IV. Let L be a line of $\mathcal{S}$ and p an affine point of L. A plane $\pi \supseteq L$ is of type IV if and only if $\mathcal{S}_\pi$ contains two lines through p, so if and only if the line $L_\infty = \pi \cap \Pi_\infty$ is secant to O_∞^p. But L_∞ contains $p_\infty = L \cap \Pi_\infty \in O_\infty^p$, so only if L_∞ contains the hole n_∞, L_∞ is not secant to O_∞^p. So there are q planes of type IV through L and one of type II. □

Lemma 4.4.4 *The set of planes of type* II *of $\mathcal{A}(O_\infty)$ is exactly the set of planes of* $\mathrm{AG}(3,q)$ *containing the hole n_∞. Every plane of type* II *contains exactly k lines of $\mathcal{A}(O_\infty)$, but no isolated points, with $k = \frac{1}{2}q$ when O_∞ is a conic and $k = q$ otherwise. Every point of Π_∞ different from the hole n_∞ occurs exactly once as the point at infinity of the k lines of $\mathcal{A}(O_\infty)$ in a plane of type* II.

Proof. Let $\mathcal{S} = \mathcal{A}(O_\infty)$. In the proof of Theorem 4.2.1 it was shown that a plane of $\mathrm{AG}(3,q)$ containing the hole n_∞ does not contain two intersecting lines of $\mathcal{A}(O_\infty)$. Let π be a plane containing n_∞, let $L_\infty = \pi \cap \Pi_\infty$, and suppose that p is a point of $\mathcal{S}$ in π. Then L_∞ contains the nucleus n_∞ of O_∞^p, so there is exactly one line in $\mathcal{S}_\pi$ through p, and there are no isolated points in $\mathcal{S}_\pi$. Since no two lines of $\mathcal{S}_\pi$ intersect, $\mathcal{S}_\pi$ consists of a number k of parallel lines. Hence a line $L \subseteq \pi$ through the hole n_∞ contains exactly k points of $\mathcal{S}$. But in the proof of Theorem 4.2.2 it was shown that a line through n_∞ contains $\frac{1}{2}q$ points of $\mathcal{S}$ if O_∞ is a conic and q otherwise. So $k = \frac{1}{2}q$ if O_∞ is a conic and $k = q$ otherwise.

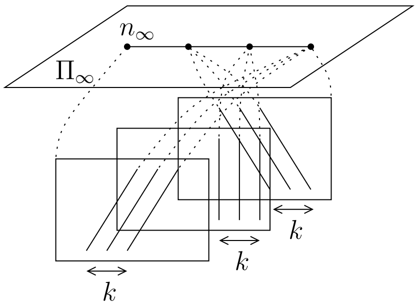

Figure 4.3: A parallel class of planes containing the hole of the geometry $\mathcal{A}(O_\infty)$.

Suppose that there is a plane π of type II which does not contain the hole n_∞. Let L be a line of $\mathcal{S}_\pi$ and let $\pi' = \langle n_\infty, L\rangle$. Then π' is of type II. But then L is contained in two planes of type II, a contradiction to Lemma 4.4.3. So the planes of type II are exactly those containing the hole n_∞.

The number of planes of type II is equal to the number of points of Π_∞ different from n_∞. Suppose that two planes π_1 and π_2 of type II contain lines of $\mathcal{S}$ that are parallel. Then there are planes containing a line of $\mathcal{S}_{\pi_1}$ and a line of $\mathcal{S}_{\pi_2}$, so two parallel lines, but not containing n_∞, a contradiction. So every point of Π_∞ different from n_∞ occurs exactly once as the point at infinity of the lines of $\mathcal{S}$ in a plane of type II. $\square$

Lemma 4.4.5 *Consider a parallel class of planes in* $\mathrm{AG}(3, q)$ *not containing the hole* n_∞ *of the geometry* $\mathcal{A}(O_\infty)$*. If* O_∞ *is a conic then half of the planes in this parallel class are of type* IV *and contain no isolated points, and the other half are of type* I *and contain* $\frac{1}{2}q(q-1)$ *points of* $\mathcal{A}(O_\infty)$*. If* O_∞ *is not a conic then every plane in the parallel class is of type* IV *and contains* $\frac{1}{2}q(q-1)$ *isolated points.*

Proof. Let $\pi_1, \dots, \pi_q$ denote the q planes of a parallel class of planes not containing the hole n_∞, and let L_∞ be the line at infinity of this parallel class. By Lemma 4.4.2 the plane π_i is not of type III and by Lemma 4.4.4 π_i is not of type II, $i = 1, \dots, q$. So if a plane π_i contains a line of $\mathcal{A}(O_\infty)$ then it is of type IV.

Assume that O_∞ is a conic. Let $\mathcal{S} = \mathcal{A}(O_\infty)$ and let $\mathcal{S}'$ be the geometry which is projectively equivalent to $\mathcal{S}$ and has as point set the complement of the point set of $\mathcal{S}$ (see Theorem 4.2.2). Let $p_\infty \in L_\infty$. Then by Lemma 4.4.4

there are exactly $\frac{1}{2}q$ lines of $\mathcal{S}$ through p_∞. No two of these lines lie in the same plane π_i since then π_i would be of type II or of type III. So there are at least $\frac{1}{2}q$ planes of type IV in the parallel class. But if π_i is of type IV then it contains a line of $\mathcal{S}$ through p_∞. So there are exactly $\frac{1}{2}q$ planes of type IV in the parallel class.

In the proof of Theorem 4.2.2 it was shown that a line that is neither a line of $\mathcal{S}$ nor of $\mathcal{S}'$ contains exactly $\frac{1}{2}q$ points of both $\mathcal{S}$ and $\mathcal{S}'$. Suppose that a plane π_i which is of type IV with respect to $\mathcal{S}$ has an isolated point p of $\mathcal{S}$. Then a line L such that $p \in L \subseteq \pi_i$ is not a line of $\mathcal{S}$, so it contains 0 or $\frac{1}{2}q$ points of $\mathcal{S}$. But L contains p and L intersects the dual oval of $\mathcal{S}_{\pi_i}$ in $\frac{1}{2}q$ points, a contradiction. So a plane of type IV with respect to $\mathcal{S}$ contains no isolated points of $\mathcal{S}$. Similarly a plane π_i of type IV with respect to $\mathcal{S}'$ contains no isolated points of $\mathcal{S}'$. This implies that, with respect to $\mathcal{S}$, there are $\frac{1}{2}q$ planes of type IV in the parallel class containing no isolated points and $\frac{1}{2}q$ planes of type I containing $\frac{1}{2}q(q-1)$ isolated points.

Now assume that O_∞ is not a conic. Let $\mathcal{S} = \mathcal{A}(O_\infty)$ and let $p_\infty \in L_\infty$. Then by Lemma 4.4.4 there are exactly q lines of $\mathcal{S}$ through p_∞. No two of these lines lie in the same plane π_i. So every plane in the parallel class is of type IV. Since every affine point is a point of $\mathcal{S}$, every plane π_i contains $\frac{1}{2}q(q-1)$ isolated points. $\square$

4.4.3 The semipartial geometry $\mathrm{TQ}(4, q)$

Consider the geometry $\mathrm{TQ}(4,q)$ fully embedded in $\mathrm{AG}(4,q)$, $q = 2^h$, constructed in Section 1.4.7. The point set of $\mathrm{TQ}(4,q)$ is the set $\mathcal{R}_4^- \setminus \Pi_\infty$. In this section $\mathrm{PG}(4,q)$ will be embedded as a hyperplane in $\mathrm{PG}(5,q)$, r will denote a point of $\mathrm{PG}(5,q)$ not in $\mathrm{PG}(4,q)$ and $\mathrm{Q}^-(5,q)$ will denote a nonsingular elliptic quadric in $\mathrm{PG}(5,q)$ such that the projection of $\mathrm{Q}^-(5,q)$ from r on $\mathrm{PG}(4,q)$ is $\mathcal{R}_4^-$. The hyperplane at infinity Π_∞ of $\mathrm{AG}(4,q)$ is then the intersection of $\mathrm{PG}(4,q)$ with the hyperplane $U_r(\mathrm{Q}^-(5,q)) = r^\beta$, where β is the symplectic polarity associated with $\mathrm{Q}^-(5,q)$ (see section 1.3.1). Recall that the set of lines through r in r^β is exactly the set of lines through r that are tangent to $\mathrm{Q}^-(5,q)$, and that $r^\beta \cap \mathrm{Q}^-(5,q)$ is a nonsingular parabolic quadric with nucleus r.

Lemma 4.4.6 *The point set of the geometry* $\mathrm{TQ}(4,q)$ *is a set of type* $(0, \frac{1}{2}q, q)$ *of size* $\frac{1}{2}q^2(q^2-1)$.

Proof. This follows immediately from the construction of $\mathrm{TQ}(4,q)$. $\square$

Lemma 4.4.7 *Consider* $\mathcal{S} = \mathrm{TQ}(4,q)$ *and let* U *be a hyperplane of* $\mathrm{AG}(4,q)$*. Then either* U *is of type* **A** *and* $\mathcal{S}_U \simeq \mathrm{HT}$*, or* U *is of type* **D** *and* $\mathcal{S}_U$ *consists of* $\frac{1}{2}q(q-1)$ *parallel lines.*

Proof. Let V be the hyperplane of $\mathrm{PG}(5,q)$ spanned by U and r. There are two possibilities for the quadric $\mathcal{Q}_4 = V \cap \mathrm{Q}^-(5,q)$. The first is that $\mathcal{Q}_4$ is a nonsingular parabolic quadric in V. Since $V \neq r^\beta$, r is not the nucleus of $\mathcal{Q}_4$, and so $\mathcal{S}_U \simeq \mathrm{HT}$. The second possibility is that $\mathcal{Q}_4$ is a cone with vertex a point $p \in r^\beta$ and base a nonsingular elliptic quadric in a 3-space not containing p. In this case p is projected from r on a point p_∞ of Π_∞, and $\mathcal{S}_U$ consists of the projection from r on $\mathrm{AG}(4,q)$ of the $q(q-1)$ lines of $\mathcal{Q}_4$ that are not in r^β. This projection yields $\frac{1}{2}q(q-1)$ lines of $\mathrm{AG}(4,q)$ which intersect Π_∞ in the point p_∞. □

Lemma 4.4.8 *Consider* $\mathcal{S} = \mathrm{TQ}(4,q)$ *and let* π *be a plane of* $\mathrm{AG}(4,q)$*. Then* π *is of type* I *and it contains* $\frac{1}{2}q(q-1)$ *points of* $\mathrm{TQ}(4,q)$*, or* π *is of type* II *and it contains* $\frac{1}{2}q$ *lines of* $\mathrm{TQ}(4,q)$ *and no isolated points, or* π *is of type* IV *and it contains no isolated points.*

The number of hyperplanes of type **A** *through* π *is* $q-1, q$ *or* $q+1$ *according to* π *being of type* I*, of type* II *or of type* IV*.*

Proof. By Lemma 4.4.7 every hyperplane through π is of type **A** or of type **D**. Since $\mathcal{S}$ has order $(q-1, q^2)$, π is contained in a hyperplane U of type **A**. By Lemma 4.4.7 $\mathcal{S}_U$ is connected and $\mathcal{S}_U \simeq \mathrm{HT}$. Now since $\mathrm{HT} \simeq \mathcal{A}(O_\infty)$ with O_∞ a conic, the first part of the lemma follows from Lemmas 4.4.2, 4.4.4 and 4.4.5.

The second part of the lemma follows from the first part and from Lemma 4.4.7 by counting the total number of lines of $\mathcal{S}$ through a point of $\mathcal{S}_\pi$. □

Lemma 4.4.9 *Consider* $\mathcal{S} = \mathrm{TQ}(4,q)$ *and let* $U_1, \ldots, U_q$ *be a parallel class of hyperplanes of* $\mathrm{AG}(4,q)$*. Then there is one hyperplane, say* U_1*, which is of type* **D***. Let* n_∞ *be the point at infinity of the lines of* $\mathrm{TQ}(4,q)$ *in* U_1*. For every* $i = 2, \ldots, q$ *the hyperplane* U_i *is of type* **A** *and* n_∞ *is the hole of* $\mathcal{S}_{U_i}$*.*

Proof. Let π_∞ be the plane at infinity of the parallel class $\{U_1, \ldots, U_q\}$. Let $W = \langle r, \pi_\infty\rangle$ and for every $i = 1, \ldots, q$, let $V_i = \langle r, U_i\rangle$. Since $r \subseteq W \subseteq r^\beta$, and since $r^\beta \cap \mathrm{Q}^-(5,q)$ is a nonsingular parabolic quadric with nucleus r, $W \cap \mathrm{Q}^-(5,q)$ is a cone $\mathcal{C}$ with vertex a point p and base a nonsingular conic in a plane not containing p.

Consider the $q+1$ hyperplanes of $\mathrm{PG}(5, q)$ through W: $r^\beta, V_1, \ldots, V_q$. Exactly one of them, namely the tangent hyperplane $T_p(\mathrm{Q}^-(5, q)) = p^\beta$ at p, intersects $\mathrm{Q}^-(5, q)$ in a cone with vertex p and base a nonsingular elliptic quadric in a 3-space not containing p. The other hyperplanes intersect $\mathrm{Q}^-(5, q)$ in nonsingular parabolic quadrics. Clearly $p^\beta \neq r^\beta$, so without loss of generality we may put $V_1 = p^\beta$. Hence U_1 is of type **D**, and the lines of $\mathcal{S}$ in U_1 intersect Π_∞ in the point $n_\infty = \langle p, r \rangle \cap \mathrm{PG}(4, q)$.

The hyperplane V_i, $i \in \{2, \ldots, q\}$, intersects $\mathrm{Q}^-(5, q)$ in a nonsingular parabolic quadric $\mathcal{Q}_4$, so U_i is of type **A**. The lines on $\mathcal{Q}_4$ that contain the point p are the lines of the cone $\mathcal{C} = W \cap \mathrm{Q}^-(5, q)$. So $\mathcal{S}_{U_i}$ has no lines which intersect Π_∞ in the point n_∞, that is, n_∞ is the hole of $\mathcal{S}_{U_i}$. $\square$

4.4.4 The $(0, 2)$-geometry $\mathcal{I}(n, q, e)$

Consider the $(0, 2)$-geometry $\mathcal{I}(n, q, e)$ fully embedded in $\mathrm{AG}(n, q)$, with $q = 2^h$ and $e \in \{1, \ldots, h-1\}$ such that $\gcd(e, h) = 1$, which was constructed in Section 4.2.2. We recall that in the construction, the hyperplane U plays a special role.

Lemma 4.4.10 *A plane π containing two intersecting lines of $\mathcal{I}(n, q, e)$ is of type* III *if $\pi \subseteq U$ and of type* IV *if $\pi \not\subseteq U$, in which case $\pi \cap U$ is a line of $\mathcal{I}(n, q, e)$.*

Proof. This was shown in the proof of Theorem 4.2.3. $\square$

Proposition 4.4.11 *There is exactly one hyperplane of type* **C** *(respectively of type* III *if $n = 3$), namely U. There are no lines of $\mathcal{I}(n, q, e)$ parallel to U except those in U. Through every affine point of U there is exactly one line of $\mathcal{I}(n, q, e)$ which is not contained in U. Through every point of Π_∞ not in U_∞ there is exactly one line of $\mathcal{I}(n, q, e)$. Hence no two lines of $\mathcal{I}(n, q, e)$ which are not contained in U are parallel.*

Proof. The first statement is a consequence of Lemma 4.4.10, the rest follows immediately from the construction of $\mathcal{I}(n, q, e)$. $\square$

Proposition 4.4.12 *The number of points of the geometry $\mathcal{I}(n, q, e)$ is $(\frac{1}{2}q)^{n-1}(q + 2^{n-1} - 1)$, and the number of lines is $q^{n-2}(q + 2^{n-1} - 1)$.*

Proof. From the construction of $\mathcal{I}(n,q,e)$ it follows that the number of lines is $|\mathcal{B}_1| + |\mathcal{B}_2| = q^{n-2}(2^{n-1}-1)+q^{n-1}$. Counting the number of flags of $\mathcal{I}(n,q,e)$ yields the number of points. $\square$

Lemma 4.4.13 *A subspace $V_\infty \subseteq U_\infty$ is fixed by φ if and only if the set $V_\infty \cap \mathcal{K}_\infty$ spans V_∞.*

Proof. If $V_\infty \cap \mathcal{K}_\infty$ spans V_∞, then clearly $V_\infty^\varphi = V_\infty$. Now suppose that φ fixes a j-dimensional subspace V_∞ of U_∞. Without loss of generality we may assume that V_∞ can be coordinatized by the coordinates $X_0,\ldots,X_j$. This means that there exist homogeneous linear functions

$$f_i : \mathrm{GF}(q)^{j+1} \to \mathrm{GF}(q), \quad i = j+1,\ldots,n-2,$$

such that every point p_∞ of V_∞ has coordinates

$$p_\infty(x_0,\ldots,x_j, f_{j+1}(x_0,\ldots,x_j),\ldots,f_{n-2}(x_0,\ldots,x_j),0,0).$$

Let $p_\infty(\varepsilon_0,\ldots,\varepsilon_j, f_{j+1}(\varepsilon_0,\ldots,\varepsilon_j),\ldots,f_{n-2}(\varepsilon_0,\ldots,\varepsilon_j),0,0) \in V_\infty$, with $\varepsilon_0,\ldots,\varepsilon_j \in \mathrm{GF}(2)$. Then

$$p_\infty^\varphi(\varepsilon_0,\ldots,\varepsilon_j, f_{j+1}(\varepsilon_0,\ldots,\varepsilon_j)^{2^e},\ldots,f_{n-2}(\varepsilon_0,\ldots,\varepsilon_j)^{2^e},0,0).$$

However, since $p_\infty^\varphi \in V_\infty^\varphi = V_\infty$,

$$p_\infty^\varphi(\varepsilon_0,\ldots,\varepsilon_j, f_{j+1}(\varepsilon_0,\ldots,\varepsilon_j),\ldots,f_{n-2}(\varepsilon_0,\ldots,\varepsilon_j),0,0).$$

So for $i = j+1,\ldots,n-2$, $f_i(\varepsilon_0,\ldots,\varepsilon_j)^{2^e} = f_i(\varepsilon_0,\ldots,\varepsilon_j)$. Hence $p_\infty^\varphi = p_\infty$. It follows that $\langle V_\infty \cap \mathcal{K}_\infty\rangle = V_\infty$. $\square$

In the following theorem V is an m-dimensional subspace of $\mathrm{AG}(n,q)$, $V' = V \cap U$, $V_\infty = V \cap \Pi_\infty$, $V'_\infty = V \cap U_\infty$ and $X'_\infty = \langle V'_\infty \cap \mathcal{K}_\infty\rangle$. Also $W_\infty = V'^\varphi \cap V_\infty$, $W' = W_\infty^{\varphi^{-1}}$ and $W'_\infty = W' \cap W_\infty$. Let l denote the dimension of X'_∞. Note that $X'_\infty \subseteq W'_\infty$. Indeed, let $p_\infty \in V'_\infty \cap \mathcal{K}_\infty$. Then $p_\infty^\varphi = p_\infty$. Since $p_\infty \in V'$, $p_\infty^\varphi = p_\infty \in V'^\varphi$. Since $p_\infty \in V_\infty$, $p_\infty \in W_\infty$. Hence $p_\infty^{\varphi^{-1}} = p_\infty \in W'$. So $p_\infty \in W'_\infty$. It follows that $V'_\infty \cap \mathcal{K}_\infty \subseteq W'_\infty$. Hence $X'_\infty \subseteq W'_\infty$. See also Figure 4.4.

Theorem 4.4.14 *Let $\mathcal{S} = \mathcal{I}(n,q,e)$, and let V be an m-dimensional subspace of $\mathrm{AG}(n,q)$, $n \geq 3$, $2 \leq m \leq n-1$. Then we have the following possibilities for V.*

1. *$V \subseteq U$. Then $\mathcal{S}_V$ is the linear representation in V of the set $V'_\infty \cap \mathcal{K}_\infty$. If $X'_\infty = V'_\infty$ then $\mathcal{S}_V$ is connected, so V is of type* **C** *(respectively of type* III *if $m = 2$), and if $X'_\infty \neq V'_\infty$ then V is of type* **D** *(respectively of type* I *or* II *if $m = 2$).*

2. *$V \nsubseteq U$ and V is parallel to U. Then V is of type* **D** *(respectively of type* I *if $m = 2$) and $\mathcal{S}_V$ does not contain any lines.*

3. *V is not parallel to U. Then we have the following possibilities.*

   (a) *$W_\infty \subseteq U_\infty$. Then V is of type* **D** *(respectively of type* I *or* II *if $m = 2$). More specifically $\mathcal{S}_V$ consists of the linear representation in V' of $V'_\infty \cap \mathcal{K}_\infty$ and possibly some isolated points in $V \setminus V'$.*

   (b) *$W_\infty \nsubseteq U_\infty$ and $V'^\varphi = V_\infty$. Then V is of type* **B** *(respectively of type* IV *if $m = 2$) and every line of $\mathcal{S}_V$ is contained in the connected component $\mathcal{S}' \simeq \mathcal{I}(m, q, e)$ (respectively in the dual oval of $\mathcal{S}_V$ if $m = 2$).*

   (c) *$W_\infty \nsubseteq U_\infty$ and $V'^\varphi \neq V_\infty$. Then V is of type* **D** *(respectively of type* I *or* II *if $m = 2$). We have the following cases.*

      i. *$l = -1$. Then the connected components of $\mathcal{S}_V$ are the lines $\langle p, p^\varphi \rangle$, for all affine points $p \in W'$, and possibly some isolated points.*

      ii. *$l = 0$. Then the connected components of $\mathcal{S}_V$ are dual ovals in the planes $\langle L, L^\varphi \rangle$, for all affine lines $L \subseteq W'$ intersecting Π_∞ in the point X'_∞, the affine lines $L \subseteq V' \setminus W'$ intersecting Π_∞ in the point X'_∞, and possibly some isolated points.*

      iii. *$l \geq 1$. Then for every subspace $X' \subseteq W'$ which intersects Π_∞ in X'_∞ there is a connected component $\mathcal{S}'$ of $\mathcal{S}_V$ in $X = \langle X', X'^\varphi \rangle$ such that $\mathcal{S}' \simeq \mathcal{I}(l + 2, q, e)$. For every subspace $X' \subseteq V' \setminus W'$ which intersects Π_∞ in X'_∞ there is a connected component $\mathcal{S}'$ of $\mathcal{S}_V$ which is the linear representation in X' of the set $V'_\infty \cap \mathcal{K}_\infty$. The other connected components of $\mathcal{S}_V$ are, possibly, some isolated points.*

Proof.

1. $V \subseteq U$. In this case the theorem holds since U is a hyperplane of type **C** (respectively of type III if $n = 3$).

2. $V \nsubseteq U$ and V is parallel to U. In this case the theorem holds since by Lemma 4.4.11 every line of $\mathcal{I}(n, q, e)$ parallel to U is contained in U.

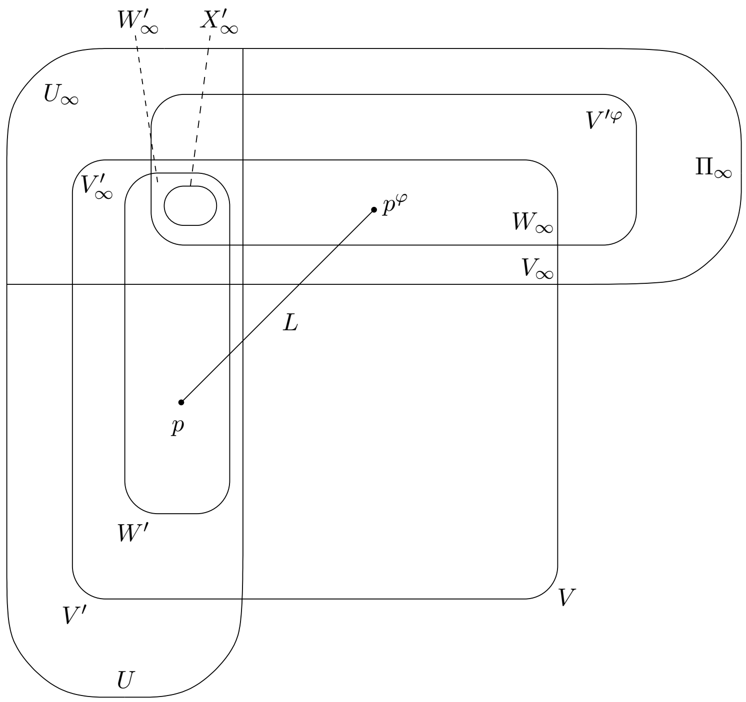

Figure 4.4: The intersection of the geometry $\mathcal{I}(n, q, e)$ with affine subspaces.

3. V is not parallel to U. Then $\dim V' = \dim V'^\varphi = \dim V_\infty = m - 1$ and $\dim V'_\infty = m - 2$. Note that $V'^\varphi \subseteq \Pi_\infty$ since $U^\varphi = \Pi_\infty$. Suppose that L is a line of $\mathcal{B}_2$ in V. If $p = L \cap U$, then $p^\varphi = L \cap \Pi_\infty$. Since $p \in V \cap U = V'$, $p^\varphi \in V'^\varphi$. But $L \subseteq V$, so $p^\varphi \in V_\infty$. Hence $p^\varphi \in W_\infty$, and $p \in W'$. Conversely if p is an affine point of W' then $p^\varphi \in W_\infty$ and the line $\langle p, p^\varphi \rangle$ is a line of $\mathcal{B}_2$ in V. So the lines of $\mathcal{B}_2$ in V are exactly the lines $\langle p, p^\varphi \rangle$, where p is an affine point of W'.

   (a) $W_\infty \subseteq U_\infty$. In this case $W' \subseteq U_\infty$, so there are no lines of $\mathcal{B}_2$ in V. Hence $\mathcal{S}_V$ consists of the linear representation in V' of $V'_\infty \cap \mathcal{K}_\infty$ and possibly some isolated points in $V \setminus V'$. So every connected component containing a line lies in a subspace of V'. It follows that V is of type **D** (respectively of type I or II if $m = 2$).

   (b) $W_\infty \nsubseteq U_\infty$ and $V'^\varphi = V_\infty$. In this case $W_\infty = V_\infty$ and $W' = V'$. So the lines of $\mathcal{B}_2$ in V are all lines $\langle p, p^\varphi \rangle$, where p is an affine point of V'. Since $V'_\infty = V' \cap U_\infty$, $V'^{\varphi}_\infty = V'^{\varphi} \cap U^{\varphi}_\infty = V_\infty \cap U_\infty = V'_\infty$, so by Lemma 4.4.13, $\langle V'_\infty \cap \mathcal{K}_\infty \rangle = V'_\infty$. Hence $X'_\infty = V'_\infty$.

Without loss of generality we may assume that V is coordinatized by the coordinates $X_0, \ldots, X_{m-2}, X_{n-1}, X_n$. Let

$$\begin{array}{rrcl} \psi: & V & \to & \mathrm{PG}(m,q) \\ & (x_0, \ldots, x_n) & \mapsto & (x_0, \ldots, x_{m-2}, x_{n-1}, x_n). \end{array}$$

Then ψ is a collineation. A line

$$\langle p(x_0, \ldots, x_{n-2}, 0, 1), p^{\varphi}(x_0^{2^e}, \ldots, x_{n-2}^{2^e}, 1, 0) \rangle$$

of $\mathcal{B}_2$ in V is mapped by ψ to the line

$$\langle (x_0, \ldots, x_{m-2}, 0, 1), (x_0^{2^e}, \ldots, x_{m-2}^{2^e}, 1, 0) \rangle$$

of $\mathrm{AG}(m,q)$. Since $\langle V'_\infty \cap \mathcal{K}_\infty \rangle = V'_\infty$ and since $X_0, \ldots, X_{m-2}, X_{n-1}, X_n$ coordinatize V, the points of $\mathcal{K}_\infty$ in V'_∞ are the points $p_\infty(\varepsilon_0, \ldots, \varepsilon_{n-2}, 0, 0)$, where $\varepsilon_0, \ldots, \varepsilon_{m-2}$ are arbitrary in $\mathrm{GF}(2)$ (but not all zero) and $\varepsilon_{m-1}, \ldots, \varepsilon_{n-2} \in \mathrm{GF}(2)$ depend on $\varepsilon_0, \ldots, \varepsilon_{m-2}$. So the points of $\mathcal{K}_\infty$ in V'_∞ are mapped by ψ to the points $(\varepsilon_0, \ldots, \varepsilon_{m-2}, 0, 0)$ where $\varepsilon_0, \ldots, \varepsilon_{m-2}$ are arbitrary in $\mathrm{GF}(2)$ (but not all zero). Now it is clear that the lines of $\mathcal{S}_V$ are mapped by ψ to the lines of $\mathcal{I}(m,q,e)$ (respectively to the lines of a dual oval if $m = 2$).

(c) $W_\infty \not\subseteq U_\infty$ and $V'^{\varphi} \neq V_\infty$. Suppose first that $l = -1$. Then $V'_\infty \cap \mathcal{K}_\infty = \emptyset$, so there are no lines of $\mathcal{B}_1$ in V. Suppose that two lines L_1 and L_2 of $\mathcal{B}_2$ in V intersect. Then by Lemma 4.4.10, the plane $\pi = \langle L_1, L_2 \rangle$ is of type IV and intersects U in a line of $\mathcal{S}$. But $\mathcal{S}_V$ does not contain any lines of $\mathcal{B}_1$, a contradiction. So the connected components of $\mathcal{S}_V$ are the lines $\langle p, p^{\varphi} \rangle$ for $p \in W'$ and some isolated points.

Now suppose that $l \geq 0$. Then $\langle V'_\infty \cap \mathcal{K}_\infty \rangle$ is not empty. Let $X' \subseteq W'$ be an $(l+1)$-dimensional subspace not contained in Π_∞, such that $X' \cap \Pi_\infty = X'_\infty$, let $X_\infty = X'^{\varphi}$ and let $X = \langle X', X_\infty \rangle$. By Lemma 4.4.13, $X'^{\varphi}_\infty = X'_\infty$, so $X' \cap X_\infty = X'_\infty$. Hence X is an $(l+2)$-dimensional subspace of $\mathrm{AG}(n,q)$ which falls under case 3b, so X is of type **B** (respectively of type IV if $l = 0$) and every line of $\mathcal{S}_X$ is contained in the connected component $\mathcal{S}' \simeq \mathcal{I}(l+2,q,e)$ (respectively in the dual oval $\mathcal{S}'$ of $\mathcal{S}_X$ if $l = 0$).

We prove that $\mathcal{S}'$ is a connected component of $\mathcal{S}_V$. Suppose that this is not so. Then there is a line $L \not\subseteq X$ of $\mathcal{S}_V$ which intersects

X in a point p of $\mathcal{S}'$. Let L' be a line of $\mathcal{B}_2$ in $\mathcal{S}'$ through p. Since $L' \not\subseteq U$, the plane $\pi = \langle L, L' \rangle \not\subseteq U$, so by Lemma 4.4.10, π is a plane of type IV and the line $M = \pi \cap U$ is in $\mathcal{B}_1$. Since $\pi \cap X = L'$, $M \not\subseteq X$. Since π is of type IV, M intersects L' and so also X in an affine point r. Now since M is a line of $\mathcal{B}_1$ in V and $M \not\subseteq X$, the point $p_\infty = M \cap \Pi_\infty$ is a point of $\mathcal{K}_\infty$ in V'_∞ but not in X'_∞. But this contradicts $X'_\infty = \langle V'_\infty \cap \mathcal{K}_\infty \rangle$.

We conclude that for every subspace $X' \subseteq W'$ of dimension $l+1$, which intersects Π_∞ in the l-space X'_∞, there is a connected component $\mathcal{S}'$ of $\mathcal{S}_V$ in $X = \langle X', X'^\varphi \rangle$ such that $\mathcal{S}' \simeq \mathcal{I}(l+2, q, e)$ (respectively such that $\mathcal{S}'$ is a dual oval if $l = 0$).

Let L denote a line of $\mathcal{B}_2$ in V. Then $p = L \cap V' \in W'$ and $L \cap V_\infty = p^\varphi \in W_\infty$. Let $X' = \langle p, X'_\infty \rangle$. Then $X' \subseteq W'$ and $L \subseteq X = \langle X', X'^\varphi \rangle$, so L is contained in the connected component $\mathcal{S}'$ of $\mathcal{S}_X$ with $\mathcal{S}' \simeq \mathcal{I}(l+2, q, e)$ (respectively with $\mathcal{S}'$ is a dual oval if $l = 0$).

The remaining connected components of $\mathcal{S}_V$ are either isolated points or they are contained in V'. Since $\mathcal{S}_{V'}$ is the linear representation of the set $V'_\infty \cap \mathcal{K}_\infty$, the rest of the theorem follows. $\square$

Corollary 4.4.15 *Consider $\mathcal{I}(n, q, e)$, $n \geq 4$, and let V' be an $(n-2)$-dimensional subspace of type* **C** *(respectively of type* III *if $n = 4$). Then there is exactly one hyperplane V of type* **B** *which contains V'. Furthermore, $\mathcal{S}_V$ is connected, so V does not contain any isolated points.*

Proof. By Theorem 4.4.14, $V' \subseteq U$, and $V'_\infty = V' \cap U_\infty$ is such that $V'_\infty \cap \mathcal{K}_\infty$ spans V'_∞. Now Lemma 4.4.13 implies that V'_∞ is fixed by φ. So $V'_\infty \subseteq V'^\varphi$ and hence $V = \langle V', V'^\varphi \rangle$ is a hyperplane. It follows from Theorem 4.4.14 that V is a hyperplane of type **B**. Since there is exactly one line of $\mathcal{B}_2$ through every affine point of V', V is the only hyperplane of type **B** containing V'.

Let $\mathcal{S}'$ be the connected component of $\mathcal{S}_V$ which is projectively equivalent to $\mathcal{I}(n-1, q, e)$. By Theorem 4.4.14, the other (if any) connected components of $\mathcal{S}_V$ are isolated points.

Let m denote the number of isolated points of $\mathcal{S}_V$. We count the number x of flags $\{p, L\}$, where p is a point of $\mathcal{S}_V$, not in V', and L is a line of $\mathcal{S}$, not contained in V, and, since $p \notin V'$, not contained in U. Through every isolated point of $\mathcal{S}_V$ pass 2^{n-1} lines of $\mathcal{S}$, none of which are contained in V. Through every point of $\mathcal{S}'$ pass 2^{n-1} lines of $\mathcal{S}$, 2^{n-2} of which are contained in V. By Proposition 4.4.12, the number of points of $\mathcal{S}'$, not in V', equals $(\frac{1}{2}q)^{n-2}(q-1)$. Hence $x = q^{n-2}(q-1) + 2^{n-1}m$.

On the other hand the number of lines of $\mathcal{S}$ which are contained in neither V nor U equals $q^{n-1}-q^{n-2}$. Hence $x \leq q^{n-2}(q-1)$. We conclude that $m=0$, so $\mathcal{S}_V$ is connected. □

Corollary 4.4.16 *Consider $\mathcal{I}(3,q,e)$, and let π be a plane which intersects U in an affine line L, which is not a line of $\mathcal{I}(3,q,e)$. Then π is of type* II *and it contains exactly one line of $\mathcal{I}(3,q,e)$.*

Proof. We use the notation of Theorem 4.4.14, with $\pi = V$. So $V' = L$, and V'_∞ is the point $L \cap U_\infty$. Now $V'_\infty \notin \mathcal{K}_\infty$ since L is not a line of $\mathcal{I}(3,q,e)$. Hence $V'_\infty \notin V'^\varphi$, and so $V'^\varphi \neq V_\infty$. The lines V'^φ and V_∞ are in the plane Π_∞, so W_∞ is a point. Since $V_\infty \cap U_\infty = V'_\infty$ and $V'_\infty \notin V'^\varphi$, $W_\infty \notin U_\infty$. So W' is an affine point of V', and by Theorem 4.4.14, π is a plane of type II containing exactly one line, namely the line $\langle W', W_\infty\rangle$. □

Lemma 4.4.17 *Let $\mathcal{S} = \mathcal{I}(n,q,e)$, let p be a point of $\mathcal{S}$ not contained in U, and let $\mathcal{T} = p^\perp \cap U$. Then $\mathcal{T} \cup \mathcal{K}_\infty$ is the point set of a projective space* $\mathrm{PG}(n-1,2)$.

Proof. Since the order of $\mathcal{S}$ is $(q-1, 2^{n-1}-1)$, $|\,\mathcal{T}\,| = 2^{n-1}$.

Suppose that three distinct points r_1, r_2, r_3 of $\mathcal{T}$ are collinear. Then the lines $\langle p, r_i\rangle$, $i=1,2,3$, are coplanar. Necessarily, the plane containing them is of type III. But this contradicts Lemma 4.4.10. So no three distinct points of $\mathcal{T}$ are collinear.

Let r_1, r_2 denote two distinct points of $\mathcal{T}$. By definition of $\mathcal{T}$, the line $L_i = \langle p, r_i\rangle$ is a line of $\mathcal{S}$, $i=1,2$. Hence $\alpha(r_1, L_2) > 0$. Since $\mathcal{S}$ is a $(0,2)$-geometry, $\alpha(r_1, L_2) = 2$, so there is a line $L \neq L_1$ of $\mathcal{S}$ through r_1 which intersects L_2 in an affine point. By Proposition 4.4.11, L_1 is the only line of $\mathcal{S}$ through r_1 which is not contained in U. Hence $L \subseteq U$. But L intersects L_2, so $L = \langle r_1, r_2\rangle$. Now since L is a line of $\mathcal{S}$, $L = \langle r_1, r_2\rangle$ intersects Π_∞ in a point of $\mathcal{K}_\infty$. This holds for any two distinct points $r_1, r_2 \in \mathcal{T}$.

Let r_1, r_2 be two distinct points of $\mathcal{T}$. Then the point $p_\infty = \langle r_1, r_2\rangle \cap \Pi_\infty$ is a point of $\mathcal{K}_\infty$. Hence we can choose a new basis in $\mathrm{PG}(n,q)$, such that, with respect to this basis,

$$\mathcal{K}_\infty = \{(\varepsilon_0, \ldots, \varepsilon_{n-2}, 0, 0) \neq (0, \ldots, 0) \,|\, \varepsilon_i \in \mathrm{GF}(2), 0 \leq i \leq n-2\},$$

$p_\infty(0,\ldots,0,1,0,0)$, $r_1(0,\ldots,0,1,0)$ and $r_2(0,\ldots,0,1,1,0)$.

Let $r \in \mathcal{T} \setminus \{r_1, r_2\}$. Then r, r_1, r_2 are not collinear. Let $p^i_\infty = \langle r, r_i\rangle \cap \Pi_\infty$, $i=1,2$. Then $p^i_\infty \in \mathcal{K}_\infty$, $i=1,2$, and p_∞, p^1_∞, p^2_∞ are collinear. So, with respect to the chosen basis, $p_\infty^1(\varepsilon_0, \ldots, \varepsilon_{n-2}, 0, 0)$, where $\varepsilon_i \in \mathrm{GF}(2), 0 \le i \le n-2$ and $(\varepsilon_0, \ldots, \varepsilon_{n-3}) \neq (0, \ldots, 0)$, and $p_\infty^2(\varepsilon_0, \ldots, \varepsilon_{n-2} + 1, 0, 0)$. Hence $r(\varepsilon_0, \ldots, \varepsilon_{n-2}, 1, 0)$.

With respect to the chosen basis, let

$$\mathcal{K} = \{(\varepsilon_0, \ldots, \varepsilon_{n-1}, 0) \neq (0, \ldots, 0) \mid \varepsilon_i \in \mathrm{GF}(2), 0 \le i \le n-1\}.$$

Then $\mathcal{K}$ is the point set of a projective space $\mathrm{PG}(n-1, 2)$, and $\mathcal{T} \cup \mathcal{K}_\infty \subseteq \mathcal{K}$. Since $|\mathcal{T} \cup \mathcal{K}_\infty| = |\mathcal{K}|$, $\mathcal{T} \cup \mathcal{K}_\infty = \mathcal{K}$. $\square$

We have deduced all the properties of the geometry $\mathcal{I}(n, q, e)$ that are needed in Chapters 5, 6. However, it is our opinion that further study is required to fully understand the geometry $\mathcal{I}(n, q, e)$. Therefore, we have added Appendix A.

Chapter 5

Classification of $(0, 2)$-geometries fully embedded in $\mathrm{AG}(3, 2^h)$

The aim of this chapter is to classify all $(0, 2)$-geometries which are fully embedded in $\mathrm{AG}(3, 2^h)$ and which have a plane of type IV. Note that the $(0, 2)$-geometries fully embedded in $\mathrm{AG}(3, 2^h)$, which do not have a plane of type IV, are already classified by Theorem 4.3.1. By Proposition, 4.3.5 we may assume that $h > 1$.

In Section 5.2, the classification is achieved under the additional assumption that there is at least one planar net. The remaining case, namely that there are no planar nets, is the subject of Section 5.3; a complete classification is again achieved. In Section 5.3.2, it is shown that the order of a $(0, 2)$-geometry which is fully embedded in $\mathrm{AG}(3, 2^h)$, such that there is a plane of type IV but no planar nets, is $(2^h - 1, 2^h)$. Finally, in Section 5.3.3, the $(0, 2)$-geometries of order $(2^h - 1, 2^h)$ fully embedded in $\mathrm{AG}(3, 2^h)$, such that there is a plane of type IV but no planar nets, are classified.

The results of Section 5.2 are published in [33]. The results of Section 5.3 are published in [32, 36].

5.1 Preliminaries

Let $\mathcal{S}$ be a $(0, 2)$-geometry fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$. In Lemma 4.1.3 it was shown that a plane π of $\mathrm{AG}(3, q)$ is of one of the following four types.

Type I. π does not contain any line of $\mathcal{S}$.

Type II. π contains a number of parallel lines of $\mathcal{S}$ and possibly some isolated points.

Type III. $\mathcal{S}_\pi$ is a planar net of order q and degree 3.

Type IV. $\mathcal{S}_\pi$ consists of a $\mathrm{pg}(q-1,1,2)$ (that is, a dual oval with nucleus the line at infinity) and possibly some isolated points.

Note that a plane of $\mathrm{AG}(3,q)$ which contains two intersecting lines of $\mathcal{S}$ is necessarily a plane of type III or IV, and that a plane of $\mathrm{AG}(3,q)$ which contains two parallel lines of $\mathcal{S}$ is necessarily a plane of type II or III.

Assume that π is a plane of type IV. Then $\mathcal{S}_\pi$ has $q+1$ lines, one in each parallel class of lines of π. Let $\mathcal{S}'$ be the connected component of $\mathcal{S}_\pi$ which is a dual oval. Then $\mathcal{S}'$ contains $\frac{1}{2}q(q+1)$ points, and through each point of $\mathcal{S}'$ there are two lines of $\mathcal{S}'$. Each affine line of π which is not a line of $\mathcal{S}'$ contains exactly $\frac{1}{2}q$ points of $\mathcal{S}'$, and so at least $\frac{1}{2}q$ points of $\mathcal{S}_\pi$.

Assume that π is a plane of type III. Then $\mathcal{S}_\pi$ has $3q$ lines, namely the lines of three parallel classes of lines in π. Every affine point of π is a point of $\mathcal{S}_\pi$.

A point p_∞ of Π_∞ is called a *hole* if there are no lines of $\mathcal{S}$ which intersect Π_∞ in the point p_∞. For every plane π of $\mathrm{AG}(3,q)$ we let $P_\infty(\pi)$ denote the set of points at infinity of all the lines of $\mathcal{S}_\pi$. So if π is a plane of type I, respectively type II, type III or type IV, then $P_\infty(\pi)$ consists of 0, respectively 1, 3 or $q+1$ points. If π is a plane of type IV and $L_\infty = \pi \cap \Pi_\infty$ we shall abuse notation and write $P_\infty(\pi) = L_\infty$.

Let L_∞ be a line of Π_∞ and let p_∞ be a point on L_∞. Then p_∞ is a hole if and only if $p_\infty \notin P_\infty(\pi)$ for every affine plane π intersecting Π_∞ in the line L_∞.

In this chapter we will make frequent use of the following argument. Let π and π' be parallel planes of $\mathrm{AG}(3,q)$. Let $(q-1,t)$ be the order of $\mathcal{S}$. If p is a point of $\mathcal{S}_\pi$, respectively $\mathcal{S}_{\pi'}$, on exactly i lines of $\mathcal{S}_\pi$, respectively $\mathcal{S}_{\pi'}$, with $0 \le i \le 3$, then there are exactly $t+1-i$ lines of $\mathcal{S}$ which intersect π, respectively π', in the point p. Let m_i, respectively m_i', be the number of points of $\mathcal{S}_\pi$, respectively $\mathcal{S}_{\pi'}$, that are on exactly i lines of $\mathcal{S}_\pi$, respectively $\mathcal{S}_{\pi'}$, for $0 \le i \le 3$. Since the number of lines of $\mathcal{S}$ which intersect π in an affine point must be equal to the number of lines of $\mathcal{S}$ which intersect π' in an affine point, we get that

$$\sum_{i=0}^{3} m_i(t+1-i) = \sum_{i=0}^{3} m_i'(t+1-i).$$

5.2 Classification in case there is a planar net

Lemma 5.2.1 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in* $\mathrm{AG}(3,q)$*, $q=2^h$, $h>1$. If p is a point of $\mathcal{S}$ not in a plane π of type* III*, then the number of lines of $\mathcal{S}$ through p which intersect π in an affine point is even.*

Proof. This follows easily from the fact that $\mathcal{S}_\pi$ contains a complete parallel class of lines of π, and that $\mathcal{S}$ is a $(0,2)$-geometry. □

Lemma 5.2.2 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in* $\mathrm{AG}(3,q)$*, $q=2^h$, $h>1$. If t is even, then any plane parallel to but different from a plane of type* III *is either a plane of type* II *without isolated points or a plane of type* III*. If t is odd, then any plane parallel to but different from a plane of type* III *is either a plane of type* I *or a plane of type* IV*.*

Proof. Assume that π is a plane of type III. Then by Lemma 5.2.1 the parity of the number of lines of $\mathcal{S}$ through a point p of $\mathcal{S}$ and parallel to π is constant for all points $p \notin \pi$ of $\mathcal{S}$. The lemma follows. □

Theorem 5.2.3 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in* $\mathrm{AG}(3,q)$*, $q=2^h$, $h>1$, such that there is a plane of type* IV *and a planar net. Then t is odd.*

Proof. Suppose that t is even. We show that every plane which is parallel to a plane of type III, is itself of type III. From this we will deduce a contradiction.

Let π be a plane of type III, and let π' be a plane parallel to π. Suppose that π' is not of type III. Then by Lemma 5.2.2, π' is a plane of type II and π' does not contain any isolated point. Let Θ denote the number of lines of $\mathcal{S}_{\pi'}$. Counting the number of lines of $\mathcal{S}$ intersecting both π and π' in an affine point yields $q^2(t-2)=q\Theta t$. Hence $t \,|\, 2q$.

Let $P_\infty(\pi)=\{x_\infty,y_\infty,z_\infty\}$ and $P_\infty(\pi')=\{u_\infty\}$. Without loss of generality, we may assume that $x_\infty \neq u_\infty \neq y_\infty$. Suppose that L is a line of $\mathcal{S}$ such that $L \cap \Pi_\infty = x_\infty$ and $L \nsubseteq \pi$. Let p be an affine point of L. By Lemma 4.1.4, θ_p spans Π_∞, so there is a line M of $\mathcal{S}$ through p intersecting π in an affine point. Let $\pi''=\langle L,M\rangle$. Then the line $\pi\cap\pi''$ is a line of $\mathcal{S}$ since it intersects Π_∞ in x_∞. Now $\mathcal{S}_{\pi''}$ contains two intersecting lines as well as two parallel lines, so π'' is a plane of type III. Hence every line in π'' intersecting Π_∞ in x_∞ is a line of $\mathcal{S}$. So also the line $\pi'\cap\pi''$ is a line of $\mathcal{S}$. But this contradicts $P_\infty(\pi') = \{u_\infty\}$. So every line of $\mathcal{S}$ through x_∞ is contained in the plane π. Analogously, every line of $\mathcal{S}$ through y_∞ is contained in π.

Let p be an affine point of π and let $L = \langle p, x_\infty\rangle$. Since every line of $\mathcal{S}$ which is parallel to L, is contained in π, π is the only plane of type III through L. Hence the number of planes of type IV through L equals the number of lines of $\mathcal{S}$ through p not contained in π, namely $t-2$. It follows that $t-2 \leq q$. Since $q > 2$, $t < 2q$. Since $t \mid 2q$, $t \leq q$. This means that there is a plane π'' of type II which contains L. Clearly $P_\infty(\pi'') = \{x_\infty\}$. Since every line of $\mathcal{S}$ parallel to L is contained in π, $\mathcal{S}''_\pi$ contains only one line. Let π''' be a plane parallel to π'' and suppose that π''' is of type IV. Let $\pi^{(4)}$ be a plane parallel to π. Then either $\pi^{(4)}$ is of type III or $\pi^{(4)}$ is of type II, $\pi^{(4)}$ contains no isolated points and $P_\infty(\pi^{(4)}) \neq \{x_\infty\}$. In any case the numbers of points of $\mathcal{S}$ on the lines $\pi'' \cap \pi^{(4)}$ and $\pi''' \cap \pi^{(4)}$ are equal. Hence $\mathcal{S}_{\pi''}$ and $\mathcal{S}_{\pi'''}$ have the same number of points, say m. Now counting the number of lines of $\mathcal{S}$ intersecting both π'' and π''' in affine points yields

$$qt + (m-q)(t+1) = \frac{1}{2}q(q+1)(t-1) + (m - \frac{1}{2}q(q+1))(t+1).$$

A contradiction follows. So π''' is not of type IV. Since the line $\pi \cap \pi'''$ is a line of $\mathcal{S}$ through x_∞ and since every line of $\mathcal{S}$ through x_∞ is contained in π, π''' is not of type III. Hence π''' is a plane of type II and $P_\infty(\pi''') = \{x_\infty\}$. Since this holds for every plane π''' parallel to π'', the line $L_\infty = \pi'' \cap \Pi_\infty$ contains only one point which is not a hole, namely x_∞.

Let $M = \langle p, y_\infty\rangle$. By Lemma 4.1.4, θ_p spans Π_∞, so there is a line N of $\mathcal{S}$ which intersects π in p. Let $\pi''' = \langle M, N\rangle$ and let $M_\infty = \pi''' \cap \Pi_\infty$. Since every line of $\mathcal{S}$ parallel to M is contained in π, π''' is not of type III. However $\mathcal{S}'''_\pi$ contains two intersecting lines, so π''' is a plane of type IV. Hence M_∞ does not contain any hole. But M_∞ intersects L_∞ in a point different from x_∞. So M_∞ contains a hole, a contradiction. So every plane parallel to a plane of type III, is of type III.

Let L_∞ be the line at infinity of a plane of type III. Let V be the parallel class of planes which intersect Π_∞ in L_∞. Then every plane of V is of type III. We prove that for any two planes $\pi, \pi' \in V$, $P_\infty(\pi) = P_\infty(\pi')$.

Let $p_\infty \in L_\infty$ and suppose that there are affine planes $\pi, \pi' \in V$ such that $p_\infty \in P_\infty(\pi) \cap P_\infty(\pi')$. Let L be a line of $\mathcal{S}$ which intersects π in an affine point, and let $\pi'' = \langle p_\infty, L\rangle$. Then $\mathcal{S}_{\pi''}$ contains two intersecting lines, namely L and $\pi \cap \pi''$, as well as two parallel lines, namely $\pi \cap \pi''$ and $\pi' \cap \pi''$. So π'' is of type III and $p_\infty \in P_\infty(\pi'')$. Hence every plane $\pi''' \in V$ contains a line of $\mathcal{S}$ through p_∞, namely $\pi'' \cap \pi'''$. So $p_\infty \in P_\infty(\pi''')$ for every plane $\pi''' \in V$.

Since $3q > q+1$ there exist two distinct planes $\pi_1, \pi_2 \in V$ such that $P_\infty(\pi_1) \cap P_\infty(\pi_2) \neq \emptyset$. Let $p^1_\infty \in P_\infty(\pi_1) \cap P_\infty(\pi_2)$. Then $p^1_\infty \in P_\infty(\pi)$ for every plane $\pi \in V$. Similarly since $2q > q$ there exist two planes $\pi_3, \pi_4 \in V$ such that $(P_\infty(\pi_3) \cap P_\infty(\pi_4)) \setminus \{p^1_\infty\} \neq \emptyset$. Let $p^2_\infty \in (P_\infty(\pi_3) \cap P_\infty(\pi_4)) \setminus \{p^1_\infty\}$. Then $p^2_\infty \in P_\infty(\pi)$ for every plane $\pi \in V$. Finally since $q > q-1$ there exist two planes $\pi_5, \pi_6 \in V$ such that $P_\infty(\pi_5) = P_\infty(\pi_6)$. Let p^3_∞ be the unique point of $P_\infty(\pi_5) \setminus \{p^1_\infty, p^2_\infty\}$. Then $p^3_\infty \in P_\infty(\pi)$ for every plane $\pi \in V$. Hence for every plane $\pi \in V$, $P_\infty(\pi) = \{p^1_\infty, p^2_\infty, p^3_\infty\}$. As a consequence every point of L_∞ except p^1_∞, p^2_∞ and p^3_∞ is a hole, and every affine line which intersects Π_∞ in p^i_∞, $1 \leq i \leq 3$, is a line of $\mathcal{S}$.

Let π be a plane of type IV. Then the line $M_\infty = \pi \cap \Pi_\infty$ contains no holes, so it intersects L_∞ in p^1_∞, p^2_∞ or p^3_∞. But then π contains q parallel lines of $\mathcal{S}$, which is impossible. We conclude that t is odd. $\square$

Theorem 5.2.4 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in* AG$(3,q)$*, $q = 2^h$, $h > 1$, such that there is a plane of type* IV *and a planar net. Then $t = 3$, there is a unique plane π of type* III*, and every plane parallel to but different from π is of type* I.

Proof. It follows from Theorem 5.2.3 that t is odd. Let π be a plane of type III. Suppose that a plane π' parallel to π is of type IV. Let $p_\infty \in P_\infty(\pi)$ and let L be the unique line of $\mathcal{S}_{\pi'}$ which intersects Π_∞ in p_∞. Any plane containing L and different from π' contains two parallel lines of $\mathcal{S}$, namely L and $\pi \cap \pi'$, so is of type II or III. Let p be an affine point of L. Then the number of planes of type III containing L equals half the number of lines of $\mathcal{S}$ through p which are not contained in π', so $\frac{1}{2}(t-1)$. Since by Lemma 4.1.4, θ_p spans Π_∞, there is at least one line of $\mathcal{S}$ through p not contained in π', so at least one plane of type III containing L. Suppose there is exactly one plane π'' of type III containing L. Then $t = 3$. Consider an affine point p' on the line $\pi \cap \pi''$. Then there are at least 5 lines of $\mathcal{S}$ through p', a contradiction. So there are at least two planes π'' and π''' of type III which contain L. Since L is a line of $\mathcal{S}$, $p_\infty \in P_\infty(\pi'') \cap P_\infty(\pi''')$. Hence a plane $\pi^{(4)}$ parallel to π but different from π and π' contains two parallel lines of $\mathcal{S}$, namely $\pi'' \cap \pi^{(4)}$ and $\pi''' \cap \pi^{(4)}$. But this implies that $\pi^{(4)}$ is of type II or III, a contradiction with Lemma 5.2.2. So no plane parallel to a plane of type III, is of type IV. By Lemma 5.2.2 every plane parallel to a plane of type III, is of type I.

Consider again a plane π of type III, let $P_\infty(\pi) = \{x_\infty, y_\infty, z_\infty\}$ and let $L_\infty = \pi \cap \Pi_\infty$. Since every plane parallel to π is of type I, there are no lines of $\mathcal{S}$ which are parallel to π except those contained in π. So every point of

L_∞ except $x_\infty, y_\infty, z_\infty$ is a hole. Let M_∞ be a line of Π_∞ which intersects L_∞ in the point x_∞, and let V be the parallel class of planes of $\mathrm{AG}(3,q)$ which intersect Π_∞ in M_∞. For every plane $\pi' \in V$, the line $\pi \cap \pi'$ intersects Π_∞ in x_∞, so it is a line of $\mathcal{S}$. Hence $x_\infty \in P_\infty(\pi')$ for every plane $\pi' \in V$. Since every line of $\mathcal{S}$ which intersects Π_∞ in x_∞ is contained in π, a plane $\pi' \in V$ cannot be of type III. So a plane $\pi' \in V$ is of type II or IV. Suppose that every plane of V is of type II. Then every point of M_∞ except x_∞ is a hole. Let L be a line of $\mathcal{S}$ which intersects π in an affine point p, and let $\pi' = \langle y_\infty, L\rangle$. Then π' contains two intersecting lines of $\mathcal{S}$, namely L and $\langle p, y_\infty\rangle$, so π' is of type III or IV. But π' cannot be of type III, otherwise there would be lines of $\mathcal{S}$ parallel to but not contained in π. So π' is of type IV and hence the line $N_\infty = \pi' \cap \Pi_\infty$ does not contain any holes. But the point $M_\infty \cap N_\infty$ is different from x_∞, so it is a hole, contradiction. Hence there is at least one plane of type IV in V, and the line M_∞ does not contain any holes. As a consequence no point of Π_∞ not on L_∞ is a hole.

Let π' be a plane of type III and let $M_\infty = \pi' \cap \Pi_\infty$. Since every plane parallel to π' is of type I, the line M_∞ contains $q-2$ holes. But since $q \geq 4$ and since no point of Π_∞ not on L_∞ is a hole, this implies that $M_\infty = L_\infty$. So π and π' are parallel and by Lemma 5.2.2, $\pi' = \pi$. So π is the only plane of type III.

Finally, let L be a line of $\mathcal{S}$ which intersects π in an affine point. The planes through L and x_∞, y_∞ and z_∞ are planes of type IV since they contain two intersecting lines of $\mathcal{S}$ and since π is the only plane of type III. Let π' be a plane containing L, different from these three planes. Then π' intersects L_∞ in a hole, so π' is not of type IV. It follows that π is of type II. Hence through every affine point of L there are exactly four lines of $\mathcal{S}$, so $t = 3$. $\square$

Theorem 5.2.5 *Let* $\mathcal{S}$ *be a* $(0,2)$*-geometry fully embedded in* $\mathrm{AG}(3,q)$*,* $q = 2^h$*,* $h > 1$*, such that there is a plane of type* IV *and a planar net. Then* $\mathcal{S} \simeq \mathcal{I}(3,q,e)$ *for some* e *such that* $\gcd(e,h) = 1$*.*

Proof. By Theorem 5.2.4, we know that the order of $\mathcal{S}$ is $(q-1,3)$, and there is exactly one plane π_0 of type III. Let $L_\infty = \pi_0 \cap \Pi_\infty$ and let $P_\infty(\pi_0) = \{x_\infty, y_\infty, z_\infty\}$. Let L be a line of $\mathcal{S}$ in π_0. Then through every affine point of L there is exactly one line of $\mathcal{S}$ not contained in π_0. Since every plane different from π_0 which contains two intersecting lines of $\mathcal{S}$ is of type IV, there is exactly one plane of type IV which contains L.

Let L_1, respectively L_2, be an affine line of π_0 which intersects Π_∞ in x_∞, respectively y_∞. Let π_1, respectively π_2, denote the unique plane of type IV which contains L_1, respectively L_2. Let $p_0 = L_1 \cap L_2$. Since $t = 3$ there is a

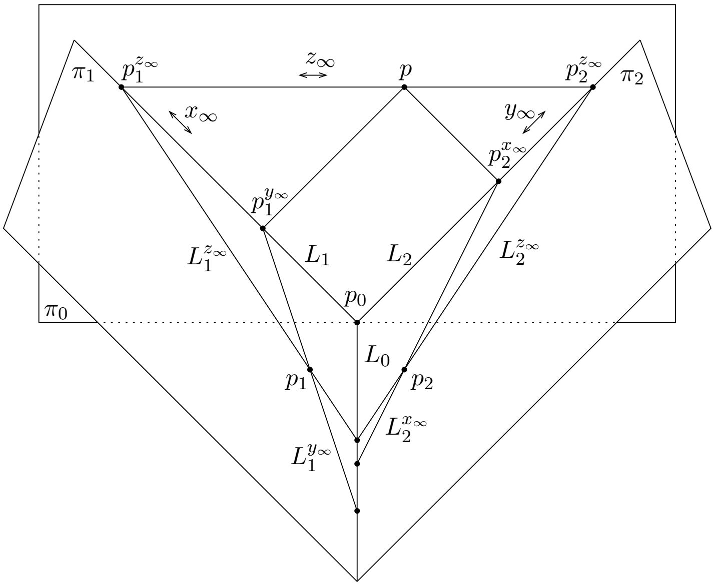

Figure 5.1: Characterization of the geometry $\mathcal{I}(3, q, e)$.

unique line L_0 of $\mathcal{S}$ through p_0 which is not contained in π_0. Necessarily, L_0 is the unique line of $\mathcal{S}$ in π_1, respectively π_2, which passes through p_0 but is not contained in π_0. So $L_0 = \pi_1 \cap \pi_2$.

Let M_1 be a line of $\mathcal{S}_{\pi_1}$, different from L_0 and L_1, and let $\pi_3 = \langle z_\infty, M_1 \rangle$. Since π_3 does not contain any of the points x_∞, y_∞, p_0 and $L_0 \cap \pi_\infty$, no four planes of $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_\infty\}$ have a point in common. Hence we can choose a basis in $\mathrm{PG}(3, q)$ such that $\pi_0 : X_2 = 0$, $\pi_1 : X_0 = 0$, $\pi_2 : X_1 = 0$, $\pi_3 : X_0 + X_1 + X_2 + X_3 = 0$ and $\pi_\infty : X_3 = 0$. It follows that $x_\infty(0, 1, 0, 0), y_\infty(1, 0, 0, 0), z_\infty(1, 1, 0, 0)$ and $p_0(0, 0, 0, 1)$, and that $L_0 : X_0 = X_1 = 0, L_1 : X_0 = X_2 = 0, L_2 : X_1 = X_2 = 0, L_\infty : X_2 = X_3 = 0$ and $M_1 : X_0 = X_1 + X_2 + X_3 = 0$. Let f be the permutation of $\mathrm{GF}(q)$ such that the set of lines of $\mathcal{S}_{\pi_1}$ other than L_1 is the set of lines having equations $X_0 = X_1 + f(\rho)X_2 + \rho X_3 = 0$, for all $\rho \in \mathrm{GF}(q)$. This is indeed a permutation of $\mathrm{GF}(q)$ since the lines of $\mathcal{S}_{\pi_1}$ form a dual oval with nucleus the line at infinity of π_0. Furthermore, the lines $L_0 : X_0 = X_1 = 0$ and $M_1 : X_0 = X_1 + X_2 + X_3 = 0$ are lines of $\mathcal{S}$ in π_0, so $f(0) = 0$ and $f(1) = 1$. This means that f is an o-polynomial.

Let $a, b \in \mathrm{GF}(q) \setminus \{0\}$ such that $a \neq b$, and let p be the point of π_0 having coordinates $p(a, b, 0, 1)$. Let $p_1^{y_\infty} = \langle p, y_\infty \rangle \cap L_1$, $p_1^{z_\infty} = \langle p, z_\infty \rangle \cap L_1$,

$p_2^{x_\infty} = \langle p, x_\infty \rangle \cap L_2$ and $p_2^{z_\infty} = \langle p, z_\infty \rangle \cap L_2$. Finally let $L_1^{y_\infty}$, respectively $L_1^{z_\infty}, L_2^{x_\infty}, L_2^{z_\infty}$, be the unique line of $\mathcal{S}$ through $p_1^{y_\infty}$, respectively $p_1^{z_\infty}, p_2^{x_\infty}, p_2^{z_\infty}$, which is not contained in π_0. With respect to the chosen basis, $p_1^{y_\infty}$ $(0,b,0,1)$, $p_1^{z_\infty}$ $(0,a+b,0,1)$, $p_2^{x_\infty}$ $(a,0,0,1)$ and $p_2^{z_\infty}$ $(a+b,0,0,1)$. Furthermore since $L_1^{y_\infty}$ and $L_1^{z_\infty}$ are lines of $\mathcal{S}$ in π_1,

$$\begin{aligned} L_1^{y_\infty} &: X_0 = X_1 + f(b)X_2 + bX_3 = 0, \\ L_1^{z_\infty} &: X_0 = X_1 + f(a+b)X_2 + (a+b)X_3 = 0. \end{aligned}$$

Consider the anti-flag $\{p_2^{z_\infty}, L_1^{z_\infty}\}$ of $\mathcal{S}$. The line $\langle p, z_\infty \rangle$ is a line of $\mathcal{S}$ through $p_2^{z_\infty}$ which intersects $L_1^{z_\infty}$, so $\alpha(p_2^{z_\infty}, L_1^{z_\infty}) = 2$. The other line of $\mathcal{S}$ through $p_2^{z_\infty}$ which intersects $L_1^{z_\infty}$ is not contained in π_0, so this line is necessarily $L_2^{z_\infty}$. But since π_2 is the unique plane of type IV through L_2, $L_2^{z_\infty} \subseteq \pi_2$. Hence $L_2^{z_\infty}$ contains the point $L_1^{z_\infty} \cap \pi_2$, which has coordinates $(0,0,a+b,f(a+b))$. This means that the line $L_2^{z_\infty}$ has equation

$$L_2^{z_\infty} : X_1 = X_0 + f(a+b)X_2 + (a+b)X_3 = 0.$$

Let $r = \langle p_2^{x_\infty}, z_\infty \rangle \cap L_1$ and let M be the unique line of $\mathcal{S}$ through r not contained in π_0. Then $r(0,a,0,1)$ and $M : X_0 = X_1 + f(a)X_2 + aX_3 = 0$. Consider the anti-flag $\{p_2^{x_\infty}, M\}$ of $\mathcal{S}$. Then by arguments similar to those used above, $L_2^{x_\infty}$ contains the point $M \cap \pi_2$. Hence

$$L_2^{x_\infty} : X_1 = X_0 + f(a)X_2 + aX_3 = 0.$$

Let L denote the unique line of $\mathcal{S}$ through p not contained in π_0. Consider the anti-flag $\{p, L_1^{y_\infty}\}$, respectively $\{p, L_1^{z_\infty}\}, \{p, L_2^{x_\infty}\}$, $\{p, L_2^{z_\infty}\}$ of $\mathcal{S}$. The line $\langle p, y_\infty \rangle$, respectively $\langle p, z_\infty \rangle, \langle p, x_\infty \rangle, \langle p, z_\infty \rangle$, is a line of $\mathcal{S}$ through p intersecting $L_1^{y_\infty}$, respectively $L_1^{z_\infty}, L_2^{x_\infty}, L_2^{z_\infty}$. This means that $\alpha(p, L_1^{y_\infty}) = 2$, respectively $\alpha(p, L_1^{z_\infty}) = \alpha(p, L_2^{x_\infty}) = \alpha(p, L_2^{z_\infty}) = 2$. The other line of $\mathcal{S}$ through p which intersects $L_1^{y_\infty}$, respectively $L_1^{z_\infty}, L_2^{x_\infty}, L_2^{z_\infty}$, is not contained in π_0, so this line is necessarily L. We conclude that L contains the points $p_1 = L_1^{y_\infty} \cap L_1^{z_\infty}$ having coordinates

$$p_1(0, bf(a+b) + f(b)(a+b), a, f(b) + f(a+b))$$

and $p_2 = L_2^{x_\infty} \cap L_2^{z_\infty}$ having coordinates

$$p_2(af(a+b) + f(a)(a+b), 0, b, f(a) + f(a+b)).$$

So the points p, p_1 and p_2 are collinear. One can verify using the coordinates of p, p_1 and p_2 that this implies that $f(a+b) = f(a) + f(b)$. It follows that $f(a+b) = f(a) + f(b)$ for every $a, b \in \mathrm{GF}(q) \setminus \{0\}$ such that $a \neq b$.

Trivially, this also holds if $ab = 0$ or if $a = b$. So $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ is an additive o-polynomial. Now by Theorem 1.3.1, there is an integer e such that $\gcd(e, h) = 1$ and $f(x) = x^{2^e}$ for every $x \in \mathrm{GF}(q)$.

Let φ be the collineation of $\mathrm{PG}(3, q)$ which maps a point $p(x_0, x_1, x_2, x_3)$ to the point $p^\varphi(x_0^{2^e}, x_1^{2^e}, x_3^{2^e}, x_2^{2^e})$. Let $\mathcal{K}_\infty$ be the set of points of $L_\infty = \pi_0 \cap \pi_\infty$ which are fixed by φ. Then since $\gcd(e, h) = 1$, $\mathcal{K}_\infty = \{x_\infty, y_\infty, z_\infty\}$.

Let $p(a, b, 0, 1)$ be an arbitrary affine point of π_0. Then there are four lines of $\mathcal{S}$ through p, namely the lines through p and an element of $\mathcal{K}_\infty$, and a line L which is not contained in π_0. If $a = 0$ or $ab(a + b) \neq 0$ then we already know the line L from the preceding paragraphs, and it turns out that $L \cap \Pi_\infty = p^\varphi(a^{2^e}, b^{2^e}, 1, 0)$. If $a \neq 0$ and $b(a + b) = 0$ then using arguments similar to those used above one can also verify that $L \cap \Pi_\infty = p^\varphi(a^{2^e}, b^{2^e}, 1, 0)$. Now it is clear that $\mathcal{S} \simeq \mathcal{I}(3, q, e)$. $\square$

5.3 Classification in case there are no planar nets

In this section we classify all $(0, 2)$-geometries fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, such that there is a plane of type IV and there are no planar nets. In Section 5.3.1 we start off with some definitions and useful lemmas. Next, in Section 5.3.2, we prove that if the order of such a $(0, 2)$-geometry is $(q - 1, t)$, then necessarily $t = q$. Finally, in Section 5.3.3, we classify the $(0, 2)$-geometries of order $(q - 1, q)$ fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, such that there is a plane of type IV and there are no planar nets.

5.3.1 Definitions and basic lemmas

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a $(0, 2)$-geometry fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, such that there are no planar nets. Then every plane containing two intersecting lines of $\mathcal{S}$ is a plane of type IV, and every plane containing two parallel lines of $\mathcal{S}$ is a plane of type II. Since any two intersecting lines of $\mathcal{S}$ are contained in a plane of type IV, we do not need to assume that there is a plane of type IV, this is automatically so. Let $(q - 1, t)$ be the order of $\mathcal{S}$. Let L be a line of $\mathcal{S}$, and let p be an affine point on L. Then the number of lines of $\mathcal{S}$ through p, different from L, equals the number of planes of type IV which contain L. So there are t planes of type IV which contain L and $q + 1 - t$ of type II.

We introduce some notation. For every point $p_\infty \in \Pi_\infty$ let $k(p_\infty)$ denote the number of lines of $\mathcal{S}$ which intersect Π_∞ in the point p_∞. So a point

$p_\infty \in \Pi_\infty$ is a hole if and only if $k(p_\infty) = 0$. Let $k_{\min}$ be the minimal value of $k(p_\infty)$ for all points $p_\infty \in \Pi_\infty$ that are not holes.

For every line $L_\infty \subseteq \Pi_\infty$ let $l(L_\infty)$ denote the number of planes of type IV which intersect Π_∞ in the line L_∞. Let $l_{\min}$ be the minimal value of $l(L_\infty)$ for all lines $L_\infty \subseteq \Pi_\infty$ such that $l(L_\infty) > 0$.

Let $\mathcal{P}_0$ denote the set of holes, let $\mathcal{P}_{\mathrm{IV}}$ denote the set of points of Π_∞ that are not holes, and let $\mathcal{P}_{\min}$ denote the set of points $p_\infty \in \Pi_\infty$ such that $k(p_\infty) = k_{\min}$. Let $\mathcal{B}_0$ denote the set of lines of Π_∞ containing at least one hole, and let $\mathcal{B}_{\mathrm{IV}}$ denote the set of lines of Π_∞ which do not contain any hole. Finally, let Π_{IV} denote the set of planes of type IV.

We prove that a line $L_\infty \subseteq \Pi_\infty$ is in $\mathcal{B}_{\mathrm{IV}}$ if and only if $l(L_\infty) > 0$. Suppose that for a line $L_\infty \subseteq \Pi_\infty$, $l(L_\infty) > 0$. So there is a plane π of type IV such that $\pi \cap \Pi_\infty = L_\infty$. Then $P_\infty(\pi) = L_\infty$, so L_∞ does not contain any hole. Hence $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Suppose on the other hand that for a line $L_\infty \subseteq \Pi_\infty$, $l(L_\infty) = 0$. Let V be the parallel class of planes which intersect Π_∞ in the line L_∞. Then every plane $\pi \in V$ is of type I or II. So for every plane $\pi \in V$ the set $P_\infty(\pi)$ contains at most one point. Since there are q planes in the set V and $q+1$ points on the line L_∞, there is at least one point $p_\infty \in L_\infty$ which does not occur in any set $P_\infty(\pi)$, $\pi \in V$. Hence p_∞ is a hole, and $L_\infty \notin \mathcal{B}_{\mathrm{IV}}$.

Proposition 5.3.1 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a* $(0,2)$*-geometry of order* $(q-1,t)$ *fully embedded in* $\mathrm{AG}(3,q)$*,* $q = 2^h$*,* $h > 1$*, such that there are no planar nets. Then* $2 \leq t \leq q$*.*

Proof. Let L be a line of $\mathcal{S}$. Then the number of planes of type IV which contain L is exactly t, so $t \leq q+1$. Suppose that $t < 2$. Then the lines of $\mathcal{S}$ through an affine point p of L do not span $\mathrm{AG}(3,q)$. But by Lemma 4.1.4, θ_p spans Π_∞, a contradiction. So $2 \leq t \leq q+1$.

Suppose that $t = q+1$. Then every line of $\mathcal{S}$ is contained in $q+1$ planes of type IV. As a consequence there are no planes of type II and hence no two lines of $\mathcal{S}$ are parallel. So $|\,\mathcal{B}\,| \leq q^2+q+1$. Let π be a plane of type IV, and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_\pi$ which is a dual oval. Then the number of lines of $\mathcal{S}$ which intersect π in a point of $\mathcal{S}'$ is $\frac{1}{2}q^2(q+1)$. So $q^2+q+1 \leq |\,\mathcal{B}\,| \leq \frac{1}{2}q^2(q+1)$, a contradiction since $q \geq 4$. $\square$

Lemma 5.3.2 *Let* $\mathcal{S}$ *be a* $(0,2)$*-geometry fully embedded in* $\mathrm{AG}(3,q)$*,* $q = 2^h$*,* $h > 1$*, such that there are no planar nets. Let* L_∞ *be a line of* Π_∞*. Then the minimal value of* $k(p_\infty)$ *for all points* $p_\infty \in L_\infty$ *is* $l(L_\infty)$*, and there are at least* $l(L_\infty)+1$ *points* $p_\infty \in L_\infty$ *such that* $k(p_\infty) = l(L_\infty)$*.*

Proof. Let V be the parallel class of planes which intersect Π_∞ in the line L_∞. Since for each point $p_\infty \in L_\infty$, every plane of type IV in V contains exactly one line of $\mathcal{S}$ which intersects Π_∞ in p_∞, $k(p_\infty) \geq l(L_\infty)$ for every point $p_\infty \in L_\infty$. Since there are at most $q - l(L_\infty)$ planes of type II in V, there are at least $l(L_\infty) + 1$ points on L_∞ which do not occur in any set $P_\infty(\pi)$, π a plane of type II in V. For such a point p_∞, $k(p_\infty) = l(L_\infty)$. □

Lemma 5.3.3 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in* AG$(3,q)$*, $q = 2^h$, $h > 1$, such that there are no planar nets. Let $p_\infty \in \mathcal{P}_{\min}$ and let π be a plane of type* II *such that $P_\infty(\pi) = \{p_\infty\}$. Then none of the planes parallel to π is of type* IV.

Proof. Let $L_\infty = \pi \cap \Pi_\infty$. By Lemma 5.3.2, there exists a point $r_\infty \in L_\infty$ such that $k(r_\infty) = l(L_\infty)$. Since $P_\infty(\pi) = \{p_\infty\}$, $k(p_\infty) > l(L_\infty)$. Since $p_\infty \in \mathcal{P}_{\min}$, $k(r_\infty) < k(p_\infty)$ implies $k(r_\infty) = 0$. So $l(L_\infty) = 0$. □

Lemma 5.3.4 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in* AG$(3,q)$*, $q = 2^h$, $h > 1$, such that there are no planar nets. Then $k_{\min} = l_{\min}$.*

Proof. Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ such that $l(L_\infty) = l_{\min}$. By Lemma 5.3.2 there is a point $p_\infty \in L_\infty$ such that $k(p_\infty) = l(L_\infty) = l_{\min}$. So $k_{\min} \leq l_{\min}$.

Let $p_\infty \in \mathcal{P}_{\min}$, let L be a line of $\mathcal{S}$ such that $L \cap \Pi_\infty = p_\infty$, let π be a plane of type IV containing L and let $L_\infty = \pi \cap \Pi_\infty$. By Lemma 5.3.3 there is no plane π' of type II parallel to π such that $P_\infty(\pi') = \{p_\infty\}$. Hence $l(L_\infty) = k(p_\infty) = k_{\min}$. So $l_{\min} \leq k_{\min}$. □

Lemma 5.3.5 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in* AG$(3,q)$*, $q = 2^h$, $h > 1$, such that there are no planar nets. Then there are at least $q+1-t$ holes.*

Proof. Let L be a line of $\mathcal{S}$ such that $p_\infty = L \cap \Pi_\infty \in \mathcal{P}_{\min}$, let π be a plane of type II which contains L and let $L_\infty = \pi \cap \Pi_\infty$. By Lemma 5.3.3, $l(L_\infty) = 0$, and by Lemma 5.3.2, there is a hole on L_∞. Now since there are $q+1-t$ planes of type II which contain L, the lemma follows. □

Lemma 5.3.6 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in* AG$(3,q)$*, $q = 2^h$, $h > 1$, such that there are no planar nets. Let $p_\infty \in \mathcal{P}_{\mathrm{IV}}$. The number of lines of $\mathcal{B}_{\mathrm{IV}}$ through p_∞ is at least t, and if $p_\infty \in \mathcal{P}_{\min}$ then equality holds. We have $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$ if and only if there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$ through every point of $\mathcal{P}_{\mathrm{IV}}$.*

Proof. Let $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ and let L be a line of $\mathcal{S}$ such that $L \cap \Pi_\infty = p_\infty$. Then the lines at infinity of the t planes of type IV containing L are lines of $\mathcal{B}_{\mathrm{IV}}$ which contain p_∞. Suppose that $p_\infty \in \mathcal{P}_{\min}$. Then by Lemma 5.3.3, none of the planes parallel to a plane of type II which contains L, is of type IV. Hence the lines at infinity of the $q+1-t$ planes of type II which contain L are in $\mathcal{B}_0$. So there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$ through p_∞.

Now it is clear that if $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$, then there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$ through every point of $\mathcal{P}_{\mathrm{IV}}$. Suppose that through every point of $\mathcal{P}_{\mathrm{IV}}$ there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$. Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Suppose that there are points $p_\infty, p'_\infty \in \mathcal{B}_{\mathrm{IV}}$ such that $k(p_\infty) > k(p'_\infty)$. Then by Lemma 5.3.2, $l(L_\infty) \leq k(p'_\infty) < k(p_\infty)$, so there is a plane π of type II such that $\pi \cap \Pi_\infty = L_\infty$ and $P_\infty(\pi) = \{p_\infty\}$. Let L be a line of $\mathcal{S}_\pi$. Then the lines at infinity of the planes of type IV which contain L are t lines of $\mathcal{B}_{\mathrm{IV}}$ through p_∞, different from L_∞. But this contradicts the fact that there are t lines of $\mathcal{B}_{\mathrm{IV}}$ through p_∞. So $k(p_\infty) = k(p'_\infty)$ for all points $p_\infty, p'_\infty \in L_\infty$, and this holds for all lines $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Now it is clear that $k(p_\infty)$ is constant for all $p_\infty \in \mathcal{P}_{\mathrm{IV}}$, so $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$. $\square$

Lemma 5.3.7 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a* $(0,2)$*-geometry of order* $(q-1,t)$ *fully embedded in* $\mathrm{AG}(3,q)$*,* $q = 2^h$*,* $h > 1$*, such that there are no planar nets. Then the following hold.*

$$
\begin{aligned}
(t+1)\,|\,\mathcal{P}\,| &= q\,|\,\mathcal{B}\,|\,; && (5.1)\\
t\,|\,\mathcal{B}\,| &= (q+1)\,|\,\Pi_{\mathrm{IV}}\,|\,; && (5.2)\\
k_{\min}\,|\,\mathcal{B}_{\mathrm{IV}}\,| &\leq |\,\Pi_{\mathrm{IV}}\,|\,; && (5.3)\\
k_{\min}\,|\,\mathcal{P}_{\mathrm{IV}}\,| &\leq |\,\mathcal{B}\,|\,; && (5.4)\\
t\,|\,\mathcal{P}_{\mathrm{IV}}\,| &\leq (q+1)\,|\,\mathcal{B}_{\mathrm{IV}}\,|\,; && (5.5)\\
qt-q+t &\leq |\,\mathcal{B}_{\mathrm{IV}}\,|\,; && (5.6)\\
(q+1)^2 - \frac{q(q+1)^2}{|\,\mathcal{B}_{\mathrm{IV}}\,| + q} &\leq |\,\mathcal{P}_{\mathrm{IV}}\,|\,. && (5.7)
\end{aligned}
$$

Equality holds in any of (5.4), (5.5), (5.6) or (5.7) if and only if $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$*. If* $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$ *then equality holds in (5.3).*

Proof. Equality (5.1) is the result of counting the flags of $\mathcal{S}$. Equality (5.2) is the result of counting the ordered pairs (L,π) where $L \in \mathcal{B}, \pi \in \Pi_{\mathrm{IV}}, L \subseteq \pi$.

We count the ordered pairs (π, L_∞) where $\pi \in \Pi_{\mathrm{IV}}$, $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ and $L_\infty = \pi \cap \Pi_\infty$. This number equals $|\,\Pi_{\mathrm{IV}}\,|$ since a plane $\pi \in \Pi_{\mathrm{IV}}$ has just one line at infinity. For every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ the number of planes π of type IV such that $L_\infty = \pi \cap \Pi_\infty$ is $l(L_\infty)$, and by Lemma 5.3.4, $l(L_\infty) \geq l_{\min} = k_{\min}$. Hence inequality (5.3). Suppose that $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$. Then for every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$, $k(p_\infty) = k_{\min}$ for every point $p_\infty \in L_\infty$. Now Lemma 5.3.2 implies that $l(L_\infty) = k_{\min}$. So $k_{\min} \,|\, \mathcal{B}_{\mathrm{IV}} \,| = |\, \Pi_{\mathrm{IV}} \,|$.

Inequality (5.4) follows from counting the ordered pairs (L, p_∞) where $L \in \mathcal{B}, p_\infty \in \mathcal{P}_{\mathrm{IV}}, p_\infty \in L$. Equality holds if and only if $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$. Inequality (5.5) follows from counting the ordered pairs (p_∞, L_∞) such that $p_\infty \in \mathcal{P}_{\mathrm{IV}}, L_\infty \in \mathcal{B}_{\mathrm{IV}}, p_\infty \in L_\infty$, using Lemma 5.3.6. Equality holds if and only if through any point $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$. By Lemma 5.3.6, this is the case if and only if $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$.

Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Then by Lemma 5.3.6, through each of the $q+1$ points of L_∞ there are at least t lines of $\mathcal{B}_{\mathrm{IV}}$ (including L_∞). This implies inequality (5.6). If there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$ through every point of $\mathcal{P}_{\mathrm{IV}}$ then clearly $|\, \mathcal{B}_{\mathrm{IV}} \,| = qt - q + t$. If $|\, \mathcal{B}_{\mathrm{IV}} \,| = qt - q + t$ then for every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ there holds that there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$ through every point of L_∞. So there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$ through every point of $\mathcal{P}_{\mathrm{IV}}$. So equality holds in (5.6) if and only if through any point $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$. By Lemma 5.3.6, this is the case if and only if $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$.

Let $\mathcal{P}_{\mathrm{IV}} = \{p_\infty^1, p_\infty^2, \ldots, p_\infty^v\}$ where $v = |\, \mathcal{P}_{\mathrm{IV}} \,|$. For every $i \in \{1, 2, \ldots, v\}$ let t_i be the number of lines of $\mathcal{B}_{\mathrm{IV}}$ through p_∞^i. Counting the ordered pairs (p_∞^i, L_∞) where $p_\infty^i \in \mathcal{P}_{\mathrm{IV}}, L_\infty \in \mathcal{B}_{\mathrm{IV}}, p_\infty^i \in L_\infty$ yields

$$\sum_{i=1}^{v} t_i = (q+1)\,|\, \mathcal{B}_{\mathrm{IV}} \,|\,.$$

Counting the ordered triples $(p_\infty^i, L_\infty, L'_\infty)$ such that $L_\infty,\ L'_\infty \in \mathcal{B}_{\mathrm{IV}}$ and $p_\infty^i = L_\infty \cap L'_\infty$ yields

$$\sum_{i=1}^{v} t_i(t_i - 1) = |\, \mathcal{B}_{\mathrm{IV}} \,|\,(\,|\, \mathcal{B}_{\mathrm{IV}} \,| - 1).$$

Let $\bar{t} = \sum_{i=1}^{v} t_i / v$. Then $\sum_{i=1}^{v} (t_i - \bar{t})^2 \geq 0$ yields

$$|\, \mathcal{B}_{\mathrm{IV}} \,|\,(\,|\, \mathcal{B}_{\mathrm{IV}} \,| - 1) + (q+1)\,|\, \mathcal{B}_{\mathrm{IV}} \,| - \frac{(q+1)^2\,|\, \mathcal{B}_{\mathrm{IV}} \,|^2}{|\, \mathcal{P}_{\mathrm{IV}} \,|} \geq 0.$$

From this, inequality (5.7) follows. Equality holds if and only if $t_i = \bar{t}$ for every point $p_\infty^i \in \mathcal{P}_{\mathrm{IV}}$. Suppose that $t_i = \bar{t}$ for every point $p_\infty^i \in \mathcal{P}_{\mathrm{IV}}$. Let $p_\infty^i \in \mathcal{P}_{\min}$. Then by Lemma 5.3.6, $t_i = t$. So $\bar{t} = t$. So equality holds in (5.7) if and only if through any point $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ there are exactly t lines of $\mathcal{B}_{\mathrm{IV}}$. By Lemma 5.3.6, this is the case if and only if $\mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$. □

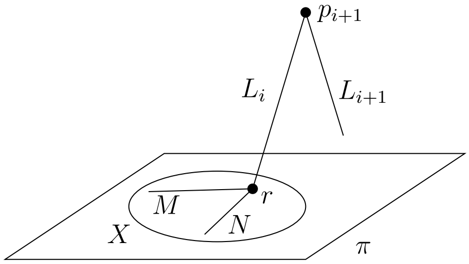

Figure 5.2: Elimination of the case $t = 2$ (Theorem 5.3.8).

5.3.2 Proof that $t = q$

In this section, we prove that every $(0,2)$-geometry which is fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets, has order $(q-1,t)$ with $t = q$. We do so by eliminating all other possible values of t. By Proposition 5.3.1, $2 \leq t \leq q$. Firstly, in Section 5.3.2.1, we eliminate the cases $t = 2$ and $t = q-1$. Then, in Section 5.3.2.2, some lemmas are proven that are valid for all $2 < t < q-1$. Next, in Section 5.3.2.3, we eliminate the case where $2 < t < q-1$ and t is odd. Finally, in Section 5.3.2.4, we eliminate the case where $2 < t < q-1$ and t is even.

5.3.2.1 Elimination of the cases $t = 2$ and $t = q-1$

Theorem 5.3.8 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then $t \neq 2$.*

Proof. Suppose that $t = 2$. Let π be a plane of type IV, let $\mathcal{S}'$ be the connected component of $\mathcal{S}_\pi$ which is a dual oval and let X be point set of $\mathcal{S}'$. Suppose that $\mathcal{S}_\pi$ has an isolated point p. Then there is a point $p_0 \in X$ and a path $(p_0, p_1, \ldots, p_n = p)$ in the point graph of $\mathcal{S}$ such that for $0 \leq i \leq n-1$, the line $L_i = \langle p_i, p_{i+1}\rangle$ is not contained in π. So $|\,L_i \cap X\,| \in \{0,1\}$ for all $0 \leq i \leq n-1$. Since $|\,L_0 \cap X\,| = 1$ and $|\,L_{n-1} \cap X\,| = 0$ there is an $i \in \{0,1,\ldots,n-2\}$ such that $|\,L_i \cap X\,| = 1$ and $|\,L_{i+1} \cap X\,| = 0$. Let $r = L_i \cap X$ and let M and N be the two lines of $\mathcal{S}'$ through r. Since $r \in L_i$ and L_i intersects L_{i+1}, $\alpha(r, L_{i+1}) = 2$. But M nor N intersects L_{i+1} since $|\,L_{i+1} \cap X\,| = 0$. So there is a line $L \notin \{L_i, M, N\}$ of $\mathcal{S}$ through r which intersects L_{i+1}. But this contradicts $t = 2$. Hence $\mathcal{S}_\pi$ has no isolated points. Note that this holds for every plane π of type IV.

We prove that any two planes of type IV intersect in a line of $\mathcal{S}$. Let π_1, π_2 be distinct planes of type IV. Note that π_i does not contain any isolated points, so through each point of $\mathcal{S}_{\pi_i}$ there are 2 lines of $\mathcal{S}_{\pi_i}$, $i = 1, 2$.

Suppose that π_1 and π_2 are parallel. Let p be a point of $\mathcal{S}_{\pi_1}$. Since $t = 2$, there is a unique line L of $\mathcal{S}$ through p which is not contained in π_1. Then L intersects π_2 in a point r of $\mathcal{S}_{\pi_2}$. Let M be a line of $\mathcal{S}_{\pi_2}$ through r. Since L is the only line of $\mathcal{S}$ through p which is not contained in π_1, $\alpha(p, M) = 1$, a contradiction.

Suppose that π_1 and π_2 intersect in a line L which is not a line of $\mathcal{S}$. Let p be a point of $\mathcal{S}_{\pi_1}$ on L. Since π_2 does not contain any isolated points, p is a point of $\mathcal{S}_{\pi_2}$. But then there are 4 lines of $\mathcal{S}$ through p, a contradiction. We conclude that any two planes of type IV intersect in a line of $\mathcal{S}$.

Let π be a plane of type IV. Since $t = 2$, every line of $\mathcal{S}_\pi$ is contained in exactly one plane of type IV other than π. Since every plane of type IV, other than π, intersects π in a line of $\mathcal{S}$, $|\,\Pi_{\mathrm{IV}}\,| = q + 2$.

Suppose that $\pi_1, \pi_2, \pi_3, \pi_4$ are distinct planes of type IV which have an affine point p in common. Then all lines $\pi_i \cap \pi_j$ with $1 \leq i < j \leq 4$ are distinct lines of $\mathcal{S}$, so there are at least 6 lines of $\mathcal{S}$ through p, a contradiction. Suppose that π_1, π_2, π_3 are distinct planes of type IV which have a point $p_\infty \in \Pi_\infty$ in common. Then all lines $\pi_i \cap \pi_j$ with $1 \leq i < j \leq 3$ are distinct lines of $\mathcal{S}$, so in particular $\mathcal{S}_{\pi_1}$ contains two parallel lines, a contradiction. We conclude that the set $\Pi_{\mathrm{IV}} \cup \{\Pi_\infty\}$ is a $(q+3)$-arc in the dual space of $\mathrm{PG}(3, q)$. But in Chapter 21 of [51], it is proven that $m(3, q)$, the size of the largest k-arc in $\mathrm{PG}(3, q)$, is $\max(5, q+1)$. So $q = 2$, a contradiction. $\square$

Theorem 5.3.9 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1, t)$ fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then $t \neq q-1$.*

Proof. Suppose that $t = q - 1$. Combining inequalities (5.6) and (5.7) from Lemma 5.3.7, we get that $|\,\mathcal{P}_{\mathrm{IV}}\,| \geq q^2 + q - 1 - 2/(q-1)$. So since $q > 2$, $|\,\mathcal{P}_0\,| \leq 2$. But Lemma 5.3.5 says $|\,\mathcal{P}_0\,| \geq 2$, so there are exactly two holes, p^1_∞ and p^2_∞.

Let $p_\infty \in \Pi_\infty$ be a point off the line $\langle p^1_\infty, p^2_\infty \rangle$. Then $k(p_\infty) \neq 0$. Suppose that $k(p_\infty) \geq 2$, and let L_1 and L_2 be two lines of $\mathcal{S}$ which intersect Π_∞ in the point p_∞. Choose $i \in \{1, 2\}$ such that p^i_∞ is not contained in the line at infinity of the plane $\langle L_1, L_2 \rangle$. Let $L_\infty = \langle p^i_\infty, p_\infty \rangle$ and let V be the parallel class of planes which intersect Π_∞ in the line L_∞. Since L_∞ contains a hole, $l(L_\infty) = 0$. Since there are q points of $\mathcal{P}_{\mathrm{IV}}$ on L_∞, for every point r_∞ of $\mathcal{P}_{\mathrm{IV}}$ on L_∞ there is exactly one plane $\pi \in V$ such that $P_\infty(\pi) = \{r_\infty\}$. But the planes $\pi_i = \langle L_i, L_\infty \rangle$, $i = 1, 2$, are in V and $P_\infty(\pi_1) = P_\infty(\pi_2) = \{p_\infty\}$, a contradiction. So $k(p_\infty) = 1$ for every point $p_\infty \in \Pi_\infty$ not on the line $\langle p^1_\infty, p^2_\infty \rangle$.

Let L_∞ be a line of Π_∞ not containing p^1_∞ or p^2_∞. Since the minimal value of $k(p_\infty)$ for all points $p_\infty \in L_\infty$ is 1, Lemma 5.3.2 implies that $l(L_\infty) = 1$. For every line L_∞ of Π_∞ containing p^1_∞ or p^2_∞, $l(L_\infty) = 0$. Hence $|\,\Pi_{\mathrm{IV}}\,| = q(q-1)$. It follows from inequality (5.2) of Lemma 5.3.7 that $|\,\mathcal{B}\,| = q(q+1)$. Let π be a plane of type IV, and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_\pi$ that is a dual oval. Then there are $q+1$ lines in $\mathcal{S}'$ and $\frac{1}{2}q(q+1)(q-2)$ lines of $\mathcal{S}$ which intersect π in a point of $\mathcal{S}'$. So $|\,\mathcal{B}\,| = q(q+1) \geq \frac{1}{2}q(q+1)(q-2) + q + 1$. But this contradicts $q > 2$. □

5.3.2.2 Some lemmas valid for all $2 < t < q-1$

Lemma 5.3.10 *Let $\mathcal{S}$ be a $(0, 2)$-geometry of order $(q-1, t)$ fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then for any line $L_\infty \subseteq \Pi_\infty$, $t \,|\, q\,l(L_\infty)$ and the number of lines of $\mathcal{S}$ which intersect Π_∞ in a point of L_∞ is equal to*

$$\frac{|\,\mathcal{B}\,|}{q+1} + \frac{q\,l(L_\infty)}{t}.$$

Furthermore,

$$l(L_\infty) \leq \frac{q^2 t(t+1)}{(q+1)(qt-q+t)} < 2t\,.$$

Proof. Let L_∞ be a line of Π_∞ and let V be the parallel class of planes which intersect Π_∞ in L_∞. Let X be the set of lines L of $\mathcal{S}$ such that $L \cap \Pi_\infty \in L_\infty$ and $\langle L, L_\infty\rangle$ is a plane of type II. We count the pairs (L, π) where $L \in X$, π is a plane of type IV and $L \subseteq \pi$. In a plane $\pi \in V$ of type IV there are no lines of X. There are $|\,\Pi_{\mathrm{IV}}\,| - l(L_\infty)$ planes of type IV not in V, and each of these planes contains exactly one line which intersects Π_∞ in a point of L_∞. However in this way we count $l(L_\infty)(q+1)(t-1)$ pairs (L, π) such that the plane $\langle L, L_\infty\rangle$ is a plane of type IV, so $L \notin X$. We obtain

$$t\,|\,X\,| = |\,\Pi_{\mathrm{IV}}\,| - l(L_\infty)(qt - q + t).$$

Hence the number of lines of $\mathcal{S}$ which intersect Π_∞ in a point of L_∞ is

$$|\,X\,| + l(L_\infty)(q+1) = \frac{|\,\mathcal{B}\,|}{q+1} + \frac{q\,l(L_\infty)}{t}.$$

Here equation (5.2) of Lemma 5.3.7 was used.

Clearly $|\,\mathcal{B}\,|/(q+1) + q\,l(L_\infty)/t$ is an integer. Lemma 5.3.5 implies that $\mathcal{B}_0$ is not empty. Let $M_\infty \in \mathcal{B}_0$. Then the number of lines of $\mathcal{S}$ which intersect

Π_∞ in a point of M_∞ is $|\,\mathcal{B}\,|/(q+1)$, so this number is an integer. Hence also $q\,l(L_\infty)/t$ is an integer, so $t\,|\,q\,l(L_\infty)$.

From equations (5.1) and (5.2) of Lemma 5.3.7 and the fact that $|\,\mathcal{P}\,| \le q^3$ it follows that

$$|\,\Pi_{\mathrm{IV}}\,| \le \frac{q^2t(t+1)}{q+1}.$$

On the other hand $|\,\Pi_{\mathrm{IV}}\,| - l(L_\infty)(qt-q+t) = t\,|\,X\,| \ge 0$, so

$$l(L_\infty) \le \frac{|\,\Pi_{\mathrm{IV}}\,|}{qt-q+t} \le \frac{q^2t(t+1)}{(q+1)(qt-q+t)}.$$

Suppose that

$$2t \le \frac{q^2t(t+1)}{(q+1)(qt-q+t)}.$$

Then $t \le (3q^2+2q)/(q^2+4q+2) < 3$, a contradiction. □

Lemma 5.3.11 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in $\mathrm{AG}(3,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Then a line $L_\infty \subseteq \Pi_\infty$ contains at most $q+1-t$ holes.*

Proof. Let L be a line of $\mathcal{S}$ such that $L \cap \Pi_\infty \notin L_\infty$, and let π be a plane of type IV which contains L. Let $M_\infty = \pi \cap \Pi_\infty$ and let $p_\infty = L_\infty \cap M_\infty$. Since π is a plane of type IV, $p_\infty \in P_\infty(\pi)$ and hence p_∞ is not a hole. Since there are t planes of type IV which contain L, there are at least t points on L_∞ which are not holes. □

Lemma 5.3.12 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, $t<q$, fully embedded in $\mathrm{AG}(3,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Then there is a plane of type* II *which contains exactly one line of $\mathcal{S}$.*

Proof. Let $\Theta_{\min}$ denote the minimal number of lines of $\mathcal{S}$ in a plane of type II. Let $p_\infty \in \mathcal{P}_{\min}$ and let L be a line of $\mathcal{S}$ which intersects Π_∞ in p_∞. Then there are $q+1-t$ planes of type II which contain L, and each of these planes contains at least $\Theta_{\min}$ lines of $\mathcal{S}$, each of which intersects Π_∞ in p_∞. It follows that $1+(q+1-t)(\Theta_{\min}-1) \le k_{\min}$.

Suppose that $\Theta_{\min} \ge 2$. Then $k_{\min} \ge q+2-t$. Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. By Lemma 5.3.4, $k_{\min} = l_{\min} \le l(L_\infty)$, and by Lemma 5.3.10,

$$k_{\min} \le \frac{q^2t(t+1)}{(q+1)(qt-q+t)}.$$

Hence $(q+1)(q+2-t)(qt-q+t) \leq q^2t(t+1)$. This inequality is quadratic in t. It is easy to check that it is satisfied when $t=1$ or $t=\frac{1}{2}q+1$ but not when $t=2$ or $t=\frac{1}{2}q$ (note that $q \geq 4$). It follows that $\frac{1}{2}q+1 \leq t \leq q-1$.

Let L_∞ be a line of Π_∞ in $\mathcal{B}_0$, and let V be the parallel class of planes which intersect Π_∞ in the line L_∞. Then $l(L_\infty)=0$, so every plane $\pi \in V$ is of type I or II. By Lemma 5.3.11, the number of points of $\mathcal{P}_{\mathrm{IV}}$ on L_∞ is at least t. For every point $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ on L_∞, $k(p_\infty)>0$, so there is at least one plane $\pi \in V$ of type II such that $P_\infty(\pi)=\{p_\infty\}$. Since $2t>q$, there is a point $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ on L_∞ such that there is exactly one plane $\pi \in V$ such that $P_\infty(\pi)=\{p_\infty\}$. So every line of $\mathcal{S}$ which intersects Π_∞ in p_∞ is contained in the plane π. Let M be a line of $\mathcal{S}$ such that $M \cap \Pi_\infty = p_\infty$, and let $\pi' \neq \pi$ be a plane of type II containing M (which exists since by assumption $t<q$). Then π' contains exactly one line of $\mathcal{S}$, so $\Theta_{\min}=1$. But this contradicts the assumption that $\Theta_{\min} \geq 2$. We conclude that $\Theta_{\min}=1$. □

Henceforth we use the following notation. For every integer $i \in \mathbb{N} \setminus \{0\}$ we let $\mathrm{odd}(i)$ denote the largest odd divisor of i.

Lemma 5.3.13 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, $2<t<q$, fully embedded in $\mathrm{AG}(3,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Let π be a plane of type* I *or* II *containing Θ lines of $\mathcal{S}$, and let $L_\infty = \pi \cap \Pi_\infty$. If $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ then*

$$\Theta \;\equiv\; q+1 \pmod{\mathrm{odd}(t+1)}.$$

If $L_\infty \in \mathcal{B}_0$ then

$$\Theta \;\equiv\; 1 \pmod{\mathrm{odd}(t+1)}.$$

As a consequence $k_{\min} \equiv 1 \pmod{\mathrm{odd}(t+1)}$.

Proof. Suppose that $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Then $l(L_\infty)>0$, so there is a plane π' of type IV parallel to π. Let m, respectively m', be the number of isolated points of $\mathcal{S}_\pi$, respectively $\mathcal{S}_{\pi'}$. Counting the number of lines of $\mathcal{S}$ which intersect π and π' in an affine point yields

$$\Theta qt + m(t+1) = \frac{1}{2}q(q+1)(t-1) + m'(t+1).$$

It follows that $t+1 \mid \frac{1}{2}q(q+1)(t-1) - \Theta qt$. Hence $\Theta \equiv q+1 \pmod{\mathrm{odd}(t+1)}$.

Now suppose that $L_\infty \in \mathcal{B}_0$. By Lemma 5.3.12, there is a plane π' of type II which contains exactly one line of $\mathcal{S}$. Let $L'_\infty = \pi' \cap \Pi_\infty$. Suppose that $L'_\infty \in \mathcal{B}_{\mathrm{IV}}$. Then by the preceding paragraph $1 \equiv q+1 \pmod{\mathrm{odd}(t+1)}$.

Since $q = 2^h$, $\mathrm{odd}(t+1) = 1$, and in this case $\Theta \equiv 1 \pmod{\mathrm{odd}(t+1)}$ holds trivially. So we may assume that $L'_\infty \in \mathcal{B}_0$. Then $l(L'_\infty) = 0$ and by Lemma 5.3.10, the number of lines of $\mathcal{S}$ which intersect Π_∞ in a point of L'_∞ equals $|\,\mathcal{B}\,|/(q+1)$. Let m' denote the number of isolated points of $\mathcal{S}_{\pi'}$. Then the number of lines of $\mathcal{S}$ which intersect π' in an affine point equals $qt+m'(t+1)$. Hence

$$|\,\mathcal{B}\,| = qt + m'(t+1) + \frac{|\,\mathcal{B}\,|}{q+1}.$$

Similarly, if m is the number of isolated points of $\mathcal{S}_\pi$,

$$|\,\mathcal{B}\,| = \Theta qt + m(t+1) + \frac{|\,\mathcal{B}\,|}{q+1}.$$

It follows that $t+1 \,|\, (\Theta - 1)qt$, so $\Theta \equiv 1 \pmod{\mathrm{odd}(t+1)}$.

Let $p_\infty \in \mathcal{P}_{\min}$, let L be a line of $\mathcal{S}$ such that $L \cap \Pi_\infty = p_\infty$, and let X be the set of all lines of $\mathcal{S}$, other than L, which intersect Π_∞ in the point p_∞. Let π be a plane containing L. If π is of type IV then π does not contain any lines of X. If π is of type II then by Lemma 5.3.3, the line $L_\infty = \pi \cap \Pi_\infty$ is in $\mathcal{B}_0$. Hence the number of lines of X in π is divisible by $\mathrm{odd}(t+1)$. So $|\,X\,| \equiv 0 \pmod{\mathrm{odd}(t+1)}$, and since $|\,X\,| = k_{\min} - 1$, $k_{\min} \equiv 1 \pmod{\mathrm{odd}(t+1)}$. $\square$

We now give a definition which may seem awkward, but which will prove to be very useful. Consider a line $L_\infty \in \mathcal{B}_0$ and a point p_∞ of $\mathcal{P}_{\mathrm{IV}}$ on L_∞, and let V be the set of planes which intersect Π_∞ in the line L_∞. Then $k(p_\infty) > 0$, so there is a plane π in V such that $p_\infty \in P_\infty(\pi)$. Since $L_\infty \in \mathcal{B}_0$, $l(L_\infty) = 0$. So π is necessarily a plane of type II and $P_\infty(\pi) = \{p_\infty\}$. It is now clear that we can choose a set Π of planes of type II, each intersecting Π_∞ in a line of $\mathcal{B}_0$, such that for every pair (p_∞, L_∞) where $p_\infty \in \mathcal{P}_{\mathrm{IV}}$, $L_\infty \in \mathcal{B}_0$, $p_\infty \in L_\infty$, there is exactly one plane $\pi \in \Pi$ such that $\pi \cap \Pi_\infty = L_\infty$ and $P_\infty(\pi) = \{p_\infty\}$. We call the elements of Π *original planes*. The set of planes of type II which intersect Π_∞ in a line of $\mathcal{B}_0$ but which are not in the set Π, is denoted by Π^*. Its elements are called *repeated planes*.

Lemma 5.3.14 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in* $\mathrm{AG}(3,q)$*, $q = 2^h$, $h > 1$, such that there are no planar nets. Then*

$$|\,\Pi^*\,| + |\,\mathcal{B}_0\,| \le (q+1)\,|\,\mathcal{P}_0\,|.$$

Proof. Let $\mathcal{B}_0 = \{L^1_\infty, L^2_\infty, \ldots, L^{|\mathcal{B}_0|}_\infty\}$. For every $L^i_\infty \in \mathcal{B}_0$ let b_i be the number of repeated planes which intersect Π_∞ in the line L^i_∞, and let c_i be the number of holes on L^i_∞.

Let $L_\infty^i \in \mathcal{B}_0$. It follows from the definition of Π that the number of points of $\mathcal{P}_{\mathrm{IV}}$ on L_∞^i equals the number of original planes through L_∞^i. The number of points of $\mathcal{P}_{\mathrm{IV}}$ on L_∞^i equals $q+1-c_i$. The number of original planes through L_∞^i is at most $q-b_i$. So for every line $L_\infty^i \in \mathcal{B}_0$ we have that $b_i+1 \leq c_i$. Hence $\sum_{i=1}^{|\mathcal{B}_0|} b_i + |\mathcal{B}_0| \leq \sum_{i=1}^{|\mathcal{B}_0|} c_i$. Clearly $\sum_{i=1}^{|\mathcal{B}_0|} b_i = |\Pi^*|$. Counting the number of pairs (p_∞, L_∞) such that $L_\infty \in \mathcal{B}_0$ and p_∞ is a point of $\mathcal{P}_0$ on L_∞ yields $\sum_{i=1}^{|\mathcal{B}_0|} c_i = (q+1)|\mathcal{P}_0|$. The lemma follows. □

Lemma 5.3.15 *Let* $\mathcal{S}$ *be a* $(0,2)$*-geometry fully embedded in* $\mathrm{AG}(3,q)$*,* $q=2^h$*,* $h>1$*, such that there are no planar nets. Let* $\mathcal{P}_{\mathrm{IV}} = \{p_\infty^1, p_\infty^2, \ldots, p_\infty^{|\mathcal{P}_{\mathrm{IV}}|}\}$*, and for every* $p_\infty^i \in \mathcal{P}_{\mathrm{IV}}$ *let* d_i *be the number of repeated planes* π *such that* $P_\infty(\pi) = \{p_\infty^i\}$*. Then*

$$|\Pi^*| = \sum_{i=1}^{|\mathcal{P}_{\mathrm{IV}}|} d_i.$$

Proof. Count the number of pairs (p_∞, π) such that $p_\infty \in \mathcal{P}_{\mathrm{IV}}$, $\pi \in \Pi^*$ and $P_\infty(\pi) = \{p_\infty\}$. □

5.3.2.3 Elimination of the case $2 < t < q-1$, t odd

Theorem 5.3.16 *Let* $\mathcal{S}$ *be a* $(0,2)$*-geometry of order* $(q-1,t)$*,* t *odd and* $2<t<q-1$*, fully embedded in* $\mathrm{AG}(3,q)$*,* $q=2^h$*,* $h>1$*, such that there are no planar nets. Then* $t=2^j-1$ *for some* $j \in \{2,\ldots,h-1\}$ *and for every line* $L_\infty \in \mathcal{B}_{\mathrm{IV}}$*,* $l(L_\infty)=t$*.*

Proof. Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Then by Lemma 5.3.10, $t \mid q\, l(L_\infty)$. Since t is odd, t and q are relatively prime. So $t \mid l(L_\infty)$. Since $L_\infty \in \mathcal{B}_{\mathrm{IV}}$, $l(L_\infty) > 0$, and by Lemma 5.3.10, $l(L_\infty) < 2t$. So $l(L_\infty) = t$, and this holds for any line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Hence $l_{\min} = t$.

By Lemma 5.3.4, $k_{\min} = l_{\min} = t$. By Lemma 5.3.13,

$$t \equiv 1 \pmod{\mathrm{odd}(t+1)}.$$

So $\mathrm{odd}(t+1) = 1$. In other words, $t = 2^j - 1$ for some integer j. Since $2<t<q-1$, $2 \leq j \leq h-1$. □

Lemma 5.3.17 *Let* $\mathcal{S}$ *be a* $(0,2)$*-geometry of order* $(q-1,t)$*,* t *odd and* $2<t<q-1$*, fully embedded in* $\mathrm{AG}(3,q)$*,* $q=2^h$*,* $h>1$*, such that there are no planar nets. Then for every point* $p_\infty \in \mathcal{P}_{\mathrm{IV}}$*, the number of lines of* $\mathcal{B}_{\mathrm{IV}}$ *through* p_∞ *equals* $k(p_\infty)$*.*

Proof. Let a denote the number of lines of $\mathcal{B}_{\mathrm{IV}}$ through a point $p_\infty \in \mathcal{P}_{\mathrm{IV}}$. Let x be the number of planes of type IV which intersect Π_∞ in a line that contains p_∞. By Theorem 5.3.16, $l(L_\infty) = t$ for every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$, so $x = at$. On the other hand every line of $\mathcal{S}$ which intersects Π_∞ in the point p_∞ is contained in t planes of type IV, so $x = k(p_\infty)t$. The lemma follows. □

Lemma 5.3.18 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, t odd and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. For every point $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ such that $k(p_\infty) \le \frac{1}{2}q - 1$, the number of repeated planes π such that $P_\infty(\pi) = \{p_\infty\}$ is at least $q+1$.*

Proof. Let $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ be such that $k(p_\infty) \le \frac{1}{2}q - 1$, and let d be the number of repeated planes π such that $P_\infty(\pi) = \{p_\infty\}$.

Theorem 5.3.16 implies that $l_{\min} = t$, so $k(p_\infty) \ge k_{\min} = l_{\min} = t$.

By Lemma 5.3.17, the set $\mathcal{L}$ of lines of $\mathcal{B}_0$ through p_∞ contains exactly $q + 1 - k(p_\infty)$ elements. Let $\mathcal{L} = \{L^1_\infty, \ldots, L^{q+1-k(p_\infty)}_\infty\}$, and for every line $L^i_\infty \in \mathcal{L}$ let e_i be the number of planes $\pi \in \Pi^*$ such that $\pi \cap \Pi_\infty = L^i_\infty$ and $P_\infty(\pi) = \{p_\infty\}$. Then clearly $d = \sum_{i=1}^{q+1-k(p_\infty)} e_i$.

Let π be an affine plane not containing p_∞, and let $L_\infty = \pi \cap \Pi_\infty$. Let $\mathcal{K}$ be the set of affine points $p \in \pi$ such that $\langle p, p_\infty \rangle$ is a line of $\mathcal{S}$. For every $L^i_\infty \in \mathcal{L}$ let $p^i_\infty = L^i_\infty \cap L_\infty$. Now for every $i \in \{1, \ldots, q+1-k(p_\infty)\}$, $e_i + 1$ equals the number of lines in π which intersect Π_∞ in the point p^i_∞ and which have a nonempty intersection with the set $\mathcal{K}$.

Since $|\mathcal{K}| = k(p_\infty) \ge t > 2$ we may choose three distinct points $p_1, p_2, p_3 \in \mathcal{K}$. Let $\mathcal{K}' = \{p_1, p_2, p_3\}$, and for every $i \in \{1, \ldots, q+1-k(p_\infty)\}$ let $e'_i + 1$ be the number of lines in π which intersect Π_∞ in the point p^i_∞ and which have a nonempty intersection with the set $\mathcal{K}'$. Clearly $e_i \ge e'_i$ for every $i \in \{1, \ldots, q+1-k(p_\infty)\}$, and since $k(p_\infty) \le \frac{1}{2}q - 1$,

$$d = \sum_{i=1}^{q+1-k(p_\infty)} e_i \ge \sum_{i=1}^{\frac{1}{2}q+2} e'_i.$$

There are two possibilities.

1. The points p_1, p_2, p_3 are collinear. Then $e'_i = 0$ for at most one $i \in \{1, \ldots, \frac{1}{2}q+2\}$, while $e'_i = 2$ for all other i. Hence $d \ge \sum_{i=1}^{\frac{1}{2}q+2} e'_i \ge q+2$.
2. The points p_1, p_2, p_3 are not collinear. Then $e'_i = 1$ for at most three $i \in \{1, \ldots, \frac{1}{2}q+2\}$, while $e'_i = 2$ for all other i. It follows that $d \ge \sum_{i=1}^{\frac{1}{2}q+2} e'_i \ge q+1$.

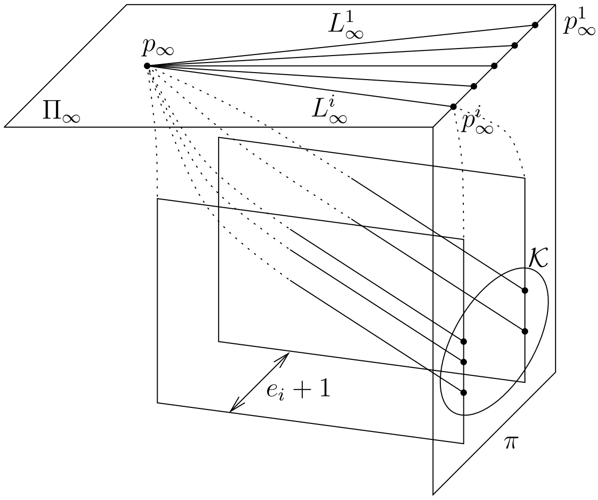

Figure 5.3: Illustration of Lemma 5.3.18.

We conclude that $d \geq q+1$. $\square$

Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, t odd and $2<t<q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. By Theorem 5.3.16, $l(L_\infty)=t$, and by Lemma 5.3.10, the number of lines of $\mathcal{S}$ which intersect Π_∞ in a point of L_∞, is $|\mathcal{B}|/(q+1)+q$. So for every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ there are $t(q+1)$ lines L of $\mathcal{S}$ such that $L\cap\Pi_\infty \in L_\infty$ and $\langle L, L_\infty\rangle$ is a plane of type IV, and $r=|\mathcal{B}|/(q+1)+q-t(q+1)$ lines L of $\mathcal{S}$ such that $L\cap\Pi_\infty\in L_\infty$ and $\langle L, L_\infty\rangle$ is a plane of type II.

Let π be a plane of type IV and let m be the number of isolated points of $\mathcal{S}_\pi$. Then there are $|\mathcal{B}|/(q+1)+q$ lines of $\mathcal{S}$ which are parallel to π (including the lines of $\mathcal{S}_\pi$). Counting the number of lines of $\mathcal{S}$ which intersect π in an affine point yields

$$\frac{1}{2}q(q+1)(t-1)+m(t+1) = |\mathcal{B}| - \left(\frac{|\mathcal{B}|}{q+1}+q\right).$$

Since $|\mathcal{B}| = (q+1)(t(q+1)+r-q)$ and since $m \leq \frac{1}{2}q(q-1)$,

$$r \;\leq\; q-t. \tag{5.8}$$

Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ and let $\{p^0_\infty,\ldots,p^q_\infty\}$ be the point set of L_∞. Then from Lemma 5.3.17 and from the fact that every line of $\mathcal{B}_{\mathrm{IV}}$ other than L_∞ intersects L_∞, it follows that $|\mathcal{B}_{\mathrm{IV}}| = \sum_{i=0}^{q}(k(p^i_\infty)-1)+1$. On the other hand, counting the number of lines of $\mathcal{S}$ which intersect Π_∞ in a point of L_∞ yields $\sum_{i=0}^{q} k(p_\infty^i) = |\mathcal{B}|/(q+1) + q = t(q+1) + r$. So $|\mathcal{B}_{\mathrm{IV}}| = qt - q + t + r$. Hence by (5.8),

$$|\mathcal{B}_{\mathrm{IV}}| \quad \leq \quad qt\,. \tag{5.9}$$

Combining inequalities (5.6) and (5.7) of Lemma 5.3.7 yields

$$|\mathcal{P}_{\mathrm{IV}}| \quad \geq \quad (q+1)\left(q+1-\frac{q}{t}\right). \tag{5.10}$$

Theorem 5.3.19 *A $(0,2)$-geometry of order $(q-1,t)$, with $2 < t < q-1$ and t odd, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets, does not exist.*

Proof. Suppose that there exists a $(0,2)$-geometry $\mathcal{S}$ of order $(q-1,t)$, t odd and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Note that these assumptions imply that $q \geq 8$.

Let $\mathcal{R}$ be the set of points $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ such that $k(p_\infty) \geq \frac{1}{2}q$. Then by Lemmas 5.3.15 and 5.3.18,

$$|\Pi^*| \quad \geq \quad (q+1)\,(\,|\mathcal{P}_{\mathrm{IV}}| - |\mathcal{R}|\,)\,. \tag{5.11}$$

Let L_∞ be a line of $\mathcal{B}_{\mathrm{IV}}$ and let $\{p_\infty^0, \ldots, p_\infty^q\}$ be the point set of L_∞. For every $0 \leq i \leq q$, let $k_i = k(p_\infty^i) - t$. By Lemma 5.3.2 and Theorem 5.3.16, $k_i \geq 0$ for all $0 \leq i \leq q$. Counting the number of lines of $\mathcal{S}$ which intersect Π_∞ in a point of L_∞ yields $\sum_{i=0}^{q} k(p_\infty^i) = |\mathcal{B}|/(q+1)+q = t(q+1)+r$. Hence $\sum_{i=0}^{q} k_i = r$. A point $p_\infty^i \in L_\infty$ is an element of $\mathcal{R}$ if and only if $k_i \geq \frac{1}{2}q - t$. So the number of points of $\mathcal{R}$ on L_∞ is at most $\sum_{i=0}^{q} k_i/(\frac{1}{2}q-t) = r/(\frac{1}{2}q-t)$. From inequality (5.8) it follows that the number of points of $\mathcal{R}$ on L_∞ is at most $(2q-2t)/(q-2t)$. This holds for every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$.

By Theorem 5.3.16, $t = 2^j - 1$ for some $j \in \{2, \ldots, h-1\}$. Hence $t \leq \frac{1}{2}q - 1$ and if $t < \frac{1}{2}q - 1$ then $t \leq \frac{1}{4}q - 1$. So there are two cases to consider.

$\boldsymbol{t = \frac{1}{2}q - 1}$**.** Let x be the number of pairs (p_∞, L_∞) where $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ and p_∞ is a point of $\mathcal{R}$ on L_∞. Since the number of points of $\mathcal{R}$ on a line of $\mathcal{B}_{\mathrm{IV}}$ is at most $(2q-2t)/(q-2t) = \frac{1}{2}q + 1$, $x \leq (\frac{1}{2}q+1)\,|\mathcal{B}_{\mathrm{IV}}|$. On the other hand Lemma 5.3.17 implies that the number of lines of $\mathcal{B}_{\mathrm{IV}}$ through a point $p_\infty \in \mathcal{R}$ is $k(p_\infty) \geq \frac{1}{2}q$. So $x \geq \frac{1}{2}q\,|\mathcal{R}|$, and $|\mathcal{R}| \leq (q+2)\,|\mathcal{B}_{\mathrm{IV}}|/q$. Now from Lemma 5.3.14 and inequality (5.11) it follows that

$$(q+1)\left(|\mathcal{P}_{\mathrm{IV}}| - \frac{q+2}{q}|\mathcal{B}_{\mathrm{IV}}|\right) \leq (q+1)\,|\mathcal{P}_0| - |\mathcal{B}_0|\,.$$

Using the fact that $\mathcal{P}_0$ is the complement of $\mathcal{P}_{\mathrm{IV}}$ in the point set of Π_∞ and that $\mathcal{B}_0$ is the complement of $\mathcal{B}_{\mathrm{IV}}$ in the line set of Π_∞, we get that

$$2(q+1)\,|\,\mathcal{P}_{\mathrm{IV}}\,| \leq q(q^2+q+1) + \frac{q^2+4q+2}{q}\,|\,\mathcal{B}_{\mathrm{IV}}\,|\,.$$

Inequalities (5.9) and (5.10) yield

$$4(q+1)^2(q+1-\frac{2q}{q-2}) \leq 2q(q^2+q+1) + (q^2+4q+2)(q-2).$$

Since $q \geq 8$, $2q/(q-2) < 3$, so $q^3-4q^2-8q-4 \leq 0$. But this contradicts $q \geq 8$.

$\boldsymbol{t \leq \frac{1}{4}q - 1}$**.** Let x be the number of pairs (p_∞, L_∞) where $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ and p_∞ is a point of $\mathcal{R}$ on L_∞. Since the number of points of $\mathcal{R}$ on a line of $\mathcal{B}_{\mathrm{IV}}$ is at most $(2q-2t)/(q-2t) < 3$, $x \leq 2\,|\,\mathcal{B}_{\mathrm{IV}}\,|$. On the other hand Lemma 5.3.17 implies that the number of lines of $\mathcal{B}_{\mathrm{IV}}$ through a point $p_\infty \in \mathcal{R}$ is $k(p_\infty) \geq \frac{1}{2}q$. So $x \geq \frac{1}{2}q\,|\,\mathcal{R}\,|$, and $|\,\mathcal{R}\,| \leq 4\,|\,\mathcal{B}_{\mathrm{IV}}\,|\,/q$. Now from Lemma 5.3.14 and inequality (5.11) it follows that

$$(q+1)\left(\,|\,\mathcal{P}_{\mathrm{IV}}\,| - \frac{4}{q}\,|\,\mathcal{B}_{\mathrm{IV}}\,|\,\right) \leq (q+1)\,|\,\mathcal{P}_0\,| - |\,\mathcal{B}_0\,|\,.$$

Using the fact that $\mathcal{P}_0$ is the complement of $\mathcal{P}_{\mathrm{IV}}$ in the point set of Π_∞ and that $\mathcal{B}_0$ is the complement of $\mathcal{B}_{\mathrm{IV}}$ in the line set of Π_∞, we get that

$$2(q+1)\,|\,\mathcal{P}_{\mathrm{IV}}\,| \leq q(q^2+q+1) + \frac{5q+4}{q}\,|\,\mathcal{B}_{\mathrm{IV}}\,|\,.$$

Inequalities (5.9) and (5.10) yield

$$(5q+4)t^2 - (q^3+5q^2+5q+2)t + 2q(q+1)^2 \geq 0\,.$$

One can verify that this contradicts $3 \leq t \leq \frac{1}{4}q - 1$ and $q \geq 8$.

We conclude that there does not exist a $(0, 2)$-geometry of order $(q-1, t)$, t odd and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. $\square$

5.3.2.4 Elimination of the case $2 < t < q-1$, t even

Theorem 5.3.20 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, t even and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then $t = 2^j$ where $j \in \{2, 3, \ldots, h-1\}$ is such that $h = jl$ with l odd. For every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$, $l(L_\infty) \in \{1, t+2\}$.*

Proof. By Lemma 5.3.13, $k_{\min} \equiv 1 \pmod{\mathrm{odd}(t+1)}$. Since t is even, $\mathrm{odd}(t+1) = t+1$. So $t+1 \mid k_{\min} - 1$. Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Then by Lemma 5.3.10, $l(L_\infty) < 2t$. Hence $k_{\min} = l_{\min} \leq l(L_\infty) < 2t$. So $k_{\min} = l_{\min} \in \{1, t+2\}$.

Let $L^1_\infty \in \mathcal{B}_{\mathrm{IV}}$ and let $L^2_\infty \in \mathcal{B}_{\mathrm{IV}}$ be such that $l(L^2_\infty) = l_{\min}$. Let π_i be a plane of type IV which intersects Π_∞ in the line L^i_∞, and let m_i be the number of isolated points of $\mathcal{S}_{\pi_i}$, $i = 1, 2$. By Lemma 5.3.10, the number of lines of $\mathcal{S}$ which intersect Π_∞ in a point of L^i_∞ is equal to $|\mathcal{B}|/(q+1) + (q\, l(L^i_\infty))/t$, $i = 1, 2$. Since every line of $\mathcal{S}$ intersects π_i either in an affine point or in a point of L^i_∞ we have, for $i = 1, 2$, that

$$|\mathcal{B}| = \frac{1}{2} q(q+1)(t-1) + m_i(t+1) + \frac{|\mathcal{B}|}{q+1} + \frac{q\, l(L^i_\infty)}{t}.$$

It follows that $(m_1 - m_2)t(t+1) = q(l(L^2_\infty) - l(L^1_\infty))$, so $t+1 \mid l(L^2_\infty) - l(L^1_\infty)$. Since $l(L^2_\infty) = l_{\min} \in \{1, t+2\}$ and since by Lemma 5.3.10, $l(L^1_\infty) < 2t$, it follows that $l(L^1_\infty) \in \{1, t+2\}$. This conclusion holds for every line $L^1_\infty \in \mathcal{B}_{\mathrm{IV}}$.

Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$. Then by Lemma 5.3.10, $t \mid q\, l(L_\infty)$. As $l(L_\infty) \in \{1, t+2\}$, $t = 2^j$ for some integer j. Since $2 < t < q-1$, $j \in \{2, \ldots, h-1\}$.

Let $L_\infty \in \mathcal{B}_0$. Then by Lemma 5.3.13, in every plane π such that $\pi \cap \Pi_\infty = L_\infty$ there are $\Theta \equiv 1 \pmod{t+1}$ lines of $\mathcal{S}$. Hence the total number of lines of $\mathcal{S}$ which intersect Π_∞ in a point of L_∞ is congruent to $q \pmod{t+1}$. But Lemma 5.3.10 implies that this number is equal to $|\mathcal{B}|/(q+1)$, hence $|\mathcal{B}| \equiv q(q+1) \pmod{t+1}$. On the other hand, equation (5.1) from Lemma 5.3.7 implies that $t+1$ divides $|\mathcal{B}|$. It follows that $t+1 \mid q(q+1)$, and since $t+1$ is odd, $t+1 \mid q+1$. So $2^j + 1 \mid 2^h + 1$. Let $l \in \mathbb{N}$ and $R \in \{0, 1, \ldots, j-1\}$ be such that $h = jl + R$. Then since $2^j + 1$ divides $2^h + 1 = (2^j + 1 - 1)^l 2^R + 1$, $2^j + 1 \mid (-1)^l 2^R + 1$. As $0 \leq R < j$ this is impossible if l is even. Hence l is odd and $R = 0$. $\square$

Corollary 5.3.21 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, t even and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Let π be a plane of type I or II containing Θ lines of $\mathcal{S}$, and let $L_\infty = \pi \cap \Pi_\infty$. If $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ then $\Theta \equiv 0 \pmod{t+1}$. If $L_\infty \in \mathcal{B}_0$ then $\Theta \equiv 1 \pmod{t+1}$. In particular if $\Theta \geq 2$ then $\Theta \geq t+1$.*

Proof. This follows immediately from Lemma 5.3.13 and the fact that $t+1 \mid q+1$, which was shown in Theorem 5.3.20. $\square$

Lemma 5.3.22 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, t even and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ contains at least $q+1-q/t$ points p_∞ such that $k(p_\infty) = l(L_\infty)$.*

Proof. Let $L_\infty \in \mathcal{B}_{\mathrm{IV}}$, and let V be the parallel class of planes which intersect Π_∞ in the line L_∞. For every point $p_\infty \in L_\infty$ let V_{p_∞} be the set of planes $\pi \in V$ of type II such that $P_\infty(\pi) = \{p_\infty\}$.

Let p_∞ be a point of L_∞ such that $k(p_\infty) > l(L_\infty)$. Then $V_{p_\infty} \neq \emptyset$. We prove that $\mid V_{p_\infty} \mid \geq t$. By Theorem 5.3.20, there are two cases to consider.

$\boldsymbol{l(L_\infty) = 1.}$ Let π be the unique plane of type IV in V, and let L be the unique line of $\mathcal{S}_\pi$ such that $L \cap \Pi_\infty = p_\infty$. Let $\pi' \in V_{p_\infty}$, let L' be a line of $\mathcal{S}_{\pi'}$, and let $\pi'' = \langle L, L' \rangle$. Then π'' is a plane of type II which contains $\Theta \geq 2$ lines of $\mathcal{S}$. By Corollary 5.3.21, $\Theta \geq t+1$. Since $P_\infty(\pi'') = \{p_\infty\}$, for every line L'' of $\mathcal{S}_{\pi''}$ which is not in π, the plane $\langle L'', L_\infty \rangle$ is in V_{p_∞}. Hence $\mid V_{p_\infty} \mid \geq t$.

$\boldsymbol{l(L_\infty) = t+2.}$ Let X be the set of lines M of $\mathcal{S}$ such that $M \cap \Pi_\infty = p_\infty$ and such that $\langle M, L_\infty \rangle$ is a plane of type IV. Then $\mid X \mid = t+2$. Let $\pi \in V_{p_\infty}$. By Corollary 5.3.21, $\mathcal{S}_\pi$ has at least $t+1$ lines. So there is a line L of $\mathcal{S}_\pi$ and a line $M \in X$ such that the plane $\pi' = \langle L, M \rangle$ contains at most one other line $M' \in X$. Since $\mathcal{S}_{\pi'}$ contains two parallel lines, $\mathcal{S}_{\pi'}$ contains at least $t+1 > 3$ parallel lines by Corollary 5.3.21. So there is a line L' of $\mathcal{S}_{\pi'}$ such that $L' \not\subseteq \pi$ and $L' \notin X$. Note that $L' \cap \Pi_\infty = p_\infty$.

Consider the planes containing L' and a line of $\mathcal{S}_\pi$. Since there are at least $t+1$ such planes and since $\mid X \mid = t+2$, there is such a plane π'' containing at most one line of X. By Corollary 5.3.21, $\mathcal{S}_{\pi''}$ has at least $t+1$ lines. Now for every line L'' of $\mathcal{S}_{\pi''}$ such that $L'' \notin X$, the plane $\langle L'', L_\infty \rangle \in V_{p_\infty}$. Since π'' contains at most one line of X, $\mid V_{p_\infty} \mid \geq t$.

By Lemma 5.3.2, $k(p_\infty) \geq l(L_\infty)$ for every point $p_\infty \in L_\infty$. For every point $p_\infty \in L_\infty$ such that $k(p_\infty) > l(L_\infty)$, $\mid V_{p_\infty} \mid \geq t$. Since there are at most $q - l(L_\infty)$ planes of type II in V, there are at most $(q - l(L_\infty))/t$ points $p_\infty \in L_\infty$ such that $k(p_\infty) > l(L_\infty)$. Hence there are at least

$$q + 1 - \frac{q - l(L_\infty)}{t} \geq q + 1 - \frac{q}{t}$$

points $p_\infty \in L_\infty$ such that $k(p_\infty) = l(L_\infty)$. $\square$

Let $\mathcal{R}$ be the set of points $p_\infty \in \mathcal{P}_{\mathrm{IV}}$ such that $k(p_\infty) = t + 2$.

Lemma 5.3.23 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, t even and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. If $p_\infty \in \mathcal{R}$ then all lines of $\mathcal{S}$ which intersect Π_∞ in p_∞ are coplanar and there are exactly t lines L_∞ of $\mathcal{B}_{\mathrm{IV}}$ through p_∞, each of them having $l(L_\infty) = t + 2$.*

Proof. Let $p_\infty \in \mathcal{R}$ and let X be the set of lines of $\mathcal{S}$ which intersect Π_∞ in the point p_∞. Then $|\,X\,| = t + 2$. Suppose that not all lines of X are coplanar. So there are lines $L, L', L'' \in X$ which are not coplanar. By Corollary 5.3.21, each of the planes $\langle L, L'\rangle$, $\langle L', L''\rangle$ and $\langle L'', L\rangle$ contains at least $t + 1$ lines of $\mathcal{S}$, each of which is in the set X. But this contradicts $|\,X\,| = t + 2$. So all lines of X are coplanar.

Let $L \in X$, and let $L^1_\infty, L^2_\infty, \ldots, L^t_\infty$ be the lines at infinity of the t planes of type IV through L. By Theorem 5.3.20, $l(L^i_\infty) \in \{1, t+2\}$ for all $i \in \{1, 2, \ldots, t\}$. Suppose that for some L^i_∞, $l(L^i_\infty) = 1$. Let $L' \in X \setminus \{L\}$. Then the plane $\pi = \langle L', L^i_\infty\rangle$ is of type II, and since $L^i_\infty \in \mathcal{B}_{\mathrm{IV}}$, Corollary 5.3.21 implies that $\mathcal{S}_\pi$ has at least $t+1$ lines, all of which are in X. But also the plane $\langle L, L^i_\infty\rangle$ contains a line of X, namely L. This contradicts the fact that all lines of X are coplanar. So $l(L^i_\infty) = t + 2$ for all $i \in \{1, 2, \ldots, t\}$. This means that for every line $L' \in X$, every plane $\langle L', L^i_\infty\rangle$, $1 \le i \le t$, is a plane of type IV.

Suppose that there is a line $L_\infty \notin \{L^1_\infty, \ldots, L^t_\infty\}$ of $\mathcal{B}_{\mathrm{IV}}$ through p_∞. Let π be a plane of type IV which intersects Π_∞ in L_∞ and let L' be the unique line of X in π. Then there are $t + 1$ planes of type IV which contain L', namely π and the planes $\langle L', L^i_\infty\rangle$, $1 \le i \le t$. Clearly this is impossible. So $L^1_\infty, \ldots, L^t_\infty$ are the only lines of $\mathcal{B}_{\mathrm{IV}}$ through p_∞. $\square$

Lemma 5.3.24 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, t even and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then either $\mathcal{R} = \emptyset$ or $|\,\mathcal{R}\,| \ge \left(\frac{t-1}{t}\right)^3 q(q+1)$.*

Proof. Suppose that $\mathcal{R} \ne \emptyset$. By Lemma 5.3.23, there is a line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ such that $l(L_\infty) = t + 2$. Let x denote the number of triples $(p_\infty, M_\infty, r_\infty)$ such that M_∞ is a line of $\mathcal{B}_{\mathrm{IV}}$ intersecting L_∞ in the point $p_\infty \in \mathcal{R}$ and such that r_∞ is a point of $\mathcal{R}$ on M_∞, different from p_∞.

By Lemma 5.3.22, there are at least $q + 1 - q/t$ points p_∞ of $\mathcal{R}$ on L_∞. By Lemma 5.3.23, through every point p_∞ of $\mathcal{R}$ on L_∞ there are exactly $t-1$

lines $M_\infty \in \mathcal{B}_{\mathrm{IV}}$ different from L_∞, and each of these lines has $l(M_\infty) = t+2$. Again by Lemma 5.3.22, there are at least $q - q/t$ points $r_\infty \in \mathcal{R}$ on M_∞ different from p_∞. Hence $x \geq (t-1)(q+1-q/t)(q-q/t)$.

Through a fixed point $r_\infty \in \mathcal{R}$ not on L_∞ there are, by Lemma 5.3.23, exactly t lines $M_\infty \in \mathcal{B}_{\mathrm{IV}}$, which may or may not intersect L_∞ in a point of $\mathcal{R}$. Hence $x \leq t\,|\,\mathcal{R}\,|$. We conclude that

$$|\,\mathcal{R}\,| \;\geq\; \frac{t-1}{t}(q+1-\frac{q}{t})(q-\frac{q}{t}) \;\geq\; \left(\frac{t-1}{t}\right)^3 q(q+1)\,.$$

□

Theorem 5.3.25 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$, t even and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then for every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$, $l(L_\infty) = 1$.*

Proof. We first remark that as a consequence of Theorem 5.3.20, $4 \leq t < \frac{1}{2}q$ and $h \geq 6$, so $q \geq 64$.

Suppose that there is a line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ such that $l(L_\infty) \neq 1$. By Theorem 5.3.20, $l(L_\infty) = t+2$. Lemma 5.3.2 implies that $\mathcal{R} \neq \emptyset$. By Lemma 5.3.24, $|\,\mathcal{R}\,| \geq \left(\frac{t-1}{t}\right)^3 q(q+1)$.

We consider again the set Π of original planes and the set Π^* of repeated planes. As before, let $\mathcal{P}_{\mathrm{IV}} = \{p_\infty^1, p_\infty^2, \ldots, p_\infty^{|\mathcal{P}_{\mathrm{IV}}|}\}$ and for every $p_\infty^i \in \mathcal{P}_{\mathrm{IV}}$ let d_i denote the number of repeated planes π such that $P_\infty(\pi) = \{p_\infty^i\}$.

Let $p_\infty^i \in \mathcal{R}$. By Lemma 5.3.23, the $t+2$ lines of $\mathcal{S}$ which intersect Π_∞ in the point p_∞^i lie in a plane π, and there are exactly $q+1-t$ lines of $\mathcal{B}_0$ through p_∞^i, one of which being the line at infinity L_∞ of π. Let M_∞ be a line of $\mathcal{B}_0$ through p_∞^i and different from L_∞. Then there are exactly $t+1$ repeated planes π' such that $\pi' \cap \Pi_\infty = M_\infty$ and $P_\infty(\pi') = \{p_\infty^i\}$. Hence $d_i = (t+1)(q-t)$, and this holds for every point $p_\infty^i \in \mathcal{R}$. By Lemma 5.3.15,

$$|\,\Pi^*\,| \;=\; \sum_{i=1}^{|\mathcal{P}_{\mathrm{IV}}|} d_i \geq \sum_{i=1}^{|\mathcal{R}|} d_i = (t+1)(q-t)\,|\,\mathcal{R}\,|\,.$$

By Lemma 5.3.14, $|\,\Pi^*\,| \;\leq (q+1)\,|\,\mathcal{P}_0\,| \;-\; |\,\mathcal{B}_0\,| \;< (q+1)\,|\,\mathcal{P}_0\,|$, so

$$(t+1)(q-t)\,|\,\mathcal{R}\,| \;\leq (q+1)\,|\,\mathcal{P}_0\,|\,.$$

Since $\mathcal{P}_0$ is the complement of $\mathcal{P}_{\mathrm{IV}}$ in the point set of Π_∞ and since $\mathcal{R} \subseteq \mathcal{P}_{\mathrm{IV}}$, $|\,\mathcal{P}_0\,| \;\leq q^2+q+1-\;|\,\mathcal{R}\,|$. Hence

$$|\,\mathcal{R}\,| \;\leq \frac{(q+1)(q^2+q+1)}{(t+1)(q-t)+q+1}.$$

Since $q \geq 64$ and $4 \leq t < \frac{1}{2}q$, $(t+1)(q-t) \geq 4(q+1)$. Hence

$$\left(\frac{t-1}{t}\right)^3 q(q+1) \leq |\,\mathcal{R}\,| \leq \frac{1}{5}(q^2+q+1)\,.$$

This clearly contradicts the fact that $t \geq 4$ and $q \geq 64$. We conclude that for every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$, $l(L_\infty) = 1$. □

Theorem 5.3.26 *A* $(0,2)$*-geometry of order* $(q-1,t)$*, with* $2 < t < q-1$ *and* t *even, fully embedded in* $\mathrm{AG}(3,q)$*,* $q = 2^h$*,* $h > 1$*, such that there are no planar nets, does not exist.*

Proof. Suppose that there exists a $(0,2)$-geometry $\mathcal{S}$ of order $(q-1,t)$, t even and $2 < t < q-1$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. By Theorem 5.3.25 and Lemma 5.3.4, $k_{\min} = 1$. Let x be the number of pairs (p_∞, L_∞) such that $p_\infty \in \mathcal{P}_{\min}$, $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ and $p_\infty \in L_\infty$. By Lemma 5.3.6, $x = t\,|\,\mathcal{P}_{\min}\,|$. By Lemma 5.3.22, on every line $L_\infty \in \mathcal{B}_{\mathrm{IV}}$ there are at least $q+1-q/t > (t-1)(q+1)/t$ points p_∞ such that $k(p_\infty) = l(L_\infty) = 1 = k_{\min}$. Hence $x > \frac{t-1}{t}(q+1)\,|\,\mathcal{B}_{\mathrm{IV}}\,|$. We conclude that

$$|\,\mathcal{P}_{\min}\,| > \frac{t-1}{t^2}(q+1)\,|\,\mathcal{B}_{\mathrm{IV}}\,|\,.$$

Let π be a plane of type IV. Then there are $\frac{1}{2}q(q+1)(t-1)$ lines of $\mathcal{S}$ which intersect π in a point of the connected component of $\mathcal{S}_\pi$ which is a dual oval. So $|\,\mathcal{B}\,| \geq \frac{1}{2}q(q+1)(t-1)$. Theorem 5.3.25 implies that $|\,\Pi_{\mathrm{IV}}\,| = |\,\mathcal{B}_{\mathrm{IV}}\,|$. Hence by equality (5.2) of Lemma 5.3.7,

$$|\,\mathcal{B}_{\mathrm{IV}}\,| = \frac{t}{q+1}\,|\,\mathcal{B}\,| \geq \frac{1}{2}qt(t-1).$$

On the other hand Lemma 5.3.5 implies that there is at least one hole, so $|\,\mathcal{P}_{\min}\,| \leq q^2+q$. Hence

$$\frac{t-1}{t^2}(q+1)\frac{1}{2}qt(t-1) < q(q+1).$$

So $(t-1)^2 < 2t$, and hence $t < 3$. But this contradicts $t > 2$.

We conclude that a $(0,2)$-geometry of order $(q-1,t)$, with $2 < t < q-1$ and t even, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets, does not exist. □

5.3.2.5 Conclusion

Theorem 5.3.27 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in $\mathrm{AG}(3,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Then $t=q$.*

Proof. This follows immediately from Proposition 5.3.1 and Theorems 5.3.8, 5.3.9, 5.3.19 and 5.3.26. □

5.3.3 Classification in case $t=q$

Theorem 5.3.28 *Let $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ be a $(0,2)$-geometry of order $(q-1,q)$ fully embedded in $\mathrm{AG}(3,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Then the following properties hold.*

1. *There is exactly one hole, which is denoted by n_∞.*
2. *$\mathcal{P}_{\mathrm{IV}}=\mathcal{P}_{\min}$. In other words, through every point of Π_∞ different from the hole n_∞, there is a constant number $k_{\min}$ of lines of $\mathcal{S}$. Let $k=k_{\min}$.*
3. *Any plane through the hole n_∞ is of type II and contains exactly k lines of $\mathcal{S}$ and no isolated points. These are the only planes of type II.*
4. *$k\in\{\frac{1}{2}q,q\}$ and $|\,\mathcal{P}\,|=kq^2$.*
5. *$k=\frac{1}{2}q$ if and only if $\mathcal{S}\simeq\mathrm{HT}$.*

Proof. Applying $t=q$ to inequalities (5.6) and (5.7) of Lemma 5.3.7 yields $|\,\mathcal{P}_{\mathrm{IV}}\,|\geq q(q+1)$, so $|\,\mathcal{P}_0\,|\leq 1$. On the other hand by Lemma 5.3.5, $\mathcal{P}_0\neq\emptyset$, so there is exactly one hole n_∞. Hence $|\,\mathcal{B}_{\mathrm{IV}}\,|=q^2=qt-q+t$. In other words, we have equality in (5.6) of Lemma 5.3.7. Hence $\mathcal{P}_{\mathrm{IV}}=\mathcal{P}_{\min}$.

Let L_∞ be a line of Π_∞ through n_∞, and let V be the parallel class of planes which intersect Π_∞ in the line L_∞. Since $n_\infty\in L_\infty$, $L_\infty\in\mathcal{B}_0$, so every plane in V is of type I or II. Let p_∞ be a point of $\mathcal{P}_{\mathrm{IV}}$ on L_∞. Then there is a line L of $\mathcal{S}$ which intersects Π_∞ in the point p_∞. Hence the plane $\pi=\langle L,L_\infty\rangle\in V$ is a plane of type II such that $P_\infty(\pi)=\{p_\infty\}$. So for every point $p_\infty\in\mathcal{P}_{\mathrm{IV}}$ on L_∞, there is at least one plane $\pi\in V$ of type II such that $P_\infty(\pi)=\{p_\infty\}$. But since $\mathcal{P}_0=\{n_\infty\}$, there are q points of $\mathcal{P}_{\mathrm{IV}}$ on L_∞. Since there are only q planes in V, it follows that for every point $p_\infty\in L_\infty\setminus\{n_\infty\}$, there is exactly one plane $\pi\in V$ of type II such that $P_\infty(\pi)=\{p_\infty\}$; moreover, π contains all k lines of $\mathcal{S}$ which intersect Π_∞ in p_∞. Hence every plane of V is of type II. We conclude that every affine plane containing the hole n_∞ is of type II and contains k lines of $\mathcal{S}$.

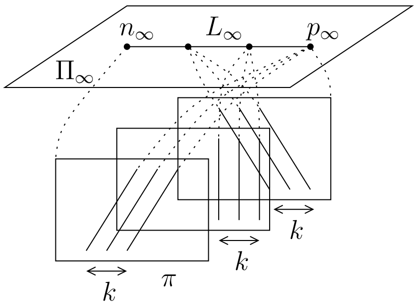

Figure 5.4: A parallel class of planes containing the unique hole in case $t = q$ (Theorem 5.3.28).

Suppose that a plane π of $\mathrm{AG}(3, q)$ which contains the hole n_∞, contains an isolated point p. Since n_∞ is a hole, the line $L = \langle n_\infty, p\rangle$ is not a line of $\mathcal{S}$. Also every plane through L is of type II, so contains at most one line of $\mathcal{S}$ through p. But π does not contain any line of $\mathcal{S}$ through p, implying that the number of lines of $\mathcal{S}$ through p is at most q, a contradiction.

Now suppose that there is a plane π of type II which does not contain the hole n_∞. Let $L_\infty = \pi \cap \Pi_\infty$ and let $p_\infty \in L_\infty$ such that $P_\infty(\pi) = \{p_\infty\}$. Then $p_\infty \neq n_\infty$, so $p_\infty \in \mathcal{P}_{\mathrm{IV}} = \mathcal{P}_{\min}$. By Lemma 5.3.3, none of the planes parallel to π is of type IV. Hence $L_\infty \in \mathcal{B}_0$, so L_∞ contains a hole. But this contradicts the fact that $n_\infty \notin L_\infty$. So every plane of type II contains the hole n_∞.

Consider a parallel class of planes which intersect Π_∞ in a line containing n_∞. Then every plane is of type II and contains k lines of $\mathcal{S}$ and no isolated points. Hence $|\,\mathcal{P}\,| = kq^2$.

Let L be a line of $\mathrm{AG}(3, q)$. Then there is at least one plane of type II which contains L, namely $\langle n_\infty, L\rangle$ if $L \cap \Pi_\infty \neq n_\infty$, and any plane through L otherwise. Since every plane of type II contains k parallel lines of $\mathcal{S}$ and no isolated points, L contains 0, k or q points of $\mathcal{S}$. We conclude that the set $\mathcal{R} = \mathcal{P} \cup \Pi_\infty$ is a set of type $(1, k + 1, q + 1)$. Furthermore, L contains q points of $\mathcal{S}$ if and only if L is a line of $\mathcal{S}$ or $k = q$.

Suppose that $k < q$, and suppose that $\mathcal{R}$ is a singular set of type $(1, k + 1, q + 1)$. Then there is a singular point $p \in \mathcal{R}$, that is, a point $p \in \mathcal{R}$ such that every line through p contains 1 or $q + 1$ points of $\mathcal{R}$. If p is an affine point, then every line L through p contains at least 2 points of $\mathcal{R}$, namely p and $L \cap \Pi_\infty$, and hence L contains $q + 1$ points of $\mathcal{R}$. So $\mathcal{R}$ is the point set of $\mathrm{PG}(3, q)$, and $|\,\mathcal{P}\,| = kq^2 = q^3$, a contradiction. If $p \in \Pi_\infty$, then since $|\,\mathcal{P}\,| = kq^2$, there are exactly kq affine lines through p which are completely contained in $\mathcal{P}$, and hence are lines of $\mathcal{S}$. But this contradicts the fact that $k(p_\infty) = k$ for every point $p_\infty \in \mathcal{P}_{\mathrm{IV}}$. So $\mathcal{R}$ is nonsingular. By Theorem 1.3.5, the following cases have to be considered.

1. $\mathcal{P}$ is a point or the complement of a point. This contradicts $|\,\mathcal{P}\,| = kq^2$.

2. $k = \frac{1}{2}q$ and $\mathcal{R} = \mathcal{R}_3$. Since every affine line which contains q points of $\mathcal{P}$ is a line of $\mathcal{S}$, it follows that $\mathcal{S} \simeq \mathrm{HT}$.

We conclude that $k \in \{\frac{1}{2}q, q\}$ and that if $k = \frac{1}{2}q$, then $\mathcal{S} \simeq \mathrm{HT}$. Clearly, if $\mathcal{S} \simeq \mathrm{HT}$, then $\mathcal{R} = \mathcal{P} \cup \Pi_\infty = \mathcal{R}_3$ is a set of type $(1, \frac{1}{2}q+1, q+1)$, so $k = \frac{1}{2}q$. □

For any point $p_\infty \in \Pi_\infty$, let π_{p_∞} denote the projective plane with as points the planes of $\mathrm{PG}(3,q)$ through p_∞ and as lines the lines of $\mathrm{PG}(3,q)$ through p_∞.

In $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, an ordered pair (Ω_∞, ψ) is called a *type* **A** *pair* if Ω_∞ is a planar oval set in Π_∞ with nucleus a point n_∞, and ψ is a collineation from π_{n_∞} to $\pi(\Omega_\infty)$ such that every line of Π_∞ through n_∞ is fixed.

Theorem 5.3.29 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a* $(0,2)$*-geometry of order* $(q-1,q)$ *fully embedded in* $\mathrm{AG}(3,q)$*,* $q = 2^h$*,* $h > 1$*, such that there are no planar nets. Let* n_∞ *be the unique hole. Then the following properties hold.*

1. *For every point* p *of* $\mathcal{S}$*,* θ_p *is an oval in* Π_∞ *with nucleus* n_∞*.*

2. *For any two points* p, p' *of* $\mathcal{S}$ *such that* p, p', n_∞ *are collinear,* $\theta_p = \theta_{p'}$*.*

3. *The set* $\Omega_\infty(\mathcal{S})$ *of all distinct sets* θ_p*,* $p \in \mathcal{P}$*, is a planar oval set in* Π_∞ *with nucleus* n_∞*.*

4. *The pair* $(\Omega_\infty(\mathcal{S}), \psi(\mathcal{S}))$*, where* $\psi(\mathcal{S})$ *maps an affine plane* π *through* n_∞ *to the point at infinity of the* k *lines of* $\mathcal{S}_\pi$*, the plane* Π_∞ *to the point* n_∞*, an affine line* L *through* n_∞ *to the set* $\theta_p \in \Omega_\infty(\mathcal{S})$*, with* p *an affine point of* L*, and a line* L_∞ *of* Π_∞ *through* n_∞ *to itself, is a type* **A** *pair.*

Proof. Let p be a point of $\mathcal{S}$, and let $L = \langle p, n_\infty \rangle$. By Theorem 5.3.28, every plane through L contains exactly one line of $\mathcal{S}$ through p. Hence every line of Π_∞ through n_∞ intersects the set θ_p in exactly one point. Furthermore, by Theorem 5.3.28, every plane through p which does not contain the line L is a plane of type I or IV, and hence contains 0 or 2 lines of $\mathcal{S}$ through

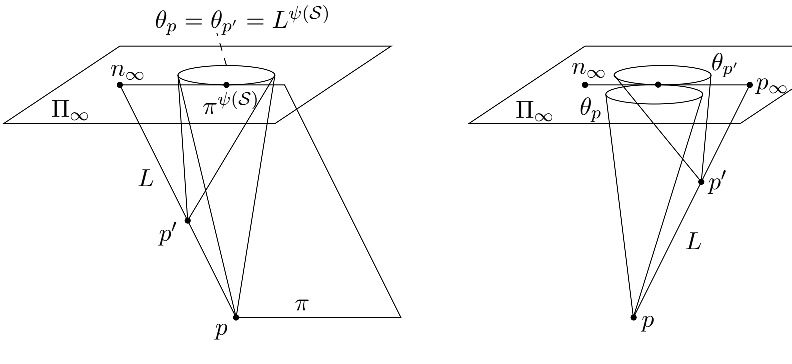

Figure 5.5: Planar oval sets and type **A** pairs (Theorem 5.3.29).

p. So every line of Π_∞ not through n_∞ intersects θ_p in 0 or 2 points. Since $|\,\theta_p\,| = q+1$, it follows that θ_p is an oval with nucleus n_∞.

Let p, p' be points of $\mathcal{S}$ such that the line $L = \langle p, p' \rangle$ intersects Π_∞ in the point n_∞. Since for every plane π through L, the lines of $\mathcal{S}$ in π through p and p' are parallel, $\theta_p = \theta_{p'}$.

Let p, p' be points of $\mathcal{S}$ such that the line $L = \langle p, p' \rangle$ intersects Π_∞ in a point p_∞ different from n_∞. Let $\pi = \langle L, n_\infty \rangle$. Then π is a plane of type II, so the line $L_\infty = \langle n_\infty, p_\infty \rangle = \pi \cap \Pi_\infty$ intersects the ovals θ_p and $\theta_{p'}$ in the same point, namely the point at infinity of the k lines of $\mathcal{S}_\pi$. Suppose that there is a point $r_\infty \in \theta_p \cap \theta_{p'}$ which is not on L_∞. Then the plane $\pi' = \langle L, r_\infty \rangle$ contains a line of $\mathcal{S}$ through p, respectively p', which intersects Π_∞ in r_∞. So π' contains two parallel lines of $\mathcal{S}$, and hence it is of type II. By Theorem 5.3.28, π' contains the hole n_∞. But $\pi' \cap \Pi_\infty = \langle p_\infty, r_\infty \rangle$, and this line does not contain n_∞, a contradiction. So θ_p and $\theta_{p'}$ intersect in exactly one point. It follows that the set $\Omega_\infty(\mathcal{S})$ consists of exactly q^2 ovals with nucleus n_∞, any two of which intersect in exactly one point. Hence $\Omega_\infty(\mathcal{S})$ is a planar oval set in Π_∞ with nucleus n_∞.

Let L be an affine line through n_∞, and let π be an affine plane containing n_∞. If $L \subseteq \pi$ then clearly $\pi^{\psi(\mathcal{S})} \in L^{\psi(\mathcal{S})}$. Assume that $L \not\subseteq \pi$. Let $L_\infty = \pi \cap \Pi_\infty$ and let $\pi' = \langle L, L_\infty \rangle$. Since for every point $p_\infty \in L_\infty \setminus \{n_\infty\}$ there is exactly one plane π'' such that $\pi'' \cap \Pi_\infty = L_\infty$ and $P_\infty(\pi'') = \{p_\infty\}$, the lines of $\mathcal{S}_\pi$ are not parallel to the lines of $\mathcal{S}_{\pi'}$. In other words, $\pi^{\psi(\mathcal{S})} \neq \pi'^{\psi(\mathcal{S})}$. Since $L \subseteq \pi'$, $\pi'^{\psi(\mathcal{S})} \in L^{\psi(\mathcal{S})}$. Since $\pi'^{\psi(\mathcal{S})}$ is the only point of $L^{\psi(\mathcal{S})}$ on L_∞, $\pi^{\psi(\mathcal{S})} \notin L^{\psi(\mathcal{S})}$. So L and π are incident in π_{n_∞} if and only if $L^{\psi(\mathcal{S})}$ and $\pi^{\psi(\mathcal{S})}$ are incident in $\pi(\Omega_\infty(\mathcal{S}))$. It is now easily verified that any point and any line of π_{n_∞} are incident if and only if their images under $\psi(\mathcal{S})$ are incident in $\pi(\Omega_\infty(\mathcal{S}))$. Since $\psi(\mathcal{S})$ fixes every line of Π_∞ through n_∞, $(\Omega_\infty(\mathcal{S}),\psi(\mathcal{S}))$ is a type **A** pair. $\square$

Lemma 5.3.30 *Consider the* $(0,2)$*-geometry* $\mathcal{S}=\mathrm{HT}=(\mathcal{P},\mathcal{B},\mathrm{I})$*, fully embedded in* $\mathrm{AG}(3,q)$*,* $q=2^h$*. Consider* $\overline{\mathcal{S}}=(\overline{\mathcal{P}},\overline{\mathcal{B}},\overline{\mathrm{I}})$*, the so-called complement of* $\mathcal{S}$*, where* $\overline{\mathcal{P}}$ *is the complement of* $\mathcal{P}$ *in the point set of* $\mathrm{AG}(3,q)$*,* $\overline{\mathcal{B}}$ *is the set of all affine lines containing* q *points of* $\overline{\mathcal{P}}$*, and* $\overline{\mathrm{I}}$ *is the natural incidence. Then* $\overline{\mathcal{S}}$ *is a* $(0,2)$*-geometry fully embedded in* $\mathrm{AG}(3,q)$*, and it is projectively equivalent to* HT*.*

Proof. From the construction of HT (see Section 1.4.7) it follows that $\mathcal{P}\cup\Pi_\infty$ is projectively equivalent to $\mathcal{R}_3$. Since $\mathcal{R}_3$ is a set of type $(1,\frac{1}{2}q+1,q+1)$, $\overline{\mathcal{P}}\cup\Pi_\infty$ is also a set of type $(1,\frac{1}{2}q+1,q+1)$. Since $\mathcal{R}_3$ is nonsingular, so is $\overline{\mathcal{P}}\cup\Pi_\infty$. By Theorem 1.3.4, $\overline{\mathcal{P}}\cup\Pi_\infty$ is projectively equivalent to $\mathcal{R}_3$. Hence $\overline{\mathcal{S}}\simeq\mathrm{HT}$. $\square$

If $\mathcal{S}\simeq\mathrm{HT}$ and $\overline{\mathcal{S}}$ is the complement of $\mathcal{S}$, then we also say that $\mathcal{S}$ and $\overline{\mathcal{S}}$ are *complementary.*

Lemma 5.3.31 *Let* $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})\simeq\mathrm{HT}$ *and let* $\overline{\mathcal{S}}=(\overline{\mathcal{P}},\overline{\mathcal{B}},\overline{\mathrm{I}})$ *be the complement of* $\mathcal{S}$*. Then the hole of* $\mathcal{S}$ *and the hole of* $\overline{\mathcal{S}}$ *coincide,* $\Omega_\infty(\mathcal{S})=\Omega_\infty(\overline{\mathcal{S}})$ *and* $\psi(\mathcal{S})=\psi(\overline{\mathcal{S}})$*. Furthermore, all the conclusions of Theorems 5.3.28 and 5.3.29 apply to the (disconnected) geometry* $\widetilde{\mathcal{S}}=(\widetilde{\mathcal{P}},\widetilde{\mathcal{B}},\widetilde{\mathrm{I}})=(\mathcal{P}\cup\overline{\mathcal{P}},\mathcal{B}\cup\overline{\mathcal{B}},\mathrm{I}\cup\overline{\mathrm{I}})$*, with* $k=q$*.*

Proof. Let n_∞ be the hole of $\mathcal{S}$. Let L be a line of $\mathrm{AG}(3,q)$ through n_∞, and let π be a plane containing L. By Lemma 4.4.4, π is a plane of type II which contains $\frac{1}{2}q$ lines of $\mathcal{S}$ but no isolated points. Hence L contains exactly $\frac{1}{2}q$ points of $\mathcal{S}$. So every affine line which intersects Π_∞ in the point n_∞, contains exactly $\frac{1}{2}q$ points of $\mathcal{S}$, and hence exactly $\frac{1}{2}q$ points of $\overline{\mathcal{S}}$. So no line of $\overline{\mathcal{S}}$ intersects Π_∞ in the point n_∞. Hence n_∞ is the hole of $\overline{\mathcal{S}}$.

Let π be a plane of type II with respect to $\mathcal{S}$. Then π contains $\frac{1}{2}q$ parallel lines of $\mathcal{S}$ but no isolated points of $\mathcal{S}$, so it contains $\frac{1}{2}q$ lines of $\overline{\mathcal{S}}$, and these lines are parallel to the lines of $\mathcal{S}$ in π. Hence $\pi^{\psi(\mathcal{S})}=\pi^{\psi(\overline{\mathcal{S}})}$.

Let L be a line of $\mathrm{AG}(3,q)$ which intersects Π_∞ in the point n_∞. Since for every plane π through L, $\pi^{\psi(\mathcal{S})}=\pi^{\psi(\overline{\mathcal{S}})}$, it follows that $L^{\psi(\mathcal{S})}=L^{\psi(\overline{\mathcal{S}})}$. This shows that $\Omega_\infty(\mathcal{S})=\Omega_\infty(\overline{\mathcal{S}})$ and $\psi(\mathcal{S})=\psi(\overline{\mathcal{S}})$.

Consider the geometry $\widetilde{\mathcal{S}}$. Since the hole of the geometries $\mathcal{S}$ and $\overline{\mathcal{S}}$ is n_∞, this point is the unique hole of $\widetilde{\mathcal{S}}$. Since through every point $p_\infty\in\Pi_\infty\setminus\{n_\infty\}$ there are exactly $\frac{1}{2}q$ lines of $\mathcal{S}$ and exactly $\frac{1}{2}q$ lines of $\overline{\mathcal{S}}$, there are exactly q

lines of $\widetilde{\mathcal{S}}$ through every point $p_\infty \in \Pi_\infty \setminus \{n_\infty\}$. Since $\overline{\mathcal{S}}$ is the complement of $\mathcal{S}$, it is immediately clear that an affine plane π is of type II with respect to $\widetilde{\mathcal{S}}$ if and only if n_∞ is on the line $\pi \cap \Pi_\infty$, and that in this case π contains q parallel lines of $\widetilde{\mathcal{S}}$ and no isolated points. Since $\overline{\mathcal{P}}$ is the complement of $\mathcal{P}$ in the point set of $\mathrm{AG}(3,q)$, $|\widetilde{\mathcal{P}}| = q^3$. So all the conclusions of Theorem 5.3.28 apply to $\widetilde{\mathcal{S}}$.

For every point p of $\mathcal{S}$, respectively $\overline{\mathcal{S}}$, $\widetilde{\mathcal{S}}$, let θ_p, respectively $\overline{\theta}_p$, $\widetilde{\theta}_p$, denote the set of points at infinity of the $q+1$ lines of $\mathcal{S}$, respectively $\overline{\mathcal{S}}$, $\widetilde{\mathcal{S}}$, through p. Let $p \in \widetilde{\mathcal{P}}$. Then either $p \in \mathcal{P}$ or $p \in \overline{\mathcal{P}}$. Since $\widetilde{\theta}_p = \theta_p$ if $p \in \mathcal{P}$ and $\widetilde{\theta}_p = \overline{\theta}_p$ if $p \in \overline{\mathcal{P}}$, and since Theorem 5.3.29 applies to both $\mathcal{S}$ and $\overline{\mathcal{S}}$, $\widetilde{\theta}_p$ is an oval of Π_∞ with nucleus n_∞.

Let p, p' be points of $\widetilde{\mathcal{S}}$ such that the line $L = \langle p, p' \rangle$ intersects Π_∞ in the point n_∞. Since for every plane π through L, the lines of $\widetilde{\mathcal{S}}$ in π through p and p' are parallel, $\widetilde{\theta}_p = \widetilde{\theta}_{p'}$.

Now it is clear that $\Omega_\infty(\widetilde{\mathcal{S}}) = \Omega_\infty(\mathcal{S}) = \Omega_\infty(\overline{\mathcal{S}})$ and $\psi(\widetilde{\mathcal{S}}) = \psi(\mathcal{S}) = \psi(\overline{\mathcal{S}})$, so $(\Omega_\infty(\widetilde{\mathcal{S}}), \psi(\widetilde{\mathcal{S}}))$ is a type **A** pair. So all the conclusions of Theorem 5.3.29 apply to $\widetilde{\mathcal{S}}$. $\square$

Let Ω_∞ be a planar oval set in Π_∞ with nucleus n_∞. Let G be the group of collineations of $\mathrm{AG}(3,q)$ which fix Ω_∞. Let X be the set of collineations ψ from π_{n_∞} to $\pi(\Omega_\infty)$ which fix every line of Π_∞ through n_∞. We define an action of G on X as follows. Let $g \in G$ and $\psi \in X$. Then since g fixes Ω_∞, g fixes n_∞. Hence g^{-1} induces a collineation of π_{n_∞}. Since g fixes Ω_∞, g induces a collineation of $\pi(\Omega_\infty)$. Hence $\psi^g = g^{-1}\psi g$ is a collineation from π_{n_∞} to $\pi(\Omega_\infty)$. Since ψ fixes every line of Π_∞ through n_∞, so does $g^{-1}\psi g$. So $\psi^g \in X$.

Lemma 5.3.32 *Let Ω_∞ be a planar oval set of Π_∞ with nucleus n_∞. Then G acts transitively on X.*

Proof. If $X = \emptyset$, then there is nothing to prove, so assume that X is not empty. Let $\psi \in X$. A collineation ψ' from π_{n_∞} to $\pi(\Omega_\infty)$ is an element of X if and only if it fixes every line of Π_∞ through n_∞. This is the case if and only if $\psi\psi'^{-1}$ is a perspectivity of the projective plane π_{n_∞} with center Π_∞. So $|X|$ is equal to the number of perspectivities of π_{n_∞} with center Π_∞, which is $q^2(q-1)$.

Let G' be the group of collineations of Π_∞ which fix Ω_∞. Since every collineation of Π_∞ can be extended to exactly $q^3(q-1)$ different collineations of $\mathrm{AG}(3,q)$, the group G has order $q^3(q-1)\,|G'|$.

Let $\psi \in X$. Suppose that a collineation $g' \in G'$ can be extended to a collineation $g \in G$ which fixes ψ. We prove that the set $V \subseteq G$ of collineations which extend g' and which fix ψ has size q. A collineation $h \in G$ is an element of V if and only if h fixes ψ and h and g have the same action on Π_∞, if and only if $f = gh^{-1} \in G$ fixes ψ and fixes every point of Π_∞, if and only if $f = gh^{-1} \in G$ is a perspectivity of $\mathrm{PG}(3,q)$ with axis Π_∞ which fixes ψ. A perspectivity f of $\mathrm{PG}(3,q)$ with axis Π_∞ (note that f fixes Ω_∞ and hence is in G) fixes ψ if and only if for every line L of $\mathrm{AG}(3,q)$ through n_∞, $L^{f\psi} = L^{\psi f}$. Since f fixes Π_∞ pointwise, $L^{\psi f} = L^\psi$ for every line L through n_∞. So f fixes ψ if and only if for every line L of $\mathrm{AG}(3,q)$ through n_∞, $L^{f\psi} = L^\psi$, if and only if f fixes every line of $\mathrm{AG}(3,q)$ through n_∞. In other words, f must be an elation with axis Π_∞ and center n_∞. Since there are exactly q such elations, $|\,V\,| = q$. This means that there are at most $q\,|\,G'\,|$ elements of G which fix ψ. If G_ψ denotes the stabilizer of ψ in G, and ψ^G denotes the orbit of ψ under G, then the orbit-stabilizer theorem yields

$$q^3(q-1)\,|\,G'\,| \;=\; |\,G\,| \;=\; |\,\psi^G\,|\,.\,|\,G_\psi\,| \;\le q\,|\,G'\,|\,.\,|\,\psi^G\,|\,.$$

Hence $|\,\psi^G\,| \ge q^2(q-1)$. Since $|\,X\,| = q^2(q-1)$, it follows that $\psi^G = X$, so G acts transitively on X. $\square$

Theorem 5.3.33 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,q)$, fully embedded in $\mathrm{AG}(3,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Let $O_\infty \in \Omega_\infty(\mathcal{S})$. Then $\mathcal{S} \simeq \mathcal{A}(O_\infty)$.*

Proof. Let $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{B}_1, \mathrm{I}_1) = \mathcal{S}$. Let n_∞ be the unique hole of $\mathcal{S}_1$. Consider the $(0,2)$-geometry $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{B}_2, \mathrm{I}_2) = \mathcal{A}(O_\infty)$. By Theorem 4.2.1, $\mathcal{S}_2$ has order $(q-1,q)$ and is fully embedded in $\mathrm{AG}(3,q)$. By Lemma 4.4.2, $\mathcal{S}_2$ has no planar nets. Hence Theorems 5.3.28 and 5.3.29 apply to $\mathcal{S}_2$. By Theorem 5.3.28, $\mathcal{S}_2$ has a unique hole. From the construction of $\mathcal{A}(O_\infty)$ (see Section 4.2.1), it is clear that none of the affine lines through n_∞, the nucleus of O_∞, is a line of $\mathcal{A}(O_\infty)$. Hence n_∞ is the unique hole of $\mathcal{S}_2$.

We define an incidence structure $\widetilde{\mathcal{S}}_i = (\widetilde{\mathcal{P}}_i, \widetilde{\mathcal{B}}_i, \widetilde{\mathrm{I}}_i)$, $i = 1,2$. If $\mathcal{S}_i \not\simeq \mathrm{HT}$, then let $\widetilde{\mathcal{S}}_i = \mathcal{S}_i$, $i = 1,2$. If $\mathcal{S}_i \simeq \mathrm{HT}$, then let $\widetilde{\mathcal{S}}_i = (\widetilde{\mathcal{P}}_i, \widetilde{\mathcal{B}}_i, \widetilde{\mathrm{I}}_i) = (\mathcal{P}_i \cup \overline{\mathcal{P}}_i, \mathcal{B}_i \cup \overline{\mathcal{B}}_i, \mathrm{I}_i \cup \overline{\mathrm{I}}_i)$, where $\overline{\mathcal{S}}_i = (\overline{\mathcal{P}}_i, \overline{\mathcal{B}}_i, \overline{\mathrm{I}}_i)$ is the complement of $\mathcal{S}_i$, $i = 1,2$. Note that in either case, $\widetilde{\mathcal{P}}_i$ is the set of all affine points, $i = 1,2$. Note also that $\widetilde{\mathcal{S}}_i$ is not necessarily connected, so it is not necessarily a $(0,2)$-geometry. However by Lemma 5.3.31, all the conclusions of Theorems 5.3.28 and 5.3.29 apply to $\widetilde{\mathcal{S}}_i$, with $k = q$, $i = 1,2$.

Let φ be a collineation from the projective plane Π_∞ to the projective plane π_{n_∞} which fixes every line of Π_∞ through n_∞. Then $\varphi\,\psi(\widetilde{\mathcal{S}}_i)$ is a collineation from Π_∞ to $\pi(\Omega_\infty(\widetilde{\mathcal{S}}_i))$ which fixes every line of Π_∞ through n_∞, $i = 1,2$. Hence $\Omega_\infty(\widetilde{\mathcal{S}}_i)$ is a regular Desarguesian planar oval set, $i = 1,2$.

We have shown in the proof of Lemma 5.3.31 that $\Omega_\infty(\widetilde{\mathcal{S}}_i) = \Omega_\infty(\mathcal{S}_i)$ if $\mathcal{S}_i \simeq \mathrm{HT}$, $i = 1, 2$. Clearly, if $\mathcal{S}_i \not\simeq \mathrm{HT}$, then also $\Omega_\infty(\widetilde{\mathcal{S}}_i) = \Omega_\infty(\mathcal{S}_i)$, $i = 1, 2$. Since $O_\infty \in \Omega_\infty(\mathcal{S}_i)$, $O_\infty \in \Omega_\infty(\widetilde{\mathcal{S}}_i)$, $i = 1, 2$. By Theorem 3.4.1, $\Omega_\infty(\widetilde{\mathcal{S}}_i)$ is the set of images of O_∞ under all elations of Π_∞ with center n_∞, $i = 1, 2$. So, in particular, $\Omega_\infty(\widetilde{\mathcal{S}}_1) = \Omega_\infty(\widetilde{\mathcal{S}}_2)$.

Let $\Omega_\infty = \Omega_\infty(\widetilde{\mathcal{S}}_1) = \Omega_\infty(\widetilde{\mathcal{S}}_2)$. Let X be the set of all collineations from π_{n_∞} to $\pi(\Omega_\infty)$ fixing every line of Π_∞ through n_∞. Then $\psi(\widetilde{\mathcal{S}}_1), \psi(\widetilde{\mathcal{S}}_2) \in X$. By Lemma 5.3.32, there is a collineation g of $\mathrm{AG}(3, q)$ which fixes Ω_∞ and which maps $\psi(\widetilde{\mathcal{S}}_1)$ to $\psi(\widetilde{\mathcal{S}}_2)$. In other words, $\psi(\widetilde{\mathcal{S}}_1)g = g\psi(\widetilde{\mathcal{S}}_2)$.

Let p be a point of $\widetilde{\mathcal{S}}_1$, and let $L = \langle p, n_\infty \rangle$. Then, by definition of $\psi(\widetilde{\mathcal{S}}_1)$, $L^{\psi(\widetilde{\mathcal{S}}_1)}$ is the set of points at infinity of the lines of $\widetilde{\mathcal{S}}_1$ through p. So the lines of $\widetilde{\mathcal{S}}_1$ through p are mapped by g to the lines $\langle p^g, p_\infty \rangle$, $p_\infty \in L^{\psi(\widetilde{\mathcal{S}}_1)g}$. Since $\widetilde{\mathcal{P}}_2$ is the set of all affine points, p^g is a point of $\widetilde{\mathcal{S}}_2$. The set of points at infinity of the lines of $\widetilde{\mathcal{S}}_2$ through p^g is $L'^{\psi(\widetilde{\mathcal{S}}_2)}$, where $L' = \langle p^g, n_\infty \rangle$. Since g fixes Ω_∞, g fixes n_∞. So $L' = L^g$. So the lines of $\widetilde{\mathcal{S}}_2$ through p^g are the lines $\langle p^g, p_\infty \rangle$, $p_\infty \in L^{g\psi(\widetilde{\mathcal{S}}_2)}$. Now since $L^{\psi(\widetilde{\mathcal{S}}_1)g} = L^{g\psi(\widetilde{\mathcal{S}}_2)}$, the set of lines of $\widetilde{\mathcal{S}}_1$ through p is mapped by g to the set of lines of $\widetilde{\mathcal{S}}_2$ through p^g. Since this is the case for every point p of $\widetilde{\mathcal{S}}_1$, we conclude that g is a collineation of $\mathrm{AG}(3, q)$ which induces an isomorphism from $\widetilde{\mathcal{S}}_1$ to $\widetilde{\mathcal{S}}_2$. So $\widetilde{\mathcal{S}}_1 \simeq \widetilde{\mathcal{S}}_2$.

If $\mathcal{S}_1 \not\simeq \mathrm{HT}$ then $\mathcal{S}_1 = \widetilde{\mathcal{S}}_1 \simeq \widetilde{\mathcal{S}}_2$, so $\widetilde{\mathcal{S}}_2$ is connected. Hence $\mathcal{S}_1 \simeq \widetilde{\mathcal{S}}_2 = \mathcal{S}_2$. If $\mathcal{S}_1 \simeq \mathrm{HT}$ then $\widetilde{\mathcal{S}}_1$ is disconnected, hence so is $\widetilde{\mathcal{S}}_2$. So $\mathcal{S}_2 \simeq \mathrm{HT}$, and here also $\mathcal{S}_1 \simeq \mathcal{S}_2$. So in any case $\mathcal{S}_1 \simeq \mathcal{S}_2$. It follows that $\mathcal{S} \simeq \mathcal{A}(O_\infty)$. □

5.4 Conclusion

Theorem 5.4.1 *Let $\mathcal{S}$ be a $(0, 2)$-geometry fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, $h > 1$, such that there is at least one plane of type IV. Then one of the following cases occurs.*

1. $\mathcal{S} \simeq \mathcal{A}(O_\infty)$.
2. $\mathcal{S} \simeq \mathcal{I}(3, q, e)$.

Proof. The theorem follows immediately from Theorems 5.2.5, 5.3.27 and 5.3.33. □

Corollary 5.4.2 *Let $\mathcal{S}$ be a $(0, \alpha)$-geometry, $\alpha > 1$, fully embedded in $\mathrm{AG}(3, q)$. Then one of the following cases occurs.*

1. $q=2$, $\alpha=2$, *and* $\mathcal{S}$ *is a* $2-(t+2,2,1)$*-design.*
2. $q=2^h$, $\alpha=2$, *and* $\mathcal{S}\simeq\mathcal{A}(O_\infty)$.
3. $q=2^h$, $\alpha=2$, *and* $\mathcal{S}\simeq\mathcal{I}(3,q,e)$.
4. $\mathcal{S}\simeq T_2^*(\mathcal{K}_\infty)$, *with* $\mathcal{K}_\infty$ *a set of type* $(0,1,\alpha+1)$ *in* Π_∞ *which spans* Π_∞.

Proof. The case $q=2$ is trivial and is solved in Proposition 4.3.5. If there are no planes of type IV, then Theorem 4.3.1 applies. If there is a plane of type IV, then $\alpha=2$ and $q=2^h$, and Theorem 5.4.1 applies. $\square$

Chapter 6

Classification of $(0, 2)$-geometries fully embedded in $\mathrm{AG}(n, 2^h)$

In Chapter 5, we classified all $(0, 2)$-geometries which are fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, $h > 1$, and which have a plane of type IV. In this chapter, we classify, using the method described in Section 4.1, all $(0, 2)$-geometries which are fully embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, $q = 2^h$, $h > 1$, and which have a plane of type IV. Thus we give the complete solution to Problem 2 (see Section 4.3).

Note that the $(0, 2)$-geometries fully embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, $q = 2^h$, which do not have a plane of type IV, are already classified by Theorem 4.3.1. Also, $h > 1$ is not really a restriction since the case $h = 1$ is trivial, and is handled in Proposition 4.3.5.

We consider in this chapter two distinct cases. Firstly, in Section 6.2, we classify the $(0, 2)$-geometries fully embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, $q = 2^h$, $h > 1$, such that there is a plane of type IV and a planar net. Secondly, in Section 6.3, we classify the $(0, 2)$-geometries fully embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, $q = 2^h$, $h > 1$, such that there is a plane of type IV, but no planar net.

The results of this chapter are published in [34].

6.1 Preliminaries

Let $\mathcal{S}$ be a $(0, 2)$-geometry fully embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, and let U be a subspace of $\mathrm{AG}(n, q)$ of dimension at least 2. Then $P_\infty(U)$ will denote the set of points $p_\infty \in U_\infty = U \cap \Pi_\infty$ such that there is a line L of $\mathcal{S}_U$ which intersects Π_∞ in the point p_∞.

Lemma 6.1.1 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 4$, $q = 2^h$, $h > 1$, and let U be a subspace of $\mathrm{AG}(n,q)$ of dimension 3. Then U is of exactly one of the following four types.*

Type A. *$\mathcal{S}_U$ contains a connected component $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$.*

Type B. *$\mathcal{S}_U$ contains a connected component $\mathcal{S}' \simeq \mathcal{I}(3,q,e)$.*

Type C. *$\mathcal{S}_U$ is a connected linear representation.*

Type D. *Every connected component of $\mathcal{S}_U$ is contained in a plane of U.*

Proof. In Section 4.4, it is proven that if U is of type **A**, **B**, **C** or **D**, then it is not of any other type. Suppose that U is not of type **D**. Then $\mathcal{S}_U$ contains a connected component $\mathcal{S}'$ which is not contained in a plane of U. By Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in the 3-dimensional affine space U. Hence Corollary 5.4.2 applies and $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$, $\mathcal{S}' \simeq \mathcal{I}(3,q,e)$ or $\mathcal{S}'$ is a linear representation. So U is of type **A**, **B** or **C**. $\square$

Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 4$, $q = 2^h$, $h > 1$, and let U be a proper subspace of $\mathrm{AG}(n,q)$ of dimension $m \geq 3$. It was proven in Section 4.4 that U is of at most one of the following four types.

Type A. $m = 3$ and $\mathcal{S}_U$ contains a connected component $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$, or $m = 4$ and $\mathcal{S}_U$ contains a connected component $\mathcal{S}' \simeq \mathrm{TQ}(4,q)$.

Type B. $\mathcal{S}_U$ contains a connected component $\mathcal{S}' \simeq \mathcal{I}(m,q,e)$.

Type C. $\mathcal{S}_U$ is a connected linear representation.

Type D. Every connected component of $\mathcal{S}_U$ is contained in a proper subspace of U.

Let U be a proper subspace of $\mathrm{AG}(n,q)$ of dimension $m \geq 3$, and let $\mathcal{S}'$ be a connected component of $\mathcal{S}_U$ which contains two intersecting lines. By Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in a subspace of U. Let V be a proper subspace of U of dimension at least 2. Consider the incidence structures $\mathcal{S}_V$ and $\mathcal{S}'_V$. These are not necessarily the same. Indeed, a point, respectively a line of $\mathcal{S}'_V$ is a point, respectively a line of $\mathcal{S}_V$, but not vice versa.

This leads to the following paradoxical situation. It is possible that, if V is a subspace of type **X**, V is of some other type **Y** with respect to $\mathcal{S}'$. For example, if V is a plane of type IV, but none of the lines of $\mathcal{S}_V$ are lines of $\mathcal{S}'$, then the plane V is of type I with respect to $\mathcal{S}'$. So, one has to keep in mind that the type of a given subspace V depends on which sub incidence structure of $\mathcal{S}$ one considers.

The following lemma gives some information about the relation between $\mathcal{S}'_V$ and $\mathcal{S}_V$.

Lemma 6.1.2 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 4$, $q = 2^h$, $h > 1$. Let U be a proper subspace of $\mathrm{AG}(n,q)$ of dimension at least 3, let $\mathcal{S}'$ be a connected component of $\mathcal{S}_U$, and let V be a proper subspace of U of dimension at least 2. Then every connected component of $\mathcal{S}'_V$ is a connected component of $\mathcal{S}_V$.*

Proof. We recall that the connected components of an incidence structure $\mathcal{T}$ are the sub incidence structures of $\mathcal{T}$ induced on the connected components of the incidence graph $\mathcal{I}(\mathcal{T})$ of $\mathcal{T}$.

Let $\mathcal{S}''$ be a connected component of $\mathcal{S}'_V$, and let x be a vertex of the incidence graph $\mathcal{I}(\mathcal{S}'')$, that is, x is a point or a line of $\mathcal{S}''$. Let $\mathcal{S}'''$ be the connected component of $\mathcal{S}_V$ which contains x. We prove that $\mathcal{S}'' = \mathcal{S}'''$.

Let y be a vertex of $\mathcal{I}(\mathcal{S}'')$, distinct from x. Since $\mathcal{S}''$ is connected, there is a path $(x_0 = x, x_1, \ldots, x_k = y)$ in $\mathcal{I}(\mathcal{S}'')$. Every point or line of $\mathcal{S}''$ is a point or line of $\mathcal{S}_V$, so $(x_0 = x, x_1, \ldots, x_k = y)$ is a path in $\mathcal{I}(\mathcal{S}_V)$. So y is in the connected component $\mathcal{S}'''$ of $\mathcal{S}_V$ which contains x.

Let y be a vertex of $\mathcal{I}(\mathcal{S}''')$, distinct from x. Since $\mathcal{S}'''$ is connected, there is a path $(x_0 = x, x_1, \ldots, x_k = y)$ in $\mathcal{I}(\mathcal{S}''')$. Every point or line of $\mathcal{S}'''$ is a point or line of $\mathcal{S}_U$, so $(x_0 = x, x_1, \ldots, x_k = y)$ is a path in $\mathcal{I}(\mathcal{S}_U)$. So x_i is in the connected component $\mathcal{S}'$ of $\mathcal{S}_U$ which contains x, $i = 1, \ldots, k$. Furthermore, x_i is contained in V, $i = 1, \ldots, k$, so $(x_0 = x, x_1, \ldots, x_k = y)$ is a path in $\mathcal{I}(\mathcal{S}'_V)$. So y is in the connected component $\mathcal{S}''$ of $\mathcal{S}'_V$ which contains x.

We conclude that the vertex sets of $\mathcal{I}(\mathcal{S}'')$ and $\mathcal{I}(\mathcal{S}''')$ are the same. Since incidence in $\mathcal{S}''$ and $\mathcal{S}'''$ is defined by incidence in $\mathrm{AG}(n,q)$, $\mathcal{I}(\mathcal{S}'') = \mathcal{I}(\mathcal{S}''')$. Hence $\mathcal{S}'' = \mathcal{S}'''$. $\square$

6.2 Classification in case there is a planar net

Lemma 6.2.1 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 4$, $q = 2^h$, $h > 1$, and suppose that U is a subspace of $\mathrm{AG}(n,q)$ of type* **C**. *Then the set $P_\infty(U)$ spans $U_\infty = U \cap \Pi_\infty$. If $V \subseteq U$ is a subspace of dimension $m \geq 2$ then $\mathcal{S}_V$ is a linear representation and $P_\infty(V) = V_\infty \cap P_\infty(U)$, where*

$V_\infty = V \cap \Pi_\infty$. *Furthermore,* V *is of type* **C** *(respectively of type* III *if* $m = 2$*) if and only if* $P_\infty(V) = V_\infty \cap P_\infty(U)$ *spans* V_∞.

Proof. Since U is of type **C**, $\mathcal{S}_U$ is the linear representation of the set $P_\infty(U)$. Since $\mathcal{S}_U$ is connected, by Proposition 1.4.12, $P_\infty(U)$ spans U_∞. Clearly since $V \subseteq U$, $\mathcal{S}_V$ is a linear representation and $P_\infty(V) = V_\infty \cap P_\infty(U)$. By Proposition 1.4.12, V is of type **C** (respectively of type III if $m = 2$) if and only if $P_\infty(V) = V_\infty \cap P_\infty(U)$ spans V_∞. □

Theorem 6.2.2 *Let* $\mathcal{S}$ *be a* $(0,2)$*-geometry fully embedded in* $\mathrm{AG}(n,q)$*,* $n \geq 3$*,* $q = 2^h$*,* $h > 1$*, such that there is a plane of type* IV*. If there is a hyperplane of type* **C** *(respectively a plane of type* III *if* $n = 3$*), and a plane of type* III*, then they do not intersect in an affine line.*

Proof. Let $n = 3$. Then by Theorem 5.4.1, $\mathcal{S} \simeq \mathcal{A}(O_\infty)$ or $\mathcal{S} \simeq \mathcal{I}(3,q,e)$. By Lemma 4.4.2, $\mathcal{A}(O_\infty)$ does not have a planar net. By Proposition 4.4.11, $\mathcal{I}(3,q,e)$ has exactly one planar net. So if $n = 3$, the theorem holds. We use induction on n to prove the theorem for $n \geq 4$.

So, suppose that $n \geq 4$, and that the theorem holds for all $3 \leq m < n$. Suppose that there exists a hyperplane U of type **C** and a plane π of type III which intersect in an affine line L.

Firstly, assume that L is a line of $\mathcal{S}$. Let U' be a hyperplane parallel to but different from U, and let $L' = \pi \cap U'$. We prove that, for every subspace $V \subseteq U$ of dimension $m - 1 \in \{2, \ldots, n-2\}$, such that $L \subseteq V$ and V is of type **C** (respectively of type III if $m = 3$), the $(m-1)$-space V' which is parallel to V and such that $L' \subseteq V' \subseteq U'$, is of type **C** (respectively of type III if $m = 3$), and $P_\infty(V') = P_\infty(V)$. Let V be such a subspace.

Consider the m-space $U'' = \langle V, \pi \rangle$, and consider the connected component $\mathcal{S}'$ of $\mathcal{S}_{U''}$ which contains the affine points of V and of π. Then $\mathcal{S}'$ is not contained in a proper subspace of U''. By Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in the m-dimensional affine space U''. Since $3 \leq m < n$ and since $\mathcal{S}'$ has an $(m-1)$-space of type **C** (respectively of type III if $m = 3$), namely V, and a plane of type III, namely π, which intersect in an affine line, namely L, it follows from the induction hypothesis that $\mathcal{S}'$ does not have any planes of type IV. Now, by Theorem 4.3.1, $\mathcal{S}'$ is a linear representation, so U'' is of type **C**.

Since U'' is of type **C** and $V' \subseteq U''$, $\mathcal{S}_{V'}$ is a linear representation. Since $V, V' \subseteq U''$ and V and V' are parallel, V' is of type **C** (respectively of type III if $m = 3$), and $P_\infty(V') = P_\infty(V)$.

Since $\mathcal{S}_U$ is a connected linear representation, and L is a line of $\mathcal{S}_U$, U contains an $(n-2)$-space V of type **C** (respectively of type III if $n = 4$) and

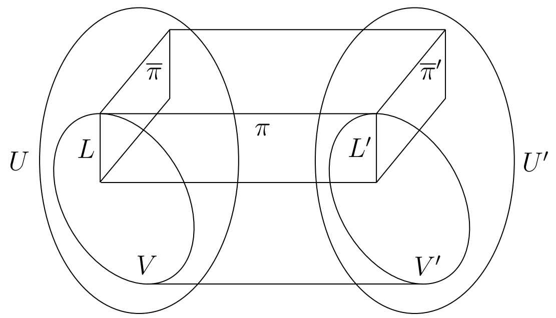

Figure 6.1: Illustration of Theorem 6.2.2.

a plane $\overline{\pi}$ of type III, such that $\overline{\pi} \cap V = L$. Let V' be the $(n-2)$-space parallel to V such that $L' \subseteq V' \subseteq U'$, and let $\overline{\pi}'$ be the plane parallel to $\overline{\pi}$ such that $L' \subseteq \overline{\pi}' \subseteq U'$. Then $\overline{\pi}' \cap V' = L'$, and, as we have shown above, V' is of type $\mathbf{C}$ (respectively of type III if $n = 4$), and $\overline{\pi}'$ is of type III.

Consider the connected component $\mathcal{S}'$ of $\mathcal{S}_{U'}$ which contains the affine points of V' and of $\overline{\pi}'$. Then, by Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in the $(n-1)$-dimensional affine space U'. Also, $\mathcal{S}'$ has an $(n-2)$-space of type $\mathbf{C}$ (respectively of type III if $n = 4$), namely V', and a plane of type III, namely $\overline{\pi}'$, which intersect in an affine line, namely L'. Now, by the induction hypothesis, $\mathcal{S}'$ does not have any planes of type IV. By Theorem 4.3.1, $\mathcal{S}'$ is a linear representation, so U' is a hyperplane of type $\mathbf{C}$.

Let $p'_\infty \in P_\infty(U) \setminus \{p_\infty\}$, and let $L_\infty = \langle p_\infty, p'_\infty \rangle$. Then $L_\infty \cap P_\infty(U)$ spans L_∞, so by Lemma 6.2.1, the plane $\overline{\pi} = \langle L, L_\infty \rangle$ is of type III, and $p_\infty, p'_\infty \in P_\infty(\overline{\pi})$. As we have shown above, the plane $\overline{\pi}'$, parallel to $\overline{\pi}$, such that $L' \subseteq \overline{\pi}' \subseteq U'$, is of type III, and $P_\infty(\overline{\pi}') = P_\infty(\overline{\pi})$. Hence $p_\infty, p'_\infty \in P_\infty(\overline{\pi}')$. Since U' is of type $\mathbf{C}$, by Lemma 6.2.1, $P_\infty(\overline{\pi}') \subseteq P_\infty(U')$. Hence $p_\infty, p'_\infty \in P_\infty(U')$. We conclude that $P_\infty(U) \subseteq P_\infty(U')$. But analogously, one proves that $P_\infty(U') \subseteq P_\infty(U)$, so $P_\infty(U') = P_\infty(U)$.

So every hyperplane U' which is parallel to U, is of type $\mathbf{C}$ and satisfies $P_\infty(U') = P_\infty(U)$. By assumption, there is a plane π' of type IV. Since a subspace of type $\mathbf{C}$ does not contain a plane of type IV, π' intersects U in an affine line L'. Since π' is of type IV, there is exactly one line L'' of $\mathcal{S}_{\pi'}$ which is parallel to L'. Let U'' be the hyperplane parallel to U which contains L''. Then the point $p'_\infty = L'' \cap \Pi_\infty$ is in $P_\infty(U'')$. Hence $p'_\infty \in P_\infty(U')$ for every hyperplane U' parallel to U. But then every line of π' which is parallel to L'' is a line of $\mathcal{S}$, which is impossible since π' is a plane of type IV. So, if we assume that L is a line of $\mathcal{S}$, we obtain a contradiction.

Secondly, assume that L is not a line of $\mathcal{S}$. Let L' be a line of $\mathcal{S}_U$ which intersects L in an affine point, and let W be the 3-space $\langle L', \pi \rangle$. Let $\mathcal{S}'$ be the connected component of $\mathcal{S}_W$ which contains the affine points of π and L'. Clearly $\mathcal{S}'$ is not contained in a plane of W. By Lemma 4.1.1 and by Corollary 5.4.2, $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$, $\mathcal{S}' \simeq \mathcal{I}(3, q, e)$ or $\mathcal{S}'$ is a linear representation. Since $\mathcal{S}'$ contains a planar net, $\mathcal{S}' \not\simeq \mathcal{A}(O_\infty)$.

Suppose that $\mathcal{S}' \simeq \mathcal{I}(3, q, e)$. By Proposition 4.4.11, π is the only plane of type III of $\mathcal{S}'$. So the plane $\pi' = \langle L, L' \rangle$ is not of type III, and since $\pi' \subseteq U$, π' is not of type IV. Since π' contains the line L' of $\mathcal{S}'$, π' is of type II. Since U is of type **C**, every line of π' parallel to L' is a line of $\mathcal{S}$, and hence of $\mathcal{S}'$. But this contradicts Proposition 4.4.11, which says that no two distinct lines of $\mathcal{S}'$, not contained in π, are parallel.

So $\mathcal{S}'$ is a linear representation. Hence there is a plane π'' of type III such that $L' \subseteq \pi'' \subseteq W$ and $\pi'' \not\subseteq U$. But now the hyperplane U of type **C** and the plane π'' of type III intersect in an affine line L', which is a line of $\mathcal{S}$. Hence we can apply the same arguments as above to find a contradiction. $\square$

Theorem 6.2.3 *Let* $\mathcal{S}$ *be a* $(0,2)$*-geometry of order* $(q-1, t)$ *fully embedded in* $\mathrm{AG}(n, q)$*,* $n \geq 3$*,* $q = 2^h$*,* $h > 1$*, such that there is a plane of type* IV*. If there is a hyperplane of type* **C** *(respectively a plane of type* III *if* $n = 3$*), then* $\mathcal{S} \simeq \mathcal{I}(n, q, e)$*.*

Proof. Let $n = 3$. Then, by Theorem 5.4.1, $\mathcal{S} \simeq \mathcal{A}(O_\infty)$ or $\mathcal{S} \simeq \mathcal{I}(3, q, e)$. By Lemma 4.4.2, $\mathcal{A}(O_\infty)$ does not have a planar net. So if $n = 3$, then the theorem holds. We use induction on n to prove the theorem for $n \geq 4$.

So, suppose that $n \geq 4$, and that the theorem holds for all $3 \leq m < n$. Suppose that there exists a hyperplane U of type **C**. Let $(q-1, \tau)$ be the order of the $(0,2)$-geometry $\mathcal{S}_U$. In other words, $|\, P_\infty(U) \,| = \tau + 1$.

Step 1: $\boldsymbol{t \leq \tau + q}$. Let $V \subseteq U$ be an $(n-2)$-space of type **C** (respectively of type III if $n = 4$), let p be an affine point of V, and let $U' \neq U$ be a hyperplane which contains V. Suppose that there are two distinct lines L_1 and L_2 of $\mathcal{S}_{U'}$ which intersect V in the point p. Let $\mathcal{S}'$ be the connected component of $\mathcal{S}_{U'}$ which contains the affine points of V, of L_1 and of L_2. Then $\mathcal{S}'$ is not contained in a proper subspace of U', so by Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in the $(n-1)$-dimensional affine space U'. Since $\mathcal{S}'$ has an $(n-2)$-space of type **C** (respectively of type III if $n = 4$), namely V, the induction hypothesis implies that either $\mathcal{S}' \simeq \mathcal{I}(n-1, q, e)$, or $\mathcal{S}'$ has no planes of type IV.

Suppose that $\mathcal{S}' \simeq \mathcal{I}(n-1, q, e)$. Then, by Proposition 4.4.11, $\mathcal{S}'$ has exactly one $(n-2)$-space of type **C**, namely V, and through every affine point of V passes exactly one line of $\mathcal{S}'$ which is not contained in V. But $L_1 \not\subseteq V$ and $L_2 \not\subseteq V$ are distinct lines of $\mathcal{S}'$ through p, a contradiction.

So $\mathcal{S}'$ has no planes of type IV. Now, by Theorem 4.3.1, $\mathcal{S}'$ is a linear representation. But then there is a plane of type III in U' which intersects U in an affine line. This contradicts Theorem 6.2.2. We conclude that for every hyperplane $U' \neq U$ which contains V, there is at most one line L of $\mathcal{S}$ such that $p \in L \subseteq U'$ and $L \not\subseteq U$. Hence $t \leq \tau + q$.

Step 2: $\boldsymbol{t = \tau + 1}$. Let p be an affine point of U. By Lemma 4.1.4, θ_p spans Π_∞, so there is a line L of $\mathcal{S}$ through p which is not contained in U. Hence $t \geq \tau + 1$.

Suppose that $t > \tau + 1$. Then there are two distinct lines L, L' of $\mathcal{S}$ through p which are not contained in U. Let M be a line of $\mathcal{S}_U$ through p. If M, L and L' are coplanar, then the plane containing them is necessarily of type III. But then there is a plane of type III which intersect U in an affine line, a contradiction to Theorem 6.2.2. So $W = \langle M, L, L' \rangle$ is a 3-space.

Let $\mathcal{S}'$ be the connected component of $\mathcal{S}_W$ containing p. Then the affine points of M, L and L' are points of $\mathcal{S}'$, so $\mathcal{S}'$ is not contained in a plane of W. Lemma 4.1.1 and Corollary 5.4.2 imply that $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$, $\mathcal{S}' \simeq \mathcal{I}(3, q, e)$ or $\mathcal{S}'$ is a linear representation. However, if $\mathcal{S}'$ is a linear representation, then there is a plane of type III in W which intersects U in an affine line, a contradiction to Theorem 6.2.2.

Suppose that $\mathcal{S}' \simeq \mathcal{I}(3, q, e)$. By Proposition 4.4.11, $\mathcal{S}'$ has exactly one plane π of type III. By Theorem 6.2.2, $\pi \subseteq U$ or π is parallel to but not contained in U. Suppose that $\pi \subseteq U$. Then π is necessarily the plane $W \cap U$. Hence $M \subseteq \pi$. But now there are at least 5 lines of $\mathcal{S}'$ through p, a contradiction since the order of $\mathcal{I}(3, q, e)$ is $(q-1, 3)$. So π is parallel to U but not contained in it. Hence M is parallel to π but not contained in it. However, this contradicts Proposition 4.4.11.

So $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$. Let π be the plane $W \cap U$. Then $M \subseteq \pi$, so π is of type II, III or IV. Since $\pi \subseteq U$, π is not of type IV. By Lemma 4.4.2, $\mathcal{A}(O_\infty)$ does not have a planar net, so π is not of type III. So π is of type II. Hence the number of lines of $\mathcal{S}'$ through p which are not contained in U, equals q.

We conclude that for every line M of $\mathcal{S}_U$ through p, $W = \langle M, L, L' \rangle$ is a 3-space, and there are exactly q lines of $\mathcal{S}_W$ through p which are not contained in U.

Let N be the line $\langle L, L' \rangle \cap U$, and let M_1 and M_2 be two distinct lines of $\mathcal{S}_U$ through p such that N, M_1 and M_2 are not coplanar. Let W_i be the 3-space $\langle M_i, L, L' \rangle$, $i = 1, 2$. Then W_1 and W_2 intersect in the plane $\langle L, L' \rangle$. Since there are exactly q lines of $\mathcal{S}_{W_i}$ through p which are not contained in U (including L and L'), $i = 1, 2$, the number of lines of $\mathcal{S}$ through p which are not contained in U is at least $2q-2$. Hence $t \geq \tau + 2q - 2$. But since $q > 2$, this contradicts $t \leq \tau + q$. So $t = \tau + 1$.

Step 3: $\tau = 2^{n-1} - 2$ and $P_\infty(U)$ is the point set of a $\mathrm{PG}(n-2,2)$. Since $t = \tau + 1$, through every affine point of U passes exactly one line of $\mathcal{S}$ which is not contained in U.

Let $V \subseteq U$ be an $(n-2)$-space of type **C**, let p' be an affine point of V, and let L' be the unique line of $\mathcal{S}$ through p' which is not contained in U. Let $U' = \langle L', V\rangle$ and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_{U'}$ which contains the affine points of V and of L'. Then, by Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in the $(n-1)$-dimensional affine space U'. Since $\mathcal{S}'$ has an $(n-2)$-space of type **C**, namely V, the induction hypothesis implies that either $\mathcal{S}'$ has no planes of type IV, or $\mathcal{S}' \simeq \mathcal{I}(n-1,q,e)$.

Suppose that $\mathcal{S}'$ has no planes of type IV. Then by Theorem 4.3.1, $\mathcal{S}'$ is a linear representation. But then there is a plane of type III which intersects U in an affine line, a contradiction to Theorem 6.2.2. So $\mathcal{S}' \simeq \mathcal{I}(n-1,q,e)$.

Let L'' be a line of $\mathcal{S}_U$ through p' which is not contained in V, and let $\pi' = \langle L', L''\rangle$. Then π' intersects U in an affine line and contains two intersecting lines of $\mathcal{S}$. Theorem 6.2.2 implies that π' is a plane of type IV. Hence there exists a line L of $\mathcal{S}$ which intersects U in an affine point $p \in U \setminus V$ and U' in an affine point $r \in U' \setminus V$, which is a point of $\mathcal{S}'$.

We recall that, for any point x of $\mathcal{S}$, $x^\perp$ denotes the set of points of $\mathcal{S}$ which are collinear to x.

Let $p'' \in r^\perp \cap V$. Since L is a line of $\mathcal{S}$ through p which intersects the line $\langle p'', r\rangle$ of $\mathcal{S}$, $\alpha(p, \langle p'', r\rangle) > 0$. Hence $\alpha(p, \langle p'', r\rangle) = 2$, so there is a line $L'' \neq L$ of $\mathcal{S}$ through p which intersects $\langle p'', r\rangle$. Since $t = \tau + 1$, L is the only line of $\mathcal{S}$ through p which is not contained in U. So $L'' \subseteq U$. But L'' intersects $\langle p'', r\rangle$, hence $L'' = \langle p, p''\rangle$. So $p'' \in p^\perp \cap V$. We conclude that $r^\perp \cap V \subseteq p^\perp \cap V$.

Let $p'' \in p^\perp \cap V$. Since $\langle p, p''\rangle$ is a line of $\mathcal{S}$ through p'' which intersects the line L of $\mathcal{S}$, $\alpha(p'', L) > 0$. Hence $\alpha(p'', L) = 2$, so there is a line $L'' \neq \langle p, p''\rangle$ of $\mathcal{S}$ through p'' which intersects L. Since $L'' \neq \langle p, p''\rangle$, L'' intersects L in a point other than p. Hence $L'' \nsubseteq U$. So L'' is the unique line of $\mathcal{S}$ through p'' which is not contained in U. However, since $\mathcal{S}' \simeq \mathcal{I}(n-1,q,e)$, there is a line of $\mathcal{S}'$ through p'' which is not contained in U. This line must be L'', so $L'' \subseteq U'$. Since $L \cap U' = r$, L'' is the line $\langle p'', r\rangle$. So $p'' \in r^\perp \cap V$. We conclude that $p^\perp \cap V = r^\perp \cap V$.

Since U is a hyperplane of type **C**, $P_\infty(U)$ is the set of points at infinity of the lines of $\mathcal{S}_U$ through p. The lines of $\mathcal{S}_U$ through p which are parallel to V intersect Π_∞ in the points of $P_\infty(V)$. Clearly, the lines of $\mathcal{S}_U$ through p which are not parallel to V, intersect V in the points of $p^\perp \cap V$. So $P_\infty(U)$

is the projection from p onto Π_∞ of the set

$$(p^\perp \cap V) \cup P_\infty(V) = (r^\perp \cap V) \cup P_\infty(V).$$

By Lemma 4.4.17, $(r^\perp \cap V) \cup P_\infty(V)$ is the point set of a projective space $\mathrm{PG}(n-2,2)$. It follows that also $P_\infty(U)$ is the point set of a projective space $\mathrm{PG}(n-2,2)$. So $\tau = 2^{n-1} - 2$.

Step 4: $\boldsymbol{\mathcal{S} \simeq \mathcal{I}(n,q,e)}$. We define V, U' and $\mathcal{S}'$ as in Step 3. Then $\mathcal{S}' \simeq \mathcal{I}(n-1,q,e)$. We recall that U is a hyperplane of type $\mathbf{C}$ such that $P_\infty(U)$ is the point set of a projective space $\mathrm{PG}(n-2,2)$.

Let $\mathcal{S}^*$ be the geometry $\mathcal{I}(n,q,e)$. By Proposition 4.4.11, $\mathcal{S}^*$ has a unique hyperplane U^* of type $\mathbf{C}$, and by definition of $\mathcal{S}^*$, $\mathcal{S}^*_{U^*}$ is the linear representation of the point set of a projective space $\mathrm{PG}(n-2,2)$. By Corollary 4.4.15, every $(n-2)$-dimensional subspace V^* of U^* of type $\mathbf{C}$ (respectively of type III if $n = 4$), is contained in a unique hyperplane U'^* of type $\mathbf{B}$, and $\mathcal{S}^*_{U'^*}$ is connected, so $\mathcal{S}^*_{U'^*} \simeq \mathcal{I}(n-1,q,e)$. Hence we can choose a basis in $\mathrm{PG}(n,q)$ such that $U^* = U$, $\mathcal{S}^*_{U^*} = \mathcal{S}_U$ (then V is of type $\mathbf{C}$, respectively of type III if $n = 4$, with respect to $\mathcal{S}^*$), and $\mathcal{S}^*_{U'} = \mathcal{S}'$.

Let p be an affine point of U, not in V. Let p_1, p_2 be distinct affine points of V such that p, p_1, p_2 are pairwise collinear in $\mathcal{S}$, and hence also in $\mathcal{S}^*$. Let L_i be the unique line of $\mathcal{S}'$, and hence the unique line of $\mathcal{S}$, which intersects U in the point p_i, $i = 1, 2$. Then since $\mathcal{S}^*_{U'} = \mathcal{S}'$, L_i is also the unique line of $\mathcal{S}^*_{U'}$, and hence the unique line of $\mathcal{S}^*$, which intersects U in the point p_i, $i = 1, 2$.

Since $\langle p_1, p_2 \rangle$ is a line of $\mathcal{S}$ through p_1 which intersects L_2, there is a second line of $\mathcal{S}$ through p_1 which intersects L_2, and this line is necessarily L_1. So L_1 and L_2 intersect in an affine point r.

Let L, respectively L^*, be the unique line of $\mathcal{S}$, respectively $\mathcal{S}^*$, which intersects U in the point p. Since $\langle p, p_i \rangle$ is a line of $\mathcal{S}$, respectively $\mathcal{S}^*$, through p which intersects L_i, there is a second line of $\mathcal{S}$, respectively $\mathcal{S}^*$, through p which intersects L_i, and this line is necessarily L, respectively L^*, $i = 1, 2$. So L, respectively L^*, intersects the lines L_1 and L_2. Hence L and L^* contain the point r, so $L = L^*$.

Let L^* be a line of $\mathcal{S}^*$. If $L^* \subseteq U$ or L^* intersects U in an affine point, then L^* is a line of $\mathcal{S}$. Since by Proposition 4.4.11, there are no lines of $\mathcal{S}^*$ which are parallel to but not contained in U, the line set of $\mathcal{S}^*$ is a subset of the line set of $\mathcal{S}$.

Let L be a line of $\mathcal{S}$. If $L \subseteq U$ or if L intersects U in an affine point, then L is a line of $\mathcal{S}^*$. Suppose that L is parallel to U but not contained in it. Let p be an affine point of L. As a consequence of Lemma 4.1.4, there is a line L' of $\mathcal{S}$ through p which is not parallel to U. Hence L' is a line of

$\mathcal{S}^*$, and p is a point of $\mathcal{S}^*$. Now the 2^{n-1} lines of $\mathcal{S}^*$ through p are lines of $\mathcal{S}$. Proposition 4.4.11 implies that L is not one of these lines. Hence there are at least $2^{n-1}+1$ lines of $\mathcal{S}$ through p, a contradiction since $t = 2^{n-1}-1$. So every line of $\mathcal{S}$ is contained in U or intersects it in an affine point, and hence is a line of $\mathcal{S}^*$. We conclude that the line sets of $\mathcal{S}$ and $\mathcal{S}^*$ are the same, so $\mathcal{S} = \mathcal{S}^*$. It follows that $\mathcal{S} \simeq \mathcal{I}(n, q, e)$. □

Theorem 6.2.4 *Let* $\mathcal{S}$ *be a* $(0,2)$*-geometry of order* $(q-1, t)$ *fully embedded in* $\mathrm{AG}(4, q)$*,* $q = 2^h$*,* $h > 1$*, such that there is a planar net and a plane of type* IV*. Then* $\mathcal{S} \simeq \mathcal{I}(4, q, e)$*.*

Proof. If there is a hyperplane of type **C**, then Theorem 6.2.3 implies that $\mathcal{S} \simeq \mathcal{I}(4, q, e)$. So, suppose that there is no hyperplane of type **C**. We deduce a contradiction.

Step 1: $t \leq q + 3$. Let π be a plane of type III, let p be an affine point of π and let U be a hyperplane containing π, such that there is a line L of $\mathcal{S}_U$ through p which is not contained in π. Let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which contains the affine points of π and L. By Lemma 4.1.1 and Corollary 5.4.2, $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$, $\mathcal{S}' \simeq \mathcal{I}(3, q, e)$, or $\mathcal{S}'$ is a linear representation.

By assumption, U is not of type **C**, so $\mathcal{S}'$ is not a linear representation. By Lemma 4.4.2, $\mathcal{A}(O_\infty)$ has no planar nets, so $\mathcal{S}' \not\simeq \mathcal{A}(O_\infty)$. It follows that $\mathcal{S}' \simeq \mathcal{I}(3, q, e)$. By Proposition 4.4.11, L is the only line of $\mathcal{S}'$, and hence the only line of $\mathcal{S}_U$, which contains p but is not contained in π.

So, for every hyperplane $U \supseteq \pi$, the number of lines of $\mathcal{S}_U$ through p, not contained in π, is at most one. Hence $t \leq q + 3$.

Step 2: there are no hyperplanes of type A. Suppose that there is a hyperplane U of type **A**. Let $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$ be a connected component of $\mathcal{S}_U$. Suppose that there is a plane $\pi \not\subseteq U$ of type IV which intersects U in a line L of $\mathcal{S}'$. Let p be an affine point of L, let $\pi' \subseteq U$ be a plane of type IV which contains L, let $U' = \langle \pi, \pi' \rangle$ and let $\mathcal{S}''$ be the connected component of $\mathcal{S}_{U'}$ which contains the point p. By Lemma 4.1.1 and Corollary 5.4.2, $\mathcal{S}'' \simeq \mathcal{A}(O_\infty)$, $\mathcal{S}'' \simeq \mathcal{I}(3, q, e)$, or $\mathcal{S}''$ is a linear representation. By assumption, U' is not of type **C**, so $\mathcal{S}''$ is not a linear representation. So $\mathcal{S}'' \simeq \mathcal{A}(O_\infty)$ or $\mathcal{S}'' \simeq \mathcal{I}(3, q, e)$. In any case, the number of lines of $\mathcal{S}''$ through p is at least 4, so there is at least one line of $\mathcal{S}_{U'}$ through p which is not contained in U or in π. Since this holds for every plane $\pi' \subseteq U$ of type IV containing L, and since by Lemma 4.4.3, there are q such planes π', we conclude that the number of lines of $\mathcal{S}$ through p is at least $2q + 2$. But this contradicts $t \leq q+3$. Hence there is no plane $\pi \not\subseteq U$ of type IV which intersects U in a line of $\mathcal{S}'$.

Let p be a point of $\mathcal{S}'$. Lemma 4.1.4 implies that there is a line L' of $\mathcal{S}$ which intersects U in the point p. Let L be a line of $\mathcal{S}'$ through p. By the preceding paragraph, the plane $\pi = \langle L, L' \rangle$ is not of type IV. Hence π is of type III, and it contains a line $L'' \neq L, L'$ of $\mathcal{S}$ through p. Since this holds for every line L of $\mathcal{S}'$ through p, there are at least $2q+3$ lines of $\mathcal{S}$ through p. But this contradicts $t \leq q+3$. So there are no hyperplanes of type **A**.

Step 3: *t* is odd, *t* > 5 and there are two planes of type III which intersect in an affine point. Let π be a plane of type IV, and let p be a point of the connected component of $\mathcal{S}_\pi$ which is a dual oval. Let L be a line of $\mathcal{S}$ which intersects π in the point p, let $U = \langle \pi, L \rangle$, and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which contains the point p. Then by Lemma 4.1.1 and Corollary 5.4.2, and since there are no hyperplanes of type **A** or **C**, $\mathcal{S}' \simeq \mathcal{I}(3, q, e)$. Hence there are exactly two lines of $\mathcal{S}'$ which intersect π in the point p.

So every hyperplane through π contains either 0 or 2 lines of $\mathcal{S}$ which intersect π in the point p. Hence t is odd. Since, by Lemma 4.1.4, θ_p spans Π_∞, there exist at least two hyperplanes U_1, U_2 such that there are two lines of $\mathcal{S}_{U_i}$, $i = 1, 2$, which intersect π in the point p. So $t \geq 5$.

Let $\mathcal{S}_i \simeq \mathcal{I}(3, q, e)$ be the connected component of $\mathcal{S}_{U_i}$ which contains the dual oval of $\mathcal{S}_\pi$, and let π_i be the unique plane of type III of $\mathcal{S}_i$, $i = 1, 2$. Suppose that for some $i = 1, 2$, π_i is parallel to π or π_i intersects π in an affine line which is not a line of $\mathcal{S}_\pi$. Then there is a line of $\mathcal{S}_\pi$, so a line of $\mathcal{S}_i$, which is parallel to but not contained in π_i, a contradiction to Proposition 4.4.11. So π_i intersects π in a line L_i of $\mathcal{S}_\pi$, $i = 1, 2$.

Suppose that $L_1 = L_2$. Then the hyperplane $U = \langle \pi_1, \pi_2 \rangle$ contains two planes of type III which intersect in an affine line. By Theorem 6.2.2, $\mathcal{S}_U$ does not contain any plane of type IV. Hence, by Theorem 4.3.1, $\mathcal{S}_U$ is a linear representation, so U is of type **C**. But this contradicts our assumption that there are no hyperplanes of type **C**. Hence $L_1 \neq L_2$, and π_1 and π_2 intersect in the affine point $L_1 \cap L_2$.

Suppose that $t = 5$. Let p be a point of $\mathcal{S}$. The set θ_p consists of six points which, by Lemma 4.1.4, span Π_∞. Let π_∞ be a plane of Π_∞ containing three noncollinear points of θ_p. Let $U = \langle p, \pi_\infty \rangle$, and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which contains p. Then $\mathcal{S}'$ contains three lines through p which are not coplanar, so $\mathcal{S}'$ is not contained in a plane of U. By Lemma 4.1.1 and Corollary 5.4.2, and since there are no hyperplanes of type **A** or **C**, $\mathcal{S}' \simeq \mathcal{I}(3, q, e)$. Hence π_∞ contains exactly four points of θ_p.

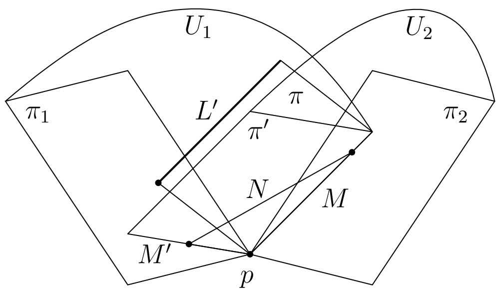

Figure 6.2: Illustration of Step 4 of Theorem 6.2.4.

Let U be a hyperplane of type **B** and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which is projectively equivalent to $\mathcal{I}(3,q,e)$. Let p be a point of $\mathcal{S}'$, not in the unique planar net of $\mathcal{S}'$. Let $\pi_\infty = U \cap \Pi_\infty$. Then π_∞ contains exactly four points p_1, p_2, p_3, p_4 of θ_p, no three of which are collinear. Let p_5, p_6 be the remaining points of θ_p. Let $i, j \in \{1,2,3,4\}$, $i \neq j$, and let $\pi_{ij} = \langle p_i, p_j, p_5 \rangle$. Since π_{ij} contains three noncollinear points of θ_p, it contains four points of θ_p. Since no three of p_1, p_2, p_3, p_4 are collinear, $p_6 \in \pi_{ij}$. So $p_6 \in \pi_{ij}$ for all $i, j \in \{1,2,3,4\}$, $i \neq j$. This implies that $p_5 = p_6$, a contradiction. So $t > 5$.

Step 4: $t = q + 2$. Let π_1 and π_2 be distinct planes of type III which intersect in an affine point p. Since $t > 5$, there is a line L of $\mathcal{S}$ through p which is contained in neither π_1 nor π_2. Let $U_1 = \langle \pi_1, L \rangle$, and let $\mathcal{S}_1$ be the connected component of $\mathcal{S}_{U_1}$ which contains the affine points of π_1 and of L. Then by Lemma 4.1.1 and Corollary 5.4.2, and since there are no hyperplanes of type **A** or **C**, $\mathcal{S}_1 \simeq \mathcal{I}(3,q,e)$.

Let M be the line $U_1 \cap \pi_2$. Then $p \in M$. Since $\mathcal{S}_1 \simeq \mathcal{I}(3,q,e)$, L is the only line of $\mathcal{S}_1$ through p which is not contained in π_1. Hence M is not a line of $\mathcal{S}_1$. By Proposition 4.4.11, there is exactly one line L' of $\mathcal{S}_1$ which is parallel to M. Let π be the plane $\langle L', M \rangle$.

Let U_2 be a hyperplane containing π_2 such that the plane $\pi' = U_1 \cap U_2$ is distinct from π. Then $M \subseteq \pi'$. Let M' be the affine line $\pi' \cap \pi_1$. Assume that M' is not a line of $\mathcal{S}$. Then M' is not a line of $\mathcal{S}_1$, so, by Corollary 4.4.16, π' contains exactly one line N of $\mathcal{S}_1$. Since L' is the unique line of $\mathcal{S}_1$ which is parallel to M, N intersects M in an affine point. If M' is a line of $\mathcal{S}_1$, then $N = M'$ is a line of $\mathcal{S}_{U_2}$ which intersects M in an affine point.

In any case, there is a line N of $\mathcal{S}_{U_2}$ which is not contained in π_2 but intersects it in an affine point. Let $\mathcal{S}_2$ be the connected component of $\mathcal{S}_{U_2}$ which contains the affine points of π_2 and of N. Then by Lemma 4.1.1

and Corollary 5.4.2, and since there are no hyperplanes of type **A** or **C**, $\mathcal{S}_2 \simeq \mathcal{I}(3, q, e)$. So there is a unique line of $\mathcal{S}_2$ through p, not contained in π_2. Since this conclusion holds for every hyperplane $U_2 \supseteq \pi_2$ such that the plane $\pi' = U_1 \cap U_2$ is distinct from π, it follows that $t \geq q + 2$.

Let $U_2 = \langle \pi_2, \pi \rangle$. Let $\mathcal{S}_2$ be the connected component of $\mathcal{S}_{U_2}$ which contains the affine points of π_2. Suppose that $\mathcal{S}_2 \simeq \mathcal{I}(3, q, e)$. As $\pi \cap \pi_2 = M$ and M is not a line of $\mathcal{S}_2$, Corollary 4.4.16 implies that π contains exactly one line N of $\mathcal{S}_2$. By Proposition 4.4.11, N is not parallel to π_2, so N intersects L' in an affine point. Hence, since L' is a line of $\mathcal{S}$, L' is a line of $\mathcal{S}_2$. But by Proposition 4.4.11, there are no lines of $\mathcal{S}_2$ which are parallel to π_2 but not contained in it, a contradiction. So $\mathcal{S}_2 \not\simeq \mathcal{I}(3, q, e)$. Since there are no hyperplanes of type **A** or **C**, $\mathcal{S}_2$ is the planar net $\mathcal{S}_{\pi_2}$. So every line of $\mathcal{S}_{U_2}$ through p is contained in the plane π_2.

It follows that $t = q + 2$. But this contradicts the fact that t is odd. We conclude that there is a hyperplane of type **C**. Now by Theorem 6.2.3, $\mathcal{S} \simeq \mathcal{I}(4, q, e)$. $\square$

Theorem 6.2.5 *Let $\mathcal{S}$ be a $(0, 2)$-geometry fully embedded in* $\mathrm{AG}(n, q)$, *$n \geq 3$, $q = 2^h$, $h > 1$, such that there is a planar net and a plane of type* IV. *Then $\mathcal{S} \simeq \mathcal{I}(n, q, e)$.*

Proof. By Corollary 5.4.2, the theorem holds for $n = 3$. By Theorem 6.2.4, the theorem holds for $n = 4$. We use induction on n to prove the theorem for $n \geq 5$.

So, suppose that $n \geq 5$ and that the theorem holds for all $3 \leq m < n$. If there is a hyperplane of type **C**, then, by Theorem 6.2.3, $\mathcal{S} \simeq \mathcal{I}(n, q, e)$. So we only need to show that there is a hyperplane of type **C**.

Let π be a plane of type III, and let p be an affine point of π. By Lemma 4.1.4, θ_p spans Π_∞. Hence we can choose a hyperplane $U_1 \supseteq \pi$ such that the lines of $\mathcal{S}_{U_1}$ through p span U_1. Let $\mathcal{S}_1$ be the connected component of $\mathcal{S}_{U_1}$ containing p. Then $\mathcal{S}_1$ is not contained in a proper subspace of U_1, so by Lemma 4.1.1, $\mathcal{S}_1$ is a $(0, 2)$-geometry fully embedded in the $(n-1)$-dimensional affine space U_1. Since π is a plane of type III with respect to $\mathcal{S}_1$, the induction hypothesis yields that either $\mathcal{S}_1 \simeq \mathcal{I}(n-1, q, e)$, or $\mathcal{S}_1$ has no planes of type IV. Suppose the latter. Then, by Theorem 4.3.1, $\mathcal{S}_1$ is a linear representation, so U_1 is of type **C**, and we are done. So we may assume that $\mathcal{S}_1 \simeq \mathcal{I}(n-1, q, e)$.

Let $V \subseteq U_1$ be the $(n-2)$-space of type **C** of $\mathcal{S}_1$. By Lemma 4.4.10, $\pi \subseteq V$, so $p \in V$. Since θ_p spans Π_∞, there is a line L of $\mathcal{S}$ through p, not contained in U_1. Let $U_2 = \langle V, L \rangle$, and let $\mathcal{S}_2$ be the connected component of

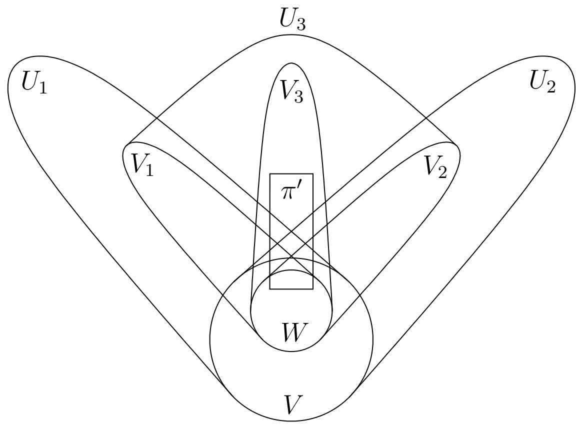

Figure 6.3: Characterization of the geometry $\mathcal{I}(n, q, e)$ (Theorem 6.2.5).

$\mathcal{S}_{U_2}$ which contains the affine points of V and of L. Then, similarly as for $\mathcal{S}_1$, one proves that either $\mathcal{S}_2$ is a linear representation or $\mathcal{S}_2 \simeq \mathcal{I}(n-1, q, e)$. In the first case, U_2 is a hyperplane of type **C**, and we are done. So we may assume that $\mathcal{S}_2 \simeq \mathcal{I}(n-1, q, e)$.

Let $W \subseteq V$ be an $(n-3)$-space of type **C** (respectively of type III if $n = 5$). By Corollary 4.4.15, there is a unique $(n-2)$-space $V_i \subseteq U_i$ through W such that the connected component $\mathcal{S}'_i$ of $\mathcal{S}_{V_i}$ which contains the affine points of W, is projectively equivalent to $\mathcal{I}(n-2, q, e)$, $i = 1, 2$.

Let $U_3 = \langle V_1, V_2 \rangle$, and let $\mathcal{S}_3$ be the connected component of $\mathcal{S}_{U_3}$ which contains the points and lines of $\mathcal{S}'_1$ and $\mathcal{S}'_2$. Then since $\mathcal{S}'_i \simeq \mathcal{I}(n-2, q, e)$ for $i = 1, 2$, $\mathcal{S}_3$ has a plane of type III and a plane of type IV. Also $\mathcal{S}_3$ is not contained in a proper subspace of U_3, so by Lemma 4.1.1, $\mathcal{S}_3$ is a $(0,2)$-geometry fully embedded in U_3. Now it follows from the induction hypothesis that $\mathcal{S}_3 \simeq \mathcal{I}(n-1, q, e)$.

Let V_3 be the $(n-2)$-space of type **C** of $\mathcal{S}_3$. By Lemma 4.4.10, every plane which is of type III with respect to $\mathcal{S}_3$, is contained in V_3. Hence $W \subseteq V_3$. Since $U_3 \cap V = W$, $V_3 \cap V = W$. It follows that there is a plane $\pi' \subseteq V_3$ of type III which intersects V in an affine line. Let $U = \langle V, \pi' \rangle$, and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which contains the affine points of V and of π'. Then by Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in U. By Theorem 6.2.2, $\mathcal{S}'$ does not have any plane of type IV. By Theorem 4.3.1, $\mathcal{S}'$ is a linear representation, so U is a hyperplane of type **C**. Hence, by Theorem 6.2.3, $\mathcal{S} \simeq \mathcal{I}(n, q, e)$. □

6.3 Classification in case there are no planar nets

In this section, we assume that $\mathcal{S}$ is a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $q = 2^h$, $h > 1$, such that there is a plane of type IV and such that there are no planar nets. Since there are no planar nets, every plane containing two intersecting lines of $\mathcal{S}$ is a plane of type IV. Hence we do not need to assume explicitly that there is a plane of type IV.

Lemma 6.3.1 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 4$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then there are no 3-spaces of type **B** or **C**, but there is always a 3-space of type **A**.*

Let U be a 3-space, p a point of $\mathcal{S}_U$, and $\mathcal{S}'$ the connected component of $\mathcal{S}_U$ which contains p. Then there are $0, 1, 2$ or $q+1$ lines of $\mathcal{S}_U$ through p. The lines of $\mathcal{S}_U$ through p span U if and only if their number is $q+1$, if and only if $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$.

Proof. There are no 3-spaces of type **B** or **C** because there are no planar nets.

If $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$, then the number of lines of $\mathcal{S}_U$ through p is $q+1$, and these lines span U. If the lines of $\mathcal{S}_U$ through p span U, then by Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in the 3-dimensional affine space U. Now, by Theorem 5.4.1, either $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$ or $\mathcal{S}' \simeq \mathcal{I}(3,q,e)$. But since there are no planar nets, $\mathcal{S}' \not\simeq \mathcal{I}(3,q,e)$, so $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$.

Suppose that there are $q+1$ lines of $\mathcal{S}_U$ through p, and that these lines do not span U. Then they are contained in some plane $\pi \subseteq U$. But this contradicts Lemma 4.1.3. So if there are $q+1$ lines of $\mathcal{S}_U$ through p, then these lines span U.

Let p' be a point of $\mathcal{S}$. Since, by Lemma 4.1.4, $\theta_{p'}$ spans Π_∞, there is a 3-space U' such that the lines of $\mathcal{S}_{U'}$ through p' span U'. Hence the connected component of $\mathcal{S}_{U'}$ which contains p', is projectively equivalent to $\mathcal{A}(O_\infty)$. So U' is of type **A**. □

Theorem 6.3.2 *Let $\mathcal{S}$ be a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in $\mathrm{AG}(4,q)$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then $t = q^2$.*

Proof. Let U be a hyperplane of type **A**, and let $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$ be a connected component of $\mathcal{S}_U$. Let L be a line of $\mathcal{S}'$, and let p be an affine point of L. By Lemma 4.1.4, there is a line M of $\mathcal{S}$ which intersects U in the point p.

Let $\pi \subseteq U$ be a plane of type IV which contains L. Then $U' = \langle \pi, M \rangle$ contains 3 distinct lines of $\mathcal{S}$ through p. By Lemma 6.3.1, there are $q+1$ lines of $\mathcal{S}_{U'}$ through p. By Lemma 4.4.3, there are q planes π of type IV in U which contain L. Hence the number of lines of $\mathcal{S}$ through p is at least $2+q(q-1)$, so $t \geq q^2-q+1$.

By Lemma 4.4.3, there is exactly one plane $\pi' \subseteq U$ through L which is of type II. Suppose that each hyperplane $U' \neq U$ which contains π', contains at most one line of $\mathcal{S}$ which intersects π' in the point p. Then the number of lines of $\mathcal{S}$ through p is at most $2q+1$. But $t \geq q^2-q+1$ and $q>2$, a contradiction. We conclude that there is a hyperplane $U' \neq U$ which contains π', and contains two distinct lines M', M'' of $\mathcal{S}$ which intersect π' in the point p.

Consider the hyperplanes of $\mathrm{AG}(4,q)$ containing the plane $\pi'' = \langle M', L \rangle$. One of these hyperplanes is U', which intersects U in the plane π'; by Lemma 4.4.3, every other hyperplane intersects U in a plane of type IV. Hence, every hyperplane which contains π'', contains at least 3 lines of $\mathcal{S}$ through p. By Lemma 6.3.1, every hyperplane which contains π'', contains precisely $q+1$ lines of $\mathcal{S}$ through p. Hence $t = q^2$. □

Lemma 6.3.3 *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ *be a* $(0,2)$*-geometry fully embedded in* $\mathrm{AG}(4,q)$*,* $q=2^h$*,* $h>1$*, such that there are no planar nets. Every plane of type* IV *contains* 0 *or* $\frac{1}{2}q(q-1)$ *isolated points. Every plane of type* II *contains* $\frac{1}{2}q$ *or* q *lines of* $\mathcal{S}$*, but no isolated points.*

Proof. Let π be a plane of type IV. Suppose that $\mathcal{S}_\pi$ has an isolated point p_1. Let p_2 be a point of the connected component of $\mathcal{S}_\pi$ which is a dual oval. By Lemma 6.3.1, for every hyperplane $U \supseteq \pi$, the number of lines of $\mathcal{S}_U$ through p_2 which are not contained in π, is at most $q-1$. Since, by Theorem 6.3.2, the number of lines of $\mathcal{S}$ through p_2 is equal to q^2+1, every hyperplane $U \supseteq \pi$ contains exactly $q+1$ lines of $\mathcal{S}$ through p_2.

Suppose that in every hyperplane $U \supseteq \pi$, there are at most 2 lines of $\mathcal{S}$ through p_1. Then, by Theorem 6.3.2, $q^2+1 \leq 2(q+1)$, a contradiction since $q>2$. So there is a hyperplane $U \supseteq \pi$ which contains at least 3 lines of $\mathcal{S}$ through p_1. By Lemma 6.3.1, the number of lines of $\mathcal{S}_U$ through p_1 is $q+1$. As we have shown, the number of lines of $\mathcal{S}_U$ through p_2 is $q+1$. By Lemma 6.3.1, the connected component $\mathcal{S}_i$ of $\mathcal{S}_U$ which contains the point p_i is projectively equivalent to $\mathcal{A}(O^i_\infty)$, $i=1,2$.

Suppose that $\mathcal{S}_1 = \mathcal{S}_2 \simeq \mathcal{A}(O_\infty)$, O_∞ a conic. By Lemma 4.4.5, every plane of type IV of $\mathcal{A}(O_\infty)$, O_∞ a conic, does not contain any isolated points.

But π is a plane of type IV with respect to $\mathcal{S}_1$ and it contains an isolated point of $\mathcal{S}_1$, a contradiction. So if $\mathcal{S}_1 = \mathcal{S}_2$, then $\mathcal{S}_1 \simeq \mathcal{A}(O_\infty)$, O_∞ not a conic. In this case every affine point of U is a point of $\mathcal{S}_1$, so π contains $\frac{1}{2}q(q-1)$ isolated points.

Suppose that $\mathcal{S}_1 \neq \mathcal{S}_2$. From the construction of $\mathcal{A}(O_\infty)$ it follows that if O_∞ is not a conic, then every point of $\mathrm{AG}(3,q)$ is a point of $\mathcal{A}(O_\infty)$, and if O_∞ is a conic, then exactly half the points of $\mathrm{AG}(3,q)$ are points of $\mathcal{A}(O_\infty)$. Hence $\mathcal{S}_1 \simeq \mathcal{S}_2 \simeq \mathcal{A}(O_\infty)$, O_∞ a conic, and $\mathcal{S}_1$ and $\mathcal{S}_2$ are complementary. Hence every affine point of U is a point of either $\mathcal{S}_1$ or $\mathcal{S}_2$, so π contains $\frac{1}{2}q(q-1)$ isolated points. We conclude that every plane of type IV contains 0 or $\frac{1}{2}q(q-1)$ isolated points.

Let π be a plane of type II. Let L_1 be a line of $\mathcal{S}_\pi$, and let p_1 be an affine point of L_1. As a consequence of Lemma 6.3.1 and Theorem 6.3.2, there are at least q hyperplanes $U \supseteq \pi$ which contain $q+1$ lines of $\mathcal{S}$ through p_1.

Suppose that π contains an isolated point p_2. As a consequence of Lemma 6.3.1 and Theorem 6.3.2, there are at least 2 hyperplanes $U \supseteq \pi$ which contain $q+1$ lines of $\mathcal{S}$ through p_2. So there is a hyperplane $U \supseteq \pi$ which contains $q+1$ lines of $\mathcal{S}$ through p_i, $i = 1,2$. By Lemma 6.3.1, the connected component $\mathcal{S}_i$ of $\mathcal{S}_U$ which contains p_i, is projectively equivalent to $\mathcal{A}(O^i_\infty)$, $i = 1,2$.

Suppose that $\mathcal{S}_1 \neq \mathcal{S}_2$. Then, analogously as above, $\mathcal{S}_1 \simeq \mathcal{S}_2 \simeq \mathcal{A}(O_\infty)$, O_∞ a conic, and $\mathcal{S}_1$ and $\mathcal{S}_2$ are complementary. Hence a plane of U is of type II with respect to $\mathcal{S}_1$ if and only if it is of type II with respect to $\mathcal{S}_2$. Since the plane π is of type II and contains the line L_1 of $\mathcal{S}_1$, it is of type II with respect to $\mathcal{S}_1$. Hence π is of type II with respect to $\mathcal{S}_2$. By Lemma 4.4.4, π does not contain any isolated points of $\mathcal{S}_2$. However p_2 is an isolated point of $\mathcal{S}_2$ in π, a contradiction.

Hence $\mathcal{S}_1 = \mathcal{S}_2$. Since the plane π is of type II and contains the line L_1 of $\mathcal{S}_1$, it is of type II with respect to $\mathcal{S}_1$. By Lemma 4.4.4, π does not contain any isolated points of $\mathcal{S}_1$. However p_2 is an isolated point of $\mathcal{S}_1$ in π, a contradiction. So π does not contain any isolated points.

Let $U \supseteq \pi$ be a hyperplane such that the connected component $\mathcal{S}_1$ of $\mathcal{S}_U$ which contains the point p_1, is projectively equivalent to $\mathcal{A}(O_\infty)$. Since π is a plane of type II and contains the line L_1 of $\mathcal{S}_1$, it is of type II with respect to $\mathcal{S}_1$. Hence the hole n_∞ of $\mathcal{S}_1$ lies on the line $L_\infty = \pi \cap \Pi_\infty$. By Lemma 4.4.5, for every line L'_∞ of the plane $\pi_\infty = U \cap \Pi_\infty$ which does not contain n_∞, there is a plane $\pi' \subseteq U$ of type IV such that $\pi' \cap \Pi_\infty = L'_\infty$. Hence there is a plane $\pi' \subseteq U$ of type IV which intersects π in an affine line L, L not a line of $\mathcal{S}$. Since π does not contain any isolated points, the number of lines of $\mathcal{S}_\pi$ equals the number of points of $\mathcal{S}$ on L. Since π' contains 0 or $\frac{1}{2}q(q-1)$ isolated points, the number of points of $\mathcal{S}$ on L equals $\frac{1}{2}q$ or q. $\square$

Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 4$, $q = 2^h$, $h > 1$, and let U be a 3-space of type **A**. We say that U is of type $\mathbf{A_1}$ if $\mathcal{S}_U$ is connected and $\mathcal{S}_U \simeq \mathrm{HT} = \mathcal{A}(O_\infty)$, O_∞ a conic, and we say that U is of type $\mathbf{A_2}$ if either $\mathcal{S}_U$ is connected and $\mathcal{S}_U \simeq \mathcal{A}(O_\infty)$, O_∞ not a conic, or $\mathcal{S}_U$ consists of two connected components which are projectively equivalent to $\mathrm{HT} = \mathcal{A}(O_\infty)$, O_∞ a conic, and which are complementary (see Lemma 5.3.30). Note that a hyperplane of type **A** is not necessarily of type $\mathbf{A_1}$ or $\mathbf{A_2}$.

Theorem 6.3.4 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a $(0,2)$-geometry fully embedded in $\mathrm{AG}(4,q)$, where $q = 2^h$, $h > 1$, such that there are no planar nets. Then $\mathcal{S} \simeq \mathrm{TQ}(4,q)$.*

Proof. Let U be a hyperplane of type **A**, let $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$ be a connected component of $\mathcal{S}_U$, and let n_∞ be the hole of $\mathcal{S}'$. Let p be a point of $\mathcal{S}_U$, and let π be a plane of U containing the line $L = \langle p, n_\infty \rangle$. Then, by Lemma 4.4.4, π is of type II with respect to $\mathcal{S}'$. Hence π is a plane of type II. By Lemma 6.3.3, π does not contain any isolated points. Hence there is a line L' of $\mathcal{S}_\pi$ through p. Note that $L' \neq L$ since L' is parallel to the lines of $\mathcal{S}'$ in π, and no line of $\mathcal{S}'$ intersects Π_∞ in the hole n_∞ of $\mathcal{S}'$. So, in every plane $\pi \subseteq U$ through the line L lies a line of $\mathcal{S}_U$, distinct from L, which contains p. Hence there are $q+1$ lines of $\mathcal{S}_U$ through p, for every point p of $\mathcal{S}_U$. By Lemma 6.3.1, every point p of $\mathcal{S}_U$ is contained in a connected component $\mathcal{S}''$ of $\mathcal{S}_U$ which is projectively equivalent to $\mathcal{A}(O_\infty)$. So U is of type $\mathbf{A_1}$ or $\mathbf{A_2}$. We conclude that every hyperplane of type **A** is of type $\mathbf{A_1}$ or $\mathbf{A_2}$.

Let π be a plane of type IV. Let p be a point of the connected component of $\mathcal{S}_\pi$ which is a dual oval. As a consequence of Lemma 6.3.1 and Theorem 6.3.2, every hyperplane $U \supseteq \pi$ contains $q+1$ lines of $\mathcal{S}$ through p, so every hyperplane $U \supseteq \pi$ is of type **A**, and hence of type $\mathbf{A_1}$ or $\mathbf{A_2}$.

Suppose that π contains an isolated point p'. Let U be a hyperplane containing π. Since π contains an isolated point and since, by Lemma 4.4.5, every plane of type IV of $\mathcal{A}(O_\infty)$, O_∞ a conic, does not contain any isolated points, U is not of type $\mathbf{A_1}$. Hence U is of type $\mathbf{A_2}$. So there are $q+1$ lines of $\mathcal{S}_U$ through p', none of which lie in π. This holds for every hyperplane $U \supseteq \pi$, so there are $(q+1)^2$ lines of $\mathcal{S}$ through p, a contradiction to Theorem 6.3.2. We conclude that π does not contain any isolated points. Note that π is an arbitrary plane of type IV.

Since π does not contain any isolated points, none of the hyperplanes containing π is of type $\mathbf{A_2}$. Hence every hyperplane containing π is of type $\mathbf{A_1}$. It follows that $|\,\mathcal{P}\,| = \frac{1}{2}q^2(q^2-1)$.

Any line L of $\mathrm{AG}(4,q)$ which contains a point of $\mathcal{S}$, lies in a plane of type II or IV. By Lemma 6.3.3, L contains $\frac{1}{2}q$ or q points of $\mathcal{S}$. So the set $\mathcal{R} = \mathcal{P} \cup \Pi_\infty$ is a set of type $(1, \frac{1}{2}q+1, q+1)$ in $\mathrm{PG}(4,q)$.

Suppose that $\mathcal{R}$ is singular. Then there is a singular point $p \in \mathcal{R}$, that is, a point $p \in \mathcal{R}$ such that every line of $\mathrm{PG}(4,q)$ contains either 1 or $q+1$ points of $\mathcal{R}$. Let π be a plane of type IV which does not contain the point p. As we have shown, every hyperplane through π is of type $\mathbf{A_1}$, so, in particular, the hyperplane $U = \langle \pi, p\rangle$ is of type $\mathbf{A_1}$. Since p is a singular point of $\mathcal{R}$, p is a singular point of the intersection of $\mathcal{R}$ with U. Since U is of type $\mathbf{A_1}$, $\mathcal{S}_U \simeq \mathrm{HT}$, so the intersection of $\mathcal{R}$ with U is projectively equivalent to the set $\mathcal{R}_3$. But $\mathcal{R}_3$ is a nonsingular set of type $(1, \frac{1}{2}q, q+1)$, a contradiction.

So $\mathcal{R}$ is a nonsingular set of type $(1, \frac{1}{2}q+1, q+1)$ in $\mathrm{PG}(4,q)$. Since $\mathcal{R}$ contains every point of the hyperplane Π_∞ of $\mathrm{PG}(4,q)$, there is no plane π of $\mathrm{PG}(4,q)$ which intersects $\mathcal{R}$ in a unital or a Baer subplane. By Theorem 1.3.4, $\mathcal{R} = \mathcal{R}_4^-$ or $\mathcal{R} = \mathcal{R}_4^+$. Since $|\,\mathcal{P}\,| = \frac{1}{2}q^2(q^2-1)$, $\mathcal{R} = \mathcal{R}_4^-$. This means that $\mathcal{P}$ is the point set of the semipartial geometry $\mathrm{TQ}(4,q)$.

Let L be a line of $\mathcal{S}$. Then every affine point of L is a point of $\mathcal{P}$, so L is a line of $\mathrm{TQ}(4,q)$. So the line set of $\mathcal{S}$ is a subset of the line set of $\mathrm{TQ}(4,q)$. Since $\mathcal{S}$ and $\mathrm{TQ}(4,q)$ have the same point set and the same order, they have equally many lines. Hence the line sets of $\mathcal{S}$ and $\mathrm{TQ}(4,q)$ are the same, and so $\mathcal{S} = \mathrm{TQ}(4,q)$. $\square$

Lemma 6.3.5 *Assume that $\mathcal{S}$ is a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 5$, $q = 2^h$, $h > 1$, such that there are no planar nets. Then there are no 4-spaces of type* **B** *or* **C**, *but there is always a 4-space of type* **A**.

Let U be a 4-space, p a point of $\mathcal{S}_U$, and $\mathcal{S}'$ the connected component of $\mathcal{S}_U$ which contains p. Then there are $0, 1, 2, q+1$ or q^2+1 lines of $\mathcal{S}_U$ through p. The lines of $\mathcal{S}_U$ through p span U if and only if their number is q^2+1, if and only if $\mathcal{S}' \simeq \mathrm{TQ}(4,q)$.

Proof. There are no 4-spaces of type **B** or **C** because there are no planar nets.

If $\mathcal{S}' \simeq \mathrm{TQ}(4,q)$, then the number of lines of $\mathcal{S}_U$ through p is q^2+1, and they span U. If the lines of $\mathcal{S}_U$ through p span U, then by Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in the 4-dimensional affine space U. By Theorem 6.3.4, $\mathcal{S}' \simeq \mathrm{TQ}(4,q)$.

Suppose that there are q^2+1 lines of $\mathcal{S}_U$ through p, and that these lines do not span U. Then they are contained in some 3-space $V \subseteq U$. But this contradicts Lemma 6.3.1. So if there are q^2+1 lines of $\mathcal{S}_U$ through p, then these lines span U.

Let p' be a point of $\mathcal{S}$. Since, by Lemma 4.1.4, $\theta_{p'}$ spans Π_∞, there is a 4-space U' such that the lines of $\mathcal{S}_{U'}$ through p' span U'. Hence the connected component $\mathcal{S}''$ of $\mathcal{S}_{U'}$ which contains p', is projectively equivalent to $\mathrm{TQ}(4,q)$. So U' is of type $\mathbf{A}$. $\square$

Lemma 6.3.6 *Assume that $\mathcal{S}$ is a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in $\mathrm{AG}(5,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Let U be a hyperplane of type $\mathbf{A}$. Then there is exactly one connected component of $\mathcal{S}_U$ which is projectively equivalent to $\mathrm{TQ}(4,q)$.*

Proof. Suppose that there are two distinct connected components $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{S}_U$ which are projectively equivalent to $\mathrm{TQ}(4,q)$. Let $\mathcal{P}_i$ be the point set of $\mathcal{S}_i$, $i=1,2$. Then $\mathcal{P}_i$ is a $\frac{1}{2}q^2(q^2-1)$-set of type $(0,\frac{1}{2}q,q)$ in U, $i=1,2$. Hence the set $\mathcal{R}$, which is the complement of $\mathcal{P}_1\cup\mathcal{P}_2$ in the set of affine points of U, is a q^2-set of type $(0,\frac{1}{2}q,q)$ in U. Let $p\in\mathcal{R}$. Then every line of U through p contains at least $\frac{1}{2}q-1$ points of $\mathcal{R}\setminus\{p\}$. Hence $q^2=|\mathcal{R}|\geq 1+(q^3+q^2+q+1)(\frac{1}{2}q-1)$, a contradiction. $\square$

Theorem 6.3.7 *Assume that $\mathcal{S}$ is a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in $\mathrm{AG}(5,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Then $t=q^3$.*

Proof. Let U be a hyperplane of type $\mathbf{A}$, let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which is projectively equivalent to $\mathrm{TQ}(4,q)$, and let π be a plane of type IV with respect to $\mathcal{S}'$. Let p be a point of the connected component of $\mathcal{S}_\pi$ which is a dual oval. Then, as a consequence of Lemma 4.1.4, there is a line L of $\mathcal{S}$ which intersects U in the point p. Let $V=\langle\pi,L\rangle$. Then the lines of $\mathcal{S}_V$ through p span V, so, by Lemma 6.3.1, their number is $q+1$.

By Lemma 4.4.8, every 3-space $W\subseteq U$ which contains π, is of type $\mathbf{A}$ with respect to $\mathcal{S}'$. Let U' be a hyperplane which contains V, and let $W=U'\cap U$. Then since $\pi\subseteq W$, W is of type $\mathbf{A}$ with respect to $\mathcal{S}'$, and so the lines of $\mathcal{S}_W$ through p span W. It follows that the lines of $\mathcal{S}_{U'}$ through p span U'. By Lemma 6.3.5, there are q^2+1 lines of $\mathcal{S}_{U'}$ through p, $q+1$ of which lie in V. Since U' was an arbitrary hyperplane containing V, it follows that the total number of lines of $\mathcal{S}$ through p is q^3+1. So $t=q^3$. $\square$

Assume that $\mathcal{S}$ is a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n\geq 5$, $q=2^h$, $h>1$, such that there are no planar nets. Let U be a 4-space of type $\mathbf{A}$, and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which is projectively equivalent to $\mathrm{TQ}(4,q)$. We say that U is of type $\mathbf{A_1}$ if $\mathcal{S}_U=\mathcal{S}'$.

Theorem 6.3.8 *There does not exist a* $(0,2)$*-geometry fully embedded in* $\mathrm{AG}(5,q)$*,* $q=2^h$*,* $h>1$*, such that there are no planar nets.*

Proof. Assume that $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ is a $(0,2)$-geometry of order $(q-1,t)$ fully embedded in $\mathrm{AG}(5,q)$, $q=2^h$, $h>1$, such that there are no planar nets. Then, by Theorem 6.3.7, $t=q^3$.

Let V be a 3-space of type $\mathbf{A}$, and let $\mathcal{S}'$ be a connected component of $\mathcal{S}_V$ which is projectively equivalent to $\mathcal{A}(O_\infty)$. Let p_1 be a point of $\mathcal{S}'$. Then there are $q+1$ lines of $\mathcal{S}_V$ through p_1. By Lemma 6.3.5, for every hyperplane $U\supseteq V$, the number of lines of $\mathcal{S}_U$ which intersect V in the point p is at most q^2-q. Since the total number of lines of $\mathcal{S}$ through p is q^3+1, for every hyperplane $U\supseteq V$, the number of lines of $\mathcal{S}_U$ through p_1 is q^2+1.

Let U be a hyperplane containing V, and let $\mathcal{S}_1$ be the connected component of $\mathcal{S}_U$ which contains the point p_1. By Lemma 6.3.5, $\mathcal{S}_1\simeq\mathrm{TQ}(4,q)$. By Lemma 4.4.7, $(\mathcal{S}_1)_V$ is connected and $(\mathcal{S}_1)_V\simeq\mathrm{HT}$. By Lemma 6.1.2, $(\mathcal{S}_1)_V$ is a connected component of $\mathcal{S}_V$. Since $\mathcal{S}'$ is also a connected component of $\mathcal{S}_V$, and $(\mathcal{S}_1)_V$ and $\mathcal{S}'$ have the point p_1 in common, $\mathcal{S}'=(\mathcal{S}_1)_V\simeq\mathrm{HT}$.

Suppose that $\mathcal{S}_V\neq\mathcal{S}'$. So there is a point p_2 of $\mathcal{S}_V$ which is not a point of $\mathcal{S}'$. Suppose that, for every hyperplane $U\supseteq V$, the number of lines of $\mathcal{S}_U$ through p_2 is at most $q+1$. Then $t+1=q^3+1\leq(q+1)^2$, a contradiction. So there is a hyperplane $U\supseteq V$ which contains strictly more than $q+1$ lines of $\mathcal{S}$ through p_2. By Lemma 6.3.5, the connected component $\mathcal{S}_2$ of $\mathcal{S}_U$ which contains p_2 is projectively equivalent to $\mathrm{TQ}(4,q)$.

As we have shown, if $\mathcal{S}_1$ is the connected component of $\mathcal{S}_U$ which contains the point p_1, then $\mathcal{S}_1\simeq\mathrm{TQ}(4,q)$ and $(\mathcal{S}_1)_V=\mathcal{S}'\simeq\mathrm{HT}$. By Lemma 6.3.6, $\mathcal{S}_1=\mathcal{S}_2$. Since p_2 is a point of $\mathcal{S}_2=\mathcal{S}_1$, and since $(\mathcal{S}_1)_V=\mathcal{S}'$, p_2 is a point of $\mathcal{S}'$, a contradiction. We conclude that $\mathcal{S}_V=\mathcal{S}'$. Since $\mathcal{S}'\simeq\mathrm{HT}$, V is of type $\mathbf{A_1}$. So every 3-space which is of type $\mathbf{A}$, is of type $\mathbf{A_1}$.

Let U be a hyperplane of type $\mathbf{A}$, and let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which is projectively equivalent to $\mathrm{TQ}(4,q)$. Suppose that $\mathcal{S}_U\neq\mathcal{S}'$. Then there is a point p of $\mathcal{S}_U$ which is not a point of $\mathcal{S}'$. Let π be a plane of U through p. By Lemma 4.4.8, there is a 3-space $V\subseteq U$ through π which is of type $\mathbf{A}$ with respect to $\mathcal{S}'$. By Lemma 4.4.7, $\mathcal{S}'_V$ is connected and projectively equivalent to HT.

Since V is of type $\mathbf{A}$, V is of type $\mathbf{A_1}$. By Lemma 6.1.2, $\mathcal{S}'_V$ is a connected component of $\mathcal{S}_V$. Since V is of type $\mathbf{A_1}$, $\mathcal{S}_V=\mathcal{S}'_V$. Since p is a point of $\mathcal{S}_V$, p is a point of $\mathcal{S}'_V$. So p is a point of $\mathcal{S}'$, a contradiction. It follows that $\mathcal{S}_U=\mathcal{S}'$, so U is of type $\mathbf{A_1}$. So every hyperplane of type $\mathbf{A}$ is of type $\mathbf{A_1}$.

Let V be a 3-space of type $\mathbf{A}$. Then V is of type $\mathbf{A_1}$. As we have shown, every hyperplane $U\supseteq V$ is of type $\mathbf{A}$, and hence of type $\mathbf{A_1}$. It follows that $|\,\mathcal{P}\,|=\frac{1}{2}q^2(q^3-q-1)$.

Let L be a line of $\mathrm{AG}(5,q)$ which is not a line of $\mathcal{S}$, but which contains a point p of $\mathcal{S}$. Let V be a 3-space which contains the line L. Suppose that every hyperplane through V contains at most $q+1$ lines of $\mathcal{S}$ through p. Then $t+1 = q^3+1 \leq (q+1)^2$, a contradiction. So there is a hyperplane $U \supseteq V$ which contains strictly more than $q+1$ lines of $\mathcal{S}$ through p. By Lemma 6.3.5, U is of type $\mathbf{A}$, and hence of type $\mathbf{A_1}$. Since the point set of $\mathrm{TQ}(4,q)$ is a set of type $(0,\frac{1}{2}q,q)$, L contains $\frac{1}{2}q$ or q points of $\mathcal{S}$.

So the set $\mathcal{R} = \mathcal{P} \cup \Pi_\infty$ is a set of type $(1,\frac{1}{2}q+1,q+1)$ in $\mathrm{PG}(5,q)$. Suppose that $\mathcal{R}$ is singular. Then there is a singular point $p \in \mathcal{R}$, that is, a point $p \in \mathcal{R}$ such that every line of $\mathrm{PG}(5,q)$ contains either 1 or $q+1$ points of $\mathcal{R}$. Let V be a 3-space of type $\mathbf{A}$ which does not contain the point p. As we have shown, every hyperplane through V is of type $\mathbf{A_1}$, so, in particular, the hyperplane $U = \langle V,p\rangle$ is of type $\mathbf{A_1}$. Since p is a singular point of $\mathcal{R}$, p is a singular point of the intersection of $\mathcal{R}$ with U. Since U is of type $\mathbf{A_1}$, $\mathcal{S}_U \simeq \mathrm{TQ}(4,q)$, so the intersection of $\mathcal{R}$ with U is projectively equivalent to the set $\mathcal{R}_4^-$. But $\mathcal{R}_4^-$ is a nonsingular set of type $(1,\frac{1}{2}q+1,q+1)$, a contradiction.

So $\mathcal{R}$ is a nonsingular set of type $(1,\frac{1}{2}q+1,q+1)$ in $\mathrm{PG}(5,q)$. Since $\mathcal{R}$ contains every point of the hyperplane Π_∞ of $\mathrm{PG}(5,q)$, there is no plane π of $\mathrm{PG}(5,q)$ which intersects $\mathcal{R}$ in a unital or a Baer subplane. By Theorem 1.3.4, $\mathcal{R} = \mathcal{R}_5$. But this contradicts $|\,\mathcal{P}\,| = \frac{1}{2}q^2(q^3-q-1)$.

We conclude that there does not exist a $(0,2)$-geometry fully embedded in $\mathrm{AG}(5,q)$, $q=2^h$, $h>1$, such that there are no planar nets. □

Theorem 6.3.9 *There does not exist a* $(0,2)$*-geometry fully embedded in* $\mathrm{AG}(n,q)$*,* $n \geq 6$*,* $q=2^h$*,* $h>1$*, such that there are no planar nets.*

Proof. Assume that $\mathcal{S} = (\mathcal{P},\mathcal{B},\mathrm{I})$ is a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 6$, $q=2^h$, $h>1$, such that there are no planar nets. Let p be a point of $\mathcal{S}$. By Lemma 4.1.4, θ_p spans Π_∞. Hence there exists a 5-space U which contains p, such that the lines of $\mathcal{S}_U$ through p span U. Let $\mathcal{S}'$ be the connected component of $\mathcal{S}_U$ which contains the point p. Then $\mathcal{S}'$ is not contained in a proper subspace of U. By Lemma 4.1.1, $\mathcal{S}'$ is a $(0,2)$-geometry fully embedded in the 5-dimensional affine space U. By Theorem 6.3.8, $\mathcal{S}'$ has a planar net. But then $\mathcal{S}$ has a planar net, a contradiction. □

6.4 Conclusion

Theorem 6.4.1 *Let $\mathcal{S}$ be a $(0,2)$-geometry fully embedded in* $\mathrm{AG}(n,q)$*, $n \geq 4$, $q = 2^h$, $h > 1$, such that there is at least one plane of type* IV*. Then one of the following cases occurs.*

1. *$n = 4$ and $\mathcal{S} \simeq \mathrm{TQ}(4,q)$.*
2. *$\mathcal{S} \simeq \mathcal{I}(n,q,e)$.*

Proof. This follows immediately from Theorems 6.2.5, 6.3.4 and 6.3.9. □

Corollary 6.4.2 *Let $\mathcal{S}$ be a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in* $\mathrm{AG}(n,q)$*, $n \geq 4$. Then one of the following cases occurs.*

1. *$q = 2$, $\alpha = 2$, and $\mathcal{S}$ is a $2-(t+2,2,1)$-design.*
2. *$n = 4$, $q = 2^h$, $\alpha = 2$, and $\mathcal{S} \simeq \mathrm{TQ}(4,q)$.*
3. *$q = 2^h$, $\alpha = 2$, and $\mathcal{S} \simeq \mathcal{I}(n,q,e)$.*
4. *$\mathcal{S} \simeq T^*_{n-1}(\mathcal{K}_\infty)$, with $\mathcal{K}_\infty$ a set of type $(0,1,\alpha+1)$ in Π_∞ which spans Π_∞.*

Proof. The case $q = 2$ is trivial and is solved in Proposition 4.3.5. If there are no planes of type IV, then Theorem 4.3.1 applies. If there is a plane of type IV, then $\alpha = 2$ and $q = 2^h$, and Theorem 6.4.1 applies. □

Chapter 7

Overview

7.1 Construction of $(0,\alpha)$-geometries fully embedded in $PG(3,q)$, and the Klein quadric

In Chapter 2, we studied $(0,\alpha)$-geometries, $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$, $q > 2$. Recall that the $(0,\alpha)$-geometries, $\alpha > 1$, fully embedded in $\mathrm{PG}(n,q)$, $n \geq 4$, $q > 2$, were classified by Debroey, De Clerck and Thas [81] (see Theorem 1.4.4).

The Plücker correspondence is used to transform the line set of a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$, into a set of points on the Klein quadric $\mathrm{Q}^+(5,q)$. By De Clerck and Thas [30], this set is a $(0,\alpha)$-set on $\mathrm{Q}^+(5,q)$, that is, every line of $\mathrm{Q}^+(5,q)$ intersects it in either 0 or α points. Conversely, when $\alpha > 1$ and $q > 2$, every $(0,\alpha)$-set on $\mathrm{Q}^+(5,q)$ corresponds to the line set of a $(0,\alpha)$-geometry fully embedded in $\mathrm{PG}(3,q)$ (see Sections 2.1, 2.2).

In Theorem 2.5.1, we prove that the $(0,2)$-set $\mathcal{E}_d$ with $d \in \{1, q+1\}$ (see Section 2.4) on the Klein quadric $\mathrm{Q}^+(5,q)$, $q = 2^h$, is essentially the union of $q+2$ nonsingular elliptic quadrics $\mathrm{Q}^-(3,q)$ on $\mathrm{Q}^+(5,q)$, whose ambient 3-spaces contain a common plane π and intersect a plane skew to π in the points of a regular hyperoval.

Next, in Section 2.6, we construct a particular set of points $\mathcal{M}_d^\alpha(A)$ on the Klein quadric $\mathrm{Q}^+(5,q)$, $q = 2^h$. Essentially, $\mathcal{M}_d^\alpha(A)$ is the union of $q\alpha - q + \alpha$ nonsingular elliptic quadrics $\mathrm{Q}^-(3,q)$ on $\mathrm{Q}^+(5,q)$, whose ambient 3-spaces share a plane π which contains d points but no lines of $\mathrm{Q}^+(5,q)$, and intersect a plane π' skew to π in the points of a maximal arc A of degree α. In Theorem 2.6.1, we show that $\mathcal{M}_d^\alpha(A)$ is a $(0,\alpha)$-set on $\mathrm{Q}^+(5,q)$ of deficiency d. Except for a few cases, the set $\mathcal{M}_d^\alpha(A)$ is a new example of a $(0,\alpha)$-set on $\mathrm{Q}^+(5,q)$.

In Theorem 2.6.2, we show that there exist $(0,\alpha)$-sets $\mathcal{M}_d^\alpha(A)$ on $\mathrm{Q}^+(5,q)$, $q=2^h$, for every $d \in \{1,q+1\}$ and $\alpha \in \{2,2^2,\ldots,2^{h-1}\}$. This implies that there exist $(0,\alpha)$-geometries of deficiency d, fully embedded in $\mathrm{PG}(3,q)$, $q=2^h$, for every $d \in \{1,q+1\}$ and $\alpha \in \{2,2^2,\ldots,2^{h-1}\}$. Again, except for a few cases, these geometries are new examples of $(0,\alpha)$-geometries fully embedded in $\mathrm{PG}(3,q)$. However, none of them is a semipartial geometry.

The $(0,\alpha)$-sets $\mathcal{M}_d^\alpha(A)$ which are not new, are the following. By Theorem 2.5.1, the $(0,2)$-set $\mathcal{E}_d$, $d \in \{1,q+1\}$, is of the form $\mathcal{M}_d^2(H)$, with H a regular hyperoval. In Section 2.6, we show that the $(0,q/2)$-set corresponding to the $(0,q/2)$-geometry $\mathrm{NQ}^+(3,q)$, $q=2^h$, is of the form $\mathcal{M}_{q+1}^{q/2}(A)$. Hence, the list of all the known distinct examples of $(0,\alpha)$-sets $\mathcal{K}$ in $\mathrm{Q}^+(5,q)$, $\alpha>1$, $q>2$, looks as follows. In this list d denotes the deficiency of the $(0,\alpha)$-set $\mathcal{K}$, and $\mathcal{S}$ denotes the corresponding $(0,\alpha)$-geometry fully embedded in $\mathrm{PG}(3,q)$.

1. $\alpha=q+1$, $d=0$, $\mathcal{K}$ is the set of all points of $\mathrm{Q}^+(5,q)$, and $\mathcal{S}$ is the design of all points and all lines of $\mathrm{PG}(3,q)$.

2. $\alpha=q$, $d=0$, $\mathcal{K}$ is the complement in $\mathrm{Q}^+(5,q)$ of a hyperplane which is not tangent to $\mathrm{Q}^+(5,q)$, and $\mathcal{S}=\overline{W(3,q)}$.

3. $\alpha=q$, $d=1$, $\mathcal{K}$ is the complement in $\mathrm{Q}^+(5,q)$ of a hyperplane which is tangent to $\mathrm{Q}^+(5,q)$, and $\mathcal{S}=H_q^3$.

4. $q=2^h$, $\alpha \in \{2,2^2,\ldots,2^{h-1}\}$, $d \in \{1,q+1\}$ and $\mathcal{K}=\mathcal{M}_d^\alpha(A)$.

5. $q=2^{2e+1}$, $\alpha=2$, $d=q\pm\sqrt{2q}+1$, and $\mathcal{K}=\mathcal{T}_{q\pm\sqrt{2q}+1}$.

7.2 Planar oval sets in PG(2, q), q even

Let O be an oval of $\mathrm{PG}(2,q)$, q even, with nucleus n. Let $\mathrm{El}(n)$ be the group of all elations of $\mathrm{PG}(2,q)$ with center n. In Section 3.1, we show that the set $\Omega(O)=\{O^e \mid e \in \mathrm{El}(n)\}$ is a regular Desarguesian planar oval set in $\mathrm{PG}(2,q)$ with nucleus n.

The main result of Chapter 3 is Theorem 3.4.1. It says that, if Ω is a regular Desarguesian planar oval set in $\mathrm{PG}(2,q)$, q even, and $O \in \Omega$, then $\Omega=\Omega(O)$.

The motivation for introducing planar oval sets is their use in the study of affine $(0,\alpha)$-geometries. Theorem 3.4.1 is crucial in the classification of $(0,2)$-geometries of order $(q-1,q)$, fully embedded in $\mathrm{AG}(3,q)$, $q=2^h$, such that there are no planar nets (see Section 5.3.3).

7.3 Affine semipartial geometries and $(0, \alpha)$-geometries

Chapters 4 to 6 are devoted to the study of affine semipartial geometries and $(0, \alpha)$-geometries, $\alpha > 1$. In Section 4.1, we explain why we study affine $(0, \alpha)$-geometries rather than affine semipartial geometries. As a consequence of Theorem 4.3.1, which is due to De Clerck and Delanote [27], and states that affine $(0, \alpha)$-geometries which have no planes of type IV are linear representations, we may restrict our attention to $(0, \alpha)$-geometries, $\alpha > 1$, fully embedded in $\mathrm{AG}(n, q)$, which have at least one plane of type IV. Hence $\alpha = 2$ and $q = 2^h$.

In Section 4.2, we construct two new examples of affine $(0, 2)$-geometries which have planes of type IV. Firstly, the geometry $\mathcal{A}(O_\infty)$ is a $(0, 2)$-geometry of order $(q - 1, q)$, fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$. The construction starts from an oval O_∞ in Π_∞. If the oval O_∞ is a conic, then $\mathcal{A}(O_\infty)$ is the geometry HT (see Section 1.4.7). Otherwise $\mathcal{A}(O_\infty)$ is a $(0, 2)$-geometry, fully embedded in $\mathrm{AG}(3, q)$, which was not known before. In both cases, there are no planar nets, and the geometry is not a semipartial geometry.

Secondly, the geometry $\mathcal{I}(n, q, e)$ is a $(0, 2)$-geometry of order $(q - 1, 2^{n-1} - 1)$, fully embedded in $\mathrm{AG}(n, q)$, $n \geq 3$, $q = 2^h$. This geometry has planar nets as well as planes of type IV. It is an example of an affine $(0, 2)$-geometry which was not known before. However, it is never a semipartial geometry.

In Chapter 5, we classify all $(0, 2)$-geometries fully embedded in $\mathrm{AG}(3, q)$, $q = 2^h$, $h > 1$, which have a plane of type IV (see Theorem 5.4.1). In Chapter 6, we classify all $(0, 2)$-geometries fully embedded in $\mathrm{AG}(n, q)$, $n \geq 4$, $q = 2^h$, $h > 1$, which have a plane of type IV (see Theorem 6.4.1). Together with Proposition 4.3.5, which deals with the trivial case $q = 2$, these results classify all affine $(0, 2)$-geometries which have at least one plane of type IV.

Theorem 7.3.1 *If $\mathcal{S}$ is a $(0, 2)$-geometry fully embedded in $\mathrm{AG}(n, q)$, $q = 2^h$, such that there is at least one plane of type* IV*, then one of the following cases holds.*

1. *$q = 2$ and $\mathcal{S}$ is a $2 - (t + 2, 2, 1)$-design.*
2. *$n = 2$ and $\mathcal{S}$ is a dual oval.*
3. *$n = 3$ and $\mathcal{S} \simeq \mathcal{A}(O_\infty)$.*
4. *$n = 4$ and $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.*

5. $n \geq 3$ *and* $\mathcal{S} \simeq \mathcal{I}(n, q, e)$.

Theorems 4.3.1 and 7.3.1 give an almost complete classification of affine $(0, \alpha)$-geometries, $\alpha > 1$.

Theorem 7.3.2 *If* $\mathcal{S}$ *is a* $(0, \alpha)$*-geometry,* $\alpha > 1$*, fully embedded in* $\mathrm{AG}(n, q)$*, then one of the following possibilities occurs.*

1. $q = 2$*,* $\alpha = 2$ *and* $\mathcal{S}$ *is a* $2 - (t + 2, 2, 1)$*-design.*

2. $n = 2$*,* $q = 2^h$*,* $\alpha = 2$ *and* $\mathcal{S}$ *is a dual oval.*

3. $n = 3$*,* $q = 2^h$*,* $\alpha = 2$ *and* $\mathcal{S} \simeq \mathcal{A}(O_\infty)$.

4. $n = 4$*,* $q = 2^h$*,* $\alpha = 2$ *and* $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.

5. $n \geq 3$*,* $q = 2^h$*,* $\alpha = 2$ *and* $\mathcal{S} \simeq \mathcal{I}(n, q, e)$.

6. $n \geq 2$ *and* $\mathcal{S} \simeq T^*_{n-1}(\mathcal{K}_\infty)$*, with* $\mathcal{K}_\infty$ *a set of type* $(0, 1, \alpha + 1)$ *in* Π_∞ *which spans* Π_∞.

Theorem 7.3.2 is probably the best result possible for affine $(0, \alpha)$-geometries, $\alpha > 1$, since a complete classification of sets of type $(0, 1, \alpha + 1)$ in $\mathrm{PG}(n, q)$ seems hopeless. For a partial classification of sets of type $(0, 1, \alpha+1)$ in $\mathrm{PG}(n, q)$, due to Ueberberg [83], see Theorem 4.3.2.

Theorem 7.3.1 yields the complete classification of affine semipartial geometries with $\alpha = 2$, which have at least one plane of type IV.

Theorem 7.3.3 *If* $\mathcal{S}$ *is a semipartial geometry with* $\alpha = 2$*, fully embedded in* $\mathrm{AG}(n, q)$*,* $q = 2^h$*, such that there is at least one plane of type* IV*, then one of the following possibilities occurs.*

1. $q = 2$ *and* $\mathcal{S}$ *is a* $2 - (t + 2, 2, 1)$*-design.*

2. $n = 2$ *and* $\mathcal{S}$ *is a dual oval.*

3. $n = 4$ *and* $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.

Theorems 4.3.1 and 7.3.3 yield the following result.

Theorem 7.3.4 *If* $\mathcal{S}$ *is a semipartial geometry,* $\alpha > 1$*, fully embedded in* $\mathrm{AG}(n, q)$*, then one of the following cases holds.*

1. $q = 2$*,* $\alpha = 2$ *and* $\mathcal{S}$ *is a* $2 - (t + 2, 2, 1)$*-design.*

2. $n = 2$*,* $q = 2^h$*,* $\alpha = 2$ *and* $\mathcal{S}$ *is a dual oval.*

3. $n = 4$, $q = 2^h$, $\alpha = 2$ *and* $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.

4. $n \geq 2$ *and* $\mathcal{S} \simeq T^*_{n-1}(\mathcal{K}_\infty)$, *with* $\mathcal{K}_\infty$ *a set of type* $(0, 1, \alpha + 1)$ *in* Π_∞, *which has two intersection numbers with respect to hyperplanes of* Π_∞.

Using Theorem 1.4.16, due to Debroey and Thas [40], and Theorem 4.3.3, due to De Winter [38], we obtain the classification of proper semipartial geometries with $\alpha > 1$, fully embedded in $\mathrm{AG}(n, q)$, $n \leq 4$.

Theorem 7.3.5 *If* $\mathcal{S}$ *is a proper semipartial geometry with* $\alpha > 1$, *fully embedded in* $\mathrm{AG}(n, q)$, $n \leq 4$, *then one of the following cases holds.*

1. $n = 3$, q *is a square and* $\mathcal{S} \simeq T^*_2(\mathcal{U}_\infty)$, *with* $\mathcal{U}_\infty$ *a unital of* Π_∞.

2. $n = 4$, $q = 2^h$ *and* $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.

3. $n \in \{3, 4\}$, q *is a square and* $\mathcal{S} \simeq T^*_{n-1}(\mathcal{B}_\infty)$, *with* $\mathcal{B}_\infty$ *a Baer subspace of* Π_∞.

The classification of linear representations of proper semipartial geometries, $\alpha > 1$, in $\mathrm{AG}(n, q)$, $n \geq 5$, is still an open problem. The only known example is $T^*_{n-1}(\mathcal{B}_\infty)$, with $\mathcal{B}_\infty$ the point set of a Baer subspace of Π_∞. Due to the recent progress regarding this problem, made by De Winter [38], we conjecture that $T^*_{n-1}(\mathcal{B}_\infty)$ is in fact the only example.

Appendix A

The geometry $\mathcal{I}(n, q, e)$

The geometry $\mathcal{I}(n,q,e)$ is a new example of a $(0,2)$-geometry fully embedded in $\mathrm{AG}(n,q)$, $n \geq 3$, $q = 2^h$. In Section 4.2.2, we gave the construction of $\mathcal{I}(n,q,e)$ and we proved that it is a $(0,2)$-geometry of order $(q-1, 2^{n-1}-1)$, fully embedded in $\mathrm{AG}(n,q)$. In Section 4.4.4, we deduced some properties of $\mathcal{I}(n,q,e)$. The focus was mainly on the incidence structure $\mathcal{S}_V$, where $\mathcal{S} = \mathcal{I}(n,q,e)$ and V is a subspace of $\mathrm{AG}(n,q)$. However, we think that, for a good understanding of the geometry $\mathcal{I}(n,q,e)$, further study is required.

In Section A.1, we study the point set of $\mathcal{I}(n,q,e)$. We give the explicit description of this set, and we determine the intersection with lines of $\mathrm{AG}(n,q)$. In Section A.2, we show that the geometry $\mathcal{I}(n,q,e)$ actually consists of two parts, and we determine their structure. In Section A.3, we use this information to obtain some isomorphisms of the geometry $\mathcal{I}(n,q,e)$.

A.1 The point set of $\mathcal{I}(n, q, e)$

Consider the geometry $\mathcal{S} = \mathcal{I}(n, 2^h, e)$ fully embedded in $\mathrm{AG}(n, 2^h)$, $n \geq 3$. We adopt here the notations and the coordinatization of $\mathrm{PG}(n, 2^h)$, used in Section 4.2.2. So $\mathcal{S} = (\mathcal{P}, \mathcal{B}_1 \cup \mathcal{B}_2, \mathrm{I})$. Let $\mathcal{P}' = \mathcal{P} \setminus U$ and let $\mathcal{R} = \mathcal{P} \cup \Pi_\infty$.

Since e and h are relatively prime, so are $2^e - 1$ and $2^h - 1$. Hence the map $\sigma' : z \mapsto z^{2^e-1}$ is a permutation of $\mathrm{GF}(2^h)$. Let σ denote the inverse of σ'.

Theorem A.1.1 *A point $p(x_0, \ldots, x_{n-1}, 1)$ of $\mathrm{AG}(n, 2^h)$, not in U, is in $\mathcal{P}'$ if and only if $\mathrm{Tr}(x_i x_{n-1}^\sigma) = 0$, for all $0 \leq i \leq n-2$.*

Proof. Let $p(x_0, \ldots, x_{n-1}, 1)$ be a point of $\mathrm{AG}(n, 2^h)$, not in U. Then $x_{n-1} \neq 0$. The point p is in $\mathcal{P}'$ if and only if it is on a line of $\mathcal{B}_2$, if and only if there is an affine point $r(y_0, \ldots, y_{n-2}, 0, 1) \in U$, such that the points p, r

and $r^{\varphi}(y_0^{2^e},\dots,y_{n-2}^{2^e},1,0)$ are collinear. Hence, $p\in\mathcal{P}'$ if and only if for every $0\le i\le n-2$, there is an $y_i\in \mathrm{GF}(2^h)$ such that $x_i+y_i=x_{n-1}y_i^{2^e}$. It follows that $p\in\mathcal{P}'$ if and only if, for every $0\le i\le n-2$, there is a solution to the equation $F_i(X)=X^{2^e}+X+x_ix_{n-1}^{\sigma}=0$. Since e and h are relatively prime, $F_i(X)=0$ has a solution if and only if $\mathrm{Tr}(x_ix_{n-1}^{\sigma})=0$ (see Liang [58] or Segre [70]). $\square$

The additive group of $\mathrm{GF}(2^h)$ is an elementary abelian group of order 2^h, so it can be seen as a vector space $V(h,2)$ of dimension h over the field $\mathrm{GF}(2)$. Let S be the set of additive subgroups of index 2 in $\mathrm{GF}(2^h)$, and let $\overline{S}$ be the set of complements in $\mathrm{GF}(2^h)$ of the elements of S. Then clearly $|\,S\,|=|\,\overline{S}\,|=2^h-1$. The elements of S are the hyperplanes of $V(h,2)$, and the elements of $\overline{S}$ the complements of these hyperplanes; in other words, the elements of $S\cup\overline{S}$ are the hyperplanes of the affine space $\mathrm{AG}(h,2)$ corresponding to $V(h,2)$, where the elements of S are those containing the element 0.

Let $\mathcal{C}_0=\{z\in\mathrm{GF}(2^h)\,|\,\mathrm{Tr}(z)=0\}$ and let $\mathcal{C}_1=\{z\in\mathrm{GF}(2^h)\,|\,\mathrm{Tr}(z)=1\}$. Then $\mathcal{C}_0\in S$ and $\mathcal{C}_1\in\overline{S}$. For every $z\in\mathrm{GF}(2^h)\setminus\{0\}$, $z\,\mathcal{C}_0\in S$. For every $z,z'\in\mathrm{GF}(2^h)\setminus\{0\}$, $z\,\mathcal{C}_0=z'\,\mathcal{C}_0$ if and only if $z=z'$ (see Theorem 2.24 of Lidl and Niederreiter [59]). Hence $S=\{z\,\mathcal{C}_0\,|\,z\in\mathrm{GF}(2^h)\setminus\{0\}\}$ and $\overline{S}=\{z\,\mathcal{C}_1\,|\,z\in\mathrm{GF}(2^h)\setminus\{0\}\}$.

Lemma A.1.2 *Let $2\le m\le h$ and let $z_1,\dots,z_m\in\mathrm{GF}(2^h)\setminus\{0\}$. Then*

$$|\bigcap_{i=1}^{m} z_i\,\mathcal{C}_0\,|\ \in\{2^{h-m},2^{h-m+1},\dots,2^{h-1}\},$$

and each of these possibilities occurs for some choice of $z_1,\dots,z_m$.

Let $2\le m\le h$ and, for $1\le i\le m$, let $z_i\in\mathrm{GF}(2^h)\setminus\{0\}$ and $\mathcal{C}^i\in\{\mathcal{C}_0,\mathcal{C}_1\}$. Then

$$|\bigcap_{i=1}^{m} z_i\,\mathcal{C}^i\,|\ \in\{0,2^{h-m},2^{h-m+1},\dots,2^{h-1}\},$$

and each of these possibilities occurs for some choice of $z_i,\mathcal{C}^i$, $1\le i\le m$.

Proof. For every $1\le i\le m$, $z_i\,\mathcal{C}_0$ is a hyperplane of the affine space $\mathrm{AG}(h,2)$ which contains the element 0. Hence $\bigcap_{i=1}^m z_i\,\mathcal{C}_0$ is a subspace of $\mathrm{AG}(h,2)$ of dimension at least $h-m$ which contains the element 0. It follows that

$$|\bigcap_{i=1}^{m} z_i\,\mathcal{C}_0\,|\ \in\{2^{h-m},2^{h-m+1},\dots,2^{h-1}\}.$$

Let $1 \le m' \le m$. Since $m' \le h$, and since $S = \{z\,\mathcal{C}_0 \mid z \in \mathrm{GF}(q) \setminus \{0\}\}$, the elements $z_1, \ldots, z_m$ may be chosen such that $z_1\,\mathcal{C}_0, \ldots, z_{m'}\,\mathcal{C}_0$ are linearly independent and $z_{m'} = \ldots = z_m$. Then $\bigcap_{i=1}^m z_i\,\mathcal{C}_0$ is a subspace of $\mathrm{AG}(h,2)$ of dimension $h - m'$, so $|\bigcap_{i=1}^m z_i\,\mathcal{C}_0| = 2^{h-m'}$. This proves the first part of the lemma.

We now prove the second part. For every $1 \le i \le m$, $z_i\,\mathcal{C}^i$ is a hyperplane of the affine space $\mathrm{AG}(h,2)$. Hence either $\bigcap_{i=1}^m z_i\,\mathcal{C}^i$ is empty, or it is a subspace of $\mathrm{AG}(h,2)$ of dimension at least $h-m$. So

$$|\bigcap_{i=1}^m z_i\,\mathcal{C}^i| \in \{0, 2^{h-m}, 2^{h-m+1}, \ldots, 2^{h-1}\}.$$

The first part of the lemma shows that each of the possibilities 2^{h-m}, $2^{h-m+1}, \ldots, 2^{h-1}$ does occur. Choose $z_i, \mathcal{C}^i$, $1 \le i \le m$ such that $z_1 = \ldots = z_m = 1$, $\mathcal{C}^1 = \mathcal{C}_0$ and $\mathcal{C}^2 = \ldots = \mathcal{C}^m = \mathcal{C}_1$. Then $\bigcap_{i=1}^m z_i\,\mathcal{C}^i = \mathcal{C}_0 \cap \mathcal{C}_1 = \emptyset$. This proves the second part of the lemma. □

Theorem A.1.3 *Consider the geometry $\mathcal{I}(n,2^h,e)$ fully embedded in $\mathrm{AG}(n,2^h)$, $n \ge 3$. Let L be a line of $\mathrm{AG}(n,2^h)$ which is parallel to but not contained in U. If $h \ge n-1$, then*

$$|L \cap \mathcal{P}| \in \{0, 2^{h-n+1}, 2^{h-n+2}, \ldots, 2^{h-1}\}.$$

If $h \le n-1$, then

$$|L \cap \mathcal{P}| \in \{0, 1, 2, \ldots, 2^{h-1}\}.$$

In either case, each of the possibilities occurs for some line L.

Proof. Let p be an affine point of L and let $p_\infty = L \cap \Pi_\infty$. Since $p \notin U$, it has coordinates $p(x_0, \ldots, x_{n-1}, 1)$, with $x_{n-1} \ne 0$. Since L is parallel to U, the point r is in U_∞ and has coordinates $r(y_0, \ldots, y_{n-2}, 0, 0)$. The affine points of L are the points $p_\rho(x_0 + \rho y_0, \ldots, x_{n-2} + \rho y_{n-2}, x_{n-1}, 1)$, $\rho \in \mathrm{GF}(2^h)$.

Since L does not contain any affine points of U, a point $p_\rho \in L$ is in $\mathcal{P}$ if and only if it is in $\mathcal{P}'$. By Theorem A.1.1, a point $p_\rho \in L$ is in $\mathcal{P}'$ if and only if $\mathrm{Tr}((x_i + \rho y_i)x_{n-1}^\sigma) = 0$ for all $0 \le i \le n-2$. For all $0 \le i \le n-2$, let

$$\mathcal{D}_i = \{\rho \in \mathrm{GF}(2^h) \mid \mathrm{Tr}((x_i + \rho y_i)x_{n-1}^\sigma) = 0\}.$$

Then $|L \cap \mathcal{P}| = |\bigcap_{i=0}^{n-2} \mathcal{D}_i|$. Let $z_i = y_i x_{n-1}^\sigma$ and let $\varepsilon_i = \mathrm{Tr}(x_i x_{n-1}^\sigma)$. If $z_i = 0$ and $\varepsilon_i = 0$, then $\mathcal{D}_i = \mathrm{GF}(2^h)$. If $z_i = 0$ and $\varepsilon_i = 1$, then $\mathcal{D}_i = \emptyset$. If $z_i \ne 0$, then $\mathcal{D}_i = z_i^{-1}\mathcal{C}_{\varepsilon_i}$.

Since $x_{n-1} \neq 0$ and not all y_i are zero, not all z_i are zero. Hence $|\bigcap_{i=0}^{n-2} \mathcal{D}_i| \leq 2^{h-1}$. Let $m = \min(h, n-1)$. By Lemma A.1.2,

$$|L \cap \mathcal{P}| = |\bigcap_{i=0}^{n-2} \mathcal{D}_i| \in \{0, 2^{h-m}, 2^{h-m+1}, \quad \ldots, 2^{h-1}\}.$$

Each of these possibilities does occur since the elements $z_0, \ldots, z_{n-2}$ can take any value in $\mathrm{GF}(2^h) \setminus \{0\}$, depending on the choice of the line L. $\square$

Theorem A.1.4 *Consider the geometry $\mathcal{I}(n, 2^h, e)$ fully embedded in $\mathrm{AG}(n, 2^h)$, $n \geq 3$. Let L be a line of $\mathrm{AG}(n, 2^h)$ which intersects U in an affine point. If $h \geq n-1$, then*

$$|L \cap \mathcal{P}| \in \{2^{h-n+1}, 2^{h-n+2}, \ldots, 2^h\}.$$

If $h \leq n-1$, then

$$|L \cap \mathcal{P}| \in \{1, 2, \ldots, 2^h\}.$$

In either case, each of the possibilities does occur, and $|L \cap \mathcal{P}| = 2^h$ if and only if L is a line of $\mathcal{I}(n, 2^h, e)$.

Proof. Let $p = L \cap U$ and let $p_\infty = L \cap \Pi_\infty$. Then the points p and p_∞ have coordinates $p(x_0, \ldots, x_{n-2}, 0, 1)$ and $p_\infty(y_0, \ldots, y_{n-2}, 1, 0)$. The affine points of L are the points $p_\rho(x_0 + \rho y_0, \ldots, x_{n-2} + \rho y_{n-2}, \rho, 1)$, $\rho \in \mathrm{GF}(2^h)$.

By Theorem A.1.1, a point $p_\rho \in L \setminus \{p\}$ is in $\mathcal{P}'$ if and only if $\mathrm{Tr}((x_i + \rho y_i)\rho^\sigma) = 0$ for all $0 \leq i \leq n-2$. The point $p = p_0$ is in $\mathcal{P}$ and $\mathrm{Tr}((x_i + \rho y_i)\rho^\sigma) = 0$ holds for $\rho = 0$. Hence a point $p_\rho \in L$ is in $\mathcal{P}$ if and only if $\mathrm{Tr}((x_i + \rho y_i)\rho^\sigma) = 0$. For all $0 \leq i \leq n-2$,

$$\begin{aligned}
\mathrm{Tr}((x_i + \rho y_i)\rho^\sigma) &= \mathrm{Tr}(x_i \rho^\sigma) + \mathrm{Tr}(y_i \rho^{1+\sigma}) \\
&= \mathrm{Tr}(x_i^{2^e} (\rho^\sigma)^{2^e - 1 + 1}) + \mathrm{Tr}(y_i \rho^{1+\sigma}) \\
&= \mathrm{Tr}((x_i^{2^e} + y_i)\rho^{1+\sigma}).
\end{aligned}$$

Since for every $z \in \mathrm{GF}(2^h)$, $z^{1+\sigma} = (z^\sigma)^{2^e}$, the map $\eta : z \mapsto z^{1+\sigma}$ is a permutation of $\mathrm{GF}(2^h)$. For every $0 \leq i \leq n-2$, let

$$\mathcal{D}_i = \{\rho \in \mathrm{GF}(2^h) \,|\, \mathrm{Tr}((x_i^{2^e} + y_i)\rho^\eta) = 0\}$$

and let

$$\mathcal{D}'_i = \{\rho \in \mathrm{GF}(2^h) \,|\, \mathrm{Tr}((x_i^{2^e} + y_i)\rho = 0\}.$$

Then $|\,L\cap\mathcal{P}\,| = |\,\bigcap_{i=0}^{n-2}\mathcal{D}_i\,|$. Since η is a permutation of $\mathrm{GF}(2^h)$, $|\,\mathcal{D}_i\,| = |\,\mathcal{D}'_i\,|$ for all $0\le i\le n-2$. Hence

$$|\,L\cap\mathcal{P}\,| = |\,\bigcap_{i=0}^{n-2}\mathcal{D}'_i\,|.$$

Let $z_i = x_i^{2^e}+y_i$ for $0\le i\le n-2$. If $z_i=0$, then $\mathcal{D}'_i=\mathrm{GF}(2^h)$. If $z_i\neq 0$, then $\mathcal{D}'_i = z_i^{-1}\mathcal{C}_0$. So $|\,L\cap\mathcal{P}\,| = 2^h$ if and only if $z_i=0$ for all $0\le i\le n-2$, if and only if $L\in\mathcal{B}_2$. Let $m=\min(h,n-1)$. If not all z_i are zero, then by Lemma A.1.2,

$$|\,L\cap\mathcal{P}\,| = |\,\bigcap_{i=0}^{n-2}\mathcal{D}'_i\,| \in \{2^{h-m},2^{h-m+1},\ldots,2^{h-1}\}.$$

Each of these possibilities does occur since the elements $z_0,\ldots,z_{n-2}$ can take any value in $\mathrm{GF}(2^h)\setminus\{0\}$, depending on the choice of the line L. □

Theorem A.1.5 *Consider the geometry $\mathcal{I}(n,2^h,e)$ fully embedded in* $\mathrm{AG}(n,2^h)$*, $n\ge 3$. If $h\ge n-1$, then the set $\mathcal{R}=\mathcal{P}\cup\Pi_\infty$ is a set of type*

$$(1,2^{h-n+1}+1,2^{h-n+2}+1,\ldots,2^h+1)$$

in $\mathrm{PG}(n,2^h)$*. If $h\le n-1$, then $\mathcal{R}$ is a set of type*

$$(1,2,3,5,9,\ldots,2^h+1)$$

in $\mathrm{PG}(n,2^h)$*. A line L of* $\mathrm{PG}(n,2^h)$ *is contained in $\mathcal{R}$ if and only if $L\subseteq U$, $L\subseteq\Pi_\infty$ or $L\in\mathcal{B}_2$.*

Proof. This follows immediately from Theorems A.1.3 and A.1.4. □

A.2 The geometry $\mathcal{I}'(n,q,e)$

The line set of the geometry $\mathcal{S}=\mathcal{I}(n,2^h,e)$ is defined as the union of two sets $\mathcal{B}_1$ and $\mathcal{B}_2$. So it is natural to consider the sub incidence structures of $\mathcal{S}$, induced on the line sets $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\mathcal{P}_1$ be the set of affine points of the hyperplane U, and let $\mathcal{P}_2$ be the complement of $\mathcal{P}_1$ in the point set $\mathcal{P}$ of $\mathcal{I}(n,2^h,e)$. Let $\mathcal{S}_i=(\mathcal{P}_i,\mathcal{B}_i,\mathrm{I}_i)$, where I_i is the natural incidence, $i=1,2$. Note that $\mathcal{P}_1$ is the set of all affine points on the lines of $\mathcal{B}_1$, but $\mathcal{P}_2$ is not the set of all affine points on the lines of $\mathcal{B}_2$. Indeed, if L is a line of $\mathcal{B}_2$, then L intersects U in an affine point which is the only affine point of L not in $\mathcal{P}_2$. Hence the geometry $\mathcal{S}_2$, which is also denoted by $\mathcal{I}'(n,2^h,e)$, is laxly embedded in $\mathrm{AG}(n,2^h)$.

Recall that $T^*_{n-1}(\mathcal{K}_\infty)$ denotes the linear representation in $\mathrm{AG}(n,q)$ of a set of points $\mathcal{K}_\infty$ in the hyperplane Π_∞ at infinity of $\mathrm{AG}(n,q)$. In this section however, we use the notation $T^*_{n-1,q}(\mathcal{K}_\infty)$. If $\mathcal{K}_\infty$ is the point set of a projective space $\mathrm{PG}(n-1,q')$, $q' \mid q$, then we also write $T^*_{n-1,q}(\mathrm{PG}(n-1,q'))$.

Proposition A.2.1 $\mathcal{S}_1 = T^*_{n-2,2^h}(\mathrm{PG}(n-2,2))$.

Proof. This follows immediately from the construction of $\mathcal{I}(n,2^h,e)$. $\square$

Consider an affine space $\mathrm{AG}(n,q^m)$ and the point set $\mathcal{K}_\infty$ of a projective space $\mathrm{PG}(n-1,q)$ in Π_∞. We assume here and in the rest of this chapter that the set $\mathcal{K}_\infty$ spans Π_∞, i. e., the points of the projective space $\mathrm{PG}(n-1,q)$ are all the points of Π_∞ whose coordinates are in the subfield $\mathrm{GF}(q)$ of $\mathrm{GF}(q^m)$, with respect to a proper basis of $\mathrm{PG}(n,q^m)$. We define an incidence structure $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1,q))$ as follows. The points of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1,q))$ are the points of $\mathrm{AG}(n,q^m)$, the lines are the sets $\mathcal{K}$ of affine points such that $\mathcal{K}\cup\mathcal{K}_\infty$ is the point set of a projective space $\mathrm{PG}(n,q)$ in $\mathrm{PG}(n,q^m)$, and incidence is defined by inclusion.

Clearly if p and p' are distinct affine points such that the line $\langle p,p'\rangle$ intersects Π_∞ in a point $p_\infty \notin \mathcal{K}_\infty$, then p and p' are not collinear in $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1,q))$. Assume that $p_\infty \in \mathcal{K}_\infty$. Choose a basis in $\mathrm{PG}(n,q^m)$ such that $\Pi_\infty : X_0 = 0$,

$$\mathcal{K}_\infty = \{(0,a_1,\ldots,a_n) \neq (0,\ldots,0) \mid a_i \in \mathrm{GF}(q), 1 \leq i \leq n\},$$

$p_\infty(0,1,0,\ldots,0)$, $p(1,0,\ldots,0)$ and $p'(1,1,0,\ldots,0)$. Then

$$\mathcal{K} = \{(1,a_1,\ldots,a_n) \mid a_i \in \mathrm{GF}(q), 1 \leq i \leq n\}$$

is a line of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1,q))$ containing p and p'. Moreover, $\mathcal{K}$ is the only line of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1,q))$ containing p and p'. Indeed, suppose that $\mathcal{K}'$ is a line containing p and p'. Then $\mathcal{K}'\cup\mathcal{K}_\infty$ is the point set of a projective space $\mathrm{PG}(n,q)$ in $\mathrm{PG}(n,q^m)$. Hence for every two distinct points r_∞, r'_∞ in $\mathcal{K}_\infty \setminus \{p_\infty\}$ such that $p_\infty, r_\infty, r'_\infty$ are collinear, the point $p'' = \langle p, r_\infty\rangle \cap \langle p', r'_\infty\rangle$ is a point of $\mathcal{K}'$. But p'' is also in $\mathcal{K}$. It follows that $\mathcal{K}' = \mathcal{K}$.

We conclude that $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1,q))$ is a partial linear space, and that two distinct affine points are collinear if and only if the line which joins them intersects Π_∞ in a point of $\mathcal{K}_\infty$.

Lemma A.2.2 *The geometry $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ is a partial linear space of order $(q^n - 1, (q^m - 1)/(q - 1) - 1)$ which has q^{mn} points and $q^{(m-1)n}(q^m - 1)/(q - 1)$ lines.*

Proof. Let $\mathcal{K}$ be a line of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$. Then $\mathcal{K} \cup \mathcal{K}_\infty$ is the point set of a projective space $\mathrm{PG}(n, q)$ in $\mathrm{PG}(n, q^m)$. So $\mathcal{K}$ is the point set of an affine space $\mathrm{AG}(n, q)$ in $\mathrm{AG}(n, q^m)$. Hence $\mathcal{K}$ contains q^n points.

Choose a point p of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$, and let $p_\infty \in \mathcal{K}_\infty$. We count the pairs $(p', \mathcal{K})$ such that $\mathcal{K}$ is a line of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ through p, and p' is a point of $\mathcal{K}$ on the line $L = \langle p, p_\infty\rangle$, distinct from p. As we have shown, for every affine point p' on L, distinct from p, there is exactly one line $\mathcal{K}$ of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ which contains p and p'. Let $\mathcal{K}$ be a line of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ containing p. Since the line L contains the points $p, p_\infty \in \mathcal{K} \cup \mathcal{K}_\infty$, it contains $q + 1$ points of $\mathcal{K} \cup \mathcal{K}_\infty$. Hence there are $q - 1$ points p' of $\mathcal{K}$ on L, distinct from p. It follows that the number of lines of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ through p is equal to $(q^m - 1)/(q - 1)$. So $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ has order $(q^n - 1, (q^m - 1)/(q - 1) - 1)$.

Clearly $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ has q^{mn} points. Counting the number of flags of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ yields the number of lines. □

Theorem A.2.3 *$\mathcal{I}'(n, 2^h, e) \cong T^\circ_{n-2,2^h}(\mathrm{PG}(n-2, 2))^D$ for all $n \geq 3$.*

Proof. Consider the geometry $\mathcal{I}(n, 2^h, e)$ and the set $\mathcal{K}_\infty$, which is the point set of a projective space $\mathrm{PG}(n-2, 2)$ in U_∞. Let $T^\circ_{n-2,2^h}(\mathrm{PG}(n-2, 2))$ be the geometry defined in U by $\mathcal{K}_\infty$.

We define a map χ on the set of points and lines of $\mathcal{I}'(n, 2^h, e)$. For every line $L \in \mathcal{B}_2$, let L^χ be the affine point $L \cap U$. Then L^χ is a point of $T^\circ_{n-2,2^h}(\mathrm{PG}(n-2, 2))$. For every point $p \in \mathcal{P}_2$, let p^χ be the set $p^\perp \cap U$, where $p^\perp$ denotes the set of points of $\mathcal{I}(n, 2^h, e)$ which are collinear to p. By Lemma 4.4.17, $p^\chi \cup \mathcal{K}_\infty$ is the point set of a projective space $\mathrm{PG}(n-1, 2)$ in U which contains $\mathcal{K}_\infty$. So p^χ is a line of $T^\circ_{n-2,2^h}(\mathrm{PG}(n-2, 2))$. Note that a point p and a line L of $\mathcal{I}'(n, 2^h, e)$ are incident if and only if p^χ and L^χ are incident in $T^\circ_{n-2,2^h}(\mathrm{PG}(n-2, 2))$.

We prove that χ is injective. By Proposition 4.4.11, no two lines of $\mathcal{I}'(n, 2^h, e)$ intersect U in the same point. Suppose that p and p' are points of $\mathcal{I}'(n, 2^h, e)$ such that $p^\chi = p'^\chi$. Let r, r' be distinct points of $p^\chi = p'^\chi$, and let L, respectively L', be the unique line of $\mathcal{B}_2$ through r, respectively r'. Then p and p' are both on the lines L and L'. Hence L and L' intersect, and $p = p'$ is the point of intersection. So χ is injective.

By Proposition 4.4.12 and Lemma A.2.2, the number of points of $\mathcal{I}'(n, 2^h, e)$ equals the number of lines of $T^{\circ}_{n-2,2^h}(\mathrm{PG}(n-2, 2))$, and the number of lines of $\mathcal{I}'(n, 2^h, e)$ equals the number of points of $T^{\circ}_{n-2,2^h}(\mathrm{PG}(n-2, 2))$. It follows that χ is a bijection from $\mathcal{P}_2 \cup \mathcal{B}_2$ to the set of points and lines of $T^{\circ}_{n-2,2^h}(\mathrm{PG}(n-2, 2))$. Since χ preserves incidence, χ is an isomorphism from $\mathcal{I}'(n, 2^h, e)$ to the dual of $T^{\circ}_{n-2,2^h}(\mathrm{PG}(n-2, 2))$. $\square$

Theorem A.2.4 *Let $n \geq 3$ and $e_1, e_2 \in \{1, \dots h-1\}$ such that $e_1 \neq e_2$ and $\gcd(e_1, h) = \gcd(e_2, h) = 1$. Then $\mathcal{I}'(n, 2^h, e_1) \cong \mathcal{I}'(n, 2^h, e_2)$, but $\mathcal{I}'(n, 2^h, e_1) \not\simeq \mathcal{I}'(n, 2^h, e_2)$.*

Proof. By Theorem A.2.3, $\mathcal{I}'(n, 2^h, e_1) \cong \mathcal{I}'(n, 2^h, e_2)$.

We may assume that $\mathcal{I}'(n, 2^h, e_1)$ and $\mathcal{I}'(n, 2^h, e_2)$ are fully embedded in the same affine space $\mathrm{AG}(n, 2^h)$, and have the same U and $\mathcal{K}_\infty$.

It follows from the proof of Theorem 4.2.3 that any plane π_i of $\mathrm{AG}(n, 2^h)$ which contains two intersecting lines of $\mathcal{I}'(n, 2^h, e_i)$, $i = 1, 2$, is of type IV with respect to $\mathcal{I}(n, 2^h, e_i)$, and intersects the hyperplane U in an affine line. Moreover, the number of lines of $\mathcal{I}'(n, 2^h, e_i)$ in π_i is 2^h, and the points in the dual plane π_i^D of π_i which correspond to these lines, have coordinates $(1, x^{2^{e_i}}, x)$, $x \in \mathrm{GF}(2^h)$, with respect to a proper basis of π_i^D. With respect to the same basis, the point of π_i^D which corresponds to the line $\pi_i \cap U$, respectively the line $\pi_i \cap \Pi_\infty$, has coordinates $(0, 1, 0)$, respectively $(0, 0, 1)$.

Suppose that $\mathcal{I}'(n, 2^h, e_1) \simeq \mathcal{I}'(n, 2^h, e_2)$. Then there is a collineation ζ of $\mathrm{AG}(n, 2^h)$ which maps the lines of $\mathcal{I}'(n, 2^h, e_1)$ onto the lines of $\mathcal{I}'(n, 2^h, e_2)$. Let π_1 be a plane which contains two intersecting lines of $\mathcal{I}'(n, 2^h, e_1)$. Then the plane $\pi_2 = \pi_1^\zeta$ contains two intersecting lines of $\mathcal{I}'(n, 2^h, e_2)$. Let S_i be the set of lines of $\mathcal{I}'(n, 2^h, e_i)$ in π_i, let $L_i = \pi_i \cap U$ and let $L^i_\infty = \pi_i \cap \Pi_\infty$, $i = 1, 2$. Then $S_1^\zeta = S_2$ since $\pi_1^\zeta = \pi_2$. Since the hyperplanes U and Π_∞ are fixed by ζ, $L_1^\zeta = L_2$ and $(L^1_\infty)^\zeta = L^2_\infty$. Now the preceding paragraph implies that there is a collineation ζ' of $\mathrm{PG}(2, 2^h)$ which fixes the points $(0, 1, 0)$ and $(0, 0, 1)$ and maps the set of points $S_1' = \{p(1, x^{2^{e_1}}, x) \mid x \in \mathrm{GF}(2^h)\}$ to the set of points $S_2' = \{p(1, x^{2^{e_2}}, x) \mid x \in \mathrm{GF}(2^h)\}$. Let A and $\theta : x \mapsto x^{2^f}$ be the matrix and field automorphism representing ζ'. Since the points $(0, 1, 0)$ and $(0, 0, 1)$ are fixed,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & b & 0 \\ c & 0 & d \end{pmatrix}.$$

Since ${S_1'}^{\zeta'} = S_2'$,

$$F(x) = bx^{2^{e_1+f}} + d^{2^{e_2}} x^{2^{e_2+f}} + c^{2^{e_2}} + a = 0, \qquad \forall x \in \mathrm{GF}(2^h).$$

Hence $F(X)$ is the zero polynomial. Since A is nonsingular, b and d are nonzero. Hence $e_1 = e_2$, a contradiction. So $\mathcal{I}'(n,2^h,e_1) \not\cong \mathcal{I}'(n,2^h,e_2)$. □

The next theorem is not new, special cases of it can be found on various places in the literature. It can be proven using Segre varieties (see for example [54], Chapter 25). However, for the sake of completeness and clarity, we give here a complete proof using coordinates.

Theorem A.2.5 *$T^{\circ}_{n-1,q^m}(\mathrm{PG}(n-1,q)) \cong T^{*}_{m-1,q^n}(\mathrm{PG}(m-1,q))$ for all $m,n \geq 2$.*

Proof. Let Π'_∞ be the hyperplane at infinity of $\mathrm{AG}(n,q^m)$ and let $\mathcal{K}'_\infty$ be the point set of the projective space $\mathrm{PG}(n-1,q)$ in Π'_∞. Choose a basis in $\mathrm{PG}(n,q^m)$ such that $\Pi'_\infty : X_0 = 0$ and

$$\mathcal{K}'_\infty = \{(0,a_1,\ldots,a_n) \neq (0,\ldots,0) \,|\, a_i \in \mathrm{GF}(q), 1 \leq i \leq n\}.$$

Let Π_∞ be the hyperplane at infinity of $\mathrm{AG}(m,q^n)$ and let $\mathcal{K}_\infty$ be the point set of the projective space $\mathrm{PG}(m-1,q)$ in Π_∞. Choose a basis in $\mathrm{PG}(m,q^n)$ such that $\Pi_\infty : X_0 = 0$ and

$$\mathcal{K}_\infty = \{(0,a_1,\ldots,a_m) \neq (0,\ldots,0) \,|\, a_i \in \mathrm{GF}(q), 1 \leq i \leq m\}.$$

The field $\mathrm{GF}(q^m)$, respectively $\mathrm{GF}(q^n)$, is an m-dimensional, respectively n-dimensional, vector space over $\mathrm{GF}(q)$, and choose a basis in this vector space containing the element 1. With respect to this basis, an element z of $\mathrm{GF}(q^m)$, respectively of $\mathrm{GF}(q^n)$, has coordinates $(a_1,\ldots,a_m)$, respectively $(a_1,\ldots,a_n)$, while an element a of $\mathrm{GF}(q)$ has coordinates $(a,0,\ldots,0)$.

Now we can express the coordinates of a point p' of $\mathrm{PG}(n,q^m)$, respectively a point p of $\mathrm{PG}(m,q^n)$, with respect to the chosen basis in $\mathrm{GF}(q^m)$, respectively $\mathrm{GF}(q^n)$. For example, if p', respectively p, is a point of $\mathrm{AG}(n,q^m)$, respectively $\mathrm{AG}(m,q^n)$, then

$$p' \begin{bmatrix} 1 \\ (a'_{11},\ldots,a'_{1m}) \\ (a'_{21},\ldots,a'_{2m}) \\ \vdots \\ (a'_{n1},\ldots,a'_{nm}) \end{bmatrix} \qquad \text{and} \qquad p \begin{bmatrix} 1 \\ (a_{11},\ldots,a_{1n}) \\ (a_{21},\ldots,a_{2n}) \\ \vdots \\ (a_{m1},\ldots,a_{mn}) \end{bmatrix},$$

where $a'_{ij}, a_{ji} \in \mathrm{GF}(q)$, $1 \leq i \leq n$, $1 \leq j \leq m$.

We define a bijection ψ from the point set of $\mathrm{AG}(m,q^n)$ to the point set of $\mathrm{AG}(n,q^m)$ as follows.

$$\psi : \qquad p\begin{bmatrix} 1 \\ (a_{11},\ldots,a_{1n}) \\ (a_{21},\ldots,a_{2n}) \\ \vdots \\ (a_{m1},\ldots,a_{mn}) \end{bmatrix} \qquad \mapsto \qquad p^{\psi}\begin{bmatrix} 1 \\ (a_{11},\ldots,a_{m1}) \\ (a_{12},\ldots,a_{m2}) \\ \vdots \\ (a_{1n},\ldots,a_{mn}) \end{bmatrix}.$$

We prove that ψ induces an isomorphism from $T^*_{m-1,q^n}(\mathrm{PG}(m-1,q))$ to $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1,q))$.

Clearly ψ is a bijection from the point set of $T^*_{m-1,q^n}(\mathrm{PG}(m-1,q))$ to the point set of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1,q))$. Let L be an arbitrary line of $T^*_{m-1,q^n}(\mathrm{PG}(m-1,q))$, let $p \in L$ and let $p_\infty = L \cap \Pi_\infty$. Let the coordinates of p and p_∞ be $p(1,x_1,\ldots,x_m)$ and $p_\infty(0,b_1,\ldots,b_m)$, or, with respect to the chosen basis in $\mathrm{GF}(q^n)$,

$$p\begin{bmatrix} 1 \\ (a_{11},\ldots,a_{1n}) \\ (a_{21},\ldots,a_{2n}) \\ \vdots \\ (a_{m1},\ldots,a_{mn}) \end{bmatrix} \qquad \text{and} \qquad p_\infty\begin{bmatrix} 0 \\ (b_1,0,\ldots,0) \\ (b_2,0,\ldots,0) \\ \vdots \\ (b_m,0,\ldots,0) \end{bmatrix}.$$

A general affine point $r \in L$ has coordinates $(1, x_1+\rho b_1,\ldots,x_m+\rho b_m)$, $\rho \in \mathrm{GF}(q^n)$. If ρ has coordinates $(\rho_1,\ldots,\rho_n)$ with respect to the chosen basis in $\mathrm{GF}(q^n)$, then we may write the coordinates of r as follows.

$$r\begin{bmatrix} 1 \\ (a_{11}+\rho_1 b_1,\ldots,a_{1n}+\rho_n b_1) \\ (a_{21}+\rho_1 b_2,\ldots,a_{2n}+\rho_n b_2) \\ \vdots \\ (a_{m1}+\rho_1 b_m,\ldots,a_{mn}+\rho_n b_m) \end{bmatrix}.$$

Let $\mathcal{K}'$ be the image of the set of affine points of L under ψ. Then a general point $r' \in \mathcal{K}'$ has coordinates

$$r'\begin{bmatrix} 1 \\ (a_{11}+\rho_1 b_1,\ldots,a_{m1}+\rho_1 b_m) \\ (a_{12}+\rho_2 b_1,\ldots,a_{m2}+\rho_2 b_m) \\ \vdots \\ (a_{1n}+\rho_n b_1,\ldots,a_{mn}+\rho_n b_m) \end{bmatrix},$$

with $\rho_1, \ldots, \rho_n \in \mathrm{GF}(q)$. For $1 \leq i \leq n$, let x'_i be the element of $\mathrm{GF}(q^m)$ which has coordinates $(a_{1i}, \ldots, a_{mi})$. Let y' be the element of $\mathrm{GF}(q^m)$ which has coordinates $(b_1, \ldots, b_m)$. Then

$$\mathcal{K}' = \{r'(1, x'_1 + \rho_1 y', \ldots, x'_n + \rho_n y') \mid \rho_1, \ldots, \rho_n \in \mathrm{GF}(q)\}.$$

Let $\mathcal{K}'_0 = \{r'(1, \rho_1, \ldots, \rho_n) \mid \rho_1, \ldots, \rho_n \in \mathrm{GF}(q)\}$. Then $\mathcal{K}'_0$ is a line of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$. The collineation of $\mathrm{PG}(n, q^m)$ with matrix

$$A = \begin{pmatrix} 1 & & & \\ x'_1 & y' & & \\ \vdots & & \ddots & \\ x'_n & & & y' \end{pmatrix}$$

fixes every point of Π'_∞ and maps $\mathcal{K}'_0$ to $\mathcal{K}'$. Hence the set $\mathcal{K}'$ is a line of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$.

We conclude that ψ maps every line of $T^*_{m-1,q^n}(\mathrm{PG}(m-1, q))$ onto a line of $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$. Since ψ is a bijection between the point sets of $T^*_{m-1,q^n}(\mathrm{PG}(m-1, q))$ and $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$, ψ acts injectively on the line set of $T^*_{m-1,q^n}(\mathrm{PG}(m-1, q))$. By Lemma A.2.2, $T^*_{m-1,q^n}(\mathrm{PG}(m-1, q))$ and $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$ have the same number of lines, so ψ induces an isomorphism from $T^*_{m-1,q^n}(\mathrm{PG}(m-1, q))$ to $T^\circ_{n-1,q^m}(\mathrm{PG}(n-1, q))$. □

Theorem A.2.6 *$\mathcal{I}'(n, 2^h, e) \cong T^*_{h-1,2^{n-1}}(\mathrm{PG}(h-1, 2))^D$ for all $n \geq 3$.*

Proof. This follows immediately from Theorems A.2.5 and A.2.3. □

Corollary A.2.7 *The geometry $\mathcal{I}'(3, 2^h, e)$ is a lax embedding in $\mathrm{PG}(3, 2^h)$ of the dual of the semipartial geometry $T^*_{h-1,4}(\mathrm{PG}(h-1, 2))$, which is an $\mathrm{spg}(3, 2^h - 2, 2, 6)$.*

Proof. By Theorem A.2.6, and since $T^*_{h-1,4}(\mathrm{PG}(h-1, 2))$ is the semipartial geometry $T^*_{h-1}(\mathcal{B}_\infty)$, with $\mathcal{B}_\infty$ the point set of the Baer subspace $\mathrm{PG}(h-1, 2)$ in the hyperplane at infinity of $\mathrm{AG}(h, 4)$. □

A.3 Isomorphisms of $\mathcal{I}(n, q, e)$

In the preceding section, we determined the structure of the two parts of $\mathcal{I}(n, 2^h, e)$, namely $\mathcal{S}_1$ and $\mathcal{S}_2 = \mathcal{I}'(n, 2^h, e)$. Here, we use these results to obtain some isomorphisms of the geometry $\mathcal{I}(n, 2^h, e)$.

Consider an affine space $\mathrm{AG}(n,q^m)$ and the point set $\mathcal{K}_\infty$ of a projective space $\mathrm{PG}(n-1,q)$ in Π_∞. Then we define an incidence structure $T^{\circledast}_{n-1,q^m}(\mathrm{PG}(n-1,q))$ as follows. The points are of two types.

(1) The points of $T^{*}_{n-1,q^m}(\mathrm{PG}(n-1,q))$.

(2) The lines of $T^{\circ}_{n-1,q^m}(\mathrm{PG}(n-1,q))$.

The lines are of two types as well.

(a) The lines of $T^{*}_{n-1,q^m}(\mathrm{PG}(n-1,q))$.

(b) The points of $T^{\circ}_{n-1,q^m}(\mathrm{PG}(n-1,q))$.

Incidence between points of type (1) and lines of the type (a), and between points of type (2) and lines of type (b), is straightforward. A point of type (2) is never incident with a line of type (a). A point of type (1) is incident with a line of type (b) if and only if the corresponding affine points are the same.

In general, the geometry $T^{\circledast}_{n-1,q^m}(\mathrm{PG}(n-1,q))$ doesn't even have an order. However, it is motivated by the following theorem.

Theorem A.3.1 *$\mathcal{I}(n,2^h,e) \cong T^{\circledast}_{n-2,2^h}(\mathrm{PG}(n-2,2))$ for all $n \geq 3$.*

Proof. Consider the set $\mathcal{K}_\infty$ which is the point set of a projective space $\mathrm{PG}(n-2,2)$ in U_∞. Let $T^{\circledast}_{n-2,2^h}(\mathrm{PG}(n-2,2))$ be the geometry defined in U by $\mathcal{K}_\infty$.

We define a map χ' on the set of points and lines of $\mathcal{I}(n,2^h,e)$. For every point $p \in \mathcal{P}_1$, let $p^{\chi'}$ be the point of type (1) of $T^{\circledast}_{n-2,2^h}(\mathrm{PG}(n-2,2))$ which corresponds to p. For every line $L \in \mathcal{B}_1$, let $L^{\chi'}$ be the line of type (a) of $T^{\circledast}_{n-2,2^h}(\mathrm{PG}(n-2,2))$ which corresponds to L. Consider the map χ from the proof of Theorem A.2.3. For every point $p \in \mathcal{P}_2$, let $p^{\chi'}$ be the point of type (2) of $T^{\circledast}_{n-2,2^h}(\mathrm{PG}(n-2,2))$ which corresponds to p^{χ}. For every line $L \in \mathcal{B}_2$, let $L^{\chi'}$ be the line of type (b) of $T^{\circledast}_{n-2,2^h}(\mathrm{PG}(n-2,2))$ which corresponds to L^{χ}. By Theorem A.2.3, χ is an isomorphism from $\mathcal{I}'(n,2^h,e)$ to the dual of $T^{\circ}_{n-2,2^h}(\mathrm{PG}(n-2,2))$. Hence χ' is an isomorphism from $\mathcal{I}(n,2^h,e)$ to $T^{\circledast}_{n-2,2^h}(\mathrm{PG}(n-2,2))$. □

Theorem A.3.2 *Let $n \geq 3$ and $e_1, e_2 \in \{1, \ldots h-1\}$ such that $e_1 \neq e_2$ and $\gcd(e_1,h) = \gcd(e_2,h) = 1$. Then $\mathcal{I}(n,2^h,e_1) \cong \mathcal{I}(n,2^h,e_2)$, but $\mathcal{I}(n,2^h,e_1) \not\simeq \mathcal{I}(n,2^h,e_2)$.*

Proof. By Theorem A.3.1, $\mathcal{I}(n,2^h,e_1) \cong \mathcal{I}(n,2^h,e_2)$.

We may assume that $\mathcal{I}(n,2^h,e_1)$ and $\mathcal{I}(n,2^h,e_2)$ are fully embedded in the same affine space $\mathrm{AG}(n,2^h)$, and have the same U and $\mathcal{K}_\infty$.

Suppose that $\mathcal{I}(n,2^h,e_1) \simeq \mathcal{I}(n,2^h,e_2)$. Then there is a collineation ζ of $\mathrm{AG}(n,2^h)$ which maps the lines of $\mathcal{I}(n,2^h,e_1)$ onto the lines of $\mathcal{I}(n,2^h,e_2)$. Clearly every planar net of $\mathcal{I}(n,2^h,e_1)$ is mapped by ζ to a planar net of $\mathcal{I}(n,2^h,e_2)$. By Lemma 4.4.10, a line of $\mathcal{I}(n,2^h,e_i)$, $i=1,2$, is contained in a planar net if and only if it is contained in U. Hence $U^\zeta = U$, so ζ induces an isomorphism from $\mathcal{I}'(n,2^h,e_1)$ to $\mathcal{I}'(n,2^h,e_2)$. But this contradicts Theorem A.2.4. We conclude that $\mathcal{I}(n,2^h,e_1) \not\simeq \mathcal{I}(n,2^h,e_2)$. □

Theorem A.3.3 $T^{\circledast}_{n-1,q^m}(\mathrm{PG}(n-1,q)) \cong T^{\circledast}_{m-1,q^n}(\mathrm{PG}(m-1,q))^D$ *for all* $m,n \geq 2$.

Proof. Consider the bijection ψ from the point set of $\mathrm{AG}(m,q^n)$ to the point set of $\mathrm{AG}(n,q^m)$, which was defined in Theorem A.2.5. Then ψ induces an isomorphism from the geometry $T^{\circledast}_{m-1,q^n}(\mathrm{PG}(m-1,q))$ to the dual of the geometry $T^{\circledast}_{n-1,q^m}(\mathrm{PG}(n-1,q))$. □

Theorem A.3.4 $\mathcal{I}(n,2^h,e) \cong \mathcal{I}(h+1,2^{n-1},e)^D$ *for all* $n \geq 3$, $h \geq 2$.

Proof. This follows immediately from Theorems A.3.1 and A.3.3. □

Corollary A.3.5 *The geometry* $\mathcal{I}(n,2^{n-1},e)$ *is self-dual for all* $n \geq 3$.

Bijlage B

Samenvatting

In deze samenvatting zullen we de belangrijkste resultaten kort bespreken. We zullen echter niet alle basisdefinities herhalen, hiervoor verwijzen we naar de Engelstalige tekst.

B.1 Inleiding

B.1.1 Incidentiestructuren

Een *partieel lineaire ruimte* $\mathcal{S}$ is een drietal $(\mathcal{P}, \mathcal{B}, \mathrm{I})$, bestaande uit een verzameling *punten* $\mathcal{P}$, een verzameling *rechten* $\mathcal{B}$, en een symmetrische *incidentierelatie* $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$, zodanig dat iedere rechte tenminste twee punten bevat, ieder punt op tenminste twee rechten ligt, en elke twee verschillende rechten hoogstens één punt gemeen hebben. Een partieel lineaire ruimte $\mathcal{S}$ heeft *orde* (s, t) als iedere rechte precies $s+1$ punten bevat, en ieder punt op precies $t+1$ rechten ligt. Een paar $\{p, L\}$, met $p \in \mathcal{P}$ en $L \in \mathcal{B}$, noemen we een *vlag* als p op L ligt; anders noemen we $\{p, L\}$ een *antivlag.* Het *incidentiegetal* $\alpha(p, L)$ van een antivlag $\{p, L\}$ is het aantal rechten van $\mathcal{S}$ die p bevatten en L snijden.

Een partieel lineaire ruimte $\mathcal{S}$ van orde (s, t) noemen we een *partiële meetkunde* als iedere antivlag van $\mathcal{S}$ hetzelfde incidentiegetal α heeft (hier is $0 < \alpha \leq \min(s+1, t+1)$). Men schrijft dan dat $\mathcal{S}$ een $\mathrm{pg}(s, t, \alpha)$ is. Partiële meetkunden werden door Bose [6] ingevoerd als een veralgemening van veralgemeende vierhoeken en (duale) netten. Een *veralgemeende vierhoek* kan dan ook gedefinieerd worden als een $\mathrm{pg}(s, t, 1)$, en een *net van orde* $s+1$ *en graad* $t+1$ als een $\mathrm{pg}(s, t, t)$.

Een samenhangende partieel lineaire ruimte $\mathcal{S}$ van orde (s, t) noemen we een $(0, \alpha)$-*meetkunde* als het incidentiegetal van elke antivlag van $\mathcal{S}$ gelijk is aan ofwel nul ofwel een constante α ($0 < \alpha \leq \min(s+1, t+1)$). Een *semipartiële meetkunde* is een $(0,\alpha)$-meetkunde $\mathcal{S}$ van orde (s,t) zodanig dat, voor elke twee niet-collineaire punten p en p' van $\mathcal{S}$, het aantal punten p'' dat collineair is met zowel p als p', gelijk is aan een constante μ ($\mu > 0$). Men schrijft dan dat $\mathcal{S}$ een spg(s,t,α,μ) is. Het puntgraaf van een semipartiële meetkunde is een sterk regulier graaf. Semipartiële meetkunden werden ingevoerd door Debroey en Thas [41] als een veralgemening van partiële meetkunden. Men kan inderdaad nagaan dat elke pg(s,t,α) een spg(s,t,α,μ) is, met $\mu = (t+1)\alpha$. Een semipartiële meetkunde die geen partiële meetkunde is, noemen we een *eigenlijke* semipartiële meetkunde.

B.1.2 Enkele begrippen uit de projectieve meetkunde

De kwadriek van Klein

De niet-singuliere hyperbolische kwadriek $\mathrm{Q}^+(5,q)$ in $\mathrm{PG}(5,q)$ wordt ook de *kwadriek van Klein* genoemd. Deze kwadriek staat op een speciale manier in verband met de projectieve meetkunde $\mathrm{PG}(3,q)$. Men kan namelijk op een natuurlijke wijze de verzameling van rechten van $\mathrm{PG}(3,q)$ identificeren met de verzameling van punten van $\mathrm{Q}^+(5,q)$. Deze identificatie maakt gebruik van de zogenaamde *Plücker coördinaten* van rechten van $\mathrm{PG}(3,q)$, en wordt daarom de *Plücker correspondentie* genoemd. Er geldt dat twee verschillende rechten van $\mathrm{PG}(3,q)$ concurrent zijn als en slechts als de corresponderende punten van $\mathrm{Q}^+(5,q)$ collineair zijn op de kwadriek van Klein.

Een *stralenwaaier* van $\mathrm{PG}(3,q)$ is de verzameling van $q+1$ rechten die door een gegeven punt p van $\mathrm{PG}(3,q)$ gaan en in een gegeven vlak $\pi \ni p$ van $\mathrm{PG}(3,q)$ liggen. De Plücker correspondentie bepaalt een bijectief verband tussen de verzameling van stralenwaaiers van $\mathrm{PG}(3,q)$ en de verzameling van rechten van $\mathrm{Q}^+(5,q)$.

Verzamelingen van type $(1, m, q+1)$

Een verzameling $\mathcal{K}$ van punten van $\mathrm{PG}(n,q)$ noemen we een *verzameling van type* $(t_1,\ldots,t_m)$ als iedere rechte van $\mathrm{PG}(n,q)$ de verzameling $\mathcal{K}$ snijdt in $t_1, t_2, \ldots$ of t_m punten.

De verzamelingen van type $(1,m,q+1)$ in $\mathrm{PG}(n,q)$ werden geclassificeerd door Hirschfeld en Thas [53, 52]. De volgende constructie geeft een belangrijk voorbeeld van een verzameling van type $(1,m,q+1)$ in $\mathrm{PG}(n,q)$.

Zij $\mathcal{Q}_{n+1}$ een niet-singuliere kwadriek in $\mathrm{PG}(n+1,q)$, $n \geq 1$, q even, en zij r een punt van $\mathrm{PG}(n+1,q)$ dat niet op de kwadriek ligt, en verschillend is van de kern als $n+1$ even is. Zij $\mathrm{PG}(n,q)$ een hypervlak van $\mathrm{PG}(n+1,q)$

dat r niet bevat, en zij $\mathcal{R}_n$ de projectie van $\mathcal{Q}_{n+1}$ vanuit r op $\mathrm{PG}(n,q)$. Dan is $\mathcal{R}_n$ een verzameling van type $(1, \frac{1}{2}q+1, q+1)$ in $\mathrm{PG}(n,q)$. Is $n+1$ oneven, en is $\mathcal{Q}_{n+1} = \mathrm{Q}^+(n+1,q)$ een hyperbolische kwadriek, dan schrijven we $\mathcal{R}_n = \mathcal{R}_n^+$. Is $n+1$ oneven, en is $\mathcal{Q}_{n+1} = \mathrm{Q}^-(n+1,q)$ een elliptische kwadriek, dan schrijven we $\mathcal{R}_n = \mathcal{R}_n^-$.

Merk op dat de verzameling $\mathcal{R}_n$ steeds een hypervlak van $\mathrm{PG}(n,q)$ bevat. Inderdaad, aangezien q even is, bestaat er een hypervlak U van $\mathrm{PG}(n+1,q)$ door r, zodanig dat de verzameling van raaklijnen aan $\mathcal{Q}_{n+1}$ door r precies de verzameling van rechten door r in U is. Bijgevolg bevat $\mathcal{R}_n$ alle punten van het hypervlak $U \cap \mathrm{PG}(n,q)$ van $\mathrm{PG}(n,q)$.

B.1.3 Projectieve en affiene incidentiestructuren

Zij R een projectieve of affiene ruimte. Een incidentiestructuur $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is *vol ingebed* (of kortweg: *ingebed*) in R als $\mathcal{B}$ een verzameling van rechten van R is, $\mathcal{P}$ de verzameling is van alle punten van R op de rechten van $\mathcal{B}$, I de restrictie is van de incidentie van R tot de punten en rechten van $\mathcal{S}$, en $\mathcal{P}$ niet bevat is in een hypervlak van R. We zeggen ook dat $\mathcal{S}$ een *projectieve* of *affiene* incidentiestructuur is, naargelang R een projectieve of affiene ruimte is.

Twee incidentiestructuren $\mathcal{S}$ en $\mathcal{S}'$ die ingebed zijn in R, worden *projectief* of *affien equivalent* genoemd als er een collineatie van R bestaat die een isomorfisme van $\mathcal{S}$ en $\mathcal{S}'$ induceert. We noteren dit als $\mathcal{S} \simeq \mathcal{S}'$. De notatie voor isomorfe incidentiestructuren is $\mathcal{S} \cong \mathcal{S}'$. Merk op dat uit $\mathcal{S} \simeq \mathcal{S}'$ volgt dat $\mathcal{S} \cong \mathcal{S}'$, maar niet omgekeerd.

Een belangrijk probleem in de eindige meetkunde is het beantwoorden van de vraag welke incidentiestructuren ingebed kunnen worden in een projectieve of affiene ruimte, en het bepalen van de structuur van deze inbeddingen. Dit probleem is reeds voor verschillende klassen van incidentiestructuren bestudeerd.

De projectieve partiële meetkunden zijn volledig geclassificeerd. In het geval van de projectieve veralgemeende vierhoeken werd het probleem opgelost door Buekenhout en Lefèvre [15]. De overige projectieve partiële meetkunden werden geclassificeerd door De Clerck en Thas [29].

De projectieve semipartiële meetkunden werden geclassificeerd door Debroey, De Clerck en Thas [81], uitgezonderd de semipartiële meetkunden ingebed in $\mathrm{PG}(n,2)$, en de projectieve semipartiële meetkunden met $\alpha = 1$ (die ook *partiële vierhoeken* genoemd worden). In feite is de classificatie van projectieve semipartiële meetkunden een direct gevolg van de classificatie door Debroey, De Clerck en Thas [30, 81] van $(0,\alpha)$-meetkunden, $\alpha > 1$, ingebed in $\mathrm{PG}(n,q)$, $n \geq 4$, $q > 2$.

De affiene partiële meetkunden, inclusief affiene veralgemeende vierhoeken, werden volledig geclassificeerd door Thas [78]. Op de affiene inbedding van semipartiële meetkunden en $(0, \alpha)$-meetkunden gaan we in Sectie B.4 dieper in.

B.2 Constructie van $(0, \alpha)$-meetkunden ingebed in PG$(3, q)$, en de kwadriek van Klein

In Hoofdstuk 2 onderzoeken we $(0, \alpha)$-meetkunden ingebed in PG$(3, q)$, aan de hand van de Plücker correspondentie. Als belangrijkste resultaat vermelden we de constructie van nieuwe $(0, \alpha)$-meetkunden ingebed in PG$(3, q)$.

Zoals gezegd, hebben Debroey, De Clerck en Thas [30, 81] de $(0, \alpha)$-meetkunden, $\alpha > 1$, ingebed in PG(n, q), $n \geq 4$, $q > 2$ geclassificeerd. Dit resultaat laat de inbedding van $(0, \alpha)$-meetkunden, $\alpha > 1$, in PG$(3, q)$ open. Toch werden in [30] sterke voorwaarden bewezen waaraan zo'n $(0, \alpha)$-meetkunde moet voldoen. In het bijzonder geldt de volgende stelling.

Stelling B.2.1 (De Clerck, Thas [30]) *Zij $\mathcal{B}$ een verzameling van rechten van* PG$(3, q)$*, $q > 2$. Zij $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P}$ de verzameling is van alle punten van* PG$(3, q)$ *die op de rechten van $\mathcal{B}$ liggen, en* I *de natuurlijke incidentie. Dan is $\mathcal{S}$ een $(0, \alpha)$-meetkunde, $\alpha > 1$, ingebed in* PG$(3, q)$*, als en slechts als elke stralenwaaier van* PG$(3, q)$ *ofwel* 0 *ofwel α rechten van $\mathcal{B}$ bevat.*

Gevolg B.2.2 *Zij $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ een $(0, \alpha)$-meetkunde, $\alpha > 1$, ingebed in* PG$(3, q)$*, $q > 2$. Dan heeft $\mathcal{S}$ orde $(q, (q + 1)(\alpha - 1))$ en bestaat er een getal $d \in \{0, \ldots, q(q+1-\alpha)/\alpha\}$, dat we de* deficiëntie *van $\mathcal{S}$ noemen, zodanig dat $|\mathcal{P}| = (q + 1)(q^2 + 1 - d)$ en $|\mathcal{B}| = (q\alpha - q + \alpha)(q^2 + 1 - d)$.*

In [30, 81] vinden we de volgende voorbeelden van $(0, \alpha)$-meetkunden, $\alpha > 1$, ingebed in PG$(3, q)$, $q > 2$.

1. Zij $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P}$ de verzameling is van alle punten van PG$(3, q)$, $\mathcal{B}$ de verzameling van alle rechten van PG$(3, q)$, en I de natuurlijke incidentie. Dan is $\mathcal{S}$ een $2-((q^4 - 1)/(q - 1), q + 1, 1)$ design, en dus een $(0, q + 1)$-meetkunde van deficiëntie 0, ingebed in PG$(3, q)$.
2. Zij $\overline{W(3, q)} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P}$ de verzameling is van alle punten van PG$(3, q)$, $\mathcal{B}$ de verzameling van rechten die niet totaal isotroop

zijn ten opzichte van een symplectische polariteit van PG$(3,q)$, en I de natuurlijke incidentie. Dan is $\overline{W(3,q)}$ een semipartiële meetkunde spg$(q,q^2-1,q,q^2(q-1))$, en dus een $(0,q)$-meetkunde van deficiëntie 0, ingebed in PG$(3,q)$.

3. Zij $H_q^3 = (\mathcal{P},\mathcal{B},\mathrm{I})$, waarbij $\mathcal{P}$ de verzameling is van alle punten die niet op een gegeven rechte L van PG$(3,q)$ liggen, $\mathcal{B}$ de verzameling van alle rechten die een lege doorsnede hebben met L, en I de natuurlijke incidentie. Dan is H_q^3 een partiële meetkunde pg(q,q^2-1,q), en dus een $(0,q)$-meetkunde van deficiëntie 1, ingebed in PG$(3,q)$.

4. Neem aan dat q even is. Zij NQ$^+(3,q) = (\mathcal{P},\mathcal{B},\mathrm{I})$, waarbij $\mathcal{P}$ de verzameling is van alle punten die niet op een gegeven niet-singuliere hyperbolische kwadriek Q$^+(3,q)$ liggen, $\mathcal{B}$ de verzameling van alle rechten die een lege intersectie hebben met Q$^+(3,q)$, en I de natuurlijke incidentie. Dan is NQ$^+(3,q)$ een $(0,q/2)$-meetkunde van deficiëntie $q+1$, ingebed in PG$(3,q)$. Deze meetkunde is geen semipartiële meetkunde.

In [30] werd het vermoeden geuit dat er geen andere voorbeelden zijn van $(0,\alpha)$-meetkunden, $\alpha > 1$, ingebed in PG$(3,q)$, $q > 2$. Zoals zal blijken is dit vermoeden niet juist. Echter, in het geval dat q oneven is, kan men wel bewijzen dat er geen andere voorbeelden zijn. Cruciaal hierbij is het resultaat van Ball, Blokhuis en Mazzocca [2], dat zegt dat er in PG$(2,q)$, q oneven, geen niet-triviale maximale bogen bestaan. Voor het vinden van nieuwe voorbeelden van $(0,\alpha)$-meetkunden, $\alpha > 1$, ingebed in PG$(3,q)$, mogen we ons dus beperken tot het geval waarbij q even is.

Zij $\mathcal{B}$ een verzameling van rechten van PG$(3,q)$ en zij $\mathcal{K}$ de verzameling van punten van de Klein kwadriek die correspondeert met $\mathcal{B}$. Dan bevat elke stralenwaaier van PG$(3,q)$ ofwel 0 ofwel α rechten van $\mathcal{B}$ als en slechts als elke rechte van Q$^+(5,q)$ ofwel 0 ofwel α punten van $\mathcal{K}$ bevat. Een verzameling van punten van de Klein kwadriek die de laatste eigenschap heeft, noemen we een *$(0,\alpha)$-verzameling op de Klein kwadriek* Q$^+(5,q)$. Wegens Stelling B.2.1 zijn de volgende objecten equivalent wanneer $q > 2$ en $\alpha > 1$.

1. Een $(0,\alpha)$-meetkunde ingebed in PG$(3,q)$.

2. Een $(0,\alpha)$-verzameling op Q$^+(5,q)$.

Zij $\mathcal{S}$ een $(0,\alpha)$-meetkunde, $\alpha > 1$, van deficiëntie d, ingebed in PG$(3,q)$, $q > 2$, en zij $\mathcal{K}$ de corresponderende $(0,\alpha)$-verzameling op Q$^+(5,q)$. Dan zeggen we ook dat d de deficiëntie van $\mathcal{K}$ is.

Beschouw de Klein kwadriek Q$^+(5,q)$, $q = 2^h$. Vermits q even is, kunnen we met Q$^+(5,q)$ een symplectische polariteit β associëren. Zij V een 3-ruimte van $\mathrm{PG}(5,q)$ die $\mathrm{Q}^+(5,q)$ snijdt in een niet-singuliere elliptische kwadriek E, en zij L de rechte V^β. Dan is L een externe rechte aan $\mathrm{Q}^+(5,q)$.

Zij O een ovoïde van V die dezelfde verzameling van raaklijnen heeft als de elliptische kwadriek E. Voor ieder punt $p \in O \setminus E$ snijdt het vlak $\pi = \langle p, L\rangle$ de Klein kwadriek in een niet-singuliere kegelsnede met kern p, en voor ieder punt $p \in O \cap E$ snijdt het vlak $\pi = \langle p, L\rangle$ de Klein kwadriek enkel in het punt p. Zij

$$\mathcal{K} = \bigcup_{p \in O\setminus E} (\mathrm{Q}^+(5,q) \cap \pi) \cup E \setminus O.$$

Dan is $\mathcal{K}$ een $(0,2)$-verzameling van deficiëntie $|\,E \cap O\,|$ op de Klein kwadriek $\mathrm{Q}^+(5,q)$. Deze constructie komt toe aan Ebert, Metsch en Szőnyi [45].

Als O een niet-singuliere elliptische kwadriek is, dan volgt uit Bruen en Hirschfeld [13] dat er twee mogelijkheden zijn voor de onderlinge ligging van E en O. Ofwel hebben E en O juist één punt gemeen, ofwel snijden E en O in de $q+1$ punten van een niet-singuliere kegelsnede in een vlak van $\mathrm{PG}(3,q)$. In het eerste geval noteren we de $(0,2)$-verzameling $\mathcal{K}$ als $\mathcal{E}_1$, in het tweede geval als $\mathcal{E}_{q+1}$.

Als $q = 2^{2e+1}$ en O een Suzuki-Tits ovoïde is, dan volgt uit Bagchi en Sastry [1] dat E en O snijden in ofwel $q+\sqrt{2q}+1$ ofwel $q-\sqrt{2q}+1$punten. In het eerste geval noteren we de $(0,2)$-verzameling $\mathcal{K}$ als $\mathcal{T}_{q+\sqrt{2q}+1}$, in het tweede geval als $\mathcal{T}_{q-\sqrt{2q}+1}$.

We kunnen de volgende stelling bewijzen over de $(0,2)$-verzamelingen $\mathcal{E}_d$, $d = 1, q+1$.

Stelling B.2.3 *Zij $\mathcal{K} \in \{\mathcal{E}_1, \mathcal{E}_{q+1}\}$, en zij π het unieke vlak van V zodanig dat $\mathrm{Q}^+(5,q) \cap \pi = E \cap O$. Dan geldt er dat*

$$\mathcal{K} = (E \cup O_1 \cup \ldots \cup O_{q+1}) \setminus \pi,$$

waarbij O_i, $1 \le i \le q+1$, een niet-singuliere 3-dimensionale elliptische kwadriek op $\mathrm{Q}^+(5,q)$ is, zodanig dat de 3-ruimte V_i die O_i bevat, V snijdt in het vlak π. De 3-ruimten $V_1, \ldots, V_{q+1}$ snijden ieder vlak $\pi' = \langle r, L\rangle$, met $r \in O \setminus E$, in de punten van de niet-singuliere kegelsnede $C' = \mathrm{Q}^+(5,q) \cap \pi'$, en V snijdt π' in de kern r van de kegelsnede C'.

Met andere woorden, $\mathcal{E}_d$, $d = 1, q+1$, kan gezien worden als de unie van 3-dimensionale niet-singuliere elliptische kwadrieken, met weglating van d punten. Dit leidt tot de volgende constructie.

Beschouw opnieuw de Klein kwadriek $\mathrm{Q}^+(5,q)$, q even. Zij π een vlak van $\mathrm{PG}(5,q)$ dat geen rechte van $\mathrm{Q}^+(5,q)$ bevat. Zij π' een vlak dat scheef is aan π, en zij $\mathcal{D}$ de verzameling van punten $p \in \pi'$, zodanig dat de 3-ruimte $V = \langle p, \pi\rangle$ de Klein kwadriek snijdt in een niet-singuliere elliptische kwadriek. Neem aan dat A een maximale boog is van graad $\alpha > 1$ in π', zodanig dat $A \subseteq \mathcal{D}$. Beschouw dan

$$\mathcal{M}^{\alpha}(A) = \bigcup_{p \in A} (\mathrm{Q}^+(5,q) \cap V) \setminus \pi.$$

Stelling B.2.4 *De verzameling* $\mathcal{M}^{\alpha}(A)$ *is een* $(0,\alpha)$*-verzameling op* $\mathrm{Q}^+(5,q)$ *van deficiëntie* $d = |\,\mathrm{Q}^+(5,q) \cap \pi\,|$.

Vermits het vlak π geen rechten van $\mathrm{Q}^+(5,q)$ bevat, bestaat $\mathrm{Q}^+(5,q) \cap \pi$ ofwel uit één enkel punt, ofwel uit de $q+1$ punten van een niet-singuliere kegelsnede. In het eerste geval heeft de $(0,\alpha)$-verzameling deficiëntie 1 en wordt ze genoteerd als $\mathcal{M}^{\alpha}_1(A)$; in het tweede geval heeft ze deficiëntie $q+1$ en wordt ze genoteerd als $\mathcal{M}^{\alpha}_{q+1}(A)$.

Als $|\,\mathrm{Q}^+(5,q) \cap \pi\,| = 1$, dan bestaat $\mathcal{D}$ uit het complement van een rechte in π'. Als $|\,\mathrm{Q}^+(5,q) \cap \pi\,| = q+1$, dan bestaat $\mathcal{D}$ uit de punten die op geen enkele rechte van een duale niet-singuliere kegelsnede van π' liggen. Hierdoor kunnen we de volgende stelling bewijzen.

Stelling B.2.5 *Er bestaan* $(0,\alpha)$*-verzamelingen op* $\mathrm{Q}^+(5,q)$*,* $q = 2^h$*, van deficiëntie* 1 *en* $q+1$*, voor alle* $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$.

Gevolg B.2.6 *Er bestaan* $(0,\alpha)$*-meetkunden ingebed in* $\mathrm{PG}(3,q)$*,* $q = 2^h$*, van deficiëntie* 1 *en* $q+1$*, voor alle* $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$.

Tot slot gaan we na welke van de gekende voorbeelden van $(0,\alpha)$-verzamelingen op $\mathrm{Q}^+(5,q)$ projectief equivalent zijn. Uit Stelling B.2.3 volgt dat de $(0,2)$-verzameling $\mathcal{E}_d$, $d = 1, q+1$, van de vorm $\mathcal{M}^2_d(H)$ is, waarbij H een reguliere hyperovaal is. Bovendien kunnen we bewijzen dat de $(0,q/2)$-verzameling op $\mathrm{Q}^+(5,q)$ die correspondeert met de $(0,q/2)$-meetkunde $\mathrm{NQ}^+(3,q)$, niets anders is dan de $(0,q/2)$-verzameling $\mathcal{M}^{q/2}_{q+1}(A)$, waarbij $A = \mathcal{D}$.

De lijst van gekende, projectief verschillende $(0,\alpha)$-verzamelingen op $\mathrm{Q}^+(5,q)$, $\alpha > 1$, $q > 2$, ziet er dus als volgt uit. In deze lijst is d de deficiëntie van de $(0,\alpha)$-verzameling $\mathcal{K}$, en $\mathcal{S}$ de corresponderende $(0,\alpha)$-meetkunde ingebed in $\mathrm{PG}(3,q)$.

1. $\alpha = q+1$, $d = 0$, en $\mathcal{K}$ is de verzameling van alle punten van $\mathrm{Q}^+(5,q)$.
2. $\alpha = q$, $d = 0$, en $\mathcal{S} = \overline{W(3,q)}$.
3. $\alpha = q$, $d = 1$, en $\mathcal{S} = H^3_q$.
4. $q = 2^h$, $\alpha \in \{2, 2^2, \ldots, 2^{h-1}\}$, $d \in \{1, q+1\}$, en $\mathcal{K} = \mathcal{M}^{\alpha}_d(A)$.
5. $q = 2^{2e+1}$, $\alpha = 2$, $d = q \pm \sqrt{2q} + 1$, en $\mathcal{K} = \mathcal{T}_{q \pm \sqrt{2q}+1}$.

B.3 Planaire ovaalverzamelingen in PG$(2, q)$, q even

In Hoofdstuk 3 worden bepaalde verzamelingen van ovalen in PG$(2, q)$, q even, onderzocht, de zogenaamde planaire ovaalverzamelingen. De resultaten uit dit hoofdstuk hebben we nodig in Hoofdstuk 5, bij de classificatie van $(0, 2)$-meetkunden ingebed in AG$(3, q)$.

Een *planaire ovaalverzameling* in PG$(2, q)$, $q = 2^h$, is een verzameling Ω van q^2 ovalen met gemeenschappelijke kern n, zodanig dat de incidentiestructuur $\pi(\Omega)$, met als punten de punten van PG$(2, q)$, als rechten de elementen van Ω en de rechten van PG$(2, q)$ door n, en met de natuurlijke incidentie, een projectief vlak van orde q is. Het punt n wordt ook de *kern van* Ω genoemd.

Een planaire ovaalverzameling Ω wordt een *reguliere Desarguesiaanse planaire ovaalverzameling* genoemd als er een collineatie van PG$(2, q)$ naar $\pi(\Omega)$ bestaat, die elke rechte door de kern n fixeert.

Zij O een ovaal van PG$(2, q)$, q even, en zij n de kern van O. Zij El(n) de groep van alle elaties van PG$(2, q)$ die het punt n als centrum hebben. We definiëren nu

$$\Omega(O) = \{O^e \mid e \in \mathrm{El}(n)\}.$$

Dan bewijzen we dat $\Omega(O)$ een reguliere Desarguesiaanse planaire ovaalverzameling met kern n in PG$(2, q)$ is. We kunnen nu onmiddellijk het belangrijkste resultaat van Hoofdstuk 3 formuleren.

Stelling B.3.1 *Zij Ω een reguliere Desarguesiaanse planaire ovaalverzameling in* PG$(2, q)$*, q even. Dan geldt voor elk element $O \in \Omega$ dat $\Omega = \Omega(O)$.*

Om Stelling B.3.1 te bewijzen, onderzoeken we de verzameling V van collineaties van PG$(2, q)$ naar $\pi(\Omega)$, die iedere rechte door de kern n van Ω fixeren. Meer bepaald onderzoeken we voor elk element $\xi \in V$ de verzameling Fix(ξ), die bestaat uit de punten van PG$(2, q)$, verschillend van n, die gefixeerd worden door ξ. We bewijzen dat, voor alle $\xi \in V$, de verzameling Fix(ξ) ofwel leeg is, ofwel $q + i$ punten bevat, met i een deler van q, en een verzameling van type $(0, 2, i)$ is zodanig dat elke rechte die i punten van Fix(ξ) bevat, door de kern n gaat. Vervolgens bewijzen we dat het getal i slechts de waarden 1 en q kan aannemen. Dit wil zeggen dat, als Fix(ξ) niet ledig is, Fix(ξ) ofwel een ovaal is met kern n, ofwel de unie van twee verschillende rechten door n. Met behulp van dit resultaat bewijzen we Stelling B.3.1.

B.4 Affiene semipartiële meetkunden en $(0, \alpha)$-meetkunden

Hoofdstukken 4 tot en met 6 vormen eigenlijk één geheel binnen deze thesis. We behandelen hierin de inbeddingen van semipartiële meetkunden en $(0, \alpha)$-meetkunden in affiene ruimten. De eigenlijke bewijzen van onze resultaten bevinden zich in Hoofdstukken 5 en 6. Hoofdstuk 4 is er om de lezer te oriënteren: we leggen uit waarom we niet enkel affiene semipartiële meetkunden, maar in de eerste plaats affiene $(0, \alpha)$-meetkunden onderzoeken; we leggen de methode uit die we daarbij gebruiken en we plaatsen onze resultaten tussen aanverwante resultaten. Bovendien geven we de constructie van twee nieuwe affiene $(0, \alpha)$-meetkunden. We maken de lezer tevens vertrouwd met alle gekende affiene $(0, \alpha)$-meetkunden door elk van deze meetkunden te onderwerpen aan een gedetailleerd onderzoek. Hierbij besteden we vooral aandacht aan de structuur van punten en rechten van de meetkunde, die in de verschillende deelruimten van de affiene ruimte liggen.

B.4.1 Alle voorbeelden op een rijtje

Lineaire representaties

Zij $\mathcal{K}_\infty$ een verzameling van punten van Π_∞, het hypervlak op oneindig van een affiene ruimte $\mathrm{AG}(n, q)$. De *lineaire representatie* $T^*_{n-1}(\mathcal{K}_\infty)$ van de verzameling $\mathcal{K}_\infty$ is de incidentiestructuur $(\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P}$ de verzameling is van alle punten van $\mathrm{AG}(n, q)$, $\mathcal{B}$ de verzameling van alle rechten die Π_∞ snijden in een punt van $\mathcal{K}_\infty$, en I de natuurlijke incidentie. Iedere lineaire representatie is een partieel lineaire ruimte, ingebed in $\mathrm{AG}(n, q)$.

Een lineaire representatie $T^*_{n-1}(\mathcal{K}_\infty)$ is een $(0, \alpha)$-meetkunde als en slechts als $\mathcal{K}_\infty$ niet bevat is in een echte deelruimte van Π_∞, en $\mathcal{K}_\infty$ een verzameling van type $(0, 1, \alpha + 1)$ in Π_∞ is. Daarenboven is $T^*_{n-1}(\mathcal{K}_\infty)$ een semipartiële meetkunde $\mathrm{spg}(q - 1, |\mathcal{K}_\infty| - 1, \alpha, \mu)$ als en slechts als ieder punt van Π_∞, niet in $\mathcal{K}_\infty$, op precies $\mu/(\alpha(\alpha + 1))$ rechten ligt die $\alpha + 1$ punten van $\mathcal{K}_\infty$ bevatten, als en slechts als de verzameling $\mathcal{K}_\infty$ twee intersectiegetallen heeft met betrekking tot hypervlakken van Π_∞ (zie Delsarte [43]).

Er zijn slechts twee voorbeelden gekend van lineaire representaties die semipartiële meetkunden zijn met $\alpha > 1$.

1. Beschouw $\mathrm{AG}(3, q^2)$ en zij $\mathcal{U}_\infty$ een unitaal van het vlak Π_∞. Dan is $T^*_2(\mathcal{U}_\infty)$ een $\mathrm{spg}(q^2 - 1, q^3, q, q^2(q^2 - 1))$, ingebed in $\mathrm{AG}(3, q^2)$.

2. Beschouw $\mathrm{AG}(n, q^2)$, $n \geq 2$, en zij $\mathcal{B}_\infty$ een Baer deelruimte van Π_∞. Dan is $T^*_{n-1}(\mathcal{B}_\infty)$ een $\mathrm{spg}(q^2 - 1, (q^n - q)/(q - 1), q, q(q + 1))$, ingebed

in $\mathrm{AG}(n, q^2)$.

Lineaire representaties zijn reeds lang gekend en bestudeerd. De semipartiële meetkunden $T_2^*(\mathcal{U}_\infty)$ en $T_{n-1}^*(\mathcal{B}_\infty)$ werden geïntroduceerd door Debroey en Thas [41].

De duale ovaal

Beschouw een verzameling $\mathcal{B}$ van rechten van $\mathrm{AG}(2, q)$, $q = 2^h$, zodanig dat $\mathcal{B} \cup \{\Pi_\infty\}$ een hyperovaal vormt in het duale vlak van $\mathrm{PG}(2, q)$. Zij $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P}$ de verzameling is van alle punten van $\mathrm{AG}(2, q)$ op de rechten van $\mathcal{B}$, en I de natuurlijke incidentie. Dan is $\mathcal{S}$ een (triviale) partiële meetkunde $\mathrm{pg}(q - 1, 1, 2)$, ingebed in $\mathrm{AG}(2, q)$. We noemen $\mathcal{S}$ een *duale ovaal.*

HT en $\mathrm{TQ}(4, q)$

Beschouw de verzameling $\mathcal{R}_3$ in $\mathrm{PG}(3, q)$, $q = 2^h$. Dan is er een vlak van $\mathrm{PG}(3, q)$ dat volledig in $\mathcal{R}_3$ ligt. Beschouw dit vlak als het vlak op oneindig Π_∞ van een affiene ruimte $\mathrm{AG}(3, q)$. Zij $\mathrm{HT} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P} = \mathcal{R}_3 \backslash \Pi_\infty$, $\mathcal{B}$ de verzameling van rechten van $\mathrm{AG}(3, q)$ die volledig in $\mathcal{R}_3$ liggen, en I de natuurlijke incidentie. Dan is HT een $(0, 2)$-meetkunde van orde $(q - 1, q)$, ingebed in $\mathrm{AG}(3, q)$. Deze meetkunde is geen semipartiële meetkunde.

Beschouw de verzameling $\mathcal{R}_4^-$ in $\mathrm{PG}(4, q)$, $q = 2^h$. Dan is er een hypervlak van $\mathrm{PG}(4, q)$ dat volledig in $\mathcal{R}_4^-$ ligt. Beschouw dit hypervlak als het hypervlak op oneindig Π_∞ van een affiene ruimte $\mathrm{AG}(4, q)$. Zij $\mathrm{TQ}(4, q) = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P} = \mathcal{R}_4^- \backslash \Pi_\infty$, $\mathcal{B}$ de verzameling van rechten van $\mathrm{AG}(4, q)$ die volledig in $\mathcal{R}_4^-$ liggen, en I de natuurlijke incidentie. Dan is $\mathrm{TQ}(4, q)$ een semipartiële meetkunde $\mathrm{spg}(q - 1, q^2, 2, 2q(q - 1))$, ingebed in $\mathrm{AG}(4, q)$.

De meetkunden HT en $\mathrm{TQ}(4, q)$, ingebed in $\mathrm{AG}(4, q)$, werden ontdekt door Hirschfeld en Thas [53].

$\mathcal{A}(O_\infty)$

Beschouw $\mathrm{AG}(3, q)$, $q = 2^h$. Zij O_∞ een ovaal in Π_∞ met kern n_∞. Kies een basis in $\mathrm{PG}(3, q)$ zodat $\Pi_\infty : X_3 = 0, n_\infty(1, 0, 0, 0)$ en $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(1, 1, 1, 0) \in O_\infty$. Zij f het o-polynoom zodat

$$O_\infty = \{(\rho, f(\rho), 1, 0) \mid \rho \in \mathrm{GF}(2^h)\} \cup \{(0, 1, 0, 0)\}.$$

Beschouw voor elk affien punt $p(x, y, z, 1)$ de ovaal

$$O_\infty^p = \{(y + zf(\rho) + \rho, f(\rho), 1, 0) \mid \rho \in \mathrm{GF}(2^h)\} \cup \{(z, 1, 0, 0)\},$$

en zij $\mathcal{L}_p$ de verzameling rechten door p die Π_∞ snijden in een punt van O_∞^p. Zij $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, waarbij $\mathcal{P}$ de verzameling van alle punten van $\mathrm{AG}(3,q)$ is, $\mathcal{B} = \bigcup_{p\in\mathcal{P}} \mathcal{L}_p$, en I de natuurlijke incidentie. Dan bewijzen we de volgende stelling.

Stelling B.4.1 *Iedere samenhangende component van $\mathcal{S}$ is een $(0,2)$-meetkunde van orde $(q-1,q)$, ingebed in $\mathrm{AG}(3,q)$. Als O_∞ een kegelsnede is, dan heeft $\mathcal{S}$ twee samenhangende componenten $\mathcal{S}_1$ en $\mathcal{S}_2$, en geldt er dat $\mathcal{S}_1 \simeq \mathcal{S}_2 \simeq \mathrm{HT}$. Als O_∞ geen kegelsnede is, dan is $\mathcal{S}$ samenhangend.*

Als $\mathcal{S}$ samenhangend is, dus als O_∞ geen kegelsnede is, dan definiëren we $\mathcal{A}(O_\infty) = \mathcal{S}$. Als $\mathcal{S}$ niet samenhangend is, dus als O_∞ een kegelsnede is, dan definiëren we $\mathcal{A}(O_\infty)$ als één van beide samenhangende componenten van $\mathcal{S}$. Vermits in het laatste geval beide componenten affien equivalent zijn, is $\mathcal{A}(O_\infty)$ goed gedefinieerd.

We concluderen dat $\mathcal{A}(O_\infty)$ een $(0,2)$-meetkunde is van orde $(q-1,q)$, ingebed in $\mathrm{AG}(3,q)$. Als O_∞ een kegelsnede is, dan is $\mathcal{A}(O_\infty) \simeq \mathrm{HT}$; als O_∞ geen kegelsnede is, dan is $\mathcal{A}(O_\infty)$ een nieuw voorbeeld van een affiene $(0,\alpha)$-meetkunde. De meetkunde $\mathcal{A}(O_\infty)$ is in geen geval een semipartiële meetkunde.

$\mathcal{I}(n,q,e)$

Zij U een hypervlak van $\mathrm{AG}(n,q)$, $n \geq 3$, $q = 2^h$. Kies een basis in $\mathrm{PG}(n,q)$ zodat $\Pi_\infty : X_n = 0$ en $U : X_{n-1} = 0$. Zij $e \in \{1,2,\ldots,h-1\}$ zodanig dat $\gcd(e,h) = 1$, en zij φ de collineatie van $\mathrm{PG}(n,q)$, zodanig dat

$$\varphi : p(x_0, x_1, \ldots, x_{n-1}, x_n) \mapsto p^\varphi(x_0^{2^e}, x_1^{2^e}, \ldots, x_n^{2^e}, x_{n-1}^{2^e}).$$

Zij $U_\infty = U \cap \Pi_\infty$, en zij $\mathcal{K}_\infty$ de verzameling van punten van U_∞ die gefixeerd worden door φ. Dan is

$$\mathcal{K}_\infty = \{(\varepsilon_0, \ldots, \varepsilon_{n-2}, 0, 0) \neq (0, \ldots, 0) \,|\, \varepsilon_i \in \mathrm{GF}(2), 0 \leq i \leq n-2\}.$$

Dus $\mathcal{K}_\infty$ is de puntenverzameling van een projectieve ruimte $\mathrm{PG}(n-2,2)$ in U_∞.

Zij

$$\mathcal{B}_1 = \{L \subseteq U \| L \not\subseteq \Pi_\infty, L \cap \Pi_\infty \in \mathcal{K}_\infty\},$$

en zij

$$\mathcal{B}_2 = \{\langle p, p^\varphi \rangle \| p \in U \setminus \Pi_\infty\}.$$

Definieer $\mathcal{I}(n,q,e) = (\mathcal{P}, \mathcal{B}_1 \cup \mathcal{B}_2, \mathrm{I})$, waarbij $\mathcal{P}$ de verzameling is van alle punten van $\mathrm{AG}(n,q)$ op de rechten van $\mathcal{B}_1$ en $\mathcal{B}_2$, en I de natuurlijke incidentie. Dan is $\mathcal{I}(n,q,e)$ een $(0,2)$-meetkunde van orde $(q-1, 2^{n-1}-1)$, ingebed in $\mathrm{AG}(n, q)$. Deze meetkunde is eveneens een nieuw voorbeeld van een affiene $(0, \alpha)$-meetkunde. Ze is echter in geen geval een semipartiële meetkunde.

B.4.2 Methode

Een krachtige methode om affiene semipartiële meetkunden en $(0, \alpha)$-meetkunden te onderzoeken is inductie toe te passen op de dimensie van de affiene ruimte. Aan de basis van deze methode ligt het volgende lemma.

Lemma B.4.2 *Zij $\mathcal{S}$ een $(0, \alpha)$-meetkunde, $\alpha > 1$, ingebed in $\mathrm{AG}(n, q)$, $n \geq 3$, en zij U een deelruimte van dimensie tenminste 2. Zij $\mathcal{S}_U$ de sub incidentiestructuur, geïnduceerd op de verzameling van punten en rechten van $\mathcal{S}$ die in U liggen. Dan is elke samenhangende component van $\mathcal{S}_U$ die twee snijdende rechten bevat, een $(0, \alpha)$-meetkunde ingebed in een deelruimte van U.*

Bij het onderzoeken van affiene $(0, \alpha)$-meetkunden, $\alpha > 1$, gaan we eerst de inbeddingen bekijken in affiene ruimten van lage dimensie. Lemma B.4.2 laat ons dan toe om deze resultaten te gebruiken bij het onderzoeken van inbeddingen van $(0, \alpha)$-meetkunden, $\alpha > 1$, in affiene ruimten van hogere dimensie.

Merk op dat het analogon van Lemma B.4.2 voor affiene semipartiële meetkunden niet geldig is, aangezien een samenhangende component van $\mathcal{S}_U$ niet noodzakelijk voldoet aan de μ-eigenschap. We kunnen dus de inductiemethode op de dimensie van de affiene ruimte niet toepassen op semipartiële meetkunden als dusdanig. We moeten daarom affiene $(0, \alpha)$-meetkunden, $\alpha > 1$, onderzoeken, en op die manier resultaten vinden die a forteriori gelden voor affiene semipartiële meetkunden met $\alpha > 1$.

De classificatie van $(0, \alpha)$-meetkunden, $\alpha > 1$, ingebed in $\mathrm{AG}(2, q)$, is eenvoudig, en laat zich als volgt toepassen op algemene affiene $(0, \alpha)$-meetkunden, $\alpha > 1$.

Lemma B.4.3 *Zij $\mathcal{S}$ een $(0, \alpha)$-meetkunde, $\alpha > 1$, ingebed in $\mathrm{AG}(n, q)$, $n \geq 3$, en zij π een vlak van $\mathrm{AG}(n, q)$. Dan is π van één van volgende types.*

Type I. *π bevat geen enkele rechte van $\mathcal{S}$.*

Type II. *π bevat een aantal parallelle rechten van $\mathcal{S}$ en mogelijk een aantal geïsoleerde punten.*

Type III. *$\mathcal{S}_\pi$ is een net van orde q en graad $\alpha + 1$ (dit noemen we een* planair net*).*

Type IV. *$\mathcal{S}_\pi$ bestaat uit een* $\mathrm{pg}(q-1,1,2)$ *(met andere woorden, een duale ovaal; er geldt dan dat $q = 2^h$ en $\alpha = 2$), en mogelijk een aantal geïsoleerde punten.*

B.4.3 Overzicht van de resultaten

Affiene $(0,\alpha)$-meetkunden

Wat betreft affiene $(0,\alpha)$-meetkunden, $\alpha > 1$, was voorheen slechts het volgende resultaat gekend.

Stelling B.4.4 (De Clerck, Delanote [27]) *Als $\mathcal{S}$ een $(0,\alpha)$-meetkunde is, $\alpha > 1$, ingebed in* $\mathrm{AG}(n,q)$*, en als er geen vlakken van type* IV *zijn, dan is $\mathcal{S} \simeq T^*_{n-1}(\mathcal{K}_\infty)$ een lineaire representatie van een verzameling $\mathcal{K}_\infty$ in Π_∞. Als q oneven is, of $\alpha > 2$, dan geldt dezelfde conclusie, zonder restrictie op de types van vlakken.*

Als gevolg van Stelling B.4.4 moeten we slechts de volgende twee problemen onderzoeken.

Probleem 1. Classificeer alle lineaire representaties van $(0,\alpha)$-meetkunden, $\alpha > 1$. Gelijkwaardig hiermee, classificeer alle verzamelingen van type $(0,1,k)$, $k > 2$, in $\mathrm{PG}(n,q)$.

Probleem 2. Classificeer alle $(0,2)$-meetkunden ingebed in $\mathrm{AG}(n,q)$, $q = 2^h$, die tenminste één vlak van type IV hebben.

De volledige oplossing van Probleem 1 is hopeloos. We kennen echter de volgende gedeeltelijke oplossing.

Stelling B.4.5 (Ueberberg [83]) *Zij $\mathcal{K}$ een verzameling van type $(0,1,k)$ in* $\mathrm{PG}(n,q)$*, $n \geq 2$, die niet bevat is in een hypervlak. Als $k \geq \sqrt{q}+1$, dan geldt één van de volgende gevallen.*

1. *$n = 2$ en $\mathcal{K}$ is een maximale boog.*
2. *$n = 2$, q is een kwadraat en $\mathcal{K}$ is een unitaal.*
3. *q is een kwadraat en $\mathcal{K}$ is de verzameling van punten van een Baer deelruimte.*
4. *$\mathcal{K}$ is het complement van een hypervlak van* $\mathrm{PG}(n,q)$.
5. *$\mathcal{K}$ is de verzameling van punten van* $\mathrm{PG}(n,q)$.

Probleem 2 is het onderwerp van Hoofdstukken 5 en 6. We komen tot de volledige oplossing van dit probleem.

Stelling B.4.6 *Als $\mathcal{S}$ een $(0,2)$-meetkunde is, ingebed in $\mathrm{AG}(n,q)$, $q=2^h$, zodat er tenminste één vlak van type* IV *is, dan geldt één van de volgende gevallen.*

1. $q=2$ *en* $\mathcal{S}$ *is een* $2-(t+2,2,1)$*-design.*
2. $n=2$ *en* $\mathcal{S}$ *is een duale ovaal.*
3. $n=3$ *en* $\mathcal{S}\simeq\mathcal{A}(O_\infty)$.
4. $n=4$ *en* $\mathcal{S}\simeq\mathrm{TQ}(4,q)$.
5. $n\geq 3$ *en* $\mathcal{S}\simeq\mathcal{I}(n,q,e)$.

Gevolg B.4.7 *Als $\mathcal{S}$ een $(0,\alpha)$-meetkunde is, $\alpha>1$, ingebed in $\mathrm{AG}(n,q)$, dan geldt één van de volgende gevallen.*

1. $q=2$, $\alpha=2$ *en* $\mathcal{S}$ *is een* $2-(t+2,2,1)$*-design.*
2. $n=2$, $q=2^h$, $\alpha=2$ *en* $\mathcal{S}$ *is een duale ovaal.*
3. $n=3$, $q=2^h$, $\alpha=2$ *en* $\mathcal{S}\simeq\mathcal{A}(O_\infty)$.
4. $n=4$, $q=2^h$, $\alpha=2$ *en* $\mathcal{S}\simeq\mathrm{TQ}(4,q)$.
5. $n\geq 3$ *en* $q=2^h$, $\alpha=2$ *en* $\mathcal{S}\simeq\mathcal{I}(n,q,e)$.
6. $n\geq 2$ *en* $\mathcal{S}\simeq T^*_{n-1}(\mathcal{K}_\infty)$, *met* $\mathcal{K}_\infty$ *een verzameling van type* $(0,1,\alpha+1)$ *in* Π_∞, *niet bevat een hypervlak van* Π_∞.

Affiene semipartiële meetkunden

De eigenlijke semipartiële meetkunden ingebed in $\mathrm{AG}(2,q)$ en $\mathrm{AG}(3,q)$ werden geclassificeerd door Debroey en Thas [40]. We vermelden hier enkel het geval waarbij $\alpha>1$.

Stelling B.4.8 (Debroey, Thas [40]) *Als $\mathcal{S}$ een eigenlijke semipartiële meetkunde is, $\alpha>1$, ingebed in $\mathrm{AG}(n,q)$, $n\leq 3$, dan geldt één van de volgende gevallen.*

1. $n=3$, *q is een kwadraat en* $\mathcal{S}\simeq T^*_2(\mathcal{U}_\infty)$, *met* $\mathcal{U}_\infty$ *een unitaal van* Π_∞.

2. $n = 3$, *q is een kwadraat en* $\mathcal{S} \simeq T_2^*(\mathcal{B}_\infty)$, *met* $\mathcal{B}_\infty$ *een Baer deelvlak van* Π_∞.

Stelling B.4.4 geldt a forteriori ook voor affiene semipartiële meetkunden met $\alpha > 1$. Bijgevolg hebben we ook in dit geval twee afzonderlijke problemen. Merk op dat we ons mogen beperken tot eigenlijke semipartiële meetkunden, aangezien de affiene partiële meetkunden geclassificeerd werden door Thas [78].

Probleem 1'. Classificeer alle lineaire representaties van semipartiële meetkunden, $\alpha > 1$. Gelijkwaardig hiermee, classificeer alle verzamelingen van type $(0, 1, k)$, $k > 2$, in $\mathrm{PG}(n, q)$, die twee intersectiegetallen hebben met betrekking tot hypervlakken.

Probleem 2'. Classificeer alle semipartiële meetkunden met $\alpha = 2$, ingebed in $\mathrm{AG}(n, q)$, $q = 2^h$, die tenminste één vlak van type IV hebben.

Probleem 1' is enkel opgelost voor kleine dimensies. De lineaire representatie van semipartiële meetkunden in $\mathrm{AG}(2, q)$ en $\mathrm{AG}(3, q)$ wordt opgelost door Stelling B.4.8. Voor $\mathrm{AG}(4, q)$ wordt de oplossing gegeven door de volgende stelling.

Stelling B.4.9 (De Winter [38]) *Als een lineaire representatie* $T_3^*(\mathcal{K}_\infty)$ *in* $\mathrm{AG}(4, q)$ *een eigenlijke semipartiële meetkunde is met* $\alpha > 1$, *dan is q een kwadraat en is* $\mathcal{K}_\infty$ *de verzameling van punten van een Baer deelruimte van* Π_∞.

In tegenstelling tot Probleem 1, lijkt een volledige oplossing van Probleem 1' niet ondenkbaar. Op basis van de resultaten van De Winter [38], vermoeden we dat Stelling B.4.9 ook geldt voor $\mathrm{AG}(n, q)$, $n \geq 5$.

Wat betreft Probleem 2', was voorheen enkel het volgende resultaat gekend.

Stelling B.4.10 (Brown, De Clerck, Delanote [11]) *Als* $\mathcal{S}$ *een semipartiële meetkunde* $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ *is, ingebed in* $\mathrm{AG}(4, q)$, *dan is* $q = 2^h$ *en* $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.

Aangezien Stelling B.4.6 de volledige oplossing geeft van Probleem 2, is Probleem 2' nu a forteriori ook volledig opgelost.

Stelling B.4.11 *Als* $\mathcal{S}$ *een semipartiële meetkunde is met* $\alpha = 2$, *ingebed in* $\mathrm{AG}(n, q)$, $q = 2^h$, *zodat er tenminste één vlak van type* IV *is, dan geldt één van de volgende gevallen.*

1. *$q = 2$ en $\mathcal{S}$ is een $2-(t+2,2,1)$-design.*

2. *$n = 2$ en $\mathcal{S}$ is een duale ovaal.*

3. *$n = 4$ en $\mathcal{S} \simeq \mathrm{TQ}(4,q)$.*

Stellingen B.4.4 en B.4.11 leiden tot het volgende resultaat.

Stelling B.4.12 *Als $\mathcal{S}$ een semipartiële meetkunde is, $\alpha > 1$, ingebed in $\mathrm{AG}(n,q)$, dan geldt één van de volgende gevallen.*

1. *$q = 2$, $\alpha = 2$ en $\mathcal{S}$ is een $2-(t+2,2,1)$-design.*

2. *$n = 2$, $q = 2^h$, $\alpha = 2$ en $\mathcal{S}$ is een duale ovaal.*

3. *$n = 4$, $q = 2^h$, $\alpha = 2$ en $\mathcal{S} \simeq \mathrm{TQ}(4,q)$.*

4. *$n \geq 2$ en $\mathcal{S} \simeq T^*_{n-1}(\mathcal{K}_\infty)$, met $\mathcal{K}_\infty$ een verzameling van type $(0,1,\alpha+1)$ in Π_∞, die twee intersectiegetallen heeft met betrekking tot hypervlakken van Π_∞.*

Uit Stellingen B.4.8, B.4.9 en B.4.12 volgt nu de classificatie van eigenlijke semipartiële meetkunden met $\alpha > 1$ ingebed in $\mathrm{AG}(n,q)$, $n \leq 4$.

Stelling B.4.13 *Als $\mathcal{S}$ een eigenlijke semipartiële meetkunde is, $\alpha > 1$, ingebed in $\mathrm{AG}(n,q)$, $n \leq 4$, dan geldt één van de volgende gevallen.*

1. *$n = 3$, q is een kwadraat en $\mathcal{S} \simeq T^*_2(\mathcal{U}_\infty)$, met $\mathcal{U}_\infty$ een unitaal van Π_∞.*

2. *$n = 4$, $q = 2^h$, $\alpha = 2$ en $\mathcal{S} \simeq \mathrm{TQ}(4,q)$.*

3. *$n \in \{3,4\}$, q is een kwadraat en $\mathcal{S} \simeq T^*_2(\mathcal{B}_\infty)$, met $\mathcal{B}_\infty$ een Baer deelvlak van Π_∞.*

B.5 Classificatie van $(0,2)$-meetkunden ingebed in $\mathrm{AG}(3,2^h)$

In Hoofdstukken 5 en 6 geven we het bewijs van Stelling B.4.6. We veronderstellen hierbij dat $q > 2$, aangezien het geval $q = 2$ triviaal is. We bewijzen Stelling B.4.6 door inductie toe te passen op de dimensie van de affiene ruimte. Daarom moeten we eerst het bewijs geven voor de kleinste dimensie, namelijk voor $\mathrm{AG}(3,q)$. Dit gebeurt in Hoofdstuk 5.

Stelling B.5.1 *Als $\mathcal{S}$ een $(0,2)$-meetkunde is, ingebed in* $\mathrm{AG}(3,q)$*, $q=2^h$, $h>1$, zodat er tenminste één vlak van type* IV *is, dan geldt één van de volgende gevallen.*

1. $\mathcal{S} \simeq \mathcal{A}(O_\infty)$.
2. $\mathcal{S} \simeq \mathcal{I}(3,q,e)$.

Het bewijs van Stelling B.5.1 is in essentie een gevallenonderzoek op verschillende niveaus. Op het bovenste niveau maken we een onderscheid tussen de gevallen waarbij er tenminste één planair net is, dit wil zeggen, een vlak van type III, en het geval waarbij er geen enkel planair net is.

1. In het geval er minstens één planair net is, is er geen verdere specificatie nodig en komen we vrij snel tot het besluit dat $\mathcal{S} \simeq \mathcal{I}(3,q,e)$. We steunen hierbij op een resultaat van Payne [65], dat zegt dat elke hyperovaal met een additief o-polynoom, een translatiehyperovaal is.

2. Het geval waarbij er geen enkel planair net is, is beduidend moeilijker. We gaan dan ook een gevallenonderzoek doen op het tweede niveau, nu aan de hand van de orde $(q-1,t)$ van $\mathcal{S}$.

   (a) Eerst beschouwen we het geval waarbij $t \neq q$. Dit deel van het bewijs is van puur combinatorische aard. Eerst tonen we aan dat $2 < t < q-1$. Vervolgens leiden we een aantal ongelijkheden en delingsvoorwaarden af omtrent de combinatorische structuur van $\mathcal{S}$. Dit leidt echter nog niet tot de volledige oplossing in dit geval; hiervoor moeten we opnieuw een gevallenonderzoek doen, nu op het derde niveau. We maken namelijk een onderscheid tussen de gevallen waarbij t oneven is enerzijds, en waarbij t even is anderzijds. We komen echter in beide gevallen tot een tegenstrijdigheid. Hiermee is dit geval afgehandeld.

   (b) Vervolgens beschouwen we het geval waarbij $t=q$. Zij $\mathcal{P}$ de verzameling van punten van $\mathcal{S}$. We bewijzen dat $|\,\mathcal{P}\,| = kq^2$, met $k \in \{\frac{1}{2}q, q\}$. We tonen aan dat $\mathcal{S} \simeq \mathrm{HT}$ als $k=\frac{1}{2}q$. Hierbij maken we gebruik van de classificatie van verzamelingen van type $(1, \frac{1}{2}q+1, q)$ in $\mathrm{PG}(3,q)$ (zie Hirschfeld en Thas [53] en Glynn [46]).

   Aangezien $t=q$, gaan er door elk punt van $\mathcal{S}$ juist $q+1$ rechten van $\mathcal{S}$. We bewijzen dat deze rechten Π_∞ steeds snijden in een ovaal, en dat de verzameling van al deze ovalen een reguliere Desarguesiaanse planaire ovaalverzameling in Π_∞ is. Vervolgens passen we Stelling B.3.1 toe, welke ons toelaat om aan te tonen dat $\mathcal{S} \simeq \mathcal{A}(O_\infty)$. Hiermee is het bewijs van Stelling B.5.1 compleet.

B.6 Classificatie van $(0, 2)$-meetkunden ingebed in AG$(n, 2^h)$

In Hoofdstuk 6 vervolledigen we het bewijs van Stelling B.4.6. We veronderstellen opnieuw dat $q > 2$. Aangezien het geval AG$(3, q)$ in Hoofdstuk 5 afgehandeld is, mogen we ons beperken tot het geval AG(n, q), $n \geq 4$.

Stelling B.6.1 *Als $\mathcal{S}$ een $(0, 2)$-meetkunde is, ingebed in* AG(n, q)*, $n \geq 4$, $q = 2^h$, $h > 1$, zodat er tenminste één vlak van type* IV *is, dan geldt één van de volgende gevallen.*

1. $n = 4$ *en* $\mathcal{S} \simeq$ TQ$(4, q)$.
2. $\mathcal{S} \simeq \mathcal{I}(n, q, e)$.

We maken gebruik van de volgende terminologie. Zij U een m-dimensionale deelruimte van AG(n, q), $3 \leq m < n$, en zij $\mathcal{S}_U$ de sub incidentiestructuur, geïnduceerd op de verzameling van punten en rechten van $\mathcal{S}$ die in U liggen. Dan kan U van één van volgende types zijn.

Type A. $m = 3$ en $\mathcal{S}_U$ bevat een samenhangende component $\mathcal{S}' \simeq \mathcal{A}(O_\infty)$, of $m = 4$ en $\mathcal{S}_U$ bevat een samenhangende component $\mathcal{S}' \simeq$ TQ$(4, q)$.

Type B. $\mathcal{S}_U$ bevat een samenhangende component $\mathcal{S}' \simeq \mathcal{I}(m, q, e)$.

Type C. $\mathcal{S}_U$ is een samenhangende lineaire representatie.

Type D. Elke samenhangende component van $\mathcal{S}_U$ is bevat in een echte deelruimte van U.

Uit Stelling B.5.1 volgt nu dat iedere 3-ruimte van type **A**, **B**, **C** of **D** is. Dit feit wordt dikwijls gebruikt in het bewijs van Stelling B.6.1. We maken onderscheid tussen de volgende gevallen.

1. Er is minstens één planair net. In dit geval bestaat het bewijs uit de volgende stappen.

   (a) Eerst bewijzen we dat, als er een hypervlak van type **C** is (respectievelijk een vlak van type III als $n = 4$), en een vlak van type III, dan snijden die elkaar niet in een affiene rechte. Om dit te bewijzen gebruiken we inductie op de dimensie van de affiene ruimte.

   (b) Vervolgens tonen we aan dat, als er een hypervlak van type **C** is, dan $\mathcal{S} \simeq \mathcal{I}(n, q, e)$. We steunen op de vorige stap, en gebruiken een inductie-argument op de dimensie van de affiene ruimte.

   (c) Tot slot bewijzen we, enkel vanuit de veronderstelling dat er een planair net is, dat $\mathcal{S} \simeq \mathcal{I}(n, q, e)$. Het geval $n = 4$ is hierbij het moeilijkste. We steunen opnieuw op de vorige stap, en gebruiken een inductie-argument op de dimensie van de affiene ruimte.

2. In het geval er geen planaire netten zijn, bestaat het bewijs uit de volgende stappen.

   (a) Eerst bewijzen we dat, indien $n = 4$, de verzameling $\mathcal{R} = \mathcal{P} \cup \Pi_\infty$, waarbij $\mathcal{P}$ de verzameling van punten van $\mathcal{S}$ is, een verzameling van type $(1, \frac{1}{2}q, q+1)$ in $\mathrm{PG}(4, q)$ is. Uit de classificatie van verzamelingen van type $(1, \frac{1}{2}q, q+1)$ in $\mathrm{PG}(n, q)$ (zie Hirschfeld en Thas [53, 52]), volgt dan dat $\mathcal{S} \simeq \mathrm{TQ}(4, q)$.

   (b) Vervolgens tonen we aan dat, indien $n = 5$, de verzameling $\mathcal{R}$ een verzameling van type $(1, \frac{1}{2}q, q + 1)$ in $\mathrm{PG}(5, q)$ is. Hierbij maken we gebruik van de vorige stap. Met behulp van de classificatie van verzamelingen van type $(1, \frac{1}{2}q, q+1)$ in $\mathrm{PG}(n, q)$ leiden we nu een tegenstrijdigheid af.

   (c) Tot slot veronderstellen we dat $n \geq 6$, en gebruiken we de vorige stap om een strijdigheid af te leiden. Hiermee is het bewijs van Stelling B.6.1 compleet, en bijgevolg ook dat van Stelling B.4.6.

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