An algebraic approach to the perfect Ree-Tits generalized octagons and related geometries

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Een algebraïsche aanpak van de perfecte veralgemeende achthoeken van Ree-Tits en verwante meetkundes

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Vooraf

> *Before we begin our banquet, I would like to say a few words. And here they are: Nitwit! Blubber! Oddment! Tweak!*
>
> Harry Potter and the Philosopher's Stone — J.K. Rowling

Wellicht kreeg ik de smaak voor eindige meetkunde te pakken op de tweede internationale conferentie over 'finite geometry and combinatorics' te Deinze (in juni 1992). Ik was zeer gefascineerd door de veralgemeende zeshoek op 63 punten, die me heel wat interessanter leek dan al die veralgemeende vierhoeken waarover toen in Gent zoveel te doen was.

Het leek me dan ook een logische stap om bij Hendrik Van Maldeghem aan te kloppen met de vraag naar een onderwerp voor een doctoraat. Zijn antwoord zal me altijd bijblijven: "Waarom probeer je niet liever de veralgemeende achthoeken, daar is nog niet zoveel op gewerkt en *jij hebt er toch de tijd voor*."

Met dank aan al wie bleef geloven dat het er nog wel eens van zou komen.

De rest van deze tekst is in het Engels opgesteld. Appendix B (blz. 247) bevat een Nederlandstalige samenvatting.

Contents

1 Introduction and overview

Of all generalized polygons related to algebraic groups the generalized octagons have probably been studied the least. Perhaps one of the reasons is the fact that only a single (infinite) family of examples is known and that these *Ree-Tits* octagons, and their embeddings in a projective space, are not easily constructed.

The standard way to define a Ree-Tits octagon is as a coset geometry of a Ree group [25]. This group is constructed by 'twisting' the Chevalley group of type F_4 over a suitable field K of characteristic 2. Another construction technique makes use of a special coordinatization introduced by H. Van Maldeghem [26], which is itself based on the properties of the Ree groups, in particular the commutation relations displayed in [24].

What is missing for the Ree-Tits octagon, and exists for all other classical generalized polygons, is an explicit embedding into some projective space. The main objective of this text is to establish a construction of such an embedding into a 25-dimensional space, where the points (lines) of the Ree-Tits octagon are a subset of the points (lines) of the projective space.

We will provide explicit 'formulae' which can be applied to the projective coordinates of a point to determine whether or not it is a point of the octagon. And likewise, we will provide a means to determine from the projective coordinates of two points of the octagon, whether they are collinear in the octagon, and more generally, what is their mutual distance.

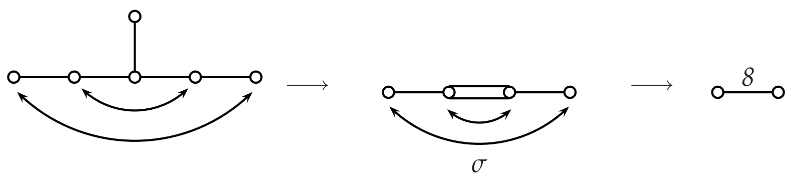
These 'formulae' are not so elementary as in the case of the other generalized polygons. For example, we will prove that a point Ke belongs to the Ree-Tits octagon if and only if both $e^2 = 0$ and $q([e, \mathbf{V}], e) = 0$. The first expression is shorthand for a system of 26 quadratic equations in 26 variables (the 26 coordinates of e). The second expression corresponds to a system of 676 polynomials of degree 3 in 26 variables, additionally involving a field automorphism σ of K. (For contrast, compare this to the formulae needed to define the points of the split Cayley hexagon : a single quadratic equation in 7 variables suffices.)

• Regretfully, the theory we will develop only works for *perfect* Ree-Tits octagons. However, there is reason to believe that most of the results we have obtained can be reformulated to also hold when K is not a perfect field.

This text looks at the octagon (and the metasymplectic space to which its points and lines belong) from three different perspectives : groups, geometries and algebras, concentrating on the latter.

We proceed in three stages: first we provide an in-depth treatment of the Lie algebra (and group) of type E_6 over a general field (Chapter 2), then we consider the subalgebra of type F_4 in general characteristic (Chapter 3) and in characteristic 2 (Chapter 4), and finally we obtain a description of the perfect Ree-Tits generalized octagon (Chapter 5).

The techniques we will use can be summarized in the following picture of Coxeter diagrams and Tits indices :



Unfortunately, a picture is only worth a thousand words. We will need four chapters to explain it in full detail.

• In Appendix A we provide an illustration of how the techniques of the main text can be applied to a much simpler example : we reduce the Lie algebra of type A_3 to one of type B_2 and finally 'twist' it to obtain a Suzuki-Tits ovoid. This appendix can be read without reference to the main text.

1.1 Generalized polygons

An *incidence geometry* (of rank 2) consists of a set $\mathcal{P}$ of points, a set $\mathcal{L}$ of lines and an incidence relation relating points to lines. The *incidence graph* of an incidence geometry is a (simple undirected) bipartite graph with vertex set $\mathcal{P} \cup \mathcal{L}$ and edges corresponding to incident point-line pairs (we assume that $\mathcal{P}$ and $\mathcal{L}$ are disjoint).

Let n be a positive integer. A *generalized n-gon* is an incidence geometry whose incidence graph Γ satisfies the following axioms [26] :

1. The diameter of Γ is n.

2. The girth of Γ (i.e., the length of the shortest cycle in Γ) is $2n$.

3. (Nondegeneracy condition.) Every vertex of Γ lies on at least 3 edges.

Interchanging points and lines of an incidence geometry yields another incidence geometry which is called the *dual* of the first. Note that the dual of a generalized n-gon is again a generalized n-gon (with the same n).

In a generalized *digon* (2-gon) every point is incident with every line (and conversely), making the structure not very interesting (see however Section 1.8).

For $n > 2$ however, two points will be incident with (are *joined* by) at most one common line, and two lines will be incident with (*intersect* in) at most one common point, making the geometry into a *partial linear space*. For $n = 3$, every pair of points are joined by exactly one line and every pair of lines intersect in exactly one point. In other words, generalized 3-gons are none other than the well-known *projective planes*.

The defining axioms for generalized n-gons lead to many interesting combinatorial properties. For example, every line of a generalized n-gon is incident with the same number of points (and dually, every point is incident with the same number of lines). In the finite case, i.e., when both $\mathcal{P}$ and $\mathcal{L}$ are finite, a theorem by Feit and Higman [16] states that only the cases $n = 2, 3, 4, 6$ and 8 may occur.

• It has already been mentioned that the smaller the n the better have the corresponding n-gons been studied. Another reason for this is probably the size of the examples : the smallest 3-, 4-, 6- and 8-gons have 7, 15, 63 and 1755 points (and at least as many lines).

The earliest examples of generalized polygons, called the *classical* generalized polygons, arise from groups of Lie type or twisted groups of Lie type. For example, the classical Desarguesian projective plane is related to the general linear group of dimension 3, i.e., a Lie group of type A_2. Generalized quadrangles (4-gons) can be obtained from the symplectic, orthogonal and unitary groups (of type B_2, C_2 and ${}^2\mathsf{A}_3$), generalized hexagons (6-gons) from groups of type G_2 and ${}^3\mathsf{D}_4$, and finally, the Ree-Tits generalized octagon (8-gon), which will be central to this text, from the Ree group of type ${}^2\mathsf{F}_4$.

• Apart from the generalized Ree-Tits octagons, no other generalized octagons are known, except for some 'free' and 'universal' constructions (cf. [26, Section 1.3.13] and [21]), which in some sense do not really count. Some experts believe that (at least in the finite case) no others exist, but this has not yet been proved. It is not even known whether all generalized octagons with 3 points on a line and 5 lines through each point (i.e., the smallest case) are necessarily isomorphic.

1.2 Root systems

Before we can talk about Lie algebras, we need to introduce one of their basic building blocks, the root systems.

Consider a finite dimensional real Euclidian vector space $\mathbf{P}$. For each vector $r \in \mathbf{P}$, $r \neq 0$, consider the reflection w_r in the hyperplane orthogonal to r, i.e.,

$$w_r(x) \stackrel{\text{def}}{=} x - \frac{2r \cdot x}{r \cdot r} r.$$

A subset Φ of $\mathbf{P}$ is called a *root system* of $\mathbf{P}$ if it satisfies the following properties [8] :

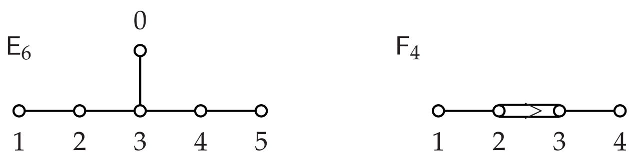
1. Φ is a finite set of non-zero vectors,
2. Φ spans $\mathbf{P}$,
3. If $r, s \in \Phi$, then $w_r(s) \in \Phi$,
4. If $r, s \in \Phi$, then $\langle r, s\rangle \overset{\text{def}}{=} 2(r \cdot s)/(r \cdot r)$ is an integer,
5. If $r, s \in \Phi$ are linearly dependent, then $r = \pm s$.

We will be interested in two specific examples of root systems. For the first one, said to be of type E_6, we will use a non-standard construction based on a 6-dimensional orthogonal real representation of the generalized quadrangle $Q^-(5,2)$. The second one, of type F_4, will be derived from the first by projecting it onto a special 4-dimensional subspace.

• In chapter 5 we will project this second example once more onto a 2-dimensional subspace. The resulting set could be called a 'root system of type ${}^2\mathsf{F}_4$', although it does not satisfy the definition above.

Every root system Φ contains a subset $\{\pi_0, \ldots, \pi_{n-1}\}$ of linearly independent vectors such that every root is a linear combination of elements of this subset with coefficients which are integral and either all non-negative or all non-positive. Such a set is called a *fundamental system of roots* and its elements are called *fundamental roots*.

Fundamental roots can be used to draw a *Dynkin diagram* which is an invariant of the root system. Although there are many different ways to select a fundamental root system in Φ, they always have isomorphic Dynkin diagrams. The diagrams for the root systems of type E_6 and F_4 are depicted below (we have numbered the nodes for future reference) :



Each node i of the Dynkin diagram corresponds to a fundamental root π_i. Two nodes in the diagram that are not connected satisfy $\langle \pi_i, \pi_j \rangle = 0$, two nodes that are connected by a single line satisfy $\langle \pi_i, \pi_j \rangle = \langle \pi_j, \pi_i \rangle = -1$ and two nodes that are connected by a double line satisfy $\{\langle \pi_i, \pi_j \rangle, \langle \pi_j, \pi_i \rangle\} = \{-1, -2\}$. Equivalently, the number of lines between two roots in the diagram indicate the angle between them : $\pi/2$ when there is no line, $2\pi/3$ when there is a single line and $3\pi/4$ when there is a double line.

In a root system of type E_6 all roots have the same length, and as a consequence $\langle r, s \rangle$ is symmetric. In a root system of type F_4 however, we distinguish between *short* and *long* roots (the lengths differ by a factor of $\sqrt{2}$.) The $>$-sign on the diagram shows that we have chosen to put the two fundamental long roots on the left, and the two fundamental short roots on the right.

The root system Φ_F of type F_4 can be obtained from the root system Φ of type E_6 by projecting it onto a suitable 4-dimensional subspace $\mathbf{P}_F$ of $\mathbf{P}$. The 24 roots of E_6 that already belong to $\mathbf{P}_F$ (and which form a root system of type D_4 in their own right) will become long roots of the new system, while the other 48 will be projected onto 24 different short roots of the new system.

The projection onto $\mathbf{P}_F$ can be written as $x \mapsto (x + \overline{x})/2$ where $\bar{\cdot}$ is an involution on $\mathbf{P}$. The long roots are precisely those roots that are left invariant by this involution. The fundamental roots of Φ_F are projections of the fundamental roots of Φ. More specifically, they have the following values :

$$\pi_0, \quad \pi_3, \quad \frac{1}{2}(\pi_2 + \pi_4), \quad \frac{1}{2}(\pi_1 + \pi_5).$$

From this it is easy to understand why this projection operation is often called a *folding* of the diagram.

With every root system Φ we may associate the *dual* root system Φ^* of roots r^* of the form

$$r^* \stackrel{\text{def}}{=} \frac{2r}{r \cdot r}, \qquad \text{with } r \in \Phi.$$

The element r^* is called the *co-root* corresponding to r. Note that $(r^*)^* = r$ and $\langle r^*, s^* \rangle = \langle s, r \rangle$. The notion of co-root is mainly useful in a root system like Φ_F where not every root has the same length.

The finite group generated by all reflections w_r for $r \in \Phi$ is called the *Weyl group* $W(\Phi)$ of that root system. The Weyl group acts regularly on the set of all fundamental systems of roots of Φ.

The Dynkin diagram for Φ can also be interpreted as a *Coxeter diagram* for $W(\Phi)$. The nodes correspond to generators $g_i = w_{\pi_i}$ satisfying $g_i^2 = 1$, with $(g_i g_j)^2 = 1$ when nodes i and j are not connected by a line in the diagram, $(g_i g_j)^3 = 1$ when connected by a single line, and $(g_i g_j)^4 = 1$ when connected by a double line.

- The 'Dynkin diagram' for the 'root system' of type ${}^2\mathsf{F}_4$ consists of two nodes connected by 4 lines. The corresponding 'Weyl group' is the dihedral group of order 16 with generators g_1, g_2 such that $g_1^2 = g_2^2 = (g_1 g_2)^8 = 1$.

An important role is also reserved for the dual of the base $\{\pi_0, \ldots, \pi_{n-1}\}$, i.e., a set of vectors $\{\pi'_0, \ldots, \pi'_{n-1}\}$ such that $\langle \pi_i, \pi'_j \rangle = \delta_{ij}$, for every $i, j \in \{0, \ldots, n-1\}$. The elements π'_i are called *fundamental weights* of the root system.

Different fundamental systems of roots yield different fundamental weights, but weights with the same index i will always belong to the same orbit $\mathcal{P}_i$ of the Weyl group. In the case of E_6, with corresponding Weyl group $W(\mathsf{E}_6)$, the orbit for π'_0 is the full set $\Phi = \mathcal{P}_0$ of 72 roots.

The orbit $\mathcal{P} = \mathcal{P}_1$ of π'_1 (with size 27) is of special interest. Its elements will be called *points* because they correspond to the points of the generalized quadrangle which we have used initially to define the root system. The symmetry of the diagram is also reflected in the properties of the weight orbits. We have

$$\mathcal{P}_0 = -\mathcal{P}_0,\ \mathcal{P}_1 = -\mathcal{P}_5,\ \mathcal{P}_2 = -\mathcal{P}_4,\ \mathcal{P}_3 = -\mathcal{P}_3.$$

For a root system of type F_4 the *long roots* are the weights that correspond to the leftmost node of the diagram and the *short roots* correspond to the rightmost node. There are 24 roots of each type. The long roots of F_4 form a subset of the roots of E_6, the short roots do not.

- The symmetry of the diagram of F_4 does not immediately provide an interpretation in terms of weight orbits. Recall that left and right hand nodes now correspond to fundamental weights of different length.

The Weyl group of the root system of type F_4 can be embedded as a subgroup $W(\mathsf{F}_4)$ into $W(\mathsf{E}_6)$. This group splits $\mathcal{P}$ into two orbits : a set L_∞ of 3 points on a line, which we will call the *line at infinity*, and a set of 24 points $\mathcal{P}^*$. If $p \in L_\infty$ then $p + \overline{p} = 0$. If $p \in \mathcal{P}^*$ then $s = (p + \overline{p})/2$ is a short root and the correspondence $p \leftrightarrow s$ is one-to-one. It turns out that for most practical purposes points of $\mathcal{P}^*$ and short roots are interchangeable.

1.3 Lie algebras

It is well known that to every root system there corresponds a *Lie algebra* over a given field K. This is a vector space $\mathbf{A}$ endowed with a bilinear product $[\cdot,\cdot]$ (called *Lie product* or *Lie bracket*) satisfying the following axioms :

1. $[A, A] = 0$, for every $A \in \mathbf{A}$.

2. (The Jacobi identity) $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$, for every $A, B, C \in \mathbf{A}$.

Typically $\mathbf{A}$ is a subspace of $\mathrm{Hom}(\mathbf{T},\mathbf{T})$, the algebra of linear transformations on some vector space $\mathbf{T}$, and then the Lie bracket is defined by $[A, B] \stackrel{\text{def}}{=} AB - BA$.

In the case of E_6 we will take $\mathbf{T}$ to be a 27-dimensional vector space $\mathbf{V}$ endowed with a special trilinear form $\langle\cdot,\cdot,\cdot\rangle$ and $\mathbf{A}$ to be the space $\mathbf{L}$ of all linear transformations A that satisfy $\langle aA, b, c\rangle + \langle a, bA, c\rangle + \langle a, b, cA\rangle = 0$, for every $a, b, c \in \mathbf{V}$. Then $\mathbf{L}$ is an algebra of dimension 78.

It is no coincidence that the dimension of $\mathbf{V}$ is the same as the number of points of $Q^-(5,2)$. There is indeed a one-to-one correspondence between these points and a set of canonical base vectors for $\mathbf{V}$. The trilinear form $\langle\cdot,\cdot,\cdot\rangle$ can be defined by means of the lines of $Q^-(5,2)$ and a regular spread Σ of such lines.

In the case of F_4 the associated Lie algebra $\mathbf{J}$ has dimension 52. It is a subalgebra of $\mathbf{L}$ consisting of those elements A for which $\infty A = 0$, where ∞ denotes a special element of $\mathbf{V}$.

Alternatively, we may use the element ∞ to define an involution $\bar{\ }$ which interchanges $\mathbf{V}$ and $\mathbf{V}^*$ (the dual vector space of $\mathbf{V}$) and easily generalizes to $\mathrm{Hom}(\mathbf{V}, \mathbf{V})$. The algebra $\mathbf{J}$ then consists of precisely those elements A of $\mathbf{L}$ that satisfy $A + \overline{A} = 0$.

In general, every Lie algebra $\mathbf{A}$ associated with a root system Φ contains a so-called *Chevalley basis*. This basis consists of elements e_r, one for each $r \in \Phi$ and elements h_{π_i}, one for each fundamental root π_i. The elements h_r are defined for all $r \in \Phi$ (not just for fundamental roots) and behave somewhat like the co-roots r^* of Φ^* (e.g., $h_{(r+s)^*} = h_{r^*} + h_{s^*}$). They generate the *torus* or *Cartan subalgebra* of the Lie algebra.

A Chevalley basis satisfies the following properties [8, Theorem 4.2.1] :

Let $r, s \in \Phi$. Then

1. $[h_r, h_s] = 0$,
2. $[h_r, e_s] = \langle r, s \rangle e_s$,
3. $[e_r, e_{-r}] = h_r$,
4. $[e_r, e_s] = 0$, when $r + s \notin \Phi$,
5. $[e_r, e_s] = N_{r,s} e_{r+s}$, when $r + s \in \Phi$,

where $N_{r,s} = \pm i$ with i the greatest integer i for which $r + s - ir \in \Phi$.

• For a Lie algebra of type E_6, $N_{r,s} = \pm 1$. For a Lie algebra of type F_4, $N_{r,s} = \pm 1$ or ± 2. Except for the unspecified sign of $N_{r,s}$, the Chevalley basis properties uniquely define the Lie algebra $\mathbf{L}$. When K has characteristic 2 the signs are immaterial. In the general case it is possible to specify exactly what the sign of $N_{r,s}$ must be for each pair r, s [8, Proposition 4.2.2].

1.4 Modules

A vector space $\mathbf{T}$ onto which $\mathbf{A}$ can act as a linear transformation (notation t^A for $t \in \mathbf{T}$, $A \in \mathbf{A}$) is called an $\mathbf{A}$-*module* (or in this context, an E_6-*module* or F_4-*module*) when $t^{[A,B]} = (t^A)^B - (t^B)^A$ for every $t \in \mathbf{T}$ and $A, B \in \mathbf{A}$.

There are many examples of $\mathbf{A}$-modules. Clearly, if $\mathbf{A}$ was initially constructed as a subalgebra of $\mathrm{Hom}(\mathbf{T}, \mathbf{T})$, then $\mathbf{T}$ is an $\mathbf{A}$-module if we set $a^A = aA$. Also $\mathbf{T}^*$ (the dual vector space of $\mathbf{T}$) is an $\mathbf{A}$-module with the action $\alpha^A = -A\alpha$. Additional examples are given by the Lie algebra $\mathbf{A}$ itself, which is an $\mathbf{A}$-module with $C^A = [C, A]$, and the base field K, which is a trivial 1-dimensional $\mathbf{A}$-module when we take $k^A = 0$ for every $k \in K$.

In fact, it is possible to associate an $\mathbf{A}$-module with the orbit of each weight. In particular, we will denote the module that corresponds to the orbit $\mathcal{P}_i$ of the fundamental weight π_i' by $\mathbf{T}_i$.

In the case of E_6 we have $\mathbf{T}_0 = \mathbf{L}$, $\mathbf{T}_1 = \mathbf{V}$, $\mathbf{T}_2 = \mathbf{V} \wedge \mathbf{V}$, $\mathbf{T}_3 = \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$, $\mathbf{T}_4 = \mathbf{V}^* \wedge \mathbf{V}^*$ and $\mathbf{T}_5 = \mathbf{V}^*$. In the case of F_4, the module that corresponds to the fundamental weight that is a long root is $\mathbf{J}$ itself, while the module for the fundamental weight that is a short root has dimension 26 and can be embedded into $\mathbf{V}$ as the hyperspace $\mathbf{W}$ of elements a that satisfy $a\overline{\infty} = 0$. (We will not explicitely establish the modules for the middle nodes of the Dynkin diagram for F_4.)

There is no easy general recipe to construct the module associated with a particular orbit of the Weyl group, hence we have to treat each case separately. The same holds for the action of the Lie algebra onto these vector spaces.

In the case of E_6, the connection between $\mathbf{T}_1 = \mathbf{V}$ and $\mathcal{P}_1 = \mathcal{P}$ is immediate : the points p of $\mathcal{P}$ serve as canonical base vectors e_p for $\mathbf{V}$. Similarly, every element $-q$ of $-\mathcal{P}$ is related to a canonical base vector η_q of $\mathbf{V}^*$. The Chevalley basis provides a way to connect $\mathbf{T}_0 = \mathbf{L}$ with $\mathcal{P}_0 = \Phi$: each of the 72 roots r does indeed define a base element e_r of $\mathbf{L}$, but we need the extra 6 elements h_{π_i} to generate the full module.

Unfortunately, for other modules we need to be more creative. In general, a module **T** associated with an orbit of the Weyl group has a canonical base vector for every element of that orbit, but these vectors do not necessarily generate the entire module as a vector space. Case in point is the 26-dimensional module **W** for F_4 : there are only 24 short roots in the orbit of the corresponding weight, so we need to find the 2 extra dimensions elsewhere. (In this particular case it was more straightforward to construct the F_4-module **W** as a subspace of the E_6-module **V**.)

1.5 Chevalley groups and isotropic elements

The general linear group $\mathrm{GL}(\mathbf{T})$ acts in a natural way on $\mathbf{T}, \mathbf{T}^*$ and $\mathrm{Hom}(\mathbf{T}, \mathbf{T})$. However, it does not always leave **A** or some of its modules, invariant. Therefore, when studying Lie algebras and their geometries, we consider only certain special subgroups of $\mathrm{GL}(\mathbf{T})$, the so-called *Chevalley groups*. If we would interpret the **A**-modules as 'generalizations' of weight orbits, then the Chevalley groups could be considered 'generalizations' of the Weyl groups.

In the case of E_6 we denote the group by $\widehat{\mathsf{E}}_6(K)$. Elements g of $\widehat{\mathsf{E}}_6(K)$ satisfy $\langle a^g, b^g, c^g \rangle = \langle a, b, c \rangle$ for every $a, b, c \in \mathbf{V}$. $\widehat{\mathsf{E}}_6(K)$ acts in a natural way on the E_6-modules introduced above. The Chevalley group $\widehat{\mathsf{F}}_4(K)$ of type F_4 is a subgroup of $\widehat{\mathsf{E}}_6(K)$. Every element of $\widehat{\mathsf{F}}_4(K)$ leaves the special element ∞ invariant.

In general, a Chevalley group has many orbits in its action on a specific module **T**. However, all canonical base vectors of **T** turn out always to belong to the same orbit. Any element of **T** that belongs to such an orbit, will be called *isotropic*. In practice we will always first give an algebraic definition of an isotropic element in a particular module, and then prove afterwards that the canonical base elements are isotropic and that the isotropic elements form a single orbit. As will be explained later, the notion of isotropic element also forms the basis for the construction of a *geometry* related to a given Lie algebra.

• As an added bonus isotropic elements also help us resolve some difficulties in cases where the base field K has small characteristic. As an example, consider again the case of E_6. For most fields the isotropic elements of $\mathbf{V} \wedge \mathbf{V}$ generate $\mathbf{V} \wedge \mathbf{V}$ (as a vector space) completely, but when char $K = 2$ this is not true. Although we will not expand on this in the text, certain operations which are of geometric interest can be defined on the space generated by the isotropic elements of $\mathbf{V} \wedge \mathbf{V}$, but this definition can not be extended to $\mathbf{V} \wedge \mathbf{V}$ itself when char $K = 2$. Similar issues arise for the module $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ when char $K = 2$ or 3.

The notion of isotropy does not carry over from algebras to subalgebras. To be sure, the isotropic elements of $\mathbf{J}$ are exactly those isotropic elements of $\mathbf{L}$ that belong to $\mathbf{J}$. Also, the isotropic elements of the F_4-module $\mathbf{W}$ are exactly those isotropic elements of the E_6-module $\mathbf{V}$ that belong to $\mathbf{W}$. It is however dangerous to generalize this to all modules. For example, we need to distinguish between *E-isotropic* elements of $\mathbf{V} \wedge \mathbf{V}$ (i.e., isotropic with respect to the algebra and Chevalley group of type E_6) and *F-isotropic* elements of the corresponding module for F_4.

There exists yet another connection between isotropic elements and Chevalley group elements. For every isotropic element E of $\mathbf{A}$, we may define a nonsingular linear transformation

$$x(E) = \exp E \stackrel{\text{def}}{=} 1 + E + \frac{1}{2}E^2 + \frac{1}{6}E^3 + \cdots ,$$

that belongs to the Chevalley group. In the case of E_6 (and hence also of F_4) we have $E^2 = 0$ whenever E is isotropic, so $x(E)$ has a simple form.

This formula generalizes to the case where E is of the form $A + \overline{A}$ where A is an isotropic element of $\mathbf{L} - \mathbf{J}$. Then $E^3 = 0$ and $x(E) = x(A)x(\overline{A}) = x(\overline{A})x(A)$. Group elements of these forms can be used to generate the full Chevalley group in both cases.

- In characteristic 2 the expression $\frac{1}{2}(A + \overline{A})^2$ should be interpreted as $A\overline{A}$.

1.6 Operations

Throughout the subsequent chapters the reader will notice that the number of (new) operations we found necessary to introduce is rather plentiful (for a complete list, see page 277). Most of these operations satisfy a number of *generic* properties which we describe below.

Consider three **A**-modules $\mathbf{T}$, $\mathbf{T}'$ and $\mathbf{T}''$, not necessarily distinct, and consider an operator $\circ$ defined for $t \in \mathbf{T}$, $t' \in \mathbf{T}'$ such that $t \circ t' \in \mathbf{T}''$. A first property is that $\circ$ is *compatible* with the Lie algebra $\mathbf{A}$ in the following sense :

$$t^A \circ t' + t \circ t'^A = (t \circ t')^A, \quad \text{for any } A \in \mathbf{A}, t \in \mathbf{T}, t' \in \mathbf{T}'.$$

Secondly, let e_p be a canonical base vector of $\mathbf{T}$ and let $e_{p'}$ be a canonical base vector of $\mathbf{T}'$ such that $p + p'$ is an element of the orbit of weights associated with $\mathbf{T}''$, then $e_p \circ e_{p'}$ is always a scalar multiple of $e_{p+p'}$. If $p + p'$ does not belong to the correct orbit, then $e_p \circ e_{p'}$ is zero, or in certain cases, undefined.

Finally, every operation is invariant for the corresponding Chevalley group, i.e., if g belongs to that group, then

$$t^g \circ t'^g = (t \circ t')^g, \qquad \text{for every } t \in \mathbf{T}, t' \in \mathbf{T}'.$$

Note that the action t^A of $A \in \mathbf{A}$ on $t \in \mathbf{T}$ is always an operation that satisfies these generic properties.

As with the modules itself, there is no easy general recipe for defining such operations, but the expected generic properties serve as helpful guidelines. As an illustration we investigate the various binary operations $a \circ \alpha$ that can be defined between elements $a \in \mathbf{V}$ and $\alpha \in \mathbf{V}^*$ in the case of E_6.

Consider the weight orbits $\mathcal{P}$ and $-\mathcal{P}$ associated with $\mathbf{V}$ and $\mathbf{V}^*$. For $p \in \mathcal{P}$ and $-q \in -\mathcal{P}$ the product $p \cdot -q$ can take one of three possible values : $-4/3$, $-1/3$ or $2/3$. This indicates that we will be able to define three binary operators. For each operator $\circ$ the value $a \circ \alpha$ will be found in the E_6-module associated with the weight $p - q$.

The first operation corresponds to the case $p \cdot -q = -4/3$. It is easily proved that this is only possible when $p = q$ and hence $p - q = 0$ is the trivial weight and the binary operation will take values in the trivial module K. A good candidate is the product $a\alpha$, the standard product between $\mathbf{V}$ and its dual $\mathbf{V}^*$. We easily verify that this product is compatible with $\mathbf{L}$,

$$a^A \alpha + a\alpha^A = (aA)\alpha - a(A\alpha) = 0 = (a\alpha)^A,$$

that it acts on the canonical base vectors in the expected way,

$$e_p \eta_q = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{otherwise,} \end{cases}$$

and that it is left invariant by $\widehat{\mathsf{E}}_6(K)$ (in fact, by $\mathrm{GL}(\mathbf{V})$ in this case).

For the second operation we need to consider the case $p \cdot -q = -1/3$. Now $p - q \in \Phi$ and we look for an operation with values in $\mathbf{L}$. In Section 2.2 we will introduce an operator $*$ which serves this purpose, and which will play a central role in the rest of the chapter. (The generic properties for this operation are proved in the text.) The third operation is the tensor product $\alpha \otimes a$ and corresponds to $p \cdot -q = 2/3$.

• Let us point out some of the difficulties that arise when we try to find an appropriate definition of a particular operation. Note for instance that $a * \alpha$ belongs to a module $\widehat{\mathbf{L}}$ which is slightly larger than the module $\mathbf{L}$ which would probably have been expected here. Because of the special case $\operatorname{char} K = 3$, the only other option here was to make $a * \alpha \in \mathbf{L}$ partially defined, i.e., only for those cases where $a\alpha = 0$.

A similar problem arises with the expression $E \times a$, with $E \in \mathbf{L}$, $a \in \mathbf{V}$. As is proved in Chapter 2, the value of this expression belongs to $\mathbf{V}^* \wedge \mathbf{V}^*$ only when $aE = 0$. Again this could be easily resolved by changing the definition of $E \times a$, but unfortunately there is no elegant way to do this when $\operatorname{char} K = 2$.

In general the solution seems to be to order the binary operations (for fixed $\mathbf{T}$ and $\mathbf{T}'$) according to increasing value of the corresponding product $p \cdot q$. A specific operation on $\mathbf{T} \times \mathbf{T}'$ need then only be defined for those elements $t \in \mathbf{T}$ and $t' \in \mathbf{T}'$ for which every 'prior' operation yields a value of zero. For example, the product $E \times a$ only needs to make sense when $aE = 0$, and $E \times F$ only when $EF = 0$, $[E, F] = 0$ and $E \cdot F = 0$, with $a \in \mathbf{V}$, $E, F \in \mathbf{L}$.

To further complicate matters, in higher dimensional modules some operations can only be defined on the submodules generated by isotropic elements, e.g., on a subspace of $\mathbf{V} \wedge \mathbf{V}$ and not on the entire space $\mathbf{V} \wedge \mathbf{V}$.

The various operations introduced in Chapter 2 (which treats E_6) are sufficient also for Chapter 3 (which treats F_4) although in many cases we combine them with the involution $\bar{\cdot}$.

For example, in the case of F_4 we will treat $a * \overline{b}$ as an operation on $\mathbf{V} \times \mathbf{V}$ (instead of $\mathbf{V} \times \mathbf{V}^*$). Even then it remains compatible with $\mathbf{J}$, i.e., $a^A * \overline{b} + a * \overline{b^A} = (a * \overline{b})^A$ for every $a, b \in \mathbf{V}$. This is because in some sense the involution is *compatible* with all operations introduced in the earlier chapters : $\overline{a * \alpha} = \overline{\alpha} * \overline{a}$, $a\alpha = \overline{\alpha}a$, etc.

• The group $\widehat{\mathsf{E}}_6(K)$ has essentially 3 orbits on pairs of isotropic elements of $\mathbf{V}$, leading to three different operations that can be defined on $\mathbf{V} \times \mathbf{V}$. The group $\widehat{\mathsf{F}}_4(K)$ has 5 orbits on pairs of isotropic elements (this time of $\mathbf{W}$) and hence we expect two new operations to emerge. These new operations turn out to correspond to $a * \overline{b}$ and $a\overline{b}$. In other words, they can be obtained from 'old' operations on $\mathbf{V} \times \mathbf{V}^*$ by applying $\bar{\cdot}$ to their second argument.

1.7 Geometry

The isotropic elements of the modules $\mathbf{T}_i$ are used as a basis for the construction of a geometry of rank n. The Dynkin diagram of the root system now serves as a *Buekenhout diagram* of the geometry, where each node corresponds to a different *type* (see also Section 1.8 below). The tables on the next page relate the various types with the node numbers in the diagram for the two Lie algebras of interest.

An element of a given type is a one-dimensional subspace of the form Kt_i, where t_i is an isotropic element of the corresponding module $\mathbf{T}_i$. Incidence between elements Kt_i and Kt_j of type i and j ($i \neq j$) can be expressed in terms of operations of $\mathbf{T}_i \times \mathbf{T}_j$. For example, a point Ka of $\mathcal{E}$ is incident with a dual point $K\alpha$ if and only if both $a\alpha = 0$ and $a * \alpha = 0$. It is incident with a simplex KE if and only if $aE = 0$ and $E \times a = 0$.

The geometry $\mathcal{E}$ of type E_6

node	type	node	type
0	simplex	3	plane
1	point	4	dual line
2	line	5	dual point

The geometry $\mathcal{F}$ of type F_4

node	type	node	type
1	hyperline	3	line
2	plane	4	point

A different way to describe the same geometry is to consider *isotropic subspaces* of **V** and $\mathbf{V}^*$. (A subspace is called isotropic when all its elements are.) Different types then correspond to isotropic subspaces of different dimensions. The following table lists the corresponding information in the case of E_6.

type	isotropic subspace
simplex	6-dimensional isotropic subspace of **V** (or $\mathbf{V}^*$)
point	1-dimensional isotropic subspace of **V**
line	2-dimensional isotropic subspace of **V**
plane	3-dimensional isotropic subspace of **V**
dual line	2-dimensional isotropic subspace of $\mathbf{V}^*$
dual point	1-dimensional isotropic subspace of $\mathbf{V}^*$

This interpretation makes some types of incidence more evident. For instance, incidence of points, lines and planes is the same as containment of the corresponding subspaces.

• A plane in $\mathcal{E}$ also corresponds to a 3-dimensional isotropic subspace of the dual space $\mathbf{V}^*$ and a dual line corresponds to a 5-dimensional isotropic subspace of **V**. Also, with every dual point we may associate a 10-dimensional subspace of $\mathbf{V}^*$ in which the isotropic points are those of a hyperbolic quadric.

We again need to be careful about the distinction between E-isotropic and F-isotropic : although points of $\mathcal{F}$ can be regarded as points of $\mathcal{E}$, the lines and planes of $\mathcal{F}$ are only a subset of those lines (and planes) of $\mathcal{E}$ all of whose points belong to $\mathcal{F}$.

In the case of F_4 we will introduce the notion of *F-isotropic* subspace (which is stronger than that of isotropic subspace) and then we have the following correspondence :

type	isotropic subspace
point	1-dimensional F-isotropic subspace of **W**
line	2-dimensional F-isotropic subspace of **W**
plane	3-dimensional F-isotropic subspace of **W**

• A hyperline does not correspond to an F-isotropic subspace. Instead with every hyperline we may associate a symplectic space which is embedded into a 6-dimensional isotropic subspace of **V**.

1.8 Geometries with a Coxeter diagram

To prove that the geometries $\mathcal{E}$ and $\mathcal{F}$ are indeed the well-known geometries of type E_6 and F_4, we need to apply the theory of *buildings*, in particular the *local approach* to buildings, as set down in [23]. It would lead us way too far to give a complete introduction to this theory here, hence we will concentrate on the points that are of interest to our main exposition.

Formally, define a *pre-geometry* $\mathcal{G}$ to be a structure consisting of

1. A set T of *types*, usually denoted by descriptive names like 'point', 'line', 'plane', ...

2. For each type t, a set of objects which are said to be *of* that type. Every object o must be of exactly one type, denoted by $t(o)$.

3. A symmetric *incidence* relation between objects of different types. (By extension, objects of the same type are said to be incident if and only if they are equal.)

We restrict ourselves to the case where T is finite, and then $|T|$ is called the *rank* of $\mathcal{G}$. A set F of pairwise incident objects is called a *flag* of $\mathcal{G}$. The type $t(F)$ of F is the set of types of all its elements. A maximal flag, i.e., a flag of type T, is called a *chamber*.

A pre-geometry $\mathcal{G}$ is called a *geometry* if it satisfies the following additional axiom :

4. Every flag of $\mathcal{G}$ can be extended to a chamber.

• Generalized polygons are examples of geometries of rank 2 (where the points and the lines are the objects of type ‘point’ and ‘line’ respectively). A projective space of dimension n is a geometry of rank n, with as objects the subspaces of dimension 0 up to $n-1$ and where the dimension serves as the type.

The *residue* of a flag F of a geometry $\mathcal{G}$ is a new geometry $\mathcal{G}_F$ with type set $T - t(F)$ and as objects all objects which are indicent with every element of F. A geometry is called *thick* if every residue of rank 1 (i.e., obtained from a flag containing objects of every type but one) contains at least 3 objects.

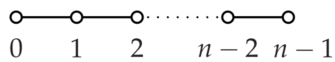
• The residue of a point in a generalized polygon consists of all lines through that point, a residue of a line consists of all points on that line. Generalized polygons as defined in Section 1.1 are always thick.

A geometry $\mathcal{G}$ is said to admit a *Coxeter diagram* if and only if for every pair $i, j \in T$ of different types and for every flag F of type $T - \{i, j\}$ the residue $\mathcal{G}_F$ is a generalized m_{ij}-gon, where $m_{ij} = m_{ji}$ depends only on the choice of i, j.

The map $\{i, j\} \mapsto m_{ij}$ is called the *Coxeter diagram* of $\mathcal{G}$. It is usually depicted as a graph with vertices representing the types and edges $\{i, j\}$ labelled with the corresponding integer m_{ij}. By convention edges with $m_{ij} = 2$ are omitted (i.e., when the residue is a digon), a single edge is drawn when $m_{ij} = 3$ (a projective plane), a double edge when $m_{ij} = 4$ (a generalized quadrangle) and a triple edge when $m_{ij} = 6$.

It is not so difficult to prove that a projective space of dimension n over a field K admits the following Coxeter diagram :

A_n

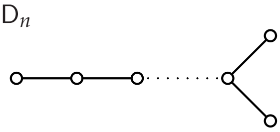


where each rank 2 residue is either a digon or a Desarguesian projective plane over K.

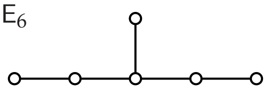
In fact, the theory of buildings teaches us that also a kind of converse is true: in a thick geometry with Coxeter diagram A_n all non-digonal rank 2 residues are isomorphic projective planes and the isomorphism type of the entire geometry is uniquely determined by the isomorphism type of these residues.

The same property holds for thick geometries with diagrams $\mathsf{D_n}$, $\mathsf{E_6}$, $\mathsf{E_7}$ (not shown) and $\mathsf{E_8}$ (not shown). It is no coincidence that the Coxeter diagrams look the same as the Dynkin diagrams with the same name.

D_n



E_6

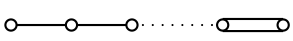


Hence, when we wish to prove that $\mathcal{E}$ is the well-known geometry (building) of type E_6 over the field K, we simply need to show that the corresponding rank 2 residues are either digonal or projective planes over K, as indicated in the diagram. We will prove this in Section 2.6.

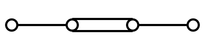
• In fact, we will take a shortcut: we only prove that the residues of the three end nodes of the diagram are the expected geometries of rank 5 (i.e., with Coxeter diagrams A_5, D_5 and D_5). As each rank 2 residue of $\mathcal{E}$ is also a rank 2 residue of at least one of these three residual geometries, this is sufficient.

When we also allow generalized quadrangles as possible residues, the situation is slightly more complicated. We will be interested in the following Coxeter diagrams :

B_n



F_4



(We will assume that the 'point' type corresponds to the leftmost node in the B_n diagram and to the rightmost node in the diagram of type F_4.)

If t is a type of a geometry, we define the *t-shadow* of an object o of that geometry to be the set of all objects of type t that are incident with o. A geometry with Coxeter diagram B_n or F_4 is said to satisfy the *intersection property* if the intersection of two point-shadows is always the point-shadow of some object of that geometry or empty and if the point-shadows of distinct elements are always distinct.

Now, the following can be proved [23] : Consider a thick geometry with Coxeter diagram B_n or F_4 which satisfies the intersection property. Let i, j be a pair of different types. Then all rank 2 residues with type set $\{i, j\}$ are isomorphic (independent of the chosen flag) and the isomorphism class of the geometry is uniquely determined by the isomorphism classes of the rank 2 residues for each i, j. Moreover, in the case of B_n, all residues with $m_{ij} = 3$ are isomorphic projective planes.

• That the intersection property is needed here is proved by the *Neumaier geometry* of the Fano planes in a fixed 7-set. This geometry has 7 points, 35 lines (triples of points) and 15 (Fano) planes on the 7-set (one of two orbits of Alt(7)). The Neumaier geometry does not satisfy the intersection property, for all planes have the entire 7-set as a point-shadow. This geometry has the same B_3 diagram as the symplectic polar space $W_5(2)$ (on 63 points) and all corresponding residues are isomorphic. The polar space however, does satisfy the intersection property.

In Section 3.5 we will show that $\mathcal{F}$ is the well-known geometry of type F_4 over the field K by proving that $\mathcal{F}$ satisfies the intersection property, that both residues with $m_{ij} = 3$ are projective planes over K and that the residue with $m_{ij} = 4$ is the symplectic quadrangle $W_2(K)$.

• We will use a similar technique as in the case of $\mathcal{E}$: we prove that the residues of both end points are the expected geometries, and that the residue of the central pair of nodes is the expected digon.

1.9 The special case of characteristic 2

It has already been mentioned that the long roots of a root system of type F_4 form a smaller root system Φ_L of type D_4. The same is true for the set Φ_S of short roots. There exists a linear transformation $\cdot^\dagger$ that transforms Φ_S into Φ_F, satisfying $x^{\dagger\dagger} = 2x$ for all $x \in \mathbf{P}_F$.

Because $\cdot^{\dagger}$ does not leave Φ invariant (we have $\Phi^{\dagger} = \Phi^{*}$) it cannot be used to fold the root system Φ_F in the same way as we applied $\bar{\cdot}$ to Φ. As a consequence it is not clear how to fold $\mathbf{J}$ into an algebra of smaller dimension, or whether this is possible at all.

In Chapter 4 we show that at least something can be done when the base field K has characteristic 2. In that case the map $a \mapsto S(a) = a * \overline{\infty}$ maps the 26-dimensional module $\mathbf{W}$ onto a 26-dimensional ideal (and hence subalgebra) $S(\mathbf{W})$ of $\mathbf{J}$. The quotient $\mathbf{Q} = \mathbf{J}/S(\mathbf{W})$ again has dimension 26 and is moreover isomorphic to $S(\mathbf{W})$.

Equivalently, in characteristic 2 it is possible to introduce a bracket operation on $\mathbf{W}$ which makes it a Lie algebra isomorphic to $\mathbf{Q}$. We can use $\cdot^{\dagger}$ to formulate an explicit isomorphism $\mu : \mathbf{W} \to \mathbf{Q}$. Many of the operations defined in earlier chapters are also well-defined on $\mathbf{Q}$, and μ remains 'compatible' with many of them.

We intend to use μ to *twist* both $\widehat{\mathsf{F}}_4(K)$ and the geometry $\mathcal{F}$ (it is customary to talk about twisting instead of folding in this case). At least one hurdle needs to be overcome : in geometric terms we would like to map points onto hyperlines in such a way that (symmetric) incidence is preserved. In algebraic terms this has two consequences. Isotropic elements should be mapped to isotropic elements, and the algebraic operation that indicates incidence should be preserved.

The first problem is that isotropic elements are defined in $\mathbf{J}$ but not in $\mathbf{Q}$. (However, it can be proved that no two isotropic elements of $\mathbf{J}$ can differ only in an element of $S(\mathbf{W})$, hence this is not really an issue). Secondly, and most unfortunately, the algebraic operation that we use to check incidence is an operation on $\mathbf{W} \times \mathbf{J}$ and not on $\mathbf{W} \times \mathbf{Q}$, and there seems no easy way to make it so.

To solve this we introduce a *duality operation* $Q(\cdot)$, defined by means of μ. This operation is quadratic and not linear like μ, maps isotropic elements of $\mathbf{W}$ onto isotropic elements of $\mathbf{J}$ (and not of $\mathbf{Q}$), and most importantly, transforms the 'incidence operation' into something very much like it, sufficiently so for incidence to be preserved.

If e is an isotropic element of e, then $Q(e) = \mu(e^{\text{frob}}) \bmod S(\mathbf{W})$. In other words, applying first Q and then μ^{-1} brings us 'almost' back to the original, up to an extra application of the Frobenius endomorphism. Because of this supplementary endomorphism we will be forced in Chapter 5 to introduce a Tits endomorphism σ and restrict ourselves to fields for which such σ exists. (A Tits endomorphism satisfies $\sigma^2 = \text{frob}$.)

1.10 The final twist

The duality between isotropic elements of $\mathbf{W}$ and of $\mathbf{J}$ also extends to Chevalley group elements. With every isotropic element e of $\mathbf{W}$ we may associate an element $x(e)$ of $\widehat{\mathsf{F}}_4(K)$ (which is essentially equal to $x(S(e))$ in the notation used before). The map

$$x(e) \mapsto x(Q(e)), \qquad x(Q(e)) \mapsto x(e^{\text{frob}}),$$

now defines an automorphism of $\widehat{\mathsf{F}}_4(K)$, and this automorphism allows us to twist $\widehat{\mathsf{F}}_4(K)$ into a smaller group. The resulting *Ree group* ${}^2\widehat{\mathsf{F}}_4(K)$ is generated by elements of the form

$$y(e) \stackrel{\text{def}}{=} x(Q(e)^{\sigma^{-1}})x(e)x(eQ(e)^{\sigma^{-1}}).$$

In Chapter 5 we introduce two new operations $q(\cdot,\cdot)$ and $c(\cdot,\cdot)$ on $\mathbf{W} \times \mathbf{W}$ (with values in $\mathbf{W}$) which are invariant for ${}^2\widehat{\mathsf{F}}_4(K)$. Both operations are only partially defined : the second argument of $q(\cdot,\cdot)$ must be isotropic, and for $c(\cdot,\cdot)$ the pair of arguments must generate an isotropic subspace of $\mathbf{W}$. We also define the notion of *semi-octagonal* element (an isotropic element e such that $q(e,e) = 0$) and of *octagonal* element (a semi-octagonal element e that satisfies $e \in q(\mathbf{V}, e)$).

The set of non-zero isotropic elements of $\mathbf{W}$, which form a single orbit of $\widehat{\mathsf{F}}_4(K)$, splits into several orbits of ${}^2\widehat{\mathsf{F}}_4(K)$. The set of non-zero octagonal elements is one of these orbits, the semi-octagonal elements that are not octagonal form another.

The octagonal elements are the basis for the final step in our construction : twisting the geometry $\mathcal{F}$ into a generalized octagon $\mathcal{O}$. The points and lines of $\mathcal{O}$ are the one- and two-dimensional *octagonal subspaces* of **W** (i.e., subspaces all of whose elements are octagonal). Equivalently, points and lines of $\mathcal{O}$ can also be interpreted as 'absolute' elements of the duality Q on $\mathcal{F}$ and there also exists a connection between lines of $\mathcal{O}$ and semi-octagonal elements of **W** that are not octagonal.

The many operations which have been introduced throughout the text now also serve a useful purpose in determining properties of pairs of points of $\mathcal{O}$. For example, if Ke and Kf are two points of $\mathcal{O}$, then they will be collinear if and only if $q(e, f) = 0$. They are at mutual distance ≤ 4 if and only if $[e, f] = 0$. If they are at distance 4, then $Kc(e, f)$ represents the unique point collinear to both of them. If they are at distance 6, then $[e, f]$ is semi-octagonal and is related to the unique line at distance 3 of both points.

The group ${}^2\widehat{\mathsf{F}}_4(K)$, which acts as an automorphism group on $\mathcal{O}$, has the expected properties : it is transitive on the points of $\mathcal{O}$ and on all pairs of points at a fixed distance. Finally, we also prove that the corresponding elations and root groups have properties that make $\mathcal{O}$ a Moufang octagon, which demonstrates that $\mathcal{O}$ is indeed the well-known *Ree-Tits generalized octagon* which we set out to construct.

1.11 Bibliographical notes

Lie algebras are objects well known by mathematicians and physicists alike and their properties can be said to belong to mathematical folklore. A textbook introduction to Lie algebras can be found in HUMPHREYS [18].

Our main reference for the theory of generalized polygons is VAN MALDEGHEM [26]. This book has been our primary source of information on generalized octagons in particular.

The Chevalley groups have been well studied both from a group theoretical and a geometrical point of view. Definitions of these groups and their 'twisted' variants can be found in CARTER [8]. An in-depth treatment of Chevalley groups of type E_6 and F_4, their subgroups and the modules $\mathbf{V}$ and $\mathbf{W}$, can be found in a 4-part paper by M. Aschbacher [1, 3, 4, 5] — a paper referred to either directly or indirectly by most authors on these subjects. His basic definitions are also recorded in [2] and [11].

In ASCHBACHER [1] many concepts are introduced which are also central to this text. Many of the transitivity theorems of $\widehat{\mathsf{E}}_6(K)$ are proved (cf. Sections 2.4 and 3.3) using techniques which are however quite different from ours. COHEN and COOPERSTEIN [10] provide details on orbits of 2-spaces in $\mathbf{V}$.

Probably the first author to give an explicit description of the 27-dimensional E_6-module $\mathbf{V}$ and the related 26-dimensional F_4-module $\mathbf{W}$ (when K is the field of real numbers) is FREUDENTHAL [17]. He defines $\mathbf{V}$ to consist of all 3×3 matrices over the octonions, i.e., all matrices of the form

$$X = \begin{pmatrix} a & C & B \\ \overline{C} & b & A \\ \overline{B} & \overline{A} & c \end{pmatrix},$$

where a, b, c are real numbers and A, B, C are octonions. The subspace $\mathbf{W}$ then corresponds to the matrices of trace 0, i.e., satisfying $a + b + c = 0$. The product $X \circ Y \stackrel{\text{def}}{=} \frac{1}{2}(XY + YX)$ makes $\mathbf{V}$ into the so-called *exceptional Jordan Algebra*. Although octonions are non-associative it is still possible to define a determinant for these types of matrix, and this determinant turns out to be very much related to the trilinear product $\langle \cdot, \cdot, \cdot \rangle$ used in our definition of the Lie-algebra of type E_6.

• We chose not to use this representation as a base for our constructions because the factor $\frac{1}{2}$ makes it difficult to extend the definitions to fields of characteristic 2.

Clearly, a lot of what is written in the subsequent chapters is not new. However, our approach to the construction of the Lie algebras of types E_6 and F_4 probably contains a lot of new ideas, for instance the use of $Q^-(5,2)$ as a basis for the definition of the module $\mathbf{V}$, the construction of $\mathbf{L}$ from $\mathbf{V}$ both as a kind of 'derivation' and as the algebra generated by $a * \alpha$ for $a\alpha = 0$, and the introduction of several operations which clarify the interaction between the various modules. We also hope that our treatment of the modules $\mathbf{V} \wedge \mathbf{V}$ and

$\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ will prove to be of some use to the mathematical community.

As far as we are aware we are the first to give an explicit algebraic definition of a duality operation like $Q(\cdot)$ and hence also the first to give an explicit algebraic construction of the Ree-Tits octagon $\mathcal{O}$, which was the main goal of this text.

1.12 Some final comments

It is also possible to describe a construction of $\mathcal{O}$ without the information given in Chapters 2 and 3, as we did in [14]. In that case we begin with the Lie algebra $\mathbf{J}$ over a field of characteristic 2 as defined from its Chevalley basis properties. The module $\mathbf{W}$ then needs to be introduced as an ideal of $\mathbf{J}$ and not as a subspace of $\mathbf{V}$. However, with this starting point proofs of theorems tend to be more involved, the connection with the generalized quadrangle and the spread Σ is lost and it is no longer clear what makes characteristic 2 special.

Of course there still remains a lot of work to be done. It would for example be nice to extend the results of Chapter 5 to the case of a non-perfect field K. In a forthcoming paper [15] we do this by establishing the correspondence between the projective coordinates of the points of $\mathcal{O}$ and their Van Maldeghem coordinates as introduced in [26, Chapter 3]. This correspondence is given as a set of formulas involving σ but never its inverse, and therefore it can be formally extended to the case in which σ is a Tits endomorphism instead of an automorphism.

Other suggestions for future work are : a more profound investigation of the larger E_6-modules in characteristic 2 and 3, application of the techniques of this text to describe the embeddings of the classical generalized hexagons and the Ree ovoids in characteristic 3, construction of other exceptional Lie algebras (for E_7 an 'extension' of the generalized quadrangle $Q^-(5,2)$ can be used as a basis) and trying to reconstruct the algebra from the combinatorial properties of the generalized octagon, in the hope of using this to prove uniqueness of the smallest case. The reader may surely find some other interesting topics to add to this list.

2 The Lie algebra of type E_6 and related structures

2.1 A root system of type E_6

As stated in the introductory chapter we will establish a root system of type E_6 in a rather unconventional way. It is well known (see for instance [13]) that the Weyl group of type E_6 is isomorphic to the automorphism group of the generalized quadrangle $Q^-(5,2)$ with parameters $(s,t) = (2,4)$ and it should therefore not come as a surprise that this equivalence can be used to define the root system.

The definition of generalized n-gon from the introduction can be rephrased as follows when $n = 4$: A generalized quadrangle of order (s,t) is a point–line incidence structure such that

- every line is incident with exactly $s+1$ points and every point is incident with exactly $t+1$ lines,
- each pair of points is incident with at most one common line (and hence each pair of lines is incident with at most one common point), and
- given a point p not incident with a line L, there is exactly one point q and exactly one line M such that q is incident with both L and M and M is incident with both p and q.

Two points p,q incident with the same line are called *collinear*. We write $p \sim q$ if p,q are collinear but different, and we write $p \perp q$ when $p \sim q$ or $p = q$.

If $p \sim q$ then pq will denote the unique line incident with both p and q (i.e., *joining* p and q).

The most simple non-trivial example of a generalized quadrangle is provided by $W(2)$ of order $(2,2)$ which can be constructed in the following way :

- The points of $W(2)$ are the 15 pairs $\{i,j\}$ from the 6-set $\{0,\dots,5\}$.
- The lines of $W(2)$ are the 15 tripartitions of $\{0,\dots,5\}$, i.e., the partitions of the 6-set into three disjoint pairs.
- A point is incident with a line if and only if the corresponding pair is a pair of the corresponding tripartition.

Points of $W(2)$ are collinear if and only if the corresponding pairs are disjoint.

$W(2)$ can be extended to the generalized quadrangle $Q^-(5,2)$ of order $(2,4)$ by adding

- 12 points denoted by i', i'' with $i \in \{0,\dots,5\}$, and
- 30 lines $i'j''$ with $i,j \in \{0,\dots,5\}$, incident with the points i', j'' and $\{i,j\}$.

The two sets $\{0',\dots,5'\}$ and $\{0'',\dots,5''\}$ are called *sixes* and their union is a *double six*. Note that points within the same six are never collinear, while each point in a six is adjacent with all points of the other six, except one. As will be proved later, $Q^-(5,2)$ contains many double sixes, and removing each one always leaves a copy of $W(2)$.

We will further refer to the generalized quadrangle $Q^-(5,2)$ as $\mathcal{Q}$. By the above, $\mathcal{Q}$ has 27 points and 45 lines, with 3 points on each line and 5 lines through each point. We denote the set of points of the quadrangle by $\mathcal{P}$ and the set of lines by $\mathcal{L}$. The line containing the three points p,q,r will often be written as pqr.

• There are various other ways to define the same generalized quadrangle $\mathcal{Q}$. (It can be proved that all generalized quadrangles of order $(2,4)$ are isomorphic [20].)

The classical definition takes as points the points of an elliptic quadric in $\mathrm{PG}(5,2)$ and as lines all isotropic lines on this quadric. Intersecting the quadric with an appropriate hyperplane yields $Q(4,2)$, a generalized quadrangle isomorphic with $W(2)$.

Another construction takes as points the 27 lines on a general cubic surface, e.g., the real cubic surface with projective coordinates satisfying $x_0^3 + x_1^3 + x_2^3 + x_3^3 = (x_0 + x_1 + x_2 + x_3)^3$. There are 45 planes whose intersection with this surface consists of 3 lines. These planes may serve as lines of the quadrangle. Two points of $\mathcal{Q}$ are collinear if and only if the corresponding lines on the cubic intersect. In the given example, the 15 planes with equations of the form $x_i + \cdots + x_j = 0$ (of 1 up to 4 terms) define a subquadrangle of order $(2,2)$.

The original definition of 'double six' (by Schläfli) refers to two sets of 6 lines in real 3-dimensional projective space that are mutually skew, such that each line of the first set intersects each line of the other, except one. Each such double six lies on a unique cubic surface.

The point graph of $\mathcal{Q}$ (points are adjacent if and only if they are collinear) is strongly regular with parameters $(v,k,\lambda,\mu) = (27,10,1,5)$. The adjacency matrix A of this graph has eigenvalues 10, 1 and -5 with multiplicities 1, 20 and 6. The eigenspace of the last eigenvalue yields a 6-dimensional real orthogonal representation for this graph, which can be established using standard techniques (cf. [6]) :

Define the matrix

$$R \stackrel{\text{def}}{=} \frac{1}{15\sqrt{6}}(A-10)(A-1). \tag{2.1}$$

- $\frac{R}{\sqrt{6}}$ is the minimal idempotent for eigenvalue -5 in the Bose–Mesner algebra associated with this strongly regular graph.

It can be proved that R is a symmetric matrix with entries $4/3\sqrt{6}$ on the diagonal, $-2/3\sqrt{6}$ at positions that correspond to pairs of adjacent points and $1/3\sqrt{6}$ everywhere else.

The eigenvalues of R are of the form $\frac{1}{15\sqrt{6}}(\lambda-10)(\lambda-1)$ where λ ranges over the eigenvalues of A. So, R has eigenvalue 0 with multiplicity 21 and $\sqrt{6}$ with multiplicity 6. From this it follows that the rank of R is 6 and that $R^2 = \sqrt{6}R$.

Let V be the 27-dimensional real vector space of (row) vectors onto which R acts as a linear transformation by multiplication on the right. Denote the subspace of V generated by the rows of R by $\mathbf{P}$. By the above, $\mathbf{P}$ is a 6-dimensional (real) vector space. We will represent a point p of $\mathcal{P}$ by the corresponding row R_p of R and we will usually identify $p \in \mathcal{P}$ with its representation R_p in $\mathbf{P}$.

The standard Euclidian dot product on V when applied to p and q yields $p \cdot q = R_p R_q^T$ which is equal to the (p,q)-th entry of RR^T. We have $RR^T = R^2 = \sqrt{6}R$ and hence

$$p \cdot q = \begin{cases} 4/3, & \text{if and only if } p = q, \\ 1/3, & \text{if and only if } p \not\sim q, \\ -2/3, & \text{if and only if } p \sim q. \end{cases} \tag{2.2}$$

• For more information on strongly regular graphs and their eigenvalues, minimal idempotents and representations we refer to [6, 7]. In a note on page 34 we present an alternative way to construct $\mathbf{P}$ without the need for this theoretical background.

The map $p \mapsto R_p$ not only provides a 6-dimensional representation of $\mathcal{Q}$, but also defines a 6-dimensional orthogonal representation of the automorphism group of $\mathcal{Q}$.

Indeed, we represent an automorphism ζ of $\mathcal{Q}$ by its permutation action on the coordinates in V (a linear transformation on V). By (2.2) the value of $p \cdot q$ only depends on the mutual position of p and q, and hence $p^\zeta \cdot q^\zeta = p \cdot q$. It follows that the qth coordinate of R_p is equal to the coordinate of R_{p^ζ} with index q^ζ. In other words, ζ acting as a permutation on the coordinates of V maps the row R_p to the row R_{p^ζ}, i.e., p onto p^ζ as expected. Because it maps rows of R onto rows of R it also leaves $\mathbf{P}$ invariant and therefore it is a linear transformation of $\mathbf{P}$. It is an orthogonal transformation because by the above it also preserves the dot product on $\mathbf{P}$.

The chosen representation is also special because it provides us with a simple criterion to decide whether three points constitute a line :

Lemma 2.1 *Three points $p_1, p_2, p_3 \in \mathcal{P}$ constitute a line of $\mathcal{Q}$ if and only if $p_1 + p_2 + p_3 = 0$ (in $\mathbf{P}$).*

Proof : We have

$$\begin{aligned}(p_1 + p_2 + p_3)^2 &= (p_1 + p_2 + p_3) \cdot (p_1 + p_2 + p_3)\\ &= p_1^2 + p_2^2 + p_3^2 + 2(p_1 \cdot p_2 + p_1 \cdot p_3 + p_2 \cdot p_3)\\ &= 4 + 2(p_1 \cdot p_2 + p_1 \cdot p_3 + p_2 \cdot p_3), \qquad \text{by (2.2).}\end{aligned}$$

Also by (2.2) the right hand side of this equality can only attain zero when $p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = -2/3$, i.e., when $p_1 \sim p_2 \sim p_3 \sim p_1$. But then p_1, p_2, p_3 form a line. Because the dot product is positive definite (it is the restriction of the standard Euclidian product to $\mathbf{P}$) the left hand side is zero if and only if $p_1 + p_2 + p_3 = 0$. ∎

A vector of the form $r = q_0 - p_0$ with $p_0 \not\sim q_0$ will be called a *root*. The pair of points (p_0, q_0) is said to be *associated* with this root and p_0 is called a *base point* of r. The set of all roots will be denoted by Φ. If r is a root then by (2.2)

$$r^2 = (q_0 - p_0)^2 = q_0 \cdot q_0 - 2p_0 \cdot q_0 + p_0 \cdot p_0 = 4/3 - 2/3 + 4/3 = 2. \qquad (2.3)$$

Lemma 2.2 *Every root has exactly* 6 *base points. The six pairs $(p_0, q_0), \ldots, (p_5, q_5)$ which are associated with the same root have the following properties :*

1. *The points $p_1, \ldots, p_5$ are exactly those points of $\mathcal{P}$ adjacent to q_0 but not to p_0 (see figure 2.1 on the next page).*

2. *The points $q_1, \ldots, q_5$ are exactly those points of $\mathcal{P}$ adjacent to p_0 but not to q_0.*

3. *Of the points $q_0, \ldots, q_5$ the only point not collinear to p_i is q_i.*

4. *Of the points $p_0, \ldots, p_5$ the only point not collinear to q_i is p_i.*

5. *The* 6 *base points p_i of r are mutually non-adjacent and so are the corresponding* 6 *base points q_i of $-r$.*

Hence, $\{p_0, \ldots, p_5\}$ and $\{q_0, \ldots, q_5\}$ constitute a double six of $\mathcal{Q}$.

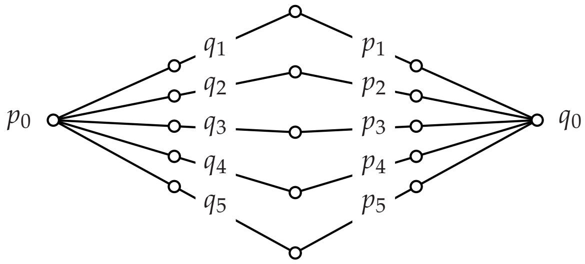


Figure 2.1: Base points of roots r and $-r$

Proof : Consider a root r associated with a given pair (p_0, q_0). Let (p, q) be any other pair associated with the same root. By (2.3) we have

$$2 = r \cdot r = (q_0 - p_0) \cdot (q - p) = q_0 \cdot q - q_0 \cdot p - p_0 \cdot q + p_0 \cdot p$$

If we assume $p \neq p_0$ (and hence $q \neq q_0$), then by (2.2) we find that the only way to make this equation work is by setting

$$p_0 \cdot p = q_0 \cdot q = 1/3, \qquad p_0 \cdot q = q_0 \cdot p = -2/3.$$

Hence $p_0 \sim q$, $q \sim p_0$, $p \not\sim p_0$ and $q \not\sim q_0$. By Lemma 2.1 the third point on the line through p_0 and q is represented by the vector $-p_0 - q$. Because $r = q_0 - p_0 = q - p$ this vector is equal to $-q_0 - p$ which corresponds to the third point on the line through q and p_0.

In other words, every base point p of r can be constructed in the following way : let L be any line through p_0 and denote the unique point on L adjacent to q_0 by a. Then q is the third point on the line through p_0 and a, and p is the third point on the line through q_0 and a. By Lemma 2.1 we have $p_0 + q + a = p + q_0 + a$ and hence $q_0 - p_0 = q - p$. So each pair (p, q) constructed in this way defines the same root r. As there are 5 different lines L, there must be 5 different pairs (p, q) apart from (p_0, q_0) that correspond to r. ∎

This lemma enables us to count the roots of Φ : for each of 27 points p_0 there are 16 points q_0 not adjacent to them. This gives a total of $27 \cdot 16 = 432$ pairs, giving $432/6 = 72$ different roots.

The following lemma explains the name 'base' point :

Lemma 2.3 *The base points* $p_0, \ldots, p_5$ *of a given root* r *form a basis for the* 6-*dimensional (real) vector space* $\mathbf{P}$. *We have* $r = -\frac{1}{3}(p_0 + \cdots + p_5)$.

Proof : The Gram-matrix of the 6 vectors p_i (i.e., the 6×6 matrix with entries $p_i \cdot p_j$) has entries $4/3$ on the diagonal and $1/3$ elsewhere. This matrix is non-singular and therefore the points p_i are represented by linearly independent vectors of $\mathbf{P}$. We already know that $\mathbf{P}$ has dimension 6 and hence they form a basis.

As a consequence, every element v of $\mathbf{P}$ is uniquely determined by the 6 values $v \cdot p_i$. Now, write $r = q_i - p_i$, then $p_i \not\perp q_i$ by Lemma 2.2, and hence using (2.2) we find $p_i \cdot r = 1/3 - 4/3 = -1$ for every i. Also

$$-\frac{1}{3}\sum_j p_j \cdot p_i = -\frac{1}{3}(4/3 + 1/3 + \cdots + 1/3) = -1,$$

for every i. Therefore r must be equal to $-\frac{1}{3}\sum_j p_j$. ∎

Instead of $p_0, \ldots, p_5$ it is more convenient to use the basis $p_0/3, \ldots, p_5/3$ to coordinatize $\mathbf{P}$. With these conventions, $p_0, \ldots, p_5$ have coordinates $(3, 0, \ldots, 0)$, ..., $(0, \ldots, 0, 3)$, and r has coordinates $(-1, \ldots, -1)$. We will denote these coordinate tuples by a short hand notation of the form 300000, ..., 000003 and 111111, where 1, 2 and 3 stand for -1, -2 and -3.

Because p_i is a base point of r we known that $q_i \stackrel{\text{def}}{=} p_i + r$ is a base point of $-r$. The points $q_0, \ldots, q_5$ then have coordinates 211111, ..., 111112. By Lemma 2.2 p_i and q_j are adjacent if $i \neq j$. Denote the third point on the line $p_i q_j$ by a_{ij}. Then $a_{ij} = -p_i - q_j = -p_i - p_j - r$ and hence $a_{ij} = a_{ji}$. As a consequence a_{ij} has coordinate 2 at positions i and j, and 1 at all other positions. It follows that the fifteen points a_{ij} with $0 \leq i < j \leq 5$ are distinct and have coordinates 221111, ..., 111122. It also follows that each of the 27 points of $\mathcal{P}$ is either of the form p_i, q_j or a_{ij}. Table 2.1 sums up this information. Note that the coordinate 'pattern' of a point p uniquely determines the value of $p \cdot r$ (with $r = 111111$).

$p_0, \ldots, p_5$ $p \cdot r = -1$	300000 030000 003000 000300 000030 000003
$q_0, \ldots, q_5$ $p \cdot r = 1$	$2\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}$ $\bar{1}2\bar{1}\bar{1}\bar{1}\bar{1}$ $\bar{1}\bar{1}2\bar{1}\bar{1}\bar{1}$ $\bar{1}\bar{1}\bar{1}2\bar{1}\bar{1}$ $\bar{1}\bar{1}\bar{1}\bar{1}2\bar{1}$ $\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}2$
$a_{01}, \ldots, a_{45}$ $p \cdot r = 0$	$\bar{2}\bar{2}1111$ $\bar{2}1\bar{2}111$ $\bar{2}11\bar{2}11$ $\bar{2}111\bar{2}1$ $\bar{2}1111\bar{2}$ $1\bar{2}\bar{2}111$ $1\overline{212}11$ $1\bar{2}11\bar{2}1$ $1\bar{2}111\bar{2}$ $11\bar{2}\bar{2}11$ $11\bar{2}1\bar{2}1$ $11\bar{2}11\bar{2}$ $111\bar{2}\bar{2}1$ $111\bar{2}1\bar{2}$ $1111\bar{2}\bar{2}$

Table 2.1: The points of $\mathcal{P}$

By the above we know that p_i, q_j, a_{ij} form a line of $\mathcal{Q}$. This already accounts for 30 of the lines. Let $\{i, j, k, l, m, n\} = \{0, \ldots, 5\}$. Then

$$a_{ij} + a_{kl} + a_{mn} = -p_i - p_j - r - p_k - p_l - r - p_m - p_n - r = -\sum p_j - 3r = 0.$$

and hence $a_{ij}a_{kl}a_{mn}$ is a line by Lemma 2.1, yielding the 15 remaining lines.

• The relation with the construction of $\mathcal{Q}$ given at the start of this section will now be clear: the points p_i, q_j, a_{ij} correspond to i', j'', $\{i, j\}$ in the earlier notation. It follows that the 15 points a_{ij} together with the 15 lines $a_{ij}a_{kl}a_{mn}$ form a subquadrangle of $\mathcal{Q}$ isomorphic to $W(2)$. The 5 points a_{ij} with fixed i constitute an ovoid in this subquadrangle. These are also the points of the subquadrangle adjacent to a fixed base point p_i of r (or q_i of $-r$).

In what follows we will use the terminology '*standard notation for points of* $\mathcal{P}$' whenever we represent the points of $\mathcal{P}$ in the form p_i, q_j and a_{ij}. Note that this notation depends on the choice of the root r.

Note that the chosen basis is not orthogonal with respect to the dot product. Indeed, we have $p_i/3 \cdot p_j/3 = \delta_{ij}/9 + 1/27$ and hence

$$c \cdot d = \frac{1}{9}\left[\sum c_i d_i + \frac{1}{3}\sum c_i \sum d_i\right], \tag{2.4}$$

when $c = (c_0, \ldots, c_5)$ and $d = (d_0, \ldots, d_5)$.

• Using (2.4) and the explicit coordinates in Table 2.1, the reader may verify that (2.2) does indeed hold. We could therefore have used Table 2.1 as a definition of the em-

Root s	Base points of s	#
$\mathtt{\overline{111111}}$ $r \cdot s = 2$	$\mathtt{300000}$ $\mathtt{030000}$ $\mathtt{003000}$ $\mathtt{000300}$ $\mathtt{000030}$ $\mathtt{000003}$	1
$\mathtt{111\overline{222}}$ $r \cdot s = 1$	$\mathtt{\overline{22}1111}$ $\mathtt{\overline{2}1\overline{2}111}$ $\mathtt{1\overline{22}111}$ $\mathtt{000300}$ $\mathtt{000030}$ $\mathtt{000003}$	20
$\mathtt{\overline{3}30000}$ $r \cdot s = 0$	$\mathtt{300000}$ $\mathtt{2\overline{11111}}$ $\mathtt{1\overline{22}111}$ $\mathtt{1\overline{2}1\overline{2}11}$ $\mathtt{1\overline{2}11\overline{2}1}$ $\mathtt{1\overline{2}111\overline{2}}$	30
$\mathtt{\overline{111}222}$ $r \cdot s = -1$	$\mathtt{2\overline{11111}}$ $\mathtt{\overline{1}2\overline{1111}}$ $\mathtt{\overline{11}2\overline{111}}$ $\mathtt{1111\overline{22}}$ $\mathtt{111\overline{2}1\overline{2}}$ $\mathtt{111\overline{22}1}$	20
$\mathtt{111111}$ $r \cdot s = -2$	$\mathtt{2\overline{11111}}$ $\mathtt{\overline{1}2\overline{1111}}$ $\mathtt{\overline{11}2\overline{111}}$ $\mathtt{\overline{111}2\overline{11}}$ $\mathtt{\overline{1111}2\overline{1}}$ $\mathtt{\overline{11111}2}$	1

$$
\begin{array}{rcccccccc}
\mathtt{\overline{111111}} & = & q_0 - p_0 & = & \cdots & = & q_5 - p_5 & = & r \\
\mathtt{111\overline{222}} & = & q_0 - a_{12} & = & q_1 - a_{02} & = & q_2 - a_{02} & & \\
 & = & a_{34} - p_5 & = & a_{45} - p_3 & = & a_{35} - p_4 & & \\
\mathtt{\overline{3}30000} & = & p_1 - p_0 & = & q_1 - q_0 & & & & \\
 & = & a_{02} - a_{12} & = & a_{03} - a_{13} & = & a_{04} - a_{14} & = & a_{05} - a_{15} \\
\mathtt{\overline{111}222} & = & a_{12} - q_0 & = & a_{20} - q_1 & = & a_{01} - q_2 & & \\
 & = & p_5 - a_{34} & = & p_3 - a_{45} & = & p_4 - a_{35} & & \\
\mathtt{111111} & = & p_0 - q_0 & = & \cdots & = & p_5 - q_5 & = & -r
\end{array}
$$

Table 2.2: Typical roots of Φ and their base points

bedding of $\mathcal{Q}$ into **P** without the need for the matrix R of (2.1). But then some more work is needed to prove that all automorphisms of $\mathcal{Q}$ can be represented by linear transformations of **P**.

It is an easy exercise to enumerate the roots of Φ by computing all differences $q - p$ between pairs of points p, q such that $p \not\perp q$. Table 2.2 lists the results in terms of coordinates. Every element of Φ can be obtained by permuting the coordinates of one of the elements listed. The last column indicates how many roots can be obtained in this way from the listed element. As with the points, the coordinate 'pattern' of a root s uniquely determines the value of $r \cdot s$ (with $r = \mathtt{\overline{111111}}$).

We also easily obtain the following results :

$r \cdot s =$	if and only if
2	$r = s$
1	$r - s \in \Phi$
0	$r - s, r + s \notin \Phi$
-1	$r + s \in \Phi$
-2	$r = -s$

$p \cdot r =$	if and only if
1	$p - r \in \mathcal{P}$
0	$p + r, p - r \notin \mathcal{P}$
-1	$p + r \in \mathcal{P}$

for all $r, s \in \Phi, p \in \mathcal{P}$. (2.5)

We will now set out to prove that the elements of Φ do indeed form a root system of type E_6. In this context it is customary to express relations between the roots not in terms of the inner product $r \cdot s$, but by means of the following binary product :

$$\langle r, s \rangle \stackrel{\text{def}}{=} 2\frac{r \cdot s}{r \cdot r}. \tag{2.6}$$

• In general $\langle r, s \rangle$ is linear only in the second argument, and not in the first, but this will only be of importance from Chapter 3 onwards, when we introduce roots of unequal lengths. For now $\langle r, s \rangle = r \cdot s$.

The binary product is often defined in terms of roots and co-roots (cf. §1.2), i.e., $\langle r, s \rangle = r^* \cdot s$. We have $r^* = r$ for al $r \in \Phi_L$.

To every root r we associate the reflection w_r about the hyperplane orthogonal to r :

$$w_r : \mathbf{P} \to \mathbf{P} : x \mapsto w_r(x) \stackrel{\text{def}}{=} x - \langle r, x \rangle r \tag{2.7}$$

As a consequence of (2.5) we see that w_r maps points onto points and roots onto roots and hence that Φ satisfies the definition of *root system* (Section 1.2).

To determine what type of root system Φ represents, we establish the following fundamental system $\{\pi_0, \ldots, \pi_5\}$ of roots for Φ :

$$\begin{array}{lclllcl}
\pi_0 & = & p_3 + p_4 + p_5 + 2r, & & \pi_3 & = & p_2 - p_3, \\
\pi_1 & = & p_0 - p_1, & & \pi_4 & = & p_3 - p_4, \\
\pi_2 & = & p_1 - p_2, & & \pi_5 & = & p_4 - p_5,
\end{array} \tag{2.8}$$

or using coordinates,

$$
\begin{array}{rclrcl}
\pi_0 & = & \texttt{\overline{222}111}, & \pi_3 & = & \texttt{003\overline{3}00}, \\
\pi_1 & = & \texttt{3\overline{3}0000}, & \pi_4 & = & \texttt{0003\overline{3}0}, \\
\pi_2 & = & \texttt{03\overline{3}000}, & \pi_5 & = & \texttt{00003\overline{3}}.
\end{array}
\tag{2.9}
$$

To show that this is indeed a fundamental system we must prove that every element of Φ can be written as a linear combination of fundamental roots in which all coefficients are either all non-negative or all non-positive.

Indeed, when $i < j$, we may write $p_i - p_j = \pi_{i-1} + \cdots + \pi_j$, and hence the property holds for every root of the form $\pm(p_i - p_j)$. When $i < j < k$ we find $p_i + p_j + p_k + 2r = \pi_0 + (p_i - p_3) + (p_j - p_4) + (p_k - p_5)$, hence the property holds for all roots of the form $\pm(p_i + p_j + p_k + 2r)$. Finally, we leave it to the reader to verify that $r = 2\pi_0 + \pi_1 + 2\pi_2 + 3\pi_3 + 2\pi_4 + \pi_5$.

The Cartan matrix which corresponds to this fundamental system, i.e., the matrix with entries $\langle \pi_i, \pi_j \rangle$, is easily computed to have the following value :

$$
\begin{pmatrix}
2 & 0 & 0 & -1 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
-1 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}.
\tag{2.10}
$$

This proves

Proposition 2.4 *The elements of Φ form a root system of type E_6.*

Also useful is the following list of so-called *fundamental weights* π_i' :

$$
\begin{array}{rclrcl}
\pi_0' & = & r, & \pi_3' & = & q_0 + q_1 + q_2, \\
\pi_1' & = & q_0, & \pi_4' & = & -p_4 - p_5, \\
\pi_2' & = & q_0 + q_1, & \pi_5' & = & -p_5,
\end{array}
\tag{2.11}
$$

or in coordinates

$$
\begin{array}{rclrcl}
\pi'_0 & = & \overline{\mathtt{111111}}, & \pi'_3 & = & \mathtt{000}\overline{\mathtt{333}}, \\
\pi'_1 & = & \mathtt{2}\overline{\mathtt{11111}}, & \pi'_4 & = & \mathtt{0000}\overline{\mathtt{33}}, \\
\pi'_2 & = & \mathtt{11}\overline{\mathtt{2222}}, & \pi'_5 & = & \mathtt{00000}\overline{\mathtt{3}}.
\end{array}
\tag{2.12}
$$

We leave it to the reader to verify that these elements of $\mathbf{P}$ satisfy $\langle \pi_i, \pi'_j \rangle = \delta_{ij}$ for $0 \leq i, j \leq 5$.

• Observe that π'_0 is a root, π'_1 is a point and π'_5 is the negative of a point. Also π'_2 is the sum of two noncollinear points and can at the same time be written as $r - p$ where $r \in \Phi$, $p \in \mathcal{P}$ such that $p \cdot r = 0$ (take $r = \overline{\mathtt{111111}}$ and $p = \overline{\mathtt{22}}\mathtt{1111}$). Likewise π'_4 is the negative of a sum of two points and at the same time of the form $r + p$ where $r \in \Phi$, $p \in \mathcal{P}$ and $p \cdot r = 1$, i.e., $\pi'_4 = \overline{\mathtt{111111}} + \mathtt{1111}\overline{\mathtt{22}}$. Finally π'_3 is at the same time the sum of three mutually noncollinear points and the negative of such a sum ($\pi'_3 = -p_3 - p_4 - p_5$) and can also be written as the sum of two roots r, s such that $r \cdot s = -1$, i.e., $\pi'_3 = \overline{\mathtt{111111}} + \mathtt{111}\overline{\mathtt{222}}$.

The finite group $W(\mathsf{E}_6)$ generated by all w_r where r ranges through Φ, is called the *Weyl group* of Φ. It is easily proved that $W(\mathsf{E}_6)$ is a group of orthogonal transformations on $\mathbf{P}$ and therefore a group of automorphisms of $\mathcal{Q}$.

The following proposition lists some transitivity properties of $W(\mathsf{E}_6)$ that will play an important role in what follows. Define an *ordered coclique* to be an ordered tuple of mutually non-collinear points of $\mathcal{P}$.

Proposition 2.5 *The Weyl group* $W(\mathsf{E}_6)$ *acts transitively on the following sets :*

1. *the set of points* $\mathcal{P}$*,*
2. *the set of roots* Φ*,*
3. *the ordered pairs* $(p, q) \in \mathcal{P} \times \mathcal{P}$ *with* $p \sim q$*,*
4. *the ordered pairs* $(p, q) \in \mathcal{P} \times \mathcal{P}$ *with* $p \not\sim q$*,*
5. *the ordered pairs* $(p, r) \in \mathcal{P} \times \Phi$ *satisfying* $p \cdot r = -1$*,*
6. *the ordered pairs* $(p, r) \in \mathcal{P} \times \Phi$ *satisfying* $p \cdot r = 0$*,*

7. *the ordered pairs* $(p,r) \in \mathcal{P} \times \Phi$ *satisfying* $p \cdot r = 1$,
8. *the ordered pairs* $(r,s) \in \Phi \times \Phi$ *satisfying* $r \cdot s = -1$,
9. *the ordered pairs* $(r,s) \in \Phi \times \Phi$ *satisfying* $r \cdot s = 0$,
10. *the ordered pairs* $(r,s) \in \Phi \times \Phi$ *satisfying* $r \cdot s = 1$,
11. *the ordered cocliques of size* 3,
12. *the ordered cocliques of size* 4,
13. *the ordered cocliques of size* 5 *that cannot be extended to an ordered coclique of size* 6,
14. *the ordered cocliques of size* 5 *that belong to an ordered coclique of size* 6,
15. *the ordered cocliques of size* 6.

Proof : (We use the standard notation for points of $\mathcal{P}$.)

Note that $w_t \in W(\mathsf{E}_6)$ interchanges the base points p of t with the corresponding base point $p + t$ of $-t$ and leaves all other points of $\mathcal{P}$ unchanged. As a consequence it is easily seen that $W(\mathsf{E}_6)$ must be transitive on $\mathcal{P}$. Indeed, to map $p \in \mathcal{P}$ onto $q \in \mathcal{P}$ such that $p \not\sim q$, we use w_{q-p}, and when $p \sim q$, we may find an intermediate point p' such that $p \not\sim p' \not\sim q'$ and then apply $w_{p'-p}$ and $w_{q-p'}$ in succession.

Now consider any pair $(p,q) \in \mathcal{P} \times \mathcal{P}$. By the above we may map (p,q) onto a pair of the form (p_0, q') for some fixed p_0. The Weyl group elements w_s that leave p_0 invariant, are exactly those for which $p_0 \cdot s = 0$, i.e., those for which p_0 is neither a base point of s nor of $-s$. Hence s is of the form $a_{lm} - p_n$ with $n \neq 0$, $a_{lm} - q_n$ or $p_j - p_i$, $i \neq 0$. We leave it to the reader to verify that using only such w_s we may now map any point in $\mathcal{P}$ onto either p_0, q_0 or q_1. This proves statements 3 and 4 of this proposition, and also statement 2, because every root r is associated with a noncollinear pair (p,q) such that $r = q - p$.

Note that $w_s(r) = r$ whenever $r \cdot s = 0$. Consider the subgroup W_r of $W(\mathsf{E}_6)$ that is generated by such elements s. From Table 2.2 (pg. 35) we see that those elements s have coordinates that are permutations of `330000`, and it easily follows that W_r acts as the symmetric group $\mathrm{Sym}(6)$ on $\mathcal{P}$ by permuting coordinates. This proves statements 5–7, and in a similar way, statements 8–10.

Consider any coclique C of size at least 2. By the above, we may fix two of the points of C to be p_0 and q_0 and again consider the subgroup W_r for $r = q_0 - p_0$. The set S of points of $\mathcal{P}$ collinear to neither p_0 or q_0 consists of the 10 points $a_{12}, \ldots, a_{45}$. The subgroup W_{p_0,q_0} of W_r fixing both p_0 and q_0 acts on $\mathbf{P}$ as $\mathrm{Sym}(5)$ by permuting the last five coordinates in $\mathbf{P}$. Hence W_{p_0,q_0} acts on S in the same way as $\mathrm{Sym}(5)$ acts on the pairs $12, \ldots, 45$.

Mutually noncollinear points in S correspond to mutually overlapping pairs. As a consequence, the orbits of W_{p_0,q_0} on cocliques in S are as follows (we list a single representative for each orbit) :

$$\{a_{12}\}, \quad \{a_{12}, a_{13}\}, \quad \{a_{12}, a_{13}, a_{23}\}, \quad \{a_{12}, a_{13}, a_{14}\}, \quad \{a_{12}, a_{13}, a_{14}, a_{15}\}.$$

This proves statements 11–15. ∎

• The cocliques of size 6 are the sets of base points of roots. The cocliques of size 5 that cannot be extended to a coclique of size 6 are the sets of points that are collinear to two points p and q such that $p \not\perp q$. Cocliques of size 5 are ovoids in the subquadrangles of order $(2, 2)$.

If (p, q) is a coclique of $\mathcal{P}$ then the orbit of $W(\mathsf{E}_6)$ formed by the sums $p + q$ is the orbit of the weight π_2'. It is also the orbit of the differences $r - p$ when $p \cdot r = 0$. Likewise, if (p, p', p'') is a coclique of $\mathcal{P}$, then π_3' lies in the orbit of $p + p' + p''$ which is the same as that of $r + s$ where $r, s \in \Phi$ such that $r \cdot s = -1$.

We will also be interested in sets of roots of the following kind :

Proposition 2.6 *Let $(r_1, r_2, \ldots, r_k) \in \Phi^k$ be such that $\langle r_i, r_j \rangle = 1$ whenever $i \neq j$. Then $k \leq 5$ and $W(\mathsf{E}_6)$ acts transitively on all k-tuples of this kind (for a given k). Any pair (r_1, r_2) of this type has exactly 3 common base points, any k-tuple with $3 \leq k \leq 5$ has exactly 2 common base points.*

Proof : For $k \leq 2$ transitivity was already proved in Proposition 2.5. Hence, without loss of generality we may choose $r_1 = \texttt{111111}$ and $r_2 = \texttt{111222}$. It is then easily seen that the stabilizer of r_1 and r_2 in $W(\mathsf{E}_6)$ acts as the group $\mathrm{Sym}(3)^2$ which independently permutes the first three and the last three coordinates. Also, roots r which satisfy $\langle r, r_1 \rangle = 1$ are those having 3 coordinates equal to 1 and 3 coordinates equal to 2. If moreover $\langle r, r_2 \rangle = 1$, then the first three coordinates of r must contain exactly two ones.

This proves that $W(\mathsf{E}_6)$ acts transitively on triples (r_1, r_2, r_3) and without loss of generality we may choose $r_3 = \texttt{112122}$. The stabilizer of this triple now acts like $\mathrm{Sym}(2)^2$ which independently permutes the first two and last two coordinates. It is easily seen that again this stabilizer acts transitively on the four remaining roots r_4 such that $\langle r_4, r_1 \rangle = \langle r_4, r_2 \rangle = \langle r_4, r_3 \rangle = 1$. Choosing $r_4 = \texttt{121122}$ we see that exactly one possibility remains to extend the 4-tuple to a 5-tuple, i.e., $r_5 = \texttt{211122}$.

With these choices we find that $r_1, \ldots, r_5$ have common base points $\texttt{000030}$ and $\texttt{000003}$. Using Table 2.1 (pg. 34) we obtain $\texttt{000300}$ as an additional common base point for r_1 and r_2. ∎

Before we can proceed to the next section we need to introduce one more concept. In what follows, fix an automorphism ω of $\mathcal{Q}$ (and hence an orthogonal transformation of $\mathbf{P}$) of order 3 having 9 orbits on $\mathcal{P}$ each of which is a line. These lines form a *spread* Σ, i.e., a partition of the 27 points of $\mathcal{P}$ into lines.

- An example of such an automorphism is given by

$$\omega : \quad \begin{array}{lll} p_0 \mapsto p_1 + r, & p_1 \mapsto p_2 + r, & p_2 \mapsto p_0 + r, \\ p_3 \mapsto -r - p_3 - p_4, & p_4 \mapsto -r - p_4 - p_5, & p_5 \mapsto -r - p_5 - p_3, \end{array}$$

using the standard notation for points of $\mathcal{P}$.

Consider two lines $L = \{p, p', -p - p'\}$ and $M = \{q, q', -q - q'\}$ of $\mathcal{Q}$. Each point of M is collinear to exactly one point of L, say $p \sim q$, $p' \sim q'$ and hence $(-p - p') \sim (-q - q')$. For each collinear pair we may construct the third point on the corresponding line, giving the three points $-p - q$, $-p' - q'$ and $p + p' + q + q'$ (see figure 2.2). These three points sum to 0 and therefore form yet another line N. The set of lines $\{L, M, N\}$ is called a *regulus* of $\mathcal{Q}$ and the

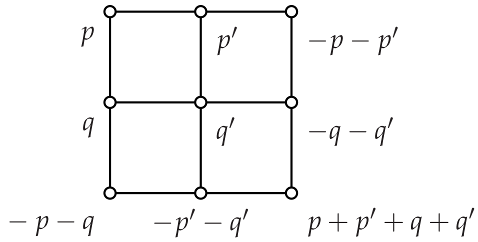


Figure 2.2: Grid generated by $p, p', q, q' \in \mathcal{P}$

configuration of 6 lines and 9 points described here is called a *grid*. A grid is made up of two reguli which are said to be *complementary*.

Given two lines $L, M \in \Sigma$, say $L = \{p, p\omega, p\omega^2\}$, $M = \{q, q\omega, q\omega^2\}$ with $p \sim q$, then the third line of the regulus through L and M is $N = \{-p-q, (-p-q)\omega, (-p-q)\omega^2\}$ and hence is a line of Σ. In other words, if two lines of a regulus belong to Σ then so does the third. We say that Σ is a *regular* spread.

Because $p + p\omega + p\omega^2 = 0$ for all $p \in \mathcal{P}$ we find that $1 + \omega + \omega^2 = 0$ over $\mathbf{P}$.

If $r \in \Phi$, then $r + r\omega + r\omega^2 = 0$ and hence $r \cdot r\omega + r \cdot r\omega^2 + r\omega \cdot r\omega^2 = -3$, which implies $r \cdot r\omega = r\omega \cdot r\omega^2 = r\omega^2 \cdot r = -1$.

Another consequence is that $p \cdot r + p \cdot r\omega + p \cdot r\omega^2 = 0$. If p is a base point of r, then $p \cdot r = -1$, hence $p \cdot r\omega + p \cdot r\omega^2 = 1$. There are two possible ways to satisfy this equality, and we will use these to define a distinction between *positive* and *negative* base points of r :

$$\begin{aligned} &p \cdot r = -1, \quad p \cdot r\omega = 1, \quad p \cdot r\omega^2 = 0, \quad \text{when } p \text{ is a positive base point of } r, \\ &p \cdot r = -1, \quad p \cdot r\omega = 0, \quad p \cdot r\omega^2 = 1, \quad \text{when } p \text{ is a negative base point of } r. \end{aligned} \tag{2.13}$$

Lemma 2.7 *If p is a positive (respectively negative) base point of r, then $p + r$ is a negative (respectively positive) base point of $-r$.*

If p is a base point of r, then p is a positive base point of r if and only if it is also a base point of $-r\omega$, and then it is a negative base point of $-r\omega$. Likewise, p is a negative base point of r if and only if it is also a base point of $-r\omega^2$ and then it is a positive base point of $-r\omega^2$.

Proof : We easily compute that $(p+r)\cdot(-r)\omega = 1 - p\cdot r\omega$ and $(p+r)\cdot(-r)\omega^2 = 1 - p\cdot r\omega^2$. Definition (2.13) then shows that $p+r$ is a negative base point of $-r$ when p is positive base point of r (and vice versa).

By (2.13) we see that $p\cdot(-r\omega) = -1$ if and only if p is a positive base point of r. In that case, because $\omega^3 = 1$, we have $p\cdot(-r\omega)\omega^2 = -p\cdot r = 1$ and $p\cdot(-r\omega)\omega = -p\cdot r\omega^2 = 0$, hence p is a negative base point of $-r\omega$. A similar argument can be used when p is a negative base point of r. ∎

Let (p,q) and (p',q') be two non-collinear pairs associated with the same root r. We will call these pairs *compatible* if and only if p, p' are both positive base points of r, or both negative base points of r.

Lemma 2.8 *Let $r \in \Phi$, $p, p' \in \mathcal{P}$. Write $q = p + r$, $q' = p' + r$.*

Then the pairs (p,q) and (p',q') are compatible if and only if either the line pq' or the line $p'q$ belongs to Σ.

Proof : Assume p is a positive base point of r. If $pq' \in \Sigma$ then either $p = q'\omega$ or $p = q'\omega^2$. Because q' is a base point of $-r$, and hence $q'\cdot r = 1$, it follows from (2.13) and the fact that ω is orthogonal, that $p = q'\omega$ and that $p' = q' - r$ is a positive base point of r. Similarly, if p is a negative base point of r, then $p = q'\omega^2$ and also p' is a negative base point. Hence in both cases (p,q) and (p',q') are compatible.

Every root r has 3 positive base points and 3 negative base points, hence there are 6 compatible pairs $\{(p,q),(p',q')\}$ in all. Likewise every of the 6 base points p of r lies on exactly one line of Σ, and gives rise to exactly one q' as in the construction above. Hence all compatible pairs associated with r are accounted for. ∎

Lemma 2.9 *Of the* 6 *lines in any grid, either* 1 *or* 3 *belong to* Σ.

Proof : The lines of a spread do not intersect, hence at most three lines of Σ lie on the grid. Because Σ is a regular spread, Σ cannot intersect the grid in 2 lines, hence we must prove that Σ intersects every grid in at least one line. We do this by counting :

For each line there are 32 lines that do not intersect it, and hence 16 grids through that line. This gives a total of $45 \times 16/6 = 120$ grids. For each line of Σ there are 4 grids through that line that intersect in 3 lines of Σ, hence there are $9 \times 4/3 = 12$ grids of that type. Through each line of Σ there remain 12 grids that intersect Σ in one line, giving 108 grids of that type. But $120 = 108 + 12$ and therefore all grids are accounted for. ∎

• We end this section with a short description of yet another way to construct the generalized quadrangle $\mathcal{Q}$ and the associated spread Σ. Points of the quadrangle are denoted by x_{ij}, y_{ij}, z_{ij} with $1 \leq i, j \leq 3$. Lines are of the form $x_{ij}y_{jk}z_{ki}$, with $1 \leq i, j, k \leq 3$ and $x_{1i}x_{2j}x_{3k}$, $y_{1i}y_{2j}y_{3k}$ or $z_{1i}z_{2j}z_{3k}$ where $\{i, j, k\} = \{1, 2, 3\}$. The spread Σ now consists of those 9 lines of the latter type for which ijk is an even permutation of 123.

2.2 The Lie algebra of type E_6 over K

In what follows let K denote any field. Let $\mathbf{V}$ be a 27-dimensional vector space over K generated by 27 base vectors e_p, each associated with a different point $p \in \mathcal{P}$. Let $\mathbf{V}^*$ denote the corresponding *dual space* and choose base vectors η_q of $\mathbf{V}^*$ for $q \in \mathcal{P}$ in such a way that

$$e_p \eta_q = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{otherwise.} \end{cases}$$

We may represent an element a of $\mathbf{V}$ as a row vector with 27 coordinates in K. We use the notation $a[p] \stackrel{\text{def}}{=} a\eta_p$ for the 'pth' coordinate of a (for $p \in \mathcal{P}$). Similarly, we may represent $\alpha \in \mathbf{V}^*$ by a column vector of 27 coordinates where $\alpha[p] \stackrel{\text{def}}{=} e_p\alpha$ denotes the 'pth' coordinate of α, taking care to use the same ordering of the coordinates for both $\mathbf{V}$ and $\mathbf{V}^*$. With these notations we find

$$a\alpha = \sum_{i \in \mathcal{P}} a[i]\alpha[i] = \sum_{i \in \mathcal{P}} (a\eta_i)(e_i\alpha) \tag{2.14}$$

• We have chosen not to use the 'functional notation' for elements of the dual vector space. Some authors would write $\alpha(a)$, or even αa, where we write $a\alpha$.

Linear transformations will be identified with matrices acting by multiplication to the right of normal vectors, or to the left of dual vectors. In other words : if A is a linear transformation on $\mathbf{V}$, then $A[p,q] \stackrel{\text{def}}{=} e_pA\eta_q$ denotes the entry at row p and column q of A.

Define the following operations :

1. A trilinear symmetric form $\langle a, b, c\rangle$ on $\mathbf{V}$, acting as follows on the base vectors e_i :

$$\langle e_i, e_j, e_k\rangle \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \{i,j,k\} \in \Sigma, \\ -1, & \text{if } \{i,j,k\} \in \mathcal{L} - \Sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{2.15}$$

2. A trilinear form $\langle \alpha, \beta, \gamma \rangle$ on $\mathbf{V}^*$, defined in a similar way :

$$\langle \eta_i, \eta_j, \eta_k \rangle \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } \{i,j,k\} \in \Sigma, \\ -1, & \text{if } \{i,j,k\} \in \mathcal{L} - \Sigma, \\ 0, & \text{otherwise.} \end{cases} \tag{2.16}$$

3. A related bilinear product $\times$ which maps a pair $a, b \in \mathbf{V}$ to an element $a \times b$ of $\mathbf{V}^*$ (and dually, a product $\alpha \times \beta \in \mathbf{V}$) using the following identities :

$$\begin{array}{lcll} c(a \times b) & \stackrel{\text{def}}{=} & \langle c, a, b \rangle, & \text{for all } c \in \mathbf{V}, \\ (\alpha \times \beta)\gamma & \stackrel{\text{def}}{=} & \langle \alpha, \beta, \gamma \rangle, & \text{for all } \gamma \in \mathbf{V}^*. \end{array} \tag{2.17}$$

4. The operations # and D with the following properties ($a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$) :

$$\begin{array}{lcllcl} a^\# & \stackrel{\text{def}}{=} & \frac{1}{2} a \times a, & \alpha^\# & \stackrel{\text{def}}{=} & \frac{1}{2} \alpha \times \alpha, \\ D(a) & \stackrel{\text{def}}{=} & \frac{1}{3} a a^\# = \frac{1}{6} \langle a, a, a \rangle, & D(\alpha) & \stackrel{\text{def}}{=} & \frac{1}{3} \alpha^\# \alpha = \frac{1}{6} \langle \alpha, \alpha, \alpha \rangle. \end{array} \tag{2.18}$$

• There is a problem with this definition when char $K = 2$ or 3, which will be resolved below.

For ease of notation we introduce the symbol $\epsilon_{ijk} \stackrel{\text{def}}{=} \langle e_i, e_j, e_k \rangle$. Note that ϵ_{ijk} is equal to 0, 1 or -1. For $a, b, c \in \mathbf{V}$ we find

$$\langle a, b, c \rangle = \sum \epsilon_{ijk} a[i] b[j] c[k], \qquad a \times b = \sum \epsilon_{ijk} a[i] b[j] \eta_k \tag{2.19}$$

where the sums may be restricted to the 270 ordered triples (i, j, k) for which $\{i, j, k\}$ is a line. Also

$$a^\# = \sum{}' \epsilon_{ijk} a[i] a[j] \eta_k \tag{2.20}$$

where the sum is taken over the 27 points k of $\mathcal{P}$, and for each k, over the 5 pairs $\{i, j\}$ such that $\{i, j, k\}$ is a line. When char $K = 2$, we take (2.20) as the definition of the operator #.

The cubic form D can be expressed as follows in terms of coordinates :

$$D(a) \stackrel{\text{def}}{=} \sum{}' \epsilon_{ijk} a[i] a[j] a[k], \tag{2.21}$$

where the sum ranges over the 45 lines $\{i, j, k\}$ of $\mathcal{L}$. When char $K = 2$ or 3, we take (2.21) as the definition of the operator D.

• As has already been apparent, we will often need to treat the cases char $K = 2$ and 3 separately. Note however that most equations which we will derive below can be expressed as polynomial identities in several coordinate variables $a[i], b[j], c[k], \alpha[i], \ldots$, where all coefficients are integers. Once these polynomial identities have been derived, they therefore immediately hold in all fields K, even when char $K = 2$ or 3, and no separate proof is needed.

We easily prove the following identities for every $a, b \in \mathbf{V}$ and $\ell \in K$:

$$\begin{aligned} (a + \ell b)^{\#} &= a^{\#} + \ell a \times b + \ell^2 b^{\#}, \\ D(a + \ell b) &= D(a) + \ell b a^{\#} + \ell^2 a b^{\#} + \ell^3 D(b). \end{aligned} \tag{2.22}$$

Similar properties hold over $\mathbf{V}^*$. (In what follows we will no longer mention such dual properties.)

Also

$$\langle a, b, \alpha \times \beta \rangle = (\alpha \times \beta)(a \times b) = \langle a \times b, \alpha, \beta \rangle, \quad \text{for } a, b \in \mathbf{V}, \alpha, \beta \in \mathbf{V}^*. \tag{2.23}$$

The following serves as a kind of fundamental axiom for the basic operations on $\mathbf{V}$ and $\mathbf{V}^*$:

Proposition 2.10 *Let $a, b, c, d \in \mathbf{V}$. Then*

$$\begin{aligned} (a \times b) \times (c \times d) \; &+ \; (a \times c) \times (b \times d) \; + \; (a \times d) \times (b \times c) \\ &= \; \langle a, b, c \rangle d \; + \; \langle b, c, d \rangle a \; + \; \langle c, d, a \rangle b \; + \; \langle d, a, b \rangle c. \end{aligned} \tag{2.24}$$

Proof : Clearly both the left and the right hand side of (2.24) are linear in a, b, c, d, hence it is sufficient to prove this identity when $a = e_i, b = e_j, c = e_k, d = e_l$ are base vectors of $\mathbf{V}$.

First assume that at least two of these base vectors are equal, say $i = l$. Then the left hand side of (2.24) evaluates to $2(e_i \times e_j) \times (e_i \times e_k)$ which is only nonzero when $i \sim j$, $i \sim k$ and $-i-j \sim -i-k$, hence when $\{i,j,k\}$ is a line. In that case the value of the left hand side is $2\epsilon_{ijk}e_i$. When $i = l$ the right hand side of (2.24) is easily computed to yield the same value.

Hence, assume $|\{i,j,k,l\}| = 4$. For the left hand side of (2.24) to be nonzero, at least one of the following conditions must be satisfied :

(a) $i \sim j$, $k \sim l$ and $-i-j \sim -k-l$.

(b) $j \sim k$, $i \sim l$ and $-j-k \sim -i-l$.

(c) $k \sim i$, $j \sim l$ and $-k-i \sim -j-l$.

Up to symmetry there are only two configurations of the four points i,j,k,l in which at least one of these conditions is met.

Case 1. $i \sim j \sim k \sim l \sim i$. It is easily seen that the right hand side of (2.24) is now zero. The left hand side is equal to

$$\begin{aligned} &\epsilon_{i,j,-i-j}\epsilon_{k,l,-k-l}\epsilon_{-i-j,-k-l,i+j+k+l}\ e_{i+j+k+l} \\ &\qquad + \epsilon_{i,l,-i-l}\epsilon_{j,k,-j-k}\epsilon_{-i-l,-j-k,i+j+k+l}\ e_{i+j+k+l} \end{aligned} \tag{2.25}$$

Now, the 6 ϵ-values in this expression correspond to the 6 lines of a grid. By lemma 2.9 an odd number of them must be equal to 1 and therefore an odd number must be equal to -1. As a consequence one of the terms in (2.25) has coefficient 1 and one has coefficient -1. Hence (2.25) is zero, as had to be proved.

Case 2. $\{i,j,k\}$ is a line and l is adjacent to exactly one point of that line, say $i \sim l$. The left hand side of (2.24) is now equal to $\epsilon_{ijk}e_l$, and this is easily proved to also be the value of the right hand side of (2.24).

Note that the right hand side of (2.24) can be nonzero only if at least 3 of the points i,j,k,l lie on a line, which was covered in case 2 above. Hence, all cases were covered. ∎

We may derive several additional equalities by setting some of the variables in Proposition 2.10 to the same symbol. Because these are polynomial identities with integral coefficients, they also hold when char $K = 2$ or 3.

$$\begin{aligned}
(a \times b) \times (a \times c) + a^{\#} \times (b \times c) &= \langle a, b, c \rangle a + (ca^{\#})b + (ba^{\#})c && (2.26)\\
(a^{\#} \times b^{\#}) + (a \times b)^{\#} &= (ab^{\#})a + (ba^{\#})b && (2.27)\\
(a \times b) \times a^{\#} &= D(a)b + (ba^{\#})a, && (2.28)\\
(a^{\#})^{\#} &= D(a)a, && (2.29)
\end{aligned}$$

for all $a, b, c \in \mathbf{V}$.

• In [2] equation (2.29) is considered fundamental instead of (2.24). It is not difficult to derive equations (2.24,2.26–2.28) from (2.29). For example, substituting $a + \ell b$ for a in (2.29) and equating coefficients of ℓ and ℓ^2 in the resulting polynomial identity, leads to (2.27) and (2.28). The others can be proved using the same technique.

Note that (2.29) proves that $D(a) = 0$ when $a^{\#} = 0$, also when char $K = 3$. For other characteristics this was already an immediate consequence of the definition (2.18). Applying (2.29) to $a^{\#}$ instead of a, yields

$$D(a^{\#}) = D(a)^2, \qquad \text{for all } a \in \mathbf{V}. \tag{2.30}$$

Lemma 2.11 *Let $a, b, c \in \mathbf{V}, \delta \in \mathbf{V}^*$. Then*

$$\begin{aligned}
&((a \times b) \times \delta) \times c + ((b \times c) \times \delta) \times a + ((c \times a) \times \delta) \times b\\
&\quad = \langle a, b, c \rangle \delta + (a\delta)(b \times c) + (b\delta)(c \times a) + (c\delta)(a \times b)
\end{aligned} \tag{2.31}$$

Proof : First note the following identity :

$$\begin{aligned}
d\,[((a \times b) \times \delta) \times c] &= \langle (a \times b) \times \delta), c, d \rangle\\
&= \langle a \times b, c \times d, \delta \rangle = [(a \times b) \times (c \times d)]\,\delta,
\end{aligned}$$

by (2.17) and (2.23). Multiplying (2.24) to the right with δ and using this equality, we obtain (2.31) multiplied to the left with d. Because d can be chosen arbitrarily, the lemma follows. ∎

By setting $a = b$ and $a = c$ in (2.31) we obtain some more identities :

$$\begin{aligned}
(a^{\#} \times \gamma) \times b + ((a \times b) \times \gamma) \times a &= (b\gamma)a^{\#} + (a\gamma)(a \times b) + (ba^{\#})\gamma, && (2.32)\\
(a^{\#} \times \beta) \times a &= (a\beta)a^{\#} + D(a)\beta. && (2.33)
\end{aligned}$$

• In [10, 11] the vector space $\mathbf{V}$ and the associated operations are defined in a different way : $\mathbf{V}$ is defined to consist of all triples $a = (X, Y, Z)$ of 3×3 matrices X, Y, Z over K and then $D(a) = \det X + \det Y + \det Z - \operatorname{Tr} XYZ$. Up to a minus sign this definition corresponds to ours if we represent $\mathcal{Q}$ as was explained in the final note of section 2.1. In [1] the trilinear form is introduced explicitely, and consists of two parts: a set of terms that correspond to the 15 lines of a subquadrangle of order $(2, 2)$ and a set of terms that correspond to the remaining lines. This is called the *Dickson presentation*.

Another difference lies in our distinction between the spaces $\mathbf{V}$ and $\mathbf{V}^*$. Other authors define $a^\#$ and $a \times b$ to belong to $\mathbf{V}$ instead of $\mathbf{V}^*$.

For $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ define $a * \alpha \in \operatorname{Hom}(\mathbf{V}, \mathbf{V})$ by means of the following identity :

$$b(a * \alpha)\beta \stackrel{\text{def}}{=} (b\alpha)(a\beta) - (\alpha \times \beta)(a \times b), \qquad \text{for all } b \in \mathbf{V}, \beta \in \mathbf{V}^*. \tag{2.34}$$

Note that the definition implies

$$a(b * \beta)\alpha = b(a * \alpha)\beta, \qquad \text{for all } a, b \in \mathbf{V}, \alpha, \beta \in \mathbf{V}^*. \tag{2.35}$$

We may regard $a * \alpha$ as a linear transformation on $\mathbf{V}$ (mapping b to $b(a * \alpha)$) or as a linear transformation on $\mathbf{V}^*$ (mapping β onto $(a * \alpha)\beta$). We have

$$b(a * \alpha) = (b\alpha)a - (a \times b) \times \alpha, \qquad (a * \alpha)\beta = (a\beta)\alpha - (\alpha \times \beta) \times a. \tag{2.36}$$

Also the following cases are of interest :

$$a(a * \alpha) = (a\alpha)a - 2a^\# \times \alpha, \qquad (a * \alpha)\alpha = (a\alpha)\alpha - 2a \times \alpha^\#. \tag{2.37}$$

Lemma 2.12 *Let $a, b, c \in \mathbf{V}$. Then*

$$\begin{aligned} a * a^\# + D(a)\mathbf{1} &= 0, && (2.38)\\ a * (a \times b) + b * a^\# + (ba^\#)\mathbf{1} &= 0, && (2.39)\\ a * (b \times c) + b * (c \times a) + c * (a \times b) + \langle a, b, c\rangle \mathbf{1} &= 0, && (2.40) \end{aligned}$$

where $\mathbf{1}$ denotes the identity transformation on $\mathbf{V}$.

Proof : Apply the left hand side of (2.40) to $d \in \mathbf{V}$ to obtain

$$\begin{aligned} & d[a * (b \times c) + b * (c \times a) + c * (a \times b) + \langle a, b, c\rangle \mathbf{1}] \\ = \quad & \langle d, b, c\rangle a - (a \times d) \times (b \times c) + \langle d, c, a\rangle b - (b \times d) \times (c \times a) \\ & \qquad + \langle d, a, b\rangle c - (c \times d) \times (a \times b) + \langle a, b, c\rangle d \end{aligned}$$

And this is zero by (2.24), for every $d \in \mathbf{V}$, proving (2.40). The other properties follow in a similar way from (2.26) and (2.28). ∎

We are now in a position to introduce the Lie algebra $\widehat{\mathbf{L}}$ as a subspace of $\mathrm{Hom}(\mathbf{V}, \mathbf{V})$: define $\widehat{\mathbf{L}}$ to be the set of all linear transformations A of $\mathbf{V}$ that satisfy

$$a \times aA + Aa^{\#} + \tau(A)a^{\#} = 0, \qquad \text{for all } a \in \mathbf{V}, \tag{2.41}$$

for some scalar $\tau(A) \in K$ which only depends on A. If A, A' satisfy (2.41) then so does $A + kA'$ for any $k \in K$, with $\tau(A + kA') = \tau(A) + k\tau(A')$. Hence $\widehat{\mathbf{L}}$ is a vector space and τ is a linear functional on $\widehat{\mathbf{L}}$. Also note that the identity transformation $\mathbf{1}$ belongs to $\widehat{\mathbf{L}}$ with $\tau(\mathbf{1}) = -3$.

Substituting $a + b$ for a in (2.41) and then multiplying by c, we obtain

$$aA \times b + a \times bA + A(a \times b) + \tau(A)a \times b \;=\; 0, \tag{2.42}$$

$$\langle aA, b, c\rangle + \langle a, bA, c\rangle + \langle a, b, cA\rangle + \tau(A)\langle a, b, c\rangle \;=\; 0, \tag{2.43}$$

for every $a, b, c \in \mathbf{V}$ and $A \in \widehat{\mathbf{L}}$.

The following lemma proves that $\widehat{\mathbf{L}}$ is indeed a *Lie algebra* over K with the usual definition of the 'Lie bracket' operator.

Lemma 2.13 *Let* $A, B \in \widehat{\mathbf{L}}$. *Then* $[A, B] \stackrel{\text{def}}{=} AB - BA$ *belongs to* $\widehat{\mathbf{L}}$ *and* $\tau([A, B]) = 0$.

Proof : Applying (2.42) and (2.41) we find

$$\begin{aligned} a \times aAB &= -aA \times aB - B(a \times aA) - \tau(B)(a \times aA) \\ &= -aA \times aB + BAa^{\#} + \tau(A)Ba^{\#} + \tau(B)Aa^{\#} + \tau(A)\tau(B)a^{\#}, \end{aligned}$$

and hence, interchanging A and B in this expression,

$$a \times aBA = -aB \times aA + ABa^{\#} + \tau(B)Aa^{\#} + \tau(A)Ba^{\#} + \tau(B)\tau(A)a^{\#}.$$

Subtracting these equalities, we find

$$a \times a(AB - BA) = -(AB - BA)a^{\#},$$

and therefore $[A, B] \in \widehat{\mathbf{L}}$ with $\tau([A, B]) = 0$. ∎

If we define $\mathbf{L}$ to be the kernel of τ, i.e., the subset of $A \in \widehat{\mathbf{L}}$ for which $\tau(A) = 0$, then this lemma proves that also $\mathbf{L}$ is a Lie algebra over K.

• Specialized to $\tau(A) = 0$ equations (2.42–2.43) show that the operations $\times$ and $\langle \cdot, \cdot, \cdot \rangle$ are compatible with the Lie algebra $\mathbf{L}$ (cf. Chapter 1).

Lemma 2.14 *Let $b \in \mathbf{V}$ and $\beta \in \mathbf{V}^*$. Then $b * \beta \in \widehat{\mathbf{L}}$ with $\tau(b * \beta) = b\beta$.*

Proof : Consider any $c \in \mathbf{V}$. We compute

$$\begin{aligned} c(a \times a(b * \beta)) &= \langle c, a, a(b * \beta) \rangle = a(b * \beta)(a \times c) \\ &= b(a * (a \times c))\beta, && \text{by (2.35)} \\ &= b(-c * a^{\#} - (ca^{\#})\mathbf{1})\beta, && \text{by (2.39)} \\ &= -b(c * a^{\#})\beta - (b\beta)(ca^{\#}), \\ &= -c(b * \beta)a^{\#} - (b\beta)(ca^{\#}), && \text{by (2.35)} \end{aligned}$$

This is true for every $c \in \mathbf{V}$, proving that $a \times a(b * \beta) + (b * \beta)a^{\#} + (b\beta)a^{\#}$ for every $a \in \mathbf{V}$ and hence the lemma follows. ∎

• As was mentioned in Chapter 1, we could have chosen to define $a * \alpha$ differently. For example, setting $b(a * \alpha)\beta = (b\alpha)(a\beta) + \frac{1}{3}(a\alpha)(b\beta) - (\alpha \times \beta)(a \times b)$, i.e., adding $\frac{1}{3}(a\alpha)\mathbf{1}$ to the current definition, would have made $\tau(a * \alpha) = 0$, obviating the need to introduce τ and $\widehat{\mathbf{L}}$, and allowing us to immediately define $\mathbf{L}$ from the property $a \times aA + Aa^{\#} = 0$. Unfortunately, this definition of the $*$-operator would make no sense when char $K = 3$.

The following lemmas will provide more information on the structure of $\widehat{\mathbf{L}}$ and $\mathbf{L}$.

Lemma 2.15 *Consider a root $r \in \Phi$ with positive base point p_0 and write $q_0 = p_0 + r$ as usual. Define*

$$E_r \stackrel{\text{def}}{=} e_{q_0} * \eta_{p_0}. \tag{2.44}$$

Then

$$e_p E_r = \begin{cases} e_{p+r}, & \text{when } p \text{ is a positive base point of } r, \\ -e_{p+r}, & \text{when } p \text{ is a negative base point of } r, \\ 0, & \text{for any other } p \in \mathcal{P}. \end{cases} \tag{2.45}$$

As a consequence the definition of E_r is independent of the choice of p_0.

Proof : We must evaluate $e_p(e_{q_0} * \eta_{p_0}) = (e_p\eta_{p_0})e_{q_0} - (e_p \times e_{q_0}) \times \eta_{p_0}$. If $p = p_0$ then the right hand side clearly evaluates to e_{q_0}. Otherwise this expression is nonzero only if $p \sim q_0$ and also $-p - q_0 \sim p_0$. Using the standard notation for points of $\mathcal{P}$ we see that p must be of the form p_i with $1 \leq i \leq 5$. Hence $p = p_i$ is a base point of r and $p + r = q_i$. We have $(e_p \times e_{q_0}) \times \eta_{p_0} = \pm\eta_{-p-q_0} = \pm e_{p+q_0-p_0} = \pm e_{p+r}$ and hence $e_{p_i}E_r = \pm e_{q_i}$. If (p_i, q_i) and (p_0, q_0) are compatible pairs, exactly one of the lines p_iq_0, p_0q_i belongs to the spread Σ (Lemma 2.8) yielding a plus sign. Otherwise, neither of them belongs to Σ, resulting in a minus sign. ∎

In a similar way it is easily proved that $e_q * \eta_p = 0$ when $p \sim q$. When we regard E_r as a matrix, we may reformulate (2.45) as follows :

$$E_r[p,q] = \begin{cases} +1, & \text{when } r = q - p \text{ and } p \text{ is a positive basepoint of } r, \\ -1, & \text{when } r = q - p \text{ and } p \text{ is a negative basepoint of } r, \\ 0, & \text{otherwise.} \end{cases} \tag{2.46}$$

Note that the matrices E_r are linearly independent, as each pair (p,q) with $p \not\perp q$ is associated with exactly one root.

If p is a positive base point of root r, then q is a negative base point of $-r$ (Lemma 2.7). As a consequence, (2.46) implies

$$E_r\eta_q = \begin{cases} -\eta_{q-r} & \text{when } q \text{ is a positive base point of } -r, \\ \eta_{q-r} & \text{when } q \text{ is a negative base point of } -r, \\ 0 & \text{for any other } q \in \mathcal{P}. \end{cases} \tag{2.47}$$

This also means that the transpose E_r^T of E_r satisfies $E_r^T = -E_{-r}$.

Lemma 2.16 *Consider a point $p \in \mathcal{P}$. Define*

$$H_p \stackrel{\text{def}}{=} e_p * \eta_p. \tag{2.48}$$

Then

$$e_q H_p = (p \cdot q - \frac{1}{3}) e_q, \tag{2.49}$$

where $p \cdot q - 1/3$, which is either equal to $1, 0$ or -1, should be regarded as an element of the prime field of K.

(The proof is very similar to the proof of lemma 2.15.)

Note that H_p is a diagonal matrix.

We write H_r for the matrix $H_{q_0} - H_{p_0}$ with $r \in \Phi$, $p_0, q_0 \in \mathcal{P}$, $r = q_0 - p_0$. From (2.49) it follows that

$$e_q H_r = (r \cdot q) e_q, \qquad \text{for all } q \in \mathcal{P}, \tag{2.50}$$

proving that the definition of H_r is independent of the choice of the base point p_0 of r.

Note that $\tau(H_p) = 1$ when $p \in \mathcal{P}$, and that $\tau(H_r) = 0$ when $r \in \Phi$.

Proposition 2.17 *A matrix A belongs to $\widehat{\mathbf{L}}$ if and only if*

1. *$A[p,q] = 0$ whenever p, q are collinear.*

2. *$A[p,q] = A[p',q']$ when (p,q) and (p',q') are compatible pairs associated with the same root.*

3. *$A[p,q] = -A[p',q']$ when (p,q) and (p',q') are incompatible pairs associated with the same root.*

4. *$A[i,i] + A[j,j] + A[k,k] + \tau(A) = 0$ for any three points i, j, k forming a line of $\mathcal{Q}$, where $\tau(A)$ is a scalar that does not depend on i, j, k.*

Proof : We first investigate the value of

$$\langle e_i A, e_j, e_k\rangle + \langle e_i, e_j A, e_k\rangle + \langle e_i, e_j, e_k A\rangle + \tau(A)\langle e_i, e_j, e_k\rangle \tag{2.51}$$

for all possible triples (i, j, k) and determine the necessary and sufficient conditions for this expression to be zero. We have

$$\langle e_i A, e_j, e_k\rangle = \begin{cases} \epsilon_{-j-k,j,k} A[i, -j-k], & \text{if } j \sim k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.52}$$

We need to consider the following cases :

1. The points i, j, k are all different and mutually noncollinear. By (2.52) each of the first three terms of (2.51) are zero, independent of the choice of A. The last term is zero by (2.15).

2. The points i, j, k are all different and exactly one pair is collinear, say $i \sim j$. Then (2.51) equals $\pm A[k, -i-j]$. Note that k must be collinear to exactly one point of the line $\{i, j, -i-j\}$, and hence $k \sim -i-j$.

3. The points i, j, k are all different and exactly two pairs (say i, j and j, k) are collinear. In this case (2.51) evaluates to $\epsilon_{-i-j,i,j} A[k, -i-j] + \epsilon_{-j-k,j,k} A[i, -j-k]$. Note that $(k, -i-j)$ and $(i, -j-k)$ are pairs of noncollinear points associated with the same root $i-j-k$. Moreover, these pairs are compatible if and only if $\epsilon_{-i-j,i,j} = -\epsilon_{-j-k,j,k}$.

4. The points i, j, k are the three points of a line. Then (2.51) equals $\epsilon_{ijk}(A[i,i] + A[j,j] + A[k,k] + \tau(A))$.

5. Exactly two of the points i, j, k are the same (say $i = k$) and $i \sim j$. Then (2.51) reduces to $\pm 2A[i, -i-j]$.

6. Exactly two of the points i, j, k are the same (say $i = k$) but $i \not\sim j$. Then (2.51) is zero, independent of A.

7. All three points are the same. Then (2.51) is again zero for every possible A.

Now, if $A \in \widehat{\mathbf{L}}$, then A satisfies (2.43) and then (2.51) must be zero for every triple i, j, k of points. If p, q are collinear points, then take a line pij not through q and set $k = q$ in case 2 above. We find $A[p, q] = 0$. If (p, q) and (p', q') are noncollinear pairs belonging to the same root, then $p \sim q'$ and $p' \sim q$ by Lemma 2.2. Taking $i = p', k = p$ and $j = -p - q' = -q - p'$ we obtain $A[p, q] = \pm A[p', q']$ from case 3, with a plus-sign if and only if the pairs are compatible. The final condition of this lemma then follows from case 4.

Conversely, for A to belong to $\widehat{\mathbf{L}}$ the following conditions, derived from (2.41) are sufficient

$$\begin{array}{lcll} e_i \times e_i A & = & 0, & \text{for all } i \in \mathcal{P}, \\ e_i \times e_j A + e_i A \times e_j + A(e_i \times e_j) + \tau(A)(e_i \times e_j) & = & 0, & \text{for all } i, j \in \mathcal{P}. \end{array} \tag{2.53}$$

The second condition is equivalent to requiring (2.51) to be zero for any triple $i, j, k \in \mathcal{P}$. From cases 1–7 above it follows that this requirement is always satisfied under the conditions of the lemma.

Finally, the first condition of the lemma implies that $e_i A$ can be written as a linear combination of base vectors e_j such that $i \not\sim j$. But for such j we always have $e_i \times e_j = 0$, hence $e_i \times e_i A = 0$ and also the first condition of (2.53) is satisfied. ∎

Note that the conditions of Proposition 2.17 are closed under transposition of A, i.e., $A \in \widehat{\mathbf{L}}$ if and only if $A^T \in \widehat{\mathbf{L}}$ (with $\tau(A^T) = \tau(A)$). It follows that also the dual properties of (2.41–2.43) are always valid :

$$\alpha \times A\alpha + \alpha^{\#} A + \tau(A)\alpha^{\#} = 0, \tag{2.54}$$

$$A\alpha \times \beta + \alpha \times A\beta + (\alpha \times \beta)A + \tau(A)(\alpha \times \beta) = 0, \tag{2.55}$$

$$\langle A\alpha, \beta, \gamma \rangle + \langle \alpha, A\beta, \gamma \rangle + \langle \alpha, \beta, A\gamma \rangle + \tau(A)\langle \alpha, \beta, \gamma \rangle = 0, \tag{2.56}$$

for every $\alpha, \beta, \gamma \in \mathbf{V}^*$ and $A \in \widehat{\mathbf{L}}$.

Define $\widehat{\mathbf{H}}$ to be the subspace of all diagonal matrices of $\widehat{\mathbf{L}}$ and write $\mathbf{H} \overset{\text{def}}{=} \widehat{\mathbf{H}} \cap \mathbf{L}$. (Equivalently, $\mathbf{H}$ is the subspace of diagonal elements of $\mathbf{L}$ and the subspace of all $H \in \widehat{\mathbf{H}}$ such that $\tau(H) = 0$.)

Lemma 2.18 *The elements $H_{\pi_0}, \ldots, H_{\pi_5}, H_{q_0}$ form a basis for $\widehat{\mathbf{H}}$ and hence $\widehat{\mathbf{H}}$ has dimension 7. The elements $H_{\pi_0}, \ldots, H_{\pi_5}$ form a basis for $\mathbf{H}$ and $\mathbf{H}$ has dimension 6.*

Proof : (We use the standard notation for points of $\mathcal{P}$.) We will first prove that $\dim \widehat{\mathbf{H}}$ can be at most 7. Note that every element A of $\widehat{\mathbf{H}}$ satisfies

$$A[i,i] + A[j,j] + A[k,k] + \tau(A) = 0, \quad \text{for any three points } i,j,k \text{ forming a line,} \tag{2.57}$$

i.e., condition 4 of Proposition 2.17. For $A \in \widehat{\mathbf{H}}$ we therefore have

$$A[a_{ij}, a_{ij}] = -A[p_i, p_i] - A[q_j, q_j] - \tau(A), \qquad \text{when } i \neq j, \tag{2.58}$$

and applying this to $(i,j) = (0,k)$ and to $(i,j) = (k,0)$ implies

$$A[q_i, q_i] = A[p_i, p_i] - A[p_0, p_0] + A[q_0, q_0], \qquad \text{for } i = 0, \ldots, 5. \tag{2.59}$$

This proves that all elements $A[x,x]$ can be expressed as linear combinations of $A[p_0,p_0], \ldots, A[p_5,p_5], A[q_0,q_0]$ and $\tau(A)$. Hence $\dim \widehat{\mathbf{H}}$ can be at most 8.

Now, consider the line $a_{01}a_{23}a_{45}$ and apply equations (2.57–2.59) :

$$\begin{aligned}
0 &= A[a_{01},a_{01}] + A[a_{23},a_{23}] + A[a_{45},a_{45}] + \tau(A) \\
&= -A[q_0,q_0] - A[p_1,p_1] - A[q_2,q_2] - A[p_3,p_3] - A[q_4,q_4] - A[p_5,p_5] \\
&\qquad - 2\tau(A) \\
&= -3A[q_0,q_0] + 2A[p_0,p_0] - A[p_1,p_1] - A[p_2,p_2] - A[p_3,p_3] \\
&\qquad - A[p_4,p_4] - A[p_5,p_5] - 2\tau(A).
\end{aligned}$$

This provides an additional linear dependency between $A[q_0,q_0]$, $A[p_0,p_0]$, $\ldots$, $A[p_5,p_5]$ and $\tau(A)$. Hence $\dim \widehat{\mathbf{H}}$ can be at most 7.

Now let $H \in \{H_{\pi_0}, \ldots, H_{\pi_5}, H_{q_0}\}$ and $p \in \{p_0, \ldots, p_5, a_{23}\}$. We will compute all 49 elements of the form $H[p,p]$ ($= e_p H \eta_p$). By (2.50) we have $e_p H_{\pi_i} \eta_p = p \cdot \pi_i$ and by (2.49) we find $e_p H_{q_0} \eta_p = p \cdot q_0 - 1/3$. We leave it to the reader to verify that the corresponding matrix of values $H[p,p]$ is the following (columns correspond to p, rows to H) :

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

This matrix has determinant -1 and hence the elements H are linearly independent (for every field characteristic). It follows that $\dim \widehat{\mathbf{H}} = 7$.

Because τ is a linear functional on $\widehat{\mathbf{H}}$, $\mathbf{H}$ has co-dimension at most 1 in $\widehat{\mathbf{H}}$. As $\tau(H_{\pi_i}) = 0$ and $\tau(H_{q_0}) = 1$ we find that $\dim \mathbf{H} = 6$ and $\{H_{\pi_0}, \ldots, H_{\pi_5}\}$ is a basis for $\mathbf{H}$. ∎

Note that $\mathbf{H}$ as a vector space over K behaves very much like $\mathbf{P}$ as a real vector space. Indeed, any root $r \in \Phi$ can be written as a linear combination of fundamental roots π_i with integral coefficients. Hence, if $r = \lambda_0\pi_0 + \cdots + \lambda_5\pi_5$ in $\mathbf{P}$, then $H_r = \lambda_0 H_{\pi_0} + \cdots + \lambda_5 H_{\pi_5}$, where the coefficients λ_i are now regarded as belonging to the prime field of K. This is an immediate consequence of (2.50).

We combine the previous results into the following theorem.

Theorem 2.19 *The algebras $\widehat{\mathbf{L}}$ (respectively $\mathbf{L}$) have dimension* 79 *(respectively* 78*) and can be written as a direct sum in the following way :*

$$\widehat{\mathbf{L}} = \widehat{\mathbf{H}} \oplus \bigoplus_{r\in\Phi} KE_r, \quad \mathbf{L} = \mathbf{H} \oplus \bigoplus_{r\in\Phi} KE_r. \tag{2.60}$$

$\widehat{\mathbf{L}}$ is generated (as a vector space) by all transformations of the form $a * \alpha$ with $a \in \mathbf{V}, \alpha \in \mathbf{V}^*$. $\mathbf{L}$ is generated by all such transformations satisfying the extra condition $a\alpha = 0$.

Proof : Clearly $\widehat{\mathbf{H}} \leq \widehat{\mathbf{L}}$ and $E_r \in \mathbf{L}$, hence it only remains to be proved that we did not leave out any elements of $\widehat{\mathbf{L}}$ in the sum. If $A \in \mathbf{L}$, then it follows from

Proposition 2.17 that the non-diagonal elements of A are completely determined by giving one value $A[p,q]$ for every root $q-p$. The dimension of the non-diagonal part of $\widehat{\mathbf{L}}$ is therefore the same as the dimension of $\bigoplus_{r\in\Phi} KE_r$. Together with Lemma 2.18 which describes the structure of the diagonal part of $\widehat{\mathbf{L}}$, this proves the first part of (2.60). The second part is a consequence of $\mathbf{H} = \widehat{\mathbf{H}} \cap \mathbf{L}$ and $\tau(E_r) = 0$.

Because every E_r, H_r and H_p is of the form $a * \alpha$, we see that $\widehat{\mathbf{L}}$ can indeed be generated by such elements.

Also note that every E_r is of the form $a * \alpha$ with $a\alpha$=0. By Lemma 2.14 it remains to be proved that also $\mathbf{H}$ can be generated by elements of this form. Consider any root $r = q - p \in \Phi$, with p a positive base point of r. We have

$$(e_q + e_p) * (\eta_q - \eta_p) = H_r + E_r - E_{-r}$$

As $(e_q + e_p)(\eta_q - \eta_p) = 0$ it follows that H_r is indeed a linear combination of elements of the required form. Because $\mathbf{H}$ is generated by $H_{\pi_0}, \ldots H_{\pi_5}$, the theorem follows. ∎

The elements E_r, $r \in \Phi$, together with $H_{\pi_0}, \ldots, H_{\pi_5}$ and H_{q_0}, will be called the *standard basis* for $\widehat{\mathbf{L}}$.

• When char $K = 3$ we have $\tau(\mathbf{1}) = 0$ and hence $\mathbf{1} \in \mathbf{H}$. Because $[\mathbf{1}, \mathbf{L}] = 0$, $\mathbf{L}$ is not simple and we may construct the Lie algebra $\mathbf{L}' \stackrel{\text{def}}{=} \mathbf{L}/K\mathbf{1}$ of dimension 77 which turns out to be simple. For all other characteristics $\mathbf{L}$ is itself a simple algebra.

Lemma 2.20 *Let $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ and $A \in \widehat{\mathbf{L}}$. Then*

$$[a * \alpha, A] = aA * \alpha - a * A\alpha \tag{2.61}$$

Proof : Let $x \in \mathbf{V}$. We apply (2.36) to obtain the following results :

$$\begin{aligned}
x(aA * \alpha) &= (x\alpha)aA - (x \times aA) \times \alpha \\
-x(a * A\alpha) &= -(xA\alpha)a + (x \times a) \times A\alpha \\
-x(a * \alpha)A &= -(x\alpha)aA + ((x \times a) \times \alpha)A \\
xA(a * \alpha) &= (xA\alpha)a - (xA \times a) \times \alpha
\end{aligned}$$

The sum of the left hand sides of these equations is $x(aA * \alpha - a * A\alpha - [a * \alpha, A])$. The sum of the right hand sides evaluates to

$$((x \times a) \times \alpha)A + (x \times a) \times A\alpha - (x \times aA) \times \alpha - (xA \times a) \times \alpha.$$

Applying (2.42) and (2.55) easily proves this expression to be zero. ∎

• This lemma proves that the $*$-operator is compatible with the Lie algebra **L** (cf. Chapter 1).

We will need the following technical lemma in later proofs :

Lemma 2.21 *Let $p \in \mathcal{P}$, $r, s \in \Phi$. Then*

$$e_p E_r E_{-r} = \begin{cases} -e_p, & \text{if } p \text{ is a base point of } r, \\ 0, & \text{otherwise.} \end{cases} \tag{2.62}$$

$$E_r E_s E_r = \begin{cases} -E_r, & \text{if } r = -s, \\ 0, & \text{otherwise.} \end{cases} \tag{2.63}$$

$$E_r H_s E_r = 0. \tag{2.64}$$

Proof : Clearly $e_p E_r E_{-r}$ can only be nonzero when p is a base point of E_r. When p is a positive base point of r, we find $e_p E_r = e_{p+r}$, $e_p E_r E_{-r} = -e_p$, for then $p + r$ is a negative base point of $-r$. Likewise, when p is a negative base point, we have $e_p E_r = -e_{p+r}$ and $e_p E_r E_{-r} = -e_p$, proving (2.62).

Secondly, compute $e_p E_r E_s E_r$. Note that this expression can only be nonzero if $p \cdot r = (p + r) \cdot s = (p + r + s) \cdot r = -1$. The first and the last equality imply $(r + s) \cdot r = 0$ and hence $s \cdot r = -2$, i.e., $r = -s$. When this is the case, the first part of the lemma proves $e_p E_r E_{-r} E_r = -e_p E_r$, for every p, hence $E_r E_{-r} E_r = -E_r$.

Finally, as H_s is a diagonal matrix, $e_p E_r H_s E_r$ can only be nonzero when both $p + r$ and $p + 2r$ belong to $\mathcal{P}$. But this is impossible. ∎

Proposition 2.22 *Let* $r, s \in \Phi$. *Then*

1. $[E_r, E_{-r}] = H_r$.
2. $[E_r, E_s] = E_{r+s}$ *or* $-E_{r+s}$, *when* $\langle r, s\rangle = -1$.
3. $[E_r, E_s] = 0$ *when* $\langle r, s\rangle \geq 0$.
4. $[E_r, H_s] = \langle s, r\rangle E_r$.
5. $[H_r, H_s] = 0$.

Proof : Let $r = q - p$ with $p, q \in \mathcal{P}$, i.e., $E_r = \pm e_q * \eta_p$ where the sign depends on whether p is a positive or a negative base point of r. From Lemma 2.20 we find

$$[E_r, E_s] = \pm(e_q E_s * \eta_p - e_q * E_s \eta_p), \tag{2.65}$$

1. Let $x \in \mathcal{P}$, then using (2.62), we find

$$e_x[E_r, E_{-r}] = e_x E_r E_{-r} - e_x E_{-r} E_r = \begin{cases} -e_x, & \text{if } x \text{ is a base point of } r, \\ e_x, & \text{if } x \text{ is a base point of } -r, \\ 0, & \text{otherwise.} \end{cases}$$

Comparing this with (2.50), we obtain $e_x[E_r, E_{-r}] = e_x H_r$ for every $x \in \mathcal{P}$, and hence $[E_r, E_{-r}] = H_r$.

2. If $\langle r, s\rangle = -1$ then Table 2.2 (pg. 35) shows us that there exist three base points p of r (but not of s) such that $q = p + r$ is a base point of s. With such p, equation (2.65) yields :

$$[E_r, E_s] = \pm e_{q+s} * \eta_p = \pm E_{q+s-p} = \pm E_{r+s}.$$

3. If $\langle r, s\rangle \geq 0$ then we can always find (p, q) such that p is not a base point of $-s$, and q is not a base point of s. (When $\langle r, s\rangle = 1$, choose $p \neq p_j$ in the standard notation for points of $\mathcal{P}$, in the other cases any pair will do.) Hence, by (2.65), $[E_r, E_s] = 0$ in each case.

4. Choose a base point p of r. From Lemma 2.20 we obtain $[E_r, H_s] = e_q H_s * \eta_p - e_q * H_s \eta_p = \langle s, q\rangle e_q * \eta_p - \langle s, p\rangle e_q * \eta_p = \langle s, r\rangle E_r$.

5. Both H_r and H_s are diagonal matrices and hence commute. ∎

• Proposition 2.22 almost provides us with a Chevalley basis for $\mathbf{L}$ (cf. Chapter 1) except that the Lie-bracket arguments on the left hand side of the fourth equation are in the wrong order.

To obtain a true Chevalley basis, set $h_r = -H_r$ and for every pair $\{r, -r\}$ of opposite roots, choose $e_r = E_r$ in one case and $e_{-r} = -E_{-r}$ in the other. This small discrepancy originates from the way we define E_r using the notion of positive and negative base points of r. As a consequence some of the formulas we will develop later may at certain places differ in sign from the equivalent formulas listed for instance in [8].

It has already been stated that $\mathbf{H}$ and $\mathbf{P}$ behave very much in the same way. It will therefore not come as a surprise that also the dot product on $\mathbf{P}$ has its equivalent on $\mathbf{H}$. We define the bilinear product '$\cdot$' on $\widehat{\mathbf{H}}$ by its action on the base elements $H_{\pi_0}, \dots, H_{\pi_5}, H_{q_0}$:

$$\begin{aligned} H_{\pi_i} \cdot H_{\pi_j} &\stackrel{\text{def}}{=} \pi_i \cdot \pi_j, \\ H_{\pi_i} \cdot H_{q_0} &\stackrel{\text{def}}{=} \pi_i \cdot q_0, \\ H_{q_0} \cdot H_{\pi_j} &\stackrel{\text{def}}{=} q_0 \cdot \pi_j, \\ H_{q_0} \cdot H_{q_0} &\stackrel{\text{def}}{=} q_0 \cdot q_0 - 1/3 = 1, \end{aligned} \tag{2.66}$$

where in each case the right hand side should be regarded as an element of the prime field of K.

Because we can express every root $r, s \in \Phi$ as a linear combination of fundamental roots with integer coefficients, it immediately follows that

$$H_r \cdot H_s = r \cdot s, \qquad \text{for any } r, s \in \Phi.$$

Because every $p \in \mathcal{P}$ can be written as either $q_0 + r$ with $r \in \Phi$, or as $(q_0 + r) + r'$ with $r, r' \in \Phi$ and $q_0 + r \in \mathcal{P}$, we can prove the following identities in a similar way :

$$H_p \cdot H_r = H_r \cdot H_p = p \cdot r,\ H_p \cdot H_q = p \cdot q - 1/3, \quad \text{for any } p, q \in \mathcal{P}, r \in \Phi. \tag{2.67}$$

We now extend the definition of this dot product to $\widehat{\mathbf{L}}$ as follows (let $r, s \in \Phi$, $H \in \widehat{\mathbf{H}}$) :

$$\begin{aligned} E_r \cdot E_s &\stackrel{\text{def}}{=} \begin{cases} -1 & \text{when } r = -s, \\ 0 & \text{otherwise.} \end{cases} \\ E_r \cdot H = H \cdot E_r &\stackrel{\text{def}}{=} 0 \end{aligned} \tag{2.68}$$

As an immediate consequence we find that $A \cdot B = B \cdot A$ for every $A, B \in \widehat{\mathbf{L}}$.

Lemma 2.23 *Let* $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$, $A \in \widehat{\mathbf{L}}$. *Then*

$$(a * \alpha) \cdot A = aA\alpha. \tag{2.69}$$

Proof : It is sufficient to prove this for all $a = e_p$, $\alpha = \eta_q$ with $p, q \in \mathcal{P}$ and for all $A = E_r$ with $r \in \Phi$ and for all $A = H_x$ with $x \in \mathcal{P}$ or $x \in \Phi$.

Now, $e_p E_r \eta_q$ is the (p, q)th entry of the matrix E_r. The only r for which this is nonzero is $r = q - p$ and then the entry is 1 when p is a positive base point of r and -1 when p is a negative base point of r. Also in that case $e_p * \eta_q = \pm E_{p-q} = \pm E_{-r}$, with a plus sign if q is a positive base point of $-r$ (and hence p is a negative base point of r) and a minus sign otherwise. This proves the lemma when A is of the form E_r.

Secondly, consider $e_p H_r \eta_q$ when $r \in \Phi$. Then $e_p H_r \eta_q = (p \cdot r)(e_p \eta_q)$ equals zero if $p \neq q$ and in that case $e_p * \eta_q$ is zero or a non-diagonal matrix. When $p = q$ we find $e_p H_r \eta_p = p \cdot r = H_p \cdot H_r$ by (2.67). A similar argument holds for H_x when $x \in \mathcal{P}$. ∎

As an immediate application we find that $(a * \alpha) \cdot \mathbf{1} = a\alpha$ and hence

$$\mathbf{1} \cdot A = A \cdot \mathbf{1} = \tau(A), \qquad \text{for all } A \in \widehat{\mathbf{L}}. \tag{2.70}$$

The dot product can also be used to coordinatize $\widehat{\mathbf{L}}$. From (2.60) we know that every $A \in \widehat{\mathbf{L}}$ can be written in a unique way as

$$A = A_H + \sum_{r \in \Phi} A[r] E_r, \tag{2.71}$$

with 'coordinates' $A[r] \in K$ and the 'diagonal part' $A_H \in \widehat{\mathbf{H}}$. Multiplying this equation with E_{-r} and using (2.68), we see that $A[r] = -A \cdot E_{-r} = A \cdot E_r^T$.

Lemma 2.24 *Using the standard notation for points of $\mathcal{P}$, the following elements of $\widehat{\mathbf{L}}$*

$$H_r - \mathbf{1},\ H_{q_0} - \mathbf{1},\ H_{q_0} + H_{q_1} - \mathbf{1},\\ H_{q_0} + H_{q_1} + H_{q_2} - \mathbf{1},\ -H_{p_4} - H_{p_5} - 2 \cdot \mathbf{1},\ -H_{p_5} - \mathbf{1},\ \mathbf{1} \tag{2.72}$$

form a dual basis of $H_{\pi_0}, \ldots, H_{\pi_5}, H_{q_0}$ with respect to the dot product on $\widehat{\mathbf{H}}$. By adding the elements $-E_{-r}$ for every $r \in \Phi$ we may extend this to a dual basis for the standard basis of $\widehat{\mathbf{L}}$.

Proof : We must prove that the dot product of elements of (2.72) with the standard base vectors always yields zero, unless the elements occur at corresponding positions in the given list, in which case the dot product should be 1.

First note that the first 6 elements of the dual base are strongly related to the fundamental weights of E_6, cf. (2.11). Because the fundamental weights form a dual basis of the fundamental roots in $\mathbf{P}$, and because $\mathbf{1} \cdot H_{\pi_i} = 0$, it easily follows that these 6 elements satisfy the required properties. It only remains to compute the dot product of the first 7 elements of (2.72) with H_{q_0}. We find

$$\begin{aligned}
&H_r \cdot H_{q_0} - \mathbf{1} \cdot H_{q_0} = r \cdot q_0 - 1 = 1 - 1 = 0\\
&H_{q_0} \cdot H_{q_0} - \mathbf{1} \cdot H_{q_0} = (q_0 \cdot q_0 - 1/3) - 1 = 1 - 1 = 0\\
&H_{q_0} \cdot H_{q_0} + H_{q_1} \cdot H_{q_0} - \mathbf{1} \cdot H_{q_0}\\
&\qquad = (q_0 \cdot q_0 - 1/3) + (q_1 \cdot q_0 - 1/3) - 1 = 1 + 0 - 1 = 0\\
&H_{q_0} \cdot H_{q_0} + H_{q_1} \cdot H_{q_0} + H_{q_2} \cdot H_{q_0} - \mathbf{1} \cdot H_{q_0}\\
&\qquad = (q_0 \cdot q_0 - 1/3) + (q_1 \cdot q_0 - 1/3) + (q_2 \cdot q_0 - 1/3) - 1\\
&\qquad = 1 + 0 + 0 - 1 = 0\\
&-H_{p_4 \cdot H_{q_0}} - H_{p_5} \cdot H_{q_0} - 2 \cdot \mathbf{1} \cdot H_{q_0}\\
&\qquad = -(p_4 \cdot q_0 - 1/3) - (p_5 \cdot q_0 - 1/3) - 2 = 1 + 1 - 2 = 0\\
&-H_{p_5} \cdot H_{q_0} - \mathbf{1} \cdot H_{q_0} = -(p_5 \cdot q_0 - 1/3) - 1 = 1 - 1 = 0\\
&\mathbf{1} \cdot H_{q_0} = 1
\end{aligned}$$

It is an immediate consequence of (2.68) that this dual basis for $\widehat{\mathbf{H}}$ can be extended to a dual basis for $\widehat{\mathbf{L}}$ by adding a base element $-E_{-r}$ for every base element E_r of $\widehat{\mathbf{L}}$. ∎

We end this section by proving some additional properties of this dot product.

Lemma 2.25 *Let $A, B, C \in \widehat{\mathbf{L}}$. Then*

$$[A, B] \cdot C = [B, C] \cdot A = [C, A] \cdot B, \tag{2.73}$$

and in particular $[A, B] \cdot A = 0$.

Proof : By Theorem 2.19 it is sufficient to prove this when A is of the form $a * \alpha$, with $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$. We have

$$\begin{aligned} [A, B] \cdot C &= [a * \alpha, B] \cdot C \\ &= (aB * \alpha) \cdot C - (a * B\alpha) \cdot C && \text{by (2.61)} \\ &= aBC\alpha - aCB\alpha = a[B, C]\alpha = (a * \alpha) \cdot [B, C] && \text{by (2.69),} \end{aligned}$$

proving the first equality. The second equality follows from the first by permuting the symbols A, B and C. ∎

• This lemma proves that the dot product on $\mathbf{L}$ is compatible with the Lie algebra $\mathbf{L}$ (cf. Chapter 1).

As an immediate consequence we find $[[A, B], C] \cdot D = [C, D] \cdot [A, B]$ for every $A, B, C, D \in \widehat{\mathbf{L}}$. Applying this to the Jacobi identity $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$ for Lie algebras, we find

$$[A, B] \cdot [C, D] + [B, C] \cdot [A, D] + [C, A] \cdot [B, D] = 0. \tag{2.74}$$

Proposition 2.26 *Let $A, B, C \in \mathbf{L}$. Then*

$$\operatorname{Tr} AB = 6(A \cdot B), \qquad \operatorname{Tr} ABC = 3[A, B] \cdot C. \tag{2.75}$$

Also

$$\operatorname{Tr} X = -9\tau(X), \qquad \textit{for all } X \in \widehat{\mathbf{L}}. \tag{2.76}$$

Proof : In general, the trace of a matrix X in $\mathbf{L}$ is equal to $\sum_{p\in\mathcal{P}} X[p,p] = \sum_{p\in\mathcal{P}} e_p X \eta_p$. Hence

$$\operatorname{Tr} AB = \sum e_p AB\eta_p = (\sum e_p A * \eta_p)\cdot B,$$

and

$$\operatorname{Tr} ABC = \sum e_p ABC\eta_p = (\sum e_p AB * \eta_p)\cdot C.$$

It is therefore sufficient to prove that $\sum e_p A * \eta_p = 6A$ and $\sum e_p AB * \eta_p = 3[A,B]$, and in both cases we need to prove this only when A, B are elements of the standard basis for $\mathbf{L}$, i.e., when they are of the form E_r or H_r, with $r \in \Phi$.

If p is a base point of r, then $e_p E_r * \eta_p = E_r$ by Lemma 2.45. If p is not a base point of r then $e_p E_r = 0$. Because r has exactly 6 base points, we find $\sum e_p E_r * \eta_p = 6E_r$. Also $e_p H_r * \eta_p = (p\cdot r)H_p$ which is non-zero only if p is a base point of either r or $-r$. Partitioning these 12 points into 6 pairs (p,q) where $q = p + r$, we find for each pair

$$e_p H_r * \eta_p + e_q H_r * \eta_q = (p\cdot r)H_p - (q\cdot r)H_q = -H_p + H_q = H_r.$$

Now consider $r, s \in \Phi$. We compute $e_p E_r E_s * \eta_p$. Note that this element can only be nonzero when the following conditions are satisfied : $p\cdot r = -1$, $(p+r)\cdot s = -1$ and $(p+r+s)\cdot p = 1/3$ or $4/3$. It follows that $r\cdot s = -1$ or -2 respectively. As a consequence $\sum e_p E_r E_s * \eta_p = 0$ whenever $r\cdot s \geq 0$, but then also $[E_r, E_s] = 0$ by Proposition 2.22. We consider the remaining two cases :

- $r\cdot s = -2$, i.e., $s = -r$. Then $e_p E_r E_{-r} * \eta_p = -H_p$ if p is a base point of r and 0 otherwise, and hence $\sum e_p E_r E_{-r} * \eta_p = 3H_r$, because the 6 base points of r add up to $-3r$ in $\mathbf{P}$, by Lemma 2.3.

- $r\cdot s = -1$, i.e., $r + s \in \Phi$. Then $e_p E_r E_s * \eta_p$ is nonzero only when p is a base point of r and $p + r$ is a base point of s. From Table 2.2 (pg. 35) we see that there are exactly 3 such p. Also note that none of these p is at the same time a base point of s, hence for these p we have $e_p E_s E_r * \eta_p = 0$, and hence $e_p E_r E_s * \eta_p = e_p [E_r, E_s] * \eta_p$ and this is equal to $[E_r, E_s]$, for $[E_r, E_s] = \pm E_{r+s}$ and $e_p E_{r+s} * \eta_p = E_{r+s}$ by Lemma 2.45.

We may compute the value of $e_p H_r E_s * \eta_p$ in a similar way. Indeed $e_p H_r E_s * \eta_p = (p \cdot r) e_p E_s * \eta_p$ and this is equal to $(p \cdot r) E_s$ when p is a base point of s, and zero otherwise. Now, the sum of all base points p of s is equal to $-3s$ and hence $\sum e_p H_r E_s * \eta_p = -3(s \cdot r) E_s$ and this is equal to $3[H_r, E_s]$ by Proposition 2.22. Likewise $e_p E_r H_s * \eta_p = ((p+r) \cdot s) E_r$ when p is a base point of r (and zero otherwise). Summing over all base points of r then yields $\sum e_p E_r H_s * \eta_p = 3(r \cdot s) E_r = 3[E_r, H_s]$.

Finally consider $\sum_p e_p H_r H_s * \eta_p = \sum_p (p \cdot r)(p \cdot s) H_p$. We need to prove that this expression evaluates to $[H_r, H_s] = 0$. As before, this sum is equal to the sum of $(q \cdot s) H_q - (p \cdot s) H_p$ over the 6 pairs (p, q) with $q = p + r$ and p a base point of r. Now

$$(q \cdot s) H_q - (p \cdot s) H_p = (p \cdot s) H_q + (r \cdot s) H_q - (p \cdot s) H_p = (p \cdot s) H_r + (r \cdot s) H_q$$

Summing the first term on the right hand side over all base points of r yields $(-3r \cdot s) H_r$, while summing the second term over all base points q of $-r$ yields $(r \cdot s)(-3) H_{-r}$. Both results add up to zero.

To prove (2.76) let $X \in \widehat{\mathbf{L}}$ and consider $\operatorname{Tr} X = \sum X[p, p]$. Partitioning the 27 points of $\mathcal{P}$ into 9 lines, and applying Proposition 2.17–4, we see that this sum must be equal to 9 times $-\tau(X)$. ∎

- Also, when $A, B, C, D \in \mathbf{L}$ we have

$$\operatorname{Tr} ABCD = [A, B] \cdot [C, D] - [A, D] \cdot [B, C] + (A \cdot B)(C \cdot D) + (A \cdot C)(B \cdot D) + (A \cdot D)(B \cdot C).$$

Currently we have only been able to prove this by computer.

2.3 Isotropic elements

We call an element $e \in \mathbf{V}$ *isotropic* if and only if $e^\# = 0$. Dually, we call $\eta \in \mathbf{V}^*$ isotropic if and only if $\eta^\# = 0$. The elements e_p of $\mathbf{V}$ and η_p of $\mathbf{V}^*$ with $p \in \mathcal{P}$ serve as typical examples of isotropic elements.

A subspace of $\mathbf{V}$ or $\mathbf{V}^*$ is called *isotropic* if and only if all its elements are isotropic. From (2.22) it follows that the subspace generated by $a_1, \ldots, a_k \in \mathbf{V}$ is isotropic if and only if $a_i^\# = a_i \times a_j = 0$ for all i, j, and a similar property holds for subspaces of $\mathbf{V}^*$. Clearly Ke is isotropic if and only if $e^\# = 0$, and $Ke + Kf$ is isotropic if and only if $e^\# = f^\# = e \times f = 0$.

• In [1] isotropic elements and subspaces are called *singular*, an element $a \in \mathbf{V}$ such that $D(a) = 0$, but not necessarily $a^\# = 0$, is called *brilliant*, and other elements are called *dark*. A 2-space generated by isotropic elements e, f such that $e \times f \neq 0$ is called *hyperbolic*. In [10] the authors use the terminology *white*, *gray* ($D(a) = 0$ but $a^\# \neq 0$) and *dark*.

As was mentioned in Chapter 1, the one-dimensional isotropic subspaces Ke will later serve as points of a geometry $\mathcal{E}$ and the operations defined on $\mathbf{V}$, when applied to these points, will enable us to distinguish between various relations of pairs of such points (for example, Ke and Kf will turn out to be collinear, or equal, if and only if $e \times f = 0$).

Proposition 2.27 *Let e be an isotropic element of $\mathbf{V}$. Let $a, b \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$, $A \in \widehat{\mathbf{L}}$. Then*

1. $(e \times a) \times (e \times b) = \langle e, a, b \rangle e$,
2. $(e \times a)^\# = (ea^\#)e$ *and hence* $e \times a$ *is isotropic if both* a *and* e *are isotropic,*
3. $((e \times a) \times \alpha) \times e = (e\alpha)(e \times a)$,
4. $e(e * \alpha) = (e\alpha)e$,
5. $e * (e \times a) = 0$,
6. $e \times eA = 0$ *and hence* $eA(e \times a) = 0$.

Proof : These identities are immediate consequences of (2.26), (2.27), (2.32), (2.37), (2.39) and (2.41) specialised to the case $a^\# = 0$. ∎

An element $E \in \widehat{\mathbf{L}}$ will be called *isotropic* if and only if it can be written in the form $E = e * \eta$ with $e\eta = 0$ and both e and η isotropic. It follows that all isotropic elements of $\widehat{\mathbf{L}}$ belong to $\mathbf{L}$. The elements E_r with $r \in \Phi$ serve as typical examples of isotropic elements of $\mathbf{L}$.

As with $\mathbf{V}$ and $\mathbf{V}^*$ a subspace of $\mathbf{L}$ will be called *isotropic* if and only if all its elements are isotropic.

Proposition 2.28 *Let $E = e * \eta$ be an isotropic element of $\mathbf{L}$ (with $e \in \mathbf{V}$, $\eta \in \mathbf{V}^*$, $e^\# = 0$, $\eta^\# = 0$ and $e\eta = 0$). Let $a, b, c \in \mathbf{V}$, $\alpha, \beta \in \mathbf{V}^*$, $A \in \widehat{\mathbf{L}}$. Then*

1. *$eE = 0$ and dually $E\eta = 0$.*
2. *$E(e \times a) = 0$, and dually $(\eta \times \alpha)E = 0$.*
3. *$aE \times e = 0$, and dually $\eta \times E\alpha = 0$.*
4. *$E^2 = 0$ and $E \cdot E = 0$.*
5. *$e * E\alpha = (e\alpha)E$, and dually $aE * \eta = (a\eta)E$.*
6. *$eAE = (E \cdot A)e$, and dually $EA\eta = (E \cdot A)\eta$. In particular $e \in \mathbf{V}E$ and dually $\eta \in E\mathbf{V}^*$.*
7. *$(aE)^\# = 0$, $aE \times bE = 0$, and dually $(E\alpha)^\# = 0$, $E\alpha \times E\beta = 0$. Hence $\mathbf{V}E$ (and $E\mathbf{V}^*$) are isotropic subspaces of $\mathbf{V}$ (and $\mathbf{V}^*$).*
8. *$aE * E\alpha = (aE\alpha)E$.*
9. *$EAE = (E \cdot A)E$.*
10. *$E(aE \times b) = 0$ and dually $(\beta \times E\alpha)E = 0$.*

Proof : (We leave the proof of the dual properties to the reader.)

1. By Proposition 2.27–4 we have $eE = e(e * \eta) = 0$.

2. We have $E(e \times a) = (e * \eta)(e \times a) = \langle a, e, e\rangle\eta - ((e \times a) \times \eta) \times e = 0$, by Proposition 2.27–3.

3. We find $aE \times e = -E(a \times e) - eE \times a = 0$ by the above.

4. For every $a \in \mathbf{V}$ we have $aE^2 = aE(e * \eta) = (aE\eta)e - (aE \times \eta) \times e = 0$ by the above. Also $E \cdot E = eE\eta = 0$.

5. We compute $e * E\alpha = e * (e * \eta)\alpha = e * (e\alpha)\eta - e * (e \times (\eta \times \alpha)) = (e\alpha)E$ by Proposition 2.27–5.

6. Taking the dot product of the above with $A \in \widehat{\mathbf{L}}$ yields $eAE\alpha = (e\alpha)(E \cdot A)$. This is true for every $\alpha \in \mathbf{V}^*$, hence $eAE = (E \cdot A)e$. If $E \neq 0$ we may always find $A \in \mathbf{L}$ such that $E \cdot A \neq 0$, and then $e = eAE/(E \cdot A) \in \mathbf{V}E$.

7. We have

$$
\begin{aligned}
(aE)^\# &= [(a\eta)e - (e \times a) \times \eta]^\# \\
&= (a\eta)^2 e^\# + [(e \times a) \times \eta]^\# - (a\eta)[(a \times e) \times \eta] \times e, \quad \text{by (2.22)} \\
&= ((e \times a)^\# \eta)\eta - (a\eta)(e\eta)(e \times a), \qquad \text{by Proposition 2.27–2} \\
&= (ea^\#)(e\eta)\eta = 0.
\end{aligned}
$$

Substituting $a + b$ for a in the above proves that also $aE \times bE = 0$.

8. Consider $e' \stackrel{\text{def}}{=} aE$ and $E' \stackrel{\text{def}}{=} e' * \eta$. By case 5 above, we have $E' = (a\eta)E$. Note that $(e')^\# = 0$ and $e'\eta = 0$, by the above, and hence we may apply case 5 of this proposition to E' and e'. We find $e' * E'\alpha = (e'\alpha)E'$ and then $aE * E'\alpha = (aE\alpha)E'$. Because $E' = (a\eta)E$ we obtain that $aE * E\alpha = (aE\alpha)E$ whenever $a\eta \neq 0$. By duality, the same is true when $e\alpha \neq 0$.

Now consider the case $e\alpha = a\eta = 0$. Then $aE = -(e \times a) \times \eta$ and $E\alpha =$

$-e \times (\eta \times \alpha)$. We compute

$$
\begin{aligned}
&aE * E\alpha \\
&= -[(e \times a) \times \eta] * E\alpha \\
&= [(e \times a) \times E\alpha] * \eta + (\eta \times E\alpha) * (e \times a) + \langle \eta, E\alpha, e \times a \rangle \mathbf{1}, \quad \text{by (2.40)} \\
&= [(e \times a) \times E\alpha] * \eta, \qquad \text{by case 3 above} \\
&= -[(e \times a) \times (e \times (\eta \times \alpha))] * \eta \\
&= -\langle e, a, \eta \times \alpha \rangle e * \eta, \qquad \text{by Proposition 2.27–1} \\
&= -a[e \times (\eta \times \alpha)]E = (aE\alpha)E.
\end{aligned}
$$

9. Take the dot product with A of both sides of the previous property. Because this holds for every $a \in \mathbf{V}$ and $\alpha \in \mathbf{V}^*$, we find $EAE = (E \cdot A)E$.

10. Applying (2.42) yields $E(aE \times b) = -aE^2 \times b - aE \times bE$ and this is 0 by cases 4 and 7 above. ∎

• In [12] an element E of $\mathbf{L}$ is called *extremal* if and only if $[[\mathbf{L}, E], E] \leq KE$. Note that $[[A, E], E] = AE^2 - 2EAE + E^2A = -2(E \cdot A)E$ by Proposition 2.28, hence isotropic elements of $\mathbf{L}$ are extremal in this sense. A related notion is that of *sandwich* element [27], an element E which satisfies $[[\mathbf{L}, E], E] = 0$. When char $K = 2$ all isotropic elements of $\mathbf{L}$ are sandwich elements.

Proposition 2.29 *Let $e \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ be such that $e^\# = 0$. Then $E = e * \alpha$ is isotropic if and only if $e\alpha = 0$ and $e \times \alpha^\# = 0$.*

Proof : First assume $E = e * \alpha$ is isotropic. Because $E \in \mathbf{L}$ we immediately find $\tau(E) = e\alpha = 0$.

Let f denote any isotropic element of $\mathbf{V}$. It follows from Proposition 2.27–2 that $e \times f$ is isotropic. We have $fE = (f\alpha)e - (e \times f) \times \alpha$ and hence

$$
\begin{aligned}
(fE)^\# &= (f\alpha)e^\# - (f\alpha)[(e \times f) \times \alpha] \times e + [(e \times f) \times \alpha]^\#, \quad \text{by (2.22)} \\
&= [(e \times f) \times \alpha]^\#, \qquad \text{by Proposition 2.27–3} \\
&= \langle e, f, \alpha^\# \rangle (e \times f), \qquad \text{by Proposition 2.27–2}
\end{aligned}
$$

Because E is isotropic this expression must be zero and therefore $\langle e, f, \alpha^\# \rangle = f(e \times \alpha^\#) = 0$ for every f such that $e \times f$ is isotropic. In particular, this is true for every base element $f = e_p$ of $\mathbf{V}$, and hence $e \times \alpha^\# = 0$.

Conversely, assume $e\alpha = 0$ and $e * \alpha^\# = 0$. Consider an element η of the form $\eta = E\beta$ with $\beta \in \mathbf{V}^*$ such that $e\beta \neq 0$. (If no such β exists, $e = 0$ and then E is trivially isotropic.) We have $\eta = (e\beta)\alpha - (\alpha \times \beta) \times e$ and hence, using Proposition 2.27, $e\eta = 0$ and $e * \eta = (e\beta)E$. It is therefore sufficient to prove that $\eta^\# = 0$. We have

$$\eta^\# = (e\beta)^2\alpha^\# - (e\beta)[(\alpha \times \beta) \times e] \times \alpha + [(\alpha \times \beta) \times e]^\#, \tag{2.77}$$

Applying the dual of (2.32) and using $e \times \alpha^\# = 0$ we obtain

$$[(\alpha \times \beta) \times e] \times \alpha = (e\beta)\alpha^\# + (\alpha^\#\beta)e$$

and from Proposition 2.27 we derive

$$\begin{aligned} [(\alpha \times \beta) \times e]^\# &= (e(\alpha \times \beta)^\#)e \\ &= \langle e, \alpha^\#, \beta^\# \rangle e - (e\alpha)(\beta^\#\alpha)e - (e\beta)(\alpha^\#\beta)e = -(e\beta)(\alpha^\#\beta)e. \end{aligned}$$

Subsituting these results into (2.77) yields $\eta^\# = 0$. ∎

• There are many other properties of isotropic elements of $\mathbf{L}$ that maybe could have been used as alternative definitions of isotropy. For example, when E is isotropic then $\mathbf{V}E$ (respectively $E\mathbf{V}^*$) are isotropic subspaces of $\mathbf{V}$ (respectively $\mathbf{V}^*$). Also the rank of E as a matrix is minimal over all nontrivial elements of $\mathbf{L}$ (and equal to 6). Unfortunately, we have not yet been able to prove that those definitions are equivalent to ours.

When $\operatorname{char} K \neq 2$ it seems to be sufficient that $E^2 = 0$ for $E \in \mathbf{L}$ to be isotropic. Counterexamples can easily be found when $\operatorname{char} K = 2$.

2.4 Automorphisms

Let $g : \mathbf{V} \to \mathbf{V} : a \mapsto a^g$ be a non-singular linear transformation of $\mathbf{V}$. We may extend the action of g in a unique way to $\mathbf{V}^*$ by requiring

$$a^g \alpha^g = a\alpha, \qquad \text{for all } a \in \mathbf{V}, \alpha \in \mathbf{V}^*, \tag{2.78}$$

and to $\mathrm{Hom}(\mathbf{V}, \mathbf{V})$, by requiring

$$a^g A^g \alpha^g = aA\alpha, \qquad \text{for all } a \in \mathbf{V}, \alpha \in \mathbf{V}^*, A \in \mathrm{Hom}(\mathbf{V}, \mathbf{V}). \tag{2.79}$$

We will call g an *automorphism* of $\mathbf{V}$ (and $\mathbf{V}^*$) if and only if it satisfies

$$(a^g)^\# = (a^\#)^g, \quad (\alpha^g)^\# = (\alpha^\#)^g, \quad \text{for all } a \in \mathbf{V}, \alpha \in \mathbf{V}^*. \tag{2.80}$$

As an immediate consequence, every automorphism g satisfies

$$\begin{array}{lll} a^g \times b^g = (a \times b)^g, & \langle a^g, b^g, c^g \rangle = \langle a, b, c \rangle & D(a^g) = D(a), \\ \alpha^g \times \beta^g = (\alpha \times \beta)^g, & \langle \alpha^g, \beta^g, \gamma^g \rangle = \langle \alpha, \beta, \gamma \rangle, & D(\alpha^g) = D(\alpha), \\ a^g * \alpha^g = (a * \alpha)^g, & & \end{array} \tag{2.81}$$

for all $a, b, c \in \mathbf{V}$, $\alpha, \beta, \gamma \in \mathbf{V}^*$. (When $\mathrm{char}\, K = 3$, use (2.29) to prove the identities in the third column.)

Applying g to (2.41) we find that $A \in \widehat{\mathbf{L}}$ implies $A^g \in \widehat{\mathbf{L}}$ with

$$\tau(A^g) = \tau(A), \quad \text{for } A \in \widehat{\mathbf{L}} \tag{2.82}$$

and in a similar way (2.69) implies

$$A^g \cdot B^g = A \cdot B, \quad \text{for } A, B \in \widehat{\mathbf{L}}. \tag{2.83}$$

Another consequence is that automorphisms map isotropic elements of $\mathbf{V}$ ($\mathbf{V}^*$, $\mathbf{L}$, respectively) onto isotropic elements of $\mathbf{V}$ ($\mathbf{V}^*$, $\mathbf{L}$, respectively).

• The definition of automorphism given here can easily be extended to semi-linear transformations of $\mathbf{V}$. In that case we require $a^g\alpha^g$ to be equal to $(a\alpha)^\sigma$, where σ denotes the field automorphism for the semi-linear transformation.

An example is provided by letting a non-trivial field automorphism σ of K act on the individual coordinates of the element $a \in \mathbf{V}$, i.e., $a^\sigma \stackrel{\text{def}}{=} \sum (a\eta_i)^\sigma e_i$.

It is important to note that in that case σ needs to be a bijection, i.e., $K^\sigma = K$. In later chapters we will encounter similar σ which are *endomorphisms* of K, i.e., injective but not necessarily bijective.

The following proposition provides an interesting type of automorphism.

Proposition 2.30 *Let E be an isotropic element of $\mathbf{L}$, then $x(E) \stackrel{\text{def}}{=} \mathbf{1} - E$ is an automorphism of $\mathbf{V}$ (and $\mathbf{V}^*$). We have*

$$\begin{array}{rcll} a^{x(E)} & = & a - aE, & \text{for all } a \in \mathbf{V}, \\ \alpha^{x(E)} & = & \alpha + E\alpha, & \text{for all } \alpha \in \mathbf{V}^*, \\ A^{x(E)} & = & A + [E,A] - (E \cdot A)E, & \text{for all } A \in \widehat{\mathbf{L}}. \end{array} \tag{2.84}$$

For $k, k' \in K$ we have $x(kE)x(k'E) = x((k+k')E)$. In particular $x(E)x(-E) = \mathbf{1}$.

Proof : We have $(\mathbf{1} - kE)(\mathbf{1} - k'E) = \mathbf{1} - (k+k')E + kk'E^2$ and hence by Proposition 2.28–4 $x(kE)x(k'E) = x((k+k')E)$. Substituting $k = 1$, $k' = -1$ we obtain $x(E)x(-E) = \mathbf{1}$.

To prove (2.84) we need to verify (2.78–2.79). This is straightforward if we rewrite the expression for $\alpha^{x(E)}$ as $(\mathbf{1} + E)\alpha$ and the expression for $A^{x(E)}$ as $(\mathbf{1} + E)A(\mathbf{1} - E)$. (For the last formula, note that $EAE = (E \cdot A)E$ by Proposition 2.28–9.)

Finally, let $a \in \mathbf{V}$, then

$$(a^{x(E)})^\# = (a - aE)^\# = a^\# - a \times aE + (aE)^\# = a^\# + Ea^\# = (a^\#)^{x(E)}$$

by Proposition 2.28–7 and (2.41). Toghether with its dual, this proves that $x(E)$ is an automorphism, by (2.80). ∎

• When char $K \neq 2$ we may write the last identity of (2.84) as $A^{x(E)} = A + [E,A] + \frac{1}{2}[E,[E,A]]$.

The group generated by all $x(E)$ with E an isotropic element of $\mathbf{V}$ will de denoted by $\widehat{\mathsf{E}}_6(K)$ and is called a *Chevalley group* of type E_6. Note that this is a subgroup of the general linear group of $\mathbf{V}$.

• Chevalley groups come in different flavours. The *adjoint* Chevalley group of type E_6 is usually defined as acting on $\mathbf{L}$, i.e., by means of the last identity of (2.84). This group is always a simple group. Other Chevalley groups of type E_6 have the adjoint Chevalley group as a quotient.

The group $\widehat{\mathsf{E}}_6(K)$ defined here may possibly contain elements of the form $k\mathbf{1}$ with $k \in K - \{0,1\}$ (although unfortunately we have not been able to determine whether this is really the case). These elements act trivially on $\mathbf{L}$, but not on $\mathbf{V}$, and belong to the center of $\widehat{\mathsf{E}}_6(K)$.

We are therefore careful not to call $\widehat{\mathsf{E}}_6(K)$ 'the' Chevalley group of type E_6.

• In [1] the Chevalley group of type E_6 is constructed as the *isometry group* of the trilinear form on $\mathbf{V}$. This is essentially the same as our notion of automorphism.

In the special case $E = kE_r$ with $k \in K$, $r \in \Phi$, we use the notation $x_r(k) \stackrel{\text{def}}{=} x(kE_r)$.

Lemma 2.31 *Let $r, s \in \Phi$, $p \in \mathcal{P}$, $k \in K$. Then*

$$\begin{aligned} e_p^{x_r(k)} &= \begin{cases} e_p - ke_{p+r}, & \text{when } p \text{ is a positive base point of } r,\\ e_p + ke_{p+r}, & \text{when } p \text{ is a negative base point of } r,\\ e_p, & \text{otherwise.} \end{cases}\\ \eta_p^{x_r(k)} &= \begin{cases} \eta_p + k\eta_{p-r}, & \text{when } p \text{ is a positive base point of } -r,\\ \eta_p - k\eta_{p-r}, & \text{when } p \text{ is a negative base point of } -r,\\ \eta_p, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.85}$$

$$\begin{aligned} E_{-r}^{x_r(k)} &= E_{-r} + kH_r + k^2E_r, & \\ E_s^{x_r(k)} &= E_s, & \text{when } \langle r,s\rangle \geq 0,\\ E_s^{x_r(k)} &= E_s + kE_{r+s} \text{ or } E_s - kE_{r+s}, & \text{when } \langle r,s\rangle = -1,\\ H_s^{x_r(k)} &= H_s + \langle r,s\rangle kE_r. & \end{aligned} \tag{2.86}$$

Proof : (2.85) is an immediate application of (2.45), (2.47) and (2.84). Similarly, applying (2.84) to $E = kE_r$ and $A = E_s$, yields

$$E_s^{x_r(k)} = E_s + k[E_r, E_s] - k^2(E_r \cdot E_s)E_r.$$

The value of this expression may then be computed from Proposition 2.22 and (2.68). Similarly

$$H_s^{x_r(k)} = H_s + k[E_r, H_s] - k^2(E_r \cdot H_s)E_r = H_s + \langle r, s\rangle kE_r.$$

∎

• It can be proved that the set of all elements $x_r(k)$ with $k \in K$, $r \in \Phi$ generate the group $\widehat{\mathsf{E}}_6(K)$. This is often taken as the definition of $\widehat{\mathsf{E}}_6(K)$. A proof of the equivalence of both definitions will be given after Theorem 2.36.

• Properties (2.86) differ slightly from those listed in [8, Section 4.4] because the elements E_r are Chevalley basis elements only up to a sign, as explained after Proposition 2.22. For those r for which $-E_r$ is a Chevalley base element, our $x_r(k)$ corresponds to $x_r(-k)$ in [8].

When g, h are group elements, we will write $g^h \stackrel{\text{def}}{=} h^{-1}gh$.

Let E be an isotropic element of $\mathbf{L}$. Applying an automorphism g of $\mathbf{V}$ to the first statement of (2.84) we obtain $(a^{x(E)})^g = (a - aE)^g = a^g - (aE)^g = a^g - a^g E^g = (a^g)^{x(E^g)}$ and hence

$$x(E)^g = x(E^g), \tag{2.87}$$

for all isotropic elements $E \in \mathbf{L}$ and all automorphisms g of $\mathbf{V}$.

Lemma 2.32 *Let E, F be isotropic elements of $\mathbf{L}$ such that $E \cdot F = -1$. Define*

$$n(E, F) \stackrel{\text{def}}{=} x(E)x(F)x(E). \tag{2.88}$$

Then $n(E, F) = n(F, E)$ and

$$E^{n(E,F)} = F, \quad F^{n(E,F)} = E. \tag{2.89}$$

Proof : (Note that $E^2 = F^2 = 0$ and $EFE = -E$ and $FEF = -F$ by Proposition 2.28.)

We have

$$\begin{aligned} n(E,F) &= (\mathbf{1}-E)(\mathbf{1}-F)(\mathbf{1}-E) \\ &= \mathbf{1}-E-F-E+EF+FE+E^2-EFE = \mathbf{1}-E-F+EF+FE. \end{aligned}$$

This expression is symmetric in E and F, proving $n(E,F) = n(F,E)$.

We compute

$$\begin{aligned} E^{x(E)} &= E+[E,E]-(E\cdot E)E = E \\ E^{x(F)} &= E+[F,E]-(E\cdot F)F = E+F-[E,F] \\ F^{x(E)} &= F+[E,F]-(E\cdot F)E = E+F+[E,F] \\ [E,F]^{x(E)} &= [E,F]+[E,[E,F]]-(E\cdot[E,F])E \\ &= [E,F]+E^2F-2EFE+F^2E = [E,F]+2E \\ E^{n(E,F)} &= E^{x(E)x(F)x(E)} = E^{x(F)x(E)} = (E+F-[E,F])^{x(E)} \\ &= E+E+F+[E,F]-[E,F]-2E = F \end{aligned}$$

The value for $F^{n(E,F)}$ is obtained by interchanging the roles of E and F. ∎

By (2.68) the elements $E = kE_r$ and $F = k^{-1}E_{-r}$, with $k \in K - \{0\}$, $r \in \Phi$, satisfy the conditions of Lemma 2.32, and then

$$n_r(k) \stackrel{\text{def}}{=} n(kE_r, k^{-1}E_{-r}) = x_r(k)x_{-r}(k^{-1})x_r(k) \tag{2.90}$$

is well defined. These elements will play a special role in what follows. Also of interest are the following group elements, for $k \in K - \{0\}$, $r \in \Phi$:

$$h_r(k) \stackrel{\text{def}}{=} n_r(-1)n_r(k). \tag{2.91}$$

Lemma 2.33 *Let $k \in K - \{0\}$, $p, r, s \in \Phi$. Then*

$$\begin{array}{rclrcl}
e_p^{n_r(k)} & = & \pm k^{-\langle r,p\rangle} e_{w_r(p)}, & e_p^{h_r(k)} & = & k^{\langle r,p\rangle} e_p, \\
\eta_p^{n_r(k)} & = & \pm k^{\langle r,p\rangle} \eta_{w_r(p)}, & \eta_p^{h_r(k)} & = & k^{-\langle r,p\rangle} \eta_p, \\
E_s^{n_r(k)} & = & k^{-\langle r,s\rangle} E_{w_r(s)}, & E_s^{h_r(k)} & = & k^{\langle r,s\rangle} E_s, \\
H_p^{n_r(k)} & = & H_{w_r(p)}, & H_p^{h_r(k)} & = & H_p, \\
H_s^{n_r(k)} & = & H_{w_r(s)}, & H_s^{h_r(k)} & = & H_s.
\end{array} \tag{2.92}$$

where w_r denotes the Weyl group element associated with r.

Proof : From the proof of Lemma 2.32, we obtain

$$n_r(k) = \mathbf{1} - kE_r - k^{-1}E_{-r} + E_rE_{-r} + E_{-r}E_r.$$

From (2.62) it follows that $e_p + e_pE_rE_{-r} + e_pE_{-r}E_r$ equals 0 when p is a base point of either r or $-r$, and equal e_p otherwise. Hence

$$e_p^{n_r(k)} = \begin{cases} -ke_pE_r = \pm ke_{p+r}, & \text{when } p \text{ is a base point of } r, \\ -k^{-1}e_pE_{-r} = \pm k^{-1}e_{p-r}, & \text{when } p \text{ is a base point of } -r, \\ e_p, & \text{otherwise.} \end{cases}$$

Note that these three cases correspond with $\langle r, p\rangle = -1, +1$ and 0 respectively, and that in each case the base element on the right hand side is indeed equal to $e_{w_r(p)}$, by (2.7). This proves the first equality in the left column of (2.92).

The second equality follows from the fact that $e_p^{n_r(k)}\eta_p^{n_r(k)}$ must be equal to $e_p\eta_p = 1$.

For the third equality, write $s = q - p$ with p a positive base point of s. Then $E_s = e_q * \eta_p$ and hence by the above

$$\begin{aligned}
E_s^{n_r(k)} &= e_q^{n_r(k)} * \eta_p^{n_r(k)} \\
&= \pm k^{-\langle r,q\rangle} e_{w_r(q)} * \pm k^{\langle r,p\rangle} \eta_{w_r(p)} \\
&= \pm k^{-\langle r,s\rangle} e_{w_r(q)} * \eta_{w_r(p)} \\
&= \pm k^{-\langle r,s\rangle} E_{w_r(s)}.
\end{aligned}$$

Similarly, $H_p^{n_r(k)} = e_p^{n_r(k)} * \eta_p^{n_r(k)} = \pm k^{-\langle r,p\rangle} e_{w_r(p)} * \pm k^{\langle r,p\rangle} \eta_{w_r(p)}$. The sign of both factors on the right hand side must be the same as $e_p^{n_r(k)} \eta_p^{n_r(k)} = 1$. Hence $H_p^{n_r(k)} = e_{w_r(p)} * \eta_{w_r(p)} = H_{w_r(p)}$. By definition $H_s = H_q - H_p$ when $s = q - p$, and hence also the last equality in the left column follows.

Finally, if p is a base point of r, and hence $p + r$ is a base point of $-r$, then

$$e_p^{h_r(k)} = e_p^{n_r(-1)n_r(k)} = (e_p E_r)^{n_r(k)} = -k^{-1} e_p E_r E_{-r},$$

by the above. Lemma 2.21 then shows that this is equal to $k^{-1} e_p$.

Similarly, when p is a base point of $-r$, we find that $e_p^{h_r(k)} = k e_p$, and when p is not a base point of either, $e_p^{h_r(k)} = e_p$, proving the first equality in the right column of (2.92). The other equalities in the right column can then be derived using the same methods as for the left column. ∎

Lemma 2.33 shows that the element $n_r(k)$ acts on the 1-dimensional subspace Ke_p of $\mathbf{V}$ (respectively $KE_s \leq \mathbf{L}$) in the same way as the Weyl group element w_r acts on the points of $\mathcal{P}$ (respectively the roots of Φ). Hence $W(\mathsf{E}_6)$ is a quotient group of the subgroup N of $\widehat{\mathsf{E}}_6(K)$ generated by all $n_r(k)$ with $r \in \Phi$, $k \in K$. The group elements $h_r(k)$ belong to the kernel of the corresponding homomorphism.

In the remainder of this section we will establish various transitivity properties of the group $\widehat{\mathsf{E}}_6(K)$.

For this purpose we will make use of a *reduction process* that manipulates the coordinates of elements $a \in \mathbf{V}$. Recall that $a \in \mathbf{V}$ can be written as $\sum_{p \in \mathcal{P}} a[p] e_p$ with coordinates $a[p] = a\eta_p$.

The reduction process can be summarized as follows : we choose an element of the form $x_r(k)$ to map a given $a \in \mathbf{V}$ onto $a' \in \mathbf{V}$ with the property that a' has one more coordinate that is 'known' to be zero. Successive reductions decrease the number of non-zero coordinates of a while staying in the same orbit of $\widehat{\mathsf{E}}_6(K)$, until the final image of a has a sufficiently simple structure.

In general, when it is known that $a[p] \neq 0$ for some $p \in \mathcal{P}$, we choose $q \in \mathcal{P}$ such that $p \not\perp q$ and then apply an element of the form $x_t(k)$ with $t = q - p$. From (2.85) it follows that the coordinate of the image a' of a at position q is now equal to $a'[q] = a[q] \pm ka[p]$. Hence, setting $k = \mp a[q]/a[p]$ we obtain $a'[q] = 0$.

This transformation does however also affect other coordinates $a[q]$ of a and when performing successive reductions we should make sure that earlier annihilations are not 'undone' by later actions. Note that by (2.85) $x_t(k)$ only changes coordinates $a[q]$ of a at positions which satisfy $t \cdot q = 1$.

Theorem 2.34 *The orbits of $\widehat{\mathsf{E}}_6(K)$ on $\mathbf{V}$ are as follows :*

1. *The trivial orbit $\{0\}$.*
2. *The set of all elements $a \in \mathbf{V}$ with $a^\# = 0$ but $a \neq 0$. i.e., all non-zero isotropic elements of $\mathbf{V}$.*
3. *The set of all elements $a \in \mathbf{V}$ with $D(a) = 0$ but $a^\# \neq 0$.*
4. *For each $\ell \in K - \{0\}$ an orbit of elements $a \in \mathbf{V}$ with $D(a) = \ell$.*

Typical representatives for these orbits are given by 0, e_p, $e_p + e_{p'}$ and $e_p + e_{p'} + \ell e_{p''}$, where $pp'p''$ is a line of Σ.

Proof : Choose a fixed line $pp'p''$ of Σ. Assume $a \neq 0$.

At least one coordinate $a[x]$ of a is nonzero. Because $W(\mathsf{E}_6)$ acts transitively on $\mathcal{P}$ we may always find an element $w \in W(\mathsf{E}_6)$ which maps x onto p, and hence we may find a corresponding element in the subgroup N of $\widehat{\mathsf{E}}_6(K)$ which maps Ke_x onto Ke_p. Therefore, without loss of generality we may assume $a[p] = 0$.

Now, let $q \in \mathcal{P}$ such that $p \not\perp q$, i.e., $p \cdot q = 1/3$. Then we may annihilate $a[q]$ using $x_t(k)$ with $t = q - p$ as described in the introduction to this section. This transformation will affect only coordinates $a[q']$ of a at positions that satisfy $t \cdot q' = 1$, i.e., $q \cdot q' = 1 + p \cdot q'$. For $q' = q$ this implies $q \cdot q' = 1/3$ and

$p \cdot q' = -2/3$, hence $q' \sim p$. In other words, annihilating $a[q]$ does not effect any other coordinate $a[q']$ with $q' \not\perp p$. We may therefore repeat this process to annihilate all coordinates of this type, ending up with an image b of a with $b[p] \neq 0$ and $b[q] = 0$ for every $q \not\perp p$.

If also every $b[q] = 0$ with $p \sim q$, then $b \in Ke_p$. Otherwise, because $W(\mathsf{E}_6)$ is transitive on all adjacent pairs of points, we may assume without loss of generality that $b[p'] \neq 0$. We now use $x_t(k)$ with $t = q - p'$ to annihilate all remaining coordinates $b[q]$ for which $q \neq p, p', p''$. Indeed, using a similar argument as above, we see that an annihilation of this kind affects only $a[p'']$ and $a[q]$. (It does not affect coordinates $a[q]$ with $q \not\perp p$ because all of them are zero.)

This proves that every non-zero element of $\mathbf{V}$ is equivalent (under the action of $\widehat{\mathsf{E}}_6(K)$) to an element c of the form $c = ke_p + k'e_{p'} + k''e_{p''}$ with $k \neq 0$ and $k'' = 0$ when $k' = 0$. Choose q on a line through p' different from $pp'p''$. Then $r = q - p$ is a root with $\langle r, p \rangle = -1$. Applying $h_r(k)$ to c then reduces $c[p]$ to 1 by (2.92). We therefore assume without loss of generality that $k = 1$. Likewise, if $k' \neq 0$, we may choose q' on a line through p which is different from $pp'p''$ and set $r' = p' - q'$ to reduce k' to 1 without affecting $c[p]$.

Summarizing, this proves that every $a \in \mathbf{V}$ can be mapped onto 0, e_p or an element of the form $e_p + e_{p'} + \ell e_{p''}$ with $\ell \in K$. To prove that each of these elements represents a different orbit, we may compute that the corresponding values of $a^\#$ and $D(a)$ are different in each case. Because $D(a)$ and the fact that $a^\#$ is or is not equal to 0 are invariants of $\widehat{\mathsf{E}}_6(K)$, this concludes the proof. ∎

Note that also the dual of this theorem holds, and in particular, that the non-zero isotropic elements of $\mathbf{V}^*$ form a single orbit under the action of $\widehat{\mathsf{E}}_6(K)$.

• Several results on orbits of $\widehat{\mathsf{E}}_6(K)$ are already contained in [1], but with proofs that are completely different from ours. Theorem 3.3 of [1] is the 'projective version' of the theorem above, i.e., it lists the orbits on 1-spaces Ka of $\mathbf{V}$ instead of on the elements of $\mathbf{V}$ themselves.

Theorem 2.35 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{E}}_6(K)$ on ordered pairs (e, f) where both $e, f \in \mathbf{V} - \{0\}$ are isotropic. Let $x, p \in \mathcal{P}$.*

1. *An orbit with representative (e_x, e_p) where $p \sim x$.*

2. *An orbit with representative (e_x, e_p) where $p \not\sim x$.*

3. *For each $\ell \in K - \{0\}$ an orbit with representative $(e_x, \ell e_x)$.*

The following table list several properties of the corresponding pairs (e, f).

	Case 1	*Case 2*	*Case 3*
$e \times f$	*isotropic*, $\neq 0$	0	0
$f \in Ke$	*no*	*no*	$f = \ell e$
$e \in f\widehat{\mathbf{L}}$	*no*	*yes*	*yes*

Proof : Because $\widehat{\mathsf{E}}_6(K)$ is transitive on isotropic points we may choose e of the form $e = e_x$ without loss of generality.

Again we will use the reduction process to map f to an element with few nonzero coordinates using elements of type $x_t(k)$. In this case however, we must make sure that $x_t(k)$ leaves e_x invariant. For this reason we only use elements $t = q - p$ such that $\langle t, x\rangle \geq 0$, i.e., such that

$$x \cdot q \geq x \cdot p. \tag{2.93}$$

We need to consider three different cases.

Case 1. Assume there exists $p \in \mathcal{P}$ such that $p \sim x$ and $f[p] \neq 0$. In this case $x \cdot p = -2/3$ is as small as possible and hence (2.93) is trivially satisfied. As in the proof of Theorem 2.34 we obtain an element f' equivalent to f with $f[q] = 0$ for all $q \not\sim p$ and $f'[p] \neq 0$. We claim that the fact that f' is isotropic implies $f' \in Ke_p$.

Indeed, consider $p' \sim p$ and let p'' denote the third point on the line pp'. Then by (2.20) the coordinate of $f'^{\#}$ at position p'' is equal to $\pm f'[p]f'[p']$ and must be zero. Hence $f'[p'] = 0$.

Case 2. If case 1 does not apply, assume there exists $p \in \mathcal{P}$ such that $p \not\perp x$ and $f[p] \neq 0$. Now $x \cdot p = 1/3$ and to satisfy (2.93) we must avoid all q adjacent to x in the reduction process. However, $f[q] = 0$ for all such q, for otherwise case 1 would have applied. Hence, again we may reduce f' to f with $f[q] = 0$ for all $q \not\perp p$ and $f'[p] \neq 0$. Using the same argument as before, the fact that f' is isotropic implies $f' \in Ke_p$.

Case 3. If neither of the previous cases applies, then f must be a scalar multiple of e, i.e., $f = \ell e_x$ for some $\ell \in K$.

In both cases 1 and 2 we have found that f is equivalent to an element of the form ℓe_p. In a final step, we apply a group element $h_r(\ell)$ which leaves e_x invariant and maps ℓe_p to e_p. By (2.92) r must be chosen in such a way that $\langle r, p\rangle = -1$ and to leave e_x invariant we must have $\langle r, x\rangle = 0$. In both cases a root r with these properties can easily be found.

Proposition 2.5 tells us that $W(\mathsf{E}_6)$ acts transitively on adjacent and on non-adjacent pairs of $\mathcal{P}$. Hence we may freely choose x and p as long as they satisfy the listed conditions.

Finally, to check the table of properties in the statement of this theorem, note that all properties are invariant for $\widehat{\mathsf{E}}_6(K)$. It is therefore sufficient to check them for a single (representative) pair in each orbit. This is a straightforward application of the definitions and will be left to the reader. ∎

• Theorem 6.4 of [1] proves that there are two orbits on 2-spaces generated by isotropic 1-spaces (the *singular* and the *hyperbolic* lines). This is the 'projective version' of Theorem 2.35.

• The proofs of most of the 'orbit' theorems in this section (and in Chapters 3 and 5) run along the same lines as the proofs of Theorem 2.34 and 2.35. It may be useful to refer to the earlier proofs to fill in the necessary details in some of the proofs below.

Theorem 2.36 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{E}}_6(K)$ on ordered pairs (e, η) where both $e \in \mathbf{V} - \{0\}$ and $\eta \in \mathbf{V} - \{0\}$ are isotropic. Let $x, p \in \mathcal{P}$.*

1. *For every $\ell \in K - \{0\}$ an orbit with representative $(e_x, \ell\eta_x)$.*

2. *An orbit with representative (e_x, η_p) where $p \not\sim x$.*

3. *An orbit with representative (e_x, η_p) where $p \sim x$.*

The following table list several properties of the corresponding pairs (e, η).

	Case 1	*Case 2*	*Case 3*
$e\eta$	$= \ell \neq 0$	0	0
$e * \eta$	*not isotropic*	*isotropic,* $\neq 0$	0
$e \in \eta \times \mathbf{V}^*$	*no*	*no*	*yes*
$\eta \in e \times \mathbf{V}$	*no*	*no*	*yes*

Proof : We set $\eta = \eta_x$ and apply the reduction process to e, using only group elements that leave η_x invariant. Hence we only use elements $x_t(k)$ with $t = q - p$ such that $\langle t, x\rangle \leq 0$, i.e.,

$$x \cdot q \leq x \cdot p. \tag{2.94}$$

We again consider three cases.

Case 1. If $e[x] \neq 0$ then $p = x$ makes (2.94) trivially satisfied. Using the techniques of earlier theorems, we find that e is equivalent to an element of the form ℓe_x.

Case 2. If $e[x] = 0$, assume there exists $p \not\sim x$ such that $e[p] \neq 0$. By (2.94) we may apply the reduction process to all coordinates $e[q]$ of e with $q \not\sim p$, except at position $q = x$, a coordinate which is already zero. We end up with an element of the form ℓe_p.

Case 3. If neither of the previous cases apply we may find p adjacent to x such that $e[p] = 0$. The reduction process can now only be applied to $e[q]$ such that $q \sim x$. However, the other coordinates $e[q]$ are already zero, for otherwise one of the previous cases would apply. Therefore again we end up with an element of the form ℓe_p.

In the last two cases, as in the proof of Theorem 2.35, we may find r such that $h_r(\ell)$ reduces the coefficient ℓ to 1. The transitivity of $W(\mathsf{E}_6)$ (Proposition 2.5) also allows us to choose x and p at will. We leave it to the reader to check the table of properties in the statement of this theorem for a single representative pair in each orbit. Note that $H_x = e_x * \eta_x$ is not isotropic for $H_x^2 \neq 0$. ∎

It is an immediate consequence of this theorem that the non-zero isotropic elements of $\mathbf{L}$ form a single orbit of $\widehat{\mathsf{E}}_6(K)$. Note that $\dim E_r = 6$ and hence $\dim E = 6$ whenever E is a non-zero isotropic element of $\mathbf{L}$.

• There is a much shorter proof of the fact that all non-zero isotropic elements E are equivalent : Choose $r \in \Phi$ and $k \in K$ such that $E \cdot kE_r = -kE[r] = -1$. Such r and k can be found for every non-zero isotropic element E of $\mathbf{L}$, for it is easily proved that E cannot be a diagonal matrix. Now use $n(E, kE_r)$ to map E to kE_r. An appropriate $h_s(k)$ gets rid of the factor k and the fact that $W(\mathsf{E}_6)$ acts transitively on the roots of Φ (Proposition 2.5) allows us to choose r as we please.

• In the proof of Theorem 2.36 we have used only group elements of the form $x_r(k)$. As a consequence, any non-zero isotropic element E can be written as E_s^g with $s \in \Phi$ and g a product of elements of the form $x_r(k)$. Hence $x(E)$ can be written as a product of elements of that form, by (2.87), and this again proves that $\widehat{\mathsf{E}}_6(K)$ can be generated by elements of that form. Our definition of $\widehat{\mathsf{E}}_6(K)$ is therefore equivalent to the standard definition. (See the note after Lemma 2.31.)

Theorem 2.37 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{E}}_6(K)$ on ordered pairs (E, e) where $E \in \mathbf{L} - \{0\}$ and $e \in \mathbf{V} - \{0\}$ are isotropic. Let $r \in \Phi$, $p \in \mathcal{P}$.*

1. *An orbit with representative (E_r, e_p) where $\langle r, p\rangle = -1$.*

2. *An orbit with representative (E_r, e_p) where $\langle r, p\rangle = 0$.*

3. *An orbit with representative (E_r, e_p) where $\langle r, p\rangle = +1$.*

The following table list several properties of the corresponding pairs (E, e). ../..

	Case 1	*Case 2*	*Case 3*
eE	$\neq 0$	0	0
$e \in E\mathbf{V}^* \times \mathbf{V}^*$	*no*	*yes*	*yes*
$E \in \mathbf{V} * (e \times \mathbf{V})$	*no*	*yes*	*yes*
$\dim(\mathbf{V}E \times e)$	5	2	0
$\dim(E\mathbf{V}^* \cap e \times \mathbf{V})$	0	2	5
$e \in \mathbf{V}E$	*no*	*no*	*yes*
$E \in e * \mathbf{V}^*$	*no*	*no*	*yes*

If $e \in \mathbf{V}E$ *then we can always find an isotropic element* $f \in \mathbf{V}$ *such that* $e = fE$ *and an isotropic element* $\varphi \in \mathbf{V}^*$ *such that* $E = e * \varphi$*.*

Proof : We set $E = E_r$ and apply the reduction process to e, using only group elements that leave E_r invariant. Hence we use only elements $x_t(k)$ with $t = q - p$ such that $\langle t, r \rangle \geq 0$ (see (2.86)) or equivalently, such that

$$\langle r, q \rangle \geq \langle r, p \rangle. \tag{2.95}$$

We consider three cases.

Case 1. Assume we can find a base point p of r such that $e[p] \neq 0$. We have $\langle r, p \rangle = -1$ and therefore (2.95) is trivially satisfied. After reduction we end up with an element of the form ℓe_p.

Case 2. Assume $e[q] = 0$ for all base points q of r, but we may find a point p such that $e[p] \neq 0$ and $\langle r, p \rangle = 0$. In that case the reduction process can only be applied to coordinates of e at positions q which are not base points of r. However, as in earlier proofs, the 'missing' coordinates are already taken care of. Again we end up with an element of the form ℓe_p.

Case 3. Assume $e[q] = 0$ except when q is a base point for $-r$. As before we may reduce e to an element of the form ℓe_p. To take care of the constant ℓ we apply an element $h_s(\ell)$ with $\langle s, p \rangle = -1$ and $\langle s, r \rangle = 0$. It is easily proved that such s can always be found, in each of the three cases. The transitivity properties of $W(\mathsf{E}_6)$ (Proposition 2.5) allow us to freely choose $r \in \Phi$ and $x, y, z \in \mathcal{P}$ as long as they satisfy the listed conditions.

The table of properties (and the additional properties for the case $e \in \mathbf{V}E$) need only be verified for a single representative pair of each orbit. We will discuss the most difficult cases here and leave the rest to the reader. Using the standard notation for points of $\mathcal{P}$, the three cases correspond to $p = p_0$, $p = a_{01}$ and $p = q_0$. Note that $\mathbf{V}E_r$ is generated by $e_{q_0}, \ldots, e_{q_5}$, hence only in the third case we have $e \in \mathbf{V}E$. We see that $e_{p_0} \times \mathbf{V}E_r$ is generated by the 5 elements $e_{a_{01}}, \ldots, e_{a_{05}}$, that $e_{a_{01}} \times \mathbf{V}E_r$ is generated by e_{q_0} and e_{q_1} and that $e_{q_0} \times \mathbf{V}E_r$ is trivial.

Note that $e_{p_0} \times \mathbf{V}$ is generated by $\eta_{q_1}, \ldots, \eta_{q_5}$ and $\eta_{a_{01}}, \ldots, \eta_{a_{05}}$, and hence $\mathbf{V} * (e_{p_0} \times \mathbf{V})$ is generated by all E_s which have at least one of the ten elements $q_1, \ldots, a_{05}$ as a base point. The root r is not amongst them, hence $E_r \notin \mathbf{V} * (e_{p_0} \times \mathbf{V})$. On the other hand $E_r = \pm e_{q_1} * (e_{a_{01}} \times e_{q_0})$, hence E_r belongs to both $\mathbf{V} * (e_{q_0} \times \mathbf{V})$ and $\mathbf{V} * (e_{a_{01}} \times \mathbf{V})$. Similarly, e_p belongs to $E_r\mathbf{V}^* \times \mathbf{V}^*$ if and only if p is adjacent to one of the base points of r, and this is only true in the last two cases. ∎

Note that also the dual of this theorem holds, which provides a list of orbits of pairs (E, η) with both E and η isotropic and non-zero.

Theorem 2.38 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{E}}_6(K)$ on ordered pairs (E, F) where both $E, F \in \mathbf{L} - \{0\}$ are isotropic. Let $r, s \in \Phi$.*

1. *For every $\ell \in K - \{0\}$ an orbit with representative $(E_r, \ell E_{-r})$.*
2. *An orbit with representative (E_r, E_s) where $\langle r, s\rangle = -1$.*
3. *An orbit with representative (E_r, E_s) where $\langle r, s\rangle = 0$.*
4. *An orbit with representative (E_r, E_s) where $\langle r, s\rangle = +1$.*
5. *For every $\ell \in K - \{0\}$ an orbit with representative $(E_r, \ell E_r)$.*

The following tables list several properties of the corresponding pairs (E, F). ../..

	$E \cdot F$	$[E,F]$	$F \in [E, \widehat{\mathbf{L}}]$	EF	$E+F$
Case 1	$-\ell$	*not isotr.*	*no*	$\neq 0$	*not isotr.*
Case 2	0	*isotr.*, $\neq 0$	*no*	$\neq 0$	*not isotr.*
Case 3	0	0	*yes*	$= FE \neq 0$	*not isotr.*
Case 4	0	0	*yes*	0	*isotr.*
Case 5	0	0	*yes*	0	*isotr.*

	$F \in KE$	$\mathbf{V}E \cap \mathbf{V}F$	$\dim(\mathbf{V}E \cap \mathbf{V}F)$	$\dim(E\mathbf{V}^* \times F\mathbf{V}^*)$
Case 1	*no*	$\{0\}$	0	15
Case 2	*no*	$\{0\}$	0	6
Case 3	*no*	$\mathbf{V}EF = \mathbf{V}FE$	1	6
Case 4	*no*	$E\mathbf{V}^* \times F\mathbf{V}^*$	3	3
Case 5	$F = \ell E$	$\mathbf{V}E = \mathbf{V}F$	6	0

Proof : Write $F = a * \alpha$ with $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ isotropic and $a\alpha = 0$. By the dual of Theorem 2.37 we may without loss of generality assume that $E = E_r$ and $\alpha = \eta_x$ where $x \in \mathcal{P}$, $r \in \Phi$.

Now consider $e \overset{\text{def}}{=} e_x F$. It is a consequence of Proposition 2.28 that $F = e * \eta_x$, $e^\# = 0$ and $e\eta_x = 0$.

As before we will apply the reduction process to e using only group elements that leave both E_r and η_x invariant. Hence we use only elements $x_t(k)$ with $t = q - p$ such that

$$\langle r, q \rangle \geq \langle r, p \rangle \quad \text{and} \quad x \cdot q \leq x \cdot p. \tag{2.96}$$

Note that $e[p] = e\eta_p = e_x F \eta_p = F[x,p]$ is zero for every $p \in \mathcal{P}$ adjacent to x. In other words, whenever $e[p] \neq 0$ we have $p \cdot x = 1/3$. Therefore the second inequality of (2.96) is satisfied whenever $q \neq x$, and because $e[x] = 0$ this means that we essentially only have to worry about the first inequality.

Again we consider three cases.

Case 1. Assume we can find p such that $\langle r, p \rangle = 1$. Then the left hand side of (2.96) is trivially satisfied.

Case 2. Otherwise, assume we can find such p with $\langle r, p \rangle = 0$. This time the reduction process cannot handle coordinates $e[q]$ with $q \not\perp p$ and $\langle r, q \rangle = 1$. Of these q we know that $e[q] = 0$ whenever $q = x$ or $q \sim x$. The remaining cases (when $\langle r, q \rangle = 1$) are ruled out because otherwise the previous case could have been applied to reduce e.

Case 3. Otherwise, we must find p such that $\langle r, p \rangle = -1$, and a similar argument proves that e can again be reduced to a single non-zero coordinate.

These reductions prove that F is equivalent to some element of the form $\ell e_p * \eta_x$ with $p \not\perp x$, and hence to some $\pm \ell E_s$ with $s \in \Phi$. In a final step we use a group element $h_t(\pm \ell)$ which leaves E_s invariant and maps $\pm \ell E_r$ to E_r. This means that the corresponding root t must satisfy $\langle s, t \rangle = 0$ and $\langle r, t \rangle = -1$. We leave it to the reader to verify that this is always possible unless $r = s$ or $r = -s$.

Again the transitivity properties of $W(\mathsf{E}_6)$ (Proposition 2.5) allows us to freely choose $r, s \in \Phi$ as long as they satisfy the stated conditions.

As before, the table of properties need only be verified for a single representative pair of each orbit. We will discuss the most difficult cases here and leave the rest to the reader.

We will use the standard notation for points of $\mathcal{P}$. The 5 cases now correspond to $s = -r$, $s = p_0 - a_{12}$, $s = p_1 - p_0$, $s = a_{12} - p_0$ and $s = r$.

Note that $E_r E_s$ is trivial if and only if the base points of $-r$ do not intersect with the base points of s. This is equivalent to $\langle r, s \rangle > 0$. When $s = p_1 - p_0 = q_1 - q_0$, we see that $E_r E_s$ maps e_{p_0} onto $\pm e_{q_1}$ and all maps other base elements to 0. Hence $\mathbf{V} E_r E_s = K e_{q_1}$ in that case, which is then also easily seen to be equal to $\mathbf{V} E_r \cap \mathbf{V} E_s$.

In the fourth case, with $s = a_{12} - p_0$, we see that the base points of $-r$ and $-s$

intersect in $\{q_3, q_4, q_5\}$. Hence $\mathbf{V}E_r \cap \mathbf{V}E_s$ is generated by e_{q_3}, e_{q_4} and e_{q_5}. Also $E_r\mathbf{V}^*$ is generated by $\eta_{p_0}, \ldots, \eta_{p_5}$ and $E_s\mathbf{V}^*$ is generated by $\eta_{p_0}, \eta_{p_1}, \eta_{p_2}, \eta_{a_{34}}$, $\eta_{a_{45}}$ and $\eta_{a_{35}}$. Hence $E_r\mathbf{V}^* \times E_s\mathbf{V}^*$ is also generated by e_{q_3}, e_{q_4} and e_{q_5}. ∎

As a consequence of the properties listed in this theorem the subspace generated by isotropic elements $E_1, \ldots, E_k$ of $\mathbf{L}$ will itself by isotropic if and only if $E_iE_j = 0$ for all i, j.

The table of properties of Theorem 2.38 also shows that under certain conditions $E + F$ (respectively $[E, F]$) are isotropic, and then $x(E + F) \in \widehat{\mathsf{E}}_6(K)$ (respectively $x([E, F]) \in \widehat{\mathsf{E}}_6(K)$). This allows us to formulate the following *commutation relations*.

Proposition 2.39 *Let E, F be isotropic elements of $\mathbf{L}$ such that $E \cdot F = 0$. Then*

$$x(E)x(F) = x(F)x(E)x([E, F]). \tag{2.97}$$

If moreover $EF = FE = 0$ then

$$x(E)x(F) = x(F)x(E) = x(E + F). \tag{2.98}$$

Proof: Let $a \in \mathbf{V}$. We find $a^{x(-F)x(E)} = (a + aF)^{x(E)} = a + aF - aE - aFE$ and then

$$\begin{aligned} a^{x(-F)x(E)x(F)} &= a + aF - aE - aFE - aF - aF^2 + aEF + aFEF \\ &= a - aE + a[E, F] \end{aligned}$$

because $F^2 = 0$ and $FEF = (E \cdot F)F = 0$ by Proposition 2.28. Similarly

$$a^{x(E)x([E,F])} = (a - aE)^{x([E,F])} = a - aE + a[E, F] - aE[E, F]$$

And again $aE[E, F] = aE^2F - aEFE = 0$, by Proposition 2.28. This proves (2.97), because $x(-F) = x(F)^{-1}$.

Finally, when $EF = 0$, $a^{x(E)x(F)} = (a - aE)^{x(F)} = a - aE - aF + aEF = a - a(E + F)$. ∎

• When applied to $E = kE_r$, $F = k'E_s$ with $r, s \in \Phi$, $k, k' \in K$, (2.97) reduces to *Chevalley's commutator formula* [8, Theorem 5.2.2].

Theorem 2.40 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{E}}_6(K)$ on n-tuples $(e_1, \ldots, e_n) \in \mathbf{V}^n$ of linearly independent elements that satisfy $e_i^\# = e_i \times e_j = 0$, for all i, j, $1 \leq i, j \leq n$. (We use the standard notation for points of $\mathcal{P}$.)*

- *When $n = 1$, a single orbit with representative (e_{p_0}).*
- *When $n = 2$, a single orbit with representative (e_{p_0}, e_{q_0}).*
- *When $n = 3$, a single orbit with representative $(e_{p_0}, e_{q_0}, e_{a_{12}})$.*
- *When $n = 4$, a single orbit with representative $(e_{p_0}, e_{q_0}, e_{a_{12}}, e_{a_{13}})$.*
- *An orbit with representative $(e_{p_0}, e_{q_0}, e_{a_{12}}, e_{a_{13}}, e_{a_{14}})$. All 5-tuples in this orbit can be extended to a 6-tuple with the stated properties.*
- *For every $\ell \in K - \{0\}$, an orbit with representative $(e_{p_0}, e_{q_0}, e_{a_{12}}, e_{a_{13}}, \ell e_{a_{23}})$. A 5-tuple in this orbit cannot be extended to a 6-tuple with the stated properties.*
- *For every $\ell \in K - \{0\}$, an orbit with representative $(e_{p_0}, e_{q_0}, e_{a_{12}}, e_{a_{13}}, e_{a_{14}}, \ell e_{a_{15}})$.*

(Note that in the last two cases, different ℓ need not necessarily correspond to different orbits.)

Proof : By Theorem 2.35 we may set $e_1 = e_{p_0}$ and $e_2 = e_{q_0}$ for some fixed pair of noncollinear points p_0, q_0 in $\mathcal{P}$, without loss of generality. The fact that $e_1 \times e_i = e_2 \times e_i = 0$ translates to $e_i[p_1] = \cdots = e_i[p_5] = e_i[q_1] = \cdots = e_i[q_5] = e_i[a_{01}] = \cdots = e_i[a_{05}] = 0$, using the standard notation for points of $\mathcal{P}$. Note also that the fact that e_1, e_2, e_i, $i \geq 2$, are linearly independent forces at least one coordinate of the form $e_i[a_{jk}]$, with $1 \leq j, k \leq 5$ to be non-zero. We may then apply an appropriate group element $x_t(k)$ to reduce both $e_i[p_0]$ and $e_i[q_0]$ to zero (i.e., with $t = p_0 - a_{jk}$ and $q_0 - a_{jk}$).

As a consequence we may assume that each e_i, $i \geq 3$, belongs to the 10-dimensional space generated by $e_{a_{12}}, \ldots, e_{a_{45}}$. We will apply the usual reduction process using only elements $x_t(k)$ with t of the form $t = p_j - p_i$, with $1 \leq i, j \leq 5$.

Consider e_3. The transitivity properties of $W(\mathsf{E}_6)$ (Proposition 2.5) allow us to assume that $e_3[a_{12}] \neq 0$, without loss of generality. The reduction process can now be used to annihilate all coordinates at positions $a_{13}, \ldots, a_{15}, a_{23}, \ldots, a_{25}$. We leave it to the reader to prove that $e_3^\# = 0$ implies $a_{34} = a_{35} = a45 = 0$. Hence e_3 can be mapped onto a scalar multiple of $e_{a_{12}}$. Finally this factor can be removed using a group element of the form $h_s(k)$ with for example $s = p_2 - p_3$.

When $n \geq 4$ we choose $e_3 = e_{a_{12}}$. Using a similar reasoning as above, we can reduce each e_i, $i \geq 3$ to a linear combination of $e_{a_{13}}$, $e_{a_{14}}$, $e_{a_{15}}$, $e_{a_{23}}$, $e_{a_{24}}$ and $e_{a_{25}}$. Without loss of generality we may assume that $e_4[a_{13}] \neq 0$ and use this coordinate to reduce $e_4[a_{14}]$, $e_4[a_{15}]$ $e_4[a_{23}]$ to zero. The property $e_4^\# = 0$ then implies that also $e_4[a_{25}] = e_4[a_{24}] = 0$, so we end up with a multiple of $e_{a_{13}}$. Again this scalar factor can be removed using an element of the form $h_s(k)$, for example with $s = p_3 - p_4$.

When $n \geq 5$ we further choose $e_4 = e_{a_{13}}$ and now e_i can be reduced to a linear combination of $e_{a_{14}}$, $e_{a_{15}}$ and $e_{a_{23}}$. If $e[a_{23}] \neq 0$ then $e_5^\# = 0$ implies $e_4[a_{15}] = e_4[a_{14}] = 0$, and then e_4 is a multiple of $e_{a_{23}}$. (Note that in this case we cannot get rid of the extra factor as we did before.)

If on the other hand $e[a_{23}] = 0$, then without loss of generality we may assume $e[a_{14}] \neq 0$ and then we may reduce e_4 to a multiple of $e_{a_{14}}$. In this case we do find an appropriate $h_s(k)$ to get rid of the extra factor ($s = p_5 - p_4$).

Finally, when $n = 6$, we choose $e_5 = e_{a_{14}}$ and then e_6 reduces to a multiple of $e_{a_{15}}$. (It is easily seen that the alternative $e_5 = e_{a_{23}}$ leads to $e_6 = 0$.) ∎

• We do not know whether in the last two cases of Theorem 2.40 distinct ℓ also imply distinct orbits. It is rather fortunate that in what follows we will only need the 'projective version' of the theorem, in which elements e_i are substituted by one-dimensional subspaces Ke_i.

• The 'projective version' can be found as Theorem 6.5 of [1].

The following is a technical lemma which will be useful in later proofs.

Lemma 2.41 *Let $p \in \mathcal{P}$, $r \in \Phi$. Let $A \in \mathbf{L}$ and write $A = A_H + \sum_s A[s]E_s$ as in (2.71). Then*

1. *$e_pA = 0$ if and only if $A[p,p] = 0$ and $A[s] = 0$ for all $s \in \Phi$ such that $p \cdot s = -1$.*

2. *$A(e_p \times \mathbf{V}) = 0$ if and only if $A[s] = 0$ for all $s \in \Phi$ such that $p \cdot s \geq 0$ and $A_H = 0$ (except when* char $K = 2$ *in which case this latter condition is relaxed to $A_H \in K(H_p + \mathbf{1})$).*

3. *$E_rA = 0$ if and only if $A[s] = 0$ for all $s \in \Phi$ such that $r \cdot s \leq 0$ and $A_H = 0$ (except when* char $K = 3$ *in which case this latter condition is relaxed to $A_H \in K(H_r + \mathbf{1})$).*

Proof :

1. We easily compute

$$e_pA = e_pA_H + \sum_{s \in \Phi, p \cdot s = -1} \pm A[s]e_{p+s},$$

and hence $e_pA = 0$ if and only if $e_pA_H = 0$ and $A[s] = 0$ for all $s \in \Phi$ such that $p \cdot s = -1$, i.e., p is a base point of s. Because A_H is a diagonal matrix, the first requirement can also be written as $A_H[p,p] = A[p,p] = 0$.

2. Likewise $A(e_p \times \mathbf{V}) = 0$ if and only if $A(e_p \times e_{p'}) = 0$ for all $p' \in \mathcal{P}$ which is equivalent to requiring that $A\eta_q = 0$ for all $q \in \mathcal{P}$ that are adjacent to p. From the dual of the above we know that the latter is true if and only if $A[q,q] = 0$ and $A[s] = 0$ for all $s \cdot q = 1$.

From the proof of Lemma 2.18, specialising to the case $p = q_0$, the condition that $A[q,q] = 0$ for all q adjacent to p is easily shown to imply $A_H = 0$ when char $K \neq 2$, and $A_H \in K(H_p + \mathbf{1})$ when char $K = 2$.

The second condition, i.e., that $A[s] = 0$ for every $s \in \Phi$ such that $-s$ has a base point q adjacent to p, is equivalent to $A[s] = 0$ whenever $p \cdot s \geq 0$. Indeed p is adjacent to 0, 2 or 5 base points of $-s$ acoording to whether $p \cdot (-s) = -1$, 0 or 1.

3. $E_r A = 0$ if and only if $\mathbf{V} E_r A = 0$ if and only if $e_q A = 0$ for all $q \in \mathcal{P}$ such that $q \cdot r = 1$, i.e., for all base points q of $-r$.

Again from the proof of Lemma 2.18 we find that $A[q, q] = 0$ for all base points of $-r$ if and only if $A_H = 0$ when $\operatorname{char} K \neq 3$, and $A_H \in K(H_r + \mathbf{1})$ when $\operatorname{char} K = 3$. The second condition, i.e., that $A[s] = 0$ for every $s \in \Phi$ such that r has a base point q that is also a base point of $-s$, translates to $A[s] = 0$ whenever $r \cdot (-s) \geq 0$. ∎

Theorem 2.42 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{E}}_6(K)$ on n-tuples $(E_1, \ldots, E_n)$ of linearly independent isotropic elements of $\mathbf{L}$ that satisfy $E_i E_j = 0$ for all i, j, $1 \leq i, j \leq n$.*

- *When $n \leq 4$, for each n a single orbit with representative $(E_{r_1}, \ldots, E_{r_n})$,*
- *When $n = 5$, for each $\ell \in K$ an orbit with representative $(E_{r_1}, \ldots, E_{r_4}, \ell E_{r_5})$,*

with $r_1 = \overline{\mathtt{111111}}$, $r_2 = \mathtt{111}\overline{\mathtt{222}}$, $r_3 = \mathtt{11}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{22}}$, $r_4 = \mathtt{1}\overline{\mathtt{2}}\mathtt{11}\overline{\mathtt{22}}$ *and* $r_5 = \overline{\mathtt{2}}\mathtt{111}\overline{\mathtt{22}}$.

(In the case $n = 5$ different ℓ need not necessarily correspond to different orbits.)

Proof : By Theorem 2.38 without loss of generality we may take $E_1 = E_{r_1} = E_{\overline{\mathtt{111111}}}$ and $E_2 = E_{r_2} = E_{\mathtt{111}\overline{\mathtt{222}}}$. Using Lemma 2.41 we may compute that E_3 satisfies $E_1 E_3 = E_2 E_3 = 0$ if and only if E_3 is a linear combination of elements E_s, with $s = \overline{\mathtt{111111}}, \mathtt{111}\overline{\mathtt{222}}$ or one of the following :

$$\begin{array}{ccc}
\mathtt{11}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{22}} & \mathtt{11}\overline{\mathtt{222}}\mathtt{1} & \mathtt{11}\overline{\mathtt{22}}\mathtt{1}\overline{\mathtt{2}} \\
\mathtt{1}\overline{\mathtt{2}}\mathtt{11}\overline{\mathtt{22}} & \mathtt{1}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{22}}\mathtt{1} & \mathtt{1}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{2}} \\
\overline{\mathtt{2}}\mathtt{111}\overline{\mathtt{22}} & \overline{\mathtt{2}}\mathtt{11}\overline{\mathtt{22}}\mathtt{1} & \overline{\mathtt{2}}\mathtt{11}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{2}}
\end{array} \tag{2.99}$$

(Pairs of roots on the same line or column in this grid have inner product 1, other pairs have inner product 0.)

We will now use a process similar to that of other proofs in this section to reduce the number of non-zero coordinates of E_3 to a minimum. Because

E_1, E_2, E_3 are linearly independent, we may find at least one root s in (2.99) such that $E_3[s] \neq 0$. Because of the properties of the Weyl group we may choose $s = r_3 = \mathtt{11\bar{2}1\bar{2}\bar{2}}$ without loss of generality.

Applying $x_r(k) \in \widehat{\mathsf{E}}_6(K)$ to E_3 with $r = \mathtt{\bar{2}\bar{2}1\bar{2}11}$ and an appropriate $k \in K$ annihilates the coordinate of E_3 at position $\mathtt{\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}}$. Similarly $r = \mathtt{003\bar{3}00}$ can be used to annihilate the coordinate at position $\mathtt{111\bar{2}\bar{2}\bar{2}}$. We invite the reader to verify that these reductions do not affect the other coordinates of E_3 and leave both E_1 and E_2 invariant.

After further reductions with $r = \mathtt{000\bar{3}03}$, $\mathtt{000\bar{3}30}$, $\mathtt{0\bar{3}3000}$ and $\mathtt{\bar{3}03000}$ we end up with E_3 of the following form :

$$E_3 = k_0 E_{\mathtt{11\bar{2}1\bar{2}\bar{2}}} + k_1 E_{\mathtt{1\bar{2}1\bar{2}\bar{2}1}} + k_2 E_{\mathtt{\bar{2}11\bar{2}\bar{2}1}} + k_3 E_{\mathtt{1\bar{2}1\bar{2}1\bar{2}}} + k_4 E_{\mathtt{\bar{2}11\bar{2}1\bar{2}}},$$

with $k_0, \ldots, k_4 \in K$ and $k_0 \neq 0$.

Not every element of this form is isotropic. Indeed, if E_3 is isotropic then also $e_p E_3$ must be isotropic for all $p \in \mathcal{P}$. Taking $p = \mathtt{000030}$ we find

$$e_{\mathtt{000030}} E_3 = \pm k_0 e_{\mathtt{11\bar{2}11\bar{2}}} \pm k_1 e_{\mathtt{1\bar{2}1\bar{2}11}} \pm k_2 e_{\mathtt{\bar{2}11\bar{2}11}},$$

and because $k_0 \neq 0$ this element can only be isotropic if $k_1 = k_2 = 0$. The same argument applied to $p = \mathtt{000003}$ yields $k_3 = k_4 = 0$.

We have proved that E_3 can be reduced to an element of the form $k_0 E_{r_3}$ with $k_0 \in K$ using elements of $\widehat{\mathsf{E}}_6(K)$ that leave $E_1 = E_{r_1}$ and $E_2 = E_{r_2}$ invariant. To get rid of the final scalar k_0 we apply $h_r(k_0)$ with r such that $r_1 \cdot r = r_2 \cdot r = 0$ and $r_3 \cdot r = -1$, e.g., $r = \mathtt{003\bar{3}00}$.

The case $n = 4$ can be handled in a similar way. By the above we may choose $E_1 = E_{r_1}$, $E_2 = E_{r_2}$ and $E_3 = E_{r_3}$ without loss of generality and now E_4 is a linear combination of E_1, E_2, E_3 and elements E_s with s equal to one of the following :

$$\mathtt{1\bar{2}11\bar{2}\bar{2}}, \qquad \mathtt{\bar{2}111\bar{2}\bar{2}}, \qquad \mathtt{11\bar{2}\bar{2}\bar{2}1}, \qquad \mathtt{11\bar{2}\bar{2}1\bar{2}}.$$

Without loss of generality we may assume $E_4[\mathtt{1\bar{2}11\bar{2}\bar{2}}] \neq 0$. A similar reduction process as before then yields E_4 of the following form :

$$E_4 = k_5 E_{\mathtt{1\bar{2}11\bar{2}\bar{2}}} + k_6 E_{\mathtt{11\bar{2}\bar{2}\bar{2}1}} + k_7 E_{\mathtt{11\bar{2}\bar{2}1\bar{2}}},$$

with $k_5, k_6, k_7 \in K$ and $k_5 \neq 0$.

Because E_4 is isotropic, also $e_p E_4$ must be isotropic for every $p \in \mathcal{P}$. As before, applying this to $p = \mathtt{000030}$ and $p = \mathtt{000003}$ yields $k_6 = k_7 = 0$. To get rid of the factor k_5 we may apply $h_r(k_5)$ with $r = \mathtt{330000}$.

Finally, any element E_5 such that $E_1 E_5 = \cdots = E_4 E_5 = 0$, must be a linear combination of $E_1, \ldots, E_4$ and a multiple of E_{r_5}. We can annihilate the coordinates at positions $r_1, \ldots, r_4$ of E_5 in a similar way as before, hence E_5 is equivalent to an element of the form ℓE_{r_5} as stated. ∎

• As with Theorem 2.40 we do not know whether in the last case of Theorem 2.42 distinct ℓ also imply distinct orbits.

Proposition 2.43 *Let $e_1, \ldots, e_n$ be linearly independent elements of $\mathbf{V}$ that satisfy $e_i^\# = e_i \times e_j = 0$, for all i, j, $1 \leq i, j \leq n$. Let $V_n = e_1 \times \mathbf{V} \cap \cdots \cap e_n \times \mathbf{V}$. Then* $\dim V_1 = 10$, $\dim V_2 = 5$, $\dim V_3 = 3$, $\dim V_4 = 2$, $\dim V_6 = 0$ *and* $\dim V_5 = 0$ *or* 2 *according to whether $(e_1, \ldots, e_5)$ can be extended to a* 6*-tuple with the same properties or not.*

For $n > 1$ the subspace V_n is isotropic.

Proof : By Theorem 2.40 we may without loss of generality choose every e_i to be some scalar multiple of a base vector e_{p_i} with $p_i \in \mathcal{P}$.

Let $\eta \in V_n$. We will consider the coordinates $\eta[q]$ with $q \in \mathcal{P}$. Let $a \in \mathbf{V}$. Using (2.19) we easily compute

$$e_i \times a = e_{p_i} \times a = \sum_{q \sim p_i} \pm a[p_i] \eta_{-p-q}.$$

This sum contains 10 terms, and therefore $\dim V_1 = 10$. It also easily follows that in general the dimension of V_n is equal to the number of points adjacent to all of $p_1, \ldots, p_n$. We leave it to the reader to derive the corresponding dimensions, for instance by specialising to the representatives given in Theorem 2.40. Also note that the set of points of $\mathcal{P}$ adjacent to a non-adjacent pair is always a coclique. It easily follows that V_n is isotropic whenever $n \geq 2$. ∎

In the cases $n = 2$, $n = 3$ and $n = 5$ such that $\dim V_n \neq 0$ we will call the isotropic subspace V_n of $\mathbf{V}^*$ the (isotropic) *companion space* in $\mathbf{V}^*$ of the isotropic subspace generated by $e_1, \ldots, e_n$. We use the same terminology in the dual case. If V is the companion space in $\mathbf{V}^*$ of $U \leq \mathbf{V}$, then U is the companion space of V in $\mathbf{V}$.

Proposition 2.44 *Let $e_1, \ldots, e_n$ be linearly independent elements of $\mathbf{V}$ that satisfy $e_i^\# = e_i \times e_j = 0$, for all i, j, $1 \leq i, j \leq n$. Consider the subpace L_n of $\mathbf{L}$ of all elements $A \in \mathbf{L}$ that satisfy $A(e_i \times \mathbf{V}) = 0$ for all i, $1 \leq i \leq n$. Then* $\dim L_1 = 16$ *(or* 17 *when* $\operatorname{char} K = 2$*),* $\dim L_2 = 5$, $\dim L_3 = 2$, $\dim L_4 = 1$, $\dim L_6 = 1$ *and* $\dim L_5$ *is either* 1 *or* 0 *according to whether $(e_1, \ldots, e_5)$ can be extended to a* 6*-tuple with the same properties or not.*

For $n > 1$ the space L_n is an isotropic subspace of $\mathbf{L}$.

Proof : By Lemma 2.41 the space L_1 is generated by all E_s such that $p \cdot s \neq 1$, and an additional element $H_p + \mathbf{1}$ when $\operatorname{char} K = 2$. Hence $\dim L_1 = 16$ (or 17 when $\operatorname{char} K = 2$) as stated.

We may extend this observation to the case $n > 1$. For $e_i = e_{p_i}$ with $p_i \in \mathcal{P}$ the dimension of L_n is equal to the number of roots s such that $p_i \cdot s = 1$ for all i. Without loss of generalization we need only consider those points used as representatives in Theorem 2.40.

For the point $p_0 = \mathtt{300000}$ the 16 corresponding roots s are the following :

$$
\begin{array}{lllll}
\mathtt{3\overline{3}0\underline{000}} & \mathtt{30\underline{\overline{3}}0\underline{00}} & \mathtt{30\underline{0\overline{3}}0\underline{0}} & \mathtt{30\underline{00\overline{3}}0} & \mathtt{3\underline{0}00\underline{0\overline{3}}} \\
\mathtt{2\underline{2}2\underline{1}1\underline{1}} & \mathtt{2\underline{2}1\underline{21}1} & \mathtt{2\underline{21}12\underline{1}} & \mathtt{2\underline{21}1\underline{1}2} & \mathtt{2\underline{122}11} \\
\mathtt{212121} & \mathtt{212112} & \mathtt{211221} & \mathtt{211212} & \mathtt{211122} \\
\mathtt{111111} & & & &
\end{array}
$$

Of these, only the first 5 satisfy $s \cdot q_0 = 1$ (with $q_0 = \mathtt{2\overline{11111}}$). We infer that L_2 has dimension 5 and is generated by the isotropic elements E_s with s equal to one of these 5 roots. Note that for any two such roots s, s' we have $s \cdot s' = 1$, and hence $E_s E_{s'} = 0$. Therefore L_2 is isotropic, and so are its subspaces $L_3, \ldots, L_6$.

If we also require $s \cdot a_{12} = 1$ (with $a_{12} = \mathtt{1\overline{22}111}$) then only $s = \mathtt{3\overline{3}0000}$ and $\mathtt{30\overline{3}000}$ remain, yielding $\dim L_3 = 2$. If also $s \cdot a_{13} = 1$ (with $a_{13} = \mathtt{1\overline{2}1\overline{2}11}$) then s must be equal to $\mathtt{3\overline{3}0000}$. Note that for this s we have $s \cdot a_{14} = s \cdot a_{15} = 1$ (with $a_{14} = \mathtt{1\overline{2}11\overline{2}1}$ and $a_{15} = \mathtt{1\overline{2}111\overline{2}}$), implying $L_6 = L_4$ and $L_6 = L_5$ when $e_5 = p_{a_{14}}$. The remaining case, where $e_5 = e_{a_{23}}$, yields $L_5 = \{0\}$ because $s \cdot a_{23} = 0$ (with $a_{23} = \mathtt{11\overline{22}11}$). ∎

For $n = 2$, $n = 3$ and $n = 6$ we will call L_n the (isotropic) *companion* space in $\mathbf{L}$ of the isotropic subspace generated by $e_1, \ldots, e_n$.

Proposition 2.45 *Let $E_1, \ldots, E_n$ be linearly independent isotropic elements of* $\mathbf{L}$ *that satisfy $E_i E_j = 0$ for all $i, j, 1 \leq i, j \leq n$. Let $T_n = \mathbf{V}E_1 \cap \cdots \cap \mathbf{V}E_n$. Then T_n is an isotropic subspace of* $\mathbf{V}$ *and* $\dim T_1 = 6$, $\dim T_2 = 3$, $\dim T_3 = \dim T_4 = \dim T_5 = 2$. *Cases with $n > 5$ cannot occur.*

Proof : Without loss of generalization we may choose $E_1, \ldots, E_n$ to be $E_{r_1}, \ldots, E_{r_n}$ as in Theorem 2.42. The dimension of the intersections T_n is then equal to the number of common base points of $r_1, \ldots, r_n$, which can be obtained from Proposition 2.6. ∎

For $n = 1$, $n = 2$ and $n = 5$ we will call T_n the (isotropic) *companion* space in $\mathbf{V}$ of the isotropic subspace generated by $E_1, \ldots, E_n$. By Theorem 2.37 an isotropic element e belongs to the $\mathbf{V}E$ if and only if $\mathbf{V}E \times e = 0$. As a consequence T is the companion space in $\mathbf{L}$ of $U \leq \mathbf{V}$ if and only if U is the companion space in $\mathbf{V}$ of T.

2.5 Additional E_6-modules

Let L denote an element of $\mathrm{Hom}(\mathbf{V}^*, \mathbf{V})$. We will write $L\alpha$ for the image in $\mathbf{V}$ of $\alpha \in \mathbf{V}^*$ through L. Hence $L\alpha\beta \in K$ for $\alpha, \beta \in \mathbf{V}$. Dually we write $a\Lambda$ for the image of a through an element $\Lambda \in \mathrm{Hom}(\mathbf{V}, \mathbf{V}^*)$, and then $ba\Lambda \in K$ for $a, b \in \mathbf{V}$.

• Readers who prefer the 'functional notation' for elements of $\mathbf{V}^*$ would write $\beta(L\alpha)$ or $\beta(L(\alpha))$ for $L\alpha\beta$ and $\Lambda(a)(b)$ for $ba\Lambda$. In our notation we have $L\alpha\beta = (L\alpha)\beta$ and $ba\Lambda = b(a\Lambda)$.

Define $\mathbf{V} \wedge \mathbf{V}$ to be the subspace of *antisymmetric* elements of $\mathrm{Hom}(\mathbf{V}^*, \mathbf{V})$, i.e., those elements L satisfying $L\alpha\alpha = 0$ for all $\alpha \in \mathbf{V}^*$. ($\mathbf{V} \wedge \mathbf{V}$ can be identified with the space of antisymmetric bilinear forms on $\mathbf{V}^*$.) Dually, we define $\mathbf{V}^* \wedge \mathbf{V}^*$ to be the subspace antisymmetric elements of $\mathrm{Hom}(\mathbf{V}, \mathbf{V}^*)$, i.e., those Λ satisfying $aa\Lambda = 0$, for all $a \in \mathbf{V}$.

The standard example of an element of $\mathbf{V} \wedge \mathbf{V}$ is provided by the *wedge product* $a \wedge b$ of $a, b \in \mathbf{V}$:

$$a \wedge b : \mathbf{V}^* \to \mathbf{V} : \alpha \mapsto (a \wedge b)\alpha \stackrel{\text{def}}{=} (a\alpha)b - (b\alpha)a, \tag{2.100}$$

with the following immediate properties :

$$\begin{aligned} &a \wedge b = -b \wedge a, \qquad a \wedge a = 0, \\ &(a \wedge b)\alpha\beta = -(a \wedge b)\beta\alpha = \begin{vmatrix} a\alpha & a\beta \\ b\alpha & b\beta \end{vmatrix}, \\ &(a \wedge b)\alpha\alpha = 0, \end{aligned} \tag{2.101}$$

for every $a, b \in \mathbf{V}$, $\alpha, \beta \in \mathbf{V}^*$.

It is easily proved that the elements $e_i \wedge e_j$ with $i, j \in \mathcal{P}, i \neq j$ and every pair $\{i, j\}$ counted only once, form a basis for $\mathbf{V} \wedge \mathbf{V}$ and hence $\dim \mathbf{V} \wedge \mathbf{V} = 351$.

We define a dual wedge product on $\mathbf{V}^*$ in a similar way :

$$\alpha \wedge \beta : \mathbf{V} \to \mathbf{V}^* : a \mapsto a(\alpha \wedge \beta) \stackrel{\text{def}}{=} (a\beta)\alpha - (a\alpha)\beta, \tag{2.102}$$

satisfying

$$\begin{aligned} &\alpha \wedge \beta = -\beta \wedge \alpha, \qquad \alpha \wedge \alpha = 0, \\ &ab(\alpha \wedge \beta) = -ba(\alpha \wedge \beta) = \begin{vmatrix} a\alpha & a\beta \\ b\alpha & b\beta \end{vmatrix}, \\ &aa(\alpha \wedge \beta) = 0. \end{aligned} \tag{2.103}$$

The elements $\eta_i \wedge \eta_j$ with $i, j \in \mathcal{P}, i \neq j$ and every pair $\{i, j\}$ counted only once, form a basis for $\mathbf{V}^* \wedge \mathbf{V}^*$.

Now, let $\alpha \in \mathbf{V}^*$ and $A \in \mathbf{L}$. Define $\alpha \times A$ to be the following element of $\mathrm{Hom}(\mathbf{V}^*, \mathbf{V})$:

$$\alpha \times A : \gamma \mapsto (\alpha \times A)\gamma \stackrel{\text{def}}{=} \alpha \times A\gamma, \quad \text{for every } \gamma \in \mathbf{V}^* \tag{2.104}$$

and dually, for $b \in \mathbf{V}, B \in \mathbf{L}$:

$$B \times b : c \mapsto c(B \times b) \stackrel{\text{def}}{=} cB \times b, \quad \text{for every } c \in \mathbf{V}. \tag{2.105}$$

By definition, we have $(\alpha \times A)\gamma\gamma = \langle \alpha, A\gamma, \gamma \rangle = -\gamma^{\#} A\alpha$, by (2.54). Hence $\alpha \times A \in \mathbf{V} \wedge \mathbf{V}$ whenever $A\alpha = 0$, and dually $B \times b \in \mathbf{V}^* \wedge \mathbf{V}^*$ whenever $bB = 0$.

• If we change the definition to $(\alpha \times A)\gamma \stackrel{\text{def}}{=} \alpha \times A\gamma + \frac{1}{2}\gamma \times A\alpha$ then $\alpha \times A \in \mathbf{V} \wedge \mathbf{V}$ also when $A\alpha \neq 0$. Unfortunately this cannot be made to work when $\operatorname{char} K = 2$.

• The vector space $\mathbf{V} \wedge \mathbf{V}$ (and dually $\mathbf{V}^* \wedge \mathbf{V}^*$) is an $\mathbf{L}$-module with the corresponding action defined by the identity $L^A \alpha\beta = L(A\alpha)\beta + L\alpha(A\beta)$. As a consequence $(a \wedge b)^A = aA \wedge b + a \wedge bA$, proving that the wedge product is compatible with the Lie algebra $\mathbf{L}$. Similarly, $(\beta \times B)^A = -A\beta \times B + \beta \times [B, A]$, making also the cross product compatible with $\mathbf{L}$.

• When $\operatorname{char} K = 2$ the module $\mathbf{V} \wedge \mathbf{V}$ is not irreducible. Indeed, because $a \wedge b = b \wedge a$ in this case, the map $\theta : a \wedge b \mapsto a \times b$ is well-defined and provides a homomorphism from $\mathbf{V} \wedge \mathbf{V}$ into $\mathbf{V}^*$. The kernel of θ is a 324-dimensional subspace of $\mathbf{V} \wedge \mathbf{V}$ which is also an E_6-module (and turns out to be irreducible).

Also note that when $\operatorname{char} K = 2$ the elements $\times_\alpha$ defined by $\times_\alpha \beta\gamma \stackrel{\text{def}}{=} \langle \alpha, \beta, \gamma \rangle$ are antisymmetric and hence belong to $\mathbf{V} \wedge \mathbf{V}$. The vector space generated by all elements of this kind is a submodule of $\mathbf{V} \wedge \mathbf{V}$ which is isomorphic to $\mathbf{V}^*$. It can be proved that $\theta(\times_\alpha) = \alpha$.

If g is an automorphism of $\mathbf{V}$ then its action can be extended in a unique way to $\mathrm{Hom}(\mathbf{V}^*, \mathbf{V})$ by requiring

$$L^g \alpha^g \beta^g = L\alpha\beta, \qquad \text{for all } \alpha, \beta \in \mathbf{V}^*,\ L \in \mathrm{Hom}(\mathbf{V}^*, \mathbf{V}). \tag{2.106}$$

(And similarly in the dual case.)

It follows that

$$\begin{array}{rclcrcl} (a \wedge b)^g & = & a^g \wedge b^g, & \qquad & (\alpha \wedge \beta)^g & = & \alpha^g \wedge \beta^g \\ (\alpha \times A)^g & = & (\alpha^g \times A^g), & & (B \times b)^g & = & (b^g \times B^g), \end{array}$$

for $a, b \in \mathbf{V}$, $\alpha, \beta \in \mathbf{V}^*$, $A, B \in \mathbf{L}$.

Lemma 2.46 *Let $a, b \in \mathbf{V}$. Let E be an isotropic element of $\mathbf{L}$. Then*

$$aE \wedge bE = (-aE \times b) \times E = (bE \times a) \times E. \tag{2.107}$$

Proof: Let $\alpha \in \mathbf{V}^*$. Then by Proposition 2.28–8, we have

$$(aE\alpha)bE = b(aE * E\alpha) = (bE\alpha)aE - (aE \times b) \times E\alpha,$$

proving the first equality. The second equality is then obtained by interchanging the roles of a and b. ■

Lemma 2.47 *Let $a, b, e \in \mathbf{V}$ be such that e is isotropic and $a \times b = 0$. Then*

$$e \times a \wedge e \times b = (a * (e \times b)) \times e. \tag{2.108}$$

Proof: Let $c \in \mathbf{V}$. Then

$$\begin{array}{rcl} c[(a * (e \times b)) \times e] & = & c(a * (e \times b)) \times e \\ & = & \langle b, c, e \rangle a \times e - ((c \times a) \times (e \times b)) \times e \\ & = & \langle b, c, e \rangle a \times e - \langle a, c, e \rangle b \times e, \text{ by Proposition 2.27–3,} \end{array}$$

and this is equal to $c(e \times a \wedge e \times b)$. ■

Proposition 2.48 *Let* $L \in \mathbf{V} \wedge \mathbf{V}$, $L \neq 0$. *Then the following are equivalent :*

1. $L = e \wedge f$ *for some* $e, f \in \mathbf{V}$ *such that* $e^\# = f^\# = e \times f = 0$.

2. $L = \eta \times E$ *for some isotropic* $E \in \mathbf{L}$ *and isotropic* $\eta \in \mathbf{V}^*$ *such that* $E\eta = 0$ *but* $\eta \notin E\mathbf{V}^*$.

Proof : By Theorem 2.35 the pairs (e, f) satisfying the given conditions form a single orbit of $\widehat{\mathsf{E}}_6(K)$. Likewise, by the dual of Theorem 2.37, the pairs (E, η) satisfying the given conditions form also a single orbit of $\widehat{\mathsf{E}}_6(K)$. To prove the theorem it is therefore sufficient to find at least one example for which $e \wedge f = \eta \times E$. We leave it to the reader to use either (2.107) or (2.108) to find an example of this kind. ∎

An element $L \in \mathbf{V} \wedge \mathbf{V}$ which satisfies one of the statements of this proposition will be called *isotropic*. Isotropic elements of $\mathbf{V}^* \wedge \mathbf{V}^*$ are defined in a dual way. Note that $\widehat{\mathsf{E}}_6(K)$ maps isotropic elements of $\mathbf{V} \wedge \mathbf{V}$ (or $\mathbf{V}^* \wedge \mathbf{V}^*$) onto isotropic elements and it is a consequence of the proof of this proposition that the non-zero isotropic elements of $\mathbf{V} \wedge \mathbf{V}$ form a single orbit.

If $L \in \mathbf{V} \wedge \mathbf{V}$ is isotropic then $L\mathbf{V}^*$ is a isotropic subspace of $\mathbf{V}$ of dimension 2.

• A typical isotropic element of $\mathbf{V} \wedge \mathbf{V}$ is given by $e_p \wedge e_{p'}$ where $p \not\perp p'$. In fact, these elements are related to the elements of the orbit $\mathcal{P}_2$ of which the fundamental weight π_2' is an example. Note that there are only 216 elements of this type, while $\dim \mathbf{V} \wedge \mathbf{V} = 351$, hence as with $\mathbf{L}$, these 'canonical' base elements do not provide a full basis for the module. To generate all of $\mathbf{V} \wedge \mathbf{V}$ we must add one space $\mathbf{H}_p \times \eta_p$ of dimension 5 for each $p \in \mathcal{P}$. ($\mathbf{H}_p$ denotes the subspace of elements H of $\mathbf{H}$ for which $H\eta_p = 0$.)

• When $\operatorname{char} K = 2$, then the isotropic elements of $\mathbf{V} \wedge \mathbf{V}$ clearly belong to the 324-dimensional submodule of $\mathbf{V} \wedge \mathbf{V}$.

The definitions and properties which were given above for $\mathbf{V} \wedge \mathbf{V}$ can be generalized to trilinear forms. Let P denote a trilinear form on $\mathbf{V}^*$ and write $P\alpha\beta\gamma$ for its value on the triple $(\alpha, \beta, \gamma) \in \mathbf{V}^* \times \mathbf{V}^* \times \mathbf{V}^*$. Alternatively we may regard $P\alpha\beta$ as an element of $\mathbf{V}$, and $P\alpha$ as an element of $\operatorname{Hom}(\mathbf{V}^*, \mathbf{V})$, and hence P as an element of $\operatorname{Hom}(\mathbf{V}^* \times \mathbf{V}^*, \mathbf{V})$ or of $\operatorname{Hom}(\mathbf{V}^*, \operatorname{Hom}(\mathbf{V}^*, \mathbf{V}))$.

We will be interested in the vector space $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ of all *antisymmetric* trilinear forms on $\mathbf{V}^*$, i.e., elements P that satisfy

$$P\,\delta\gamma\gamma = P\,\gamma\delta\gamma = P\,\gamma\gamma\delta = 0, \quad \text{for all } \gamma, \delta \in \mathbf{V}^*. \tag{2.109}$$

Dually we will consider the vector space $\mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$ of all antisymmetric trilinear forms on $\mathbf{V}$, i.e., those Π that satisfy

$$ccd\,\Pi = cdc\,\Pi = dcc\,\Pi = 0, \quad \text{for all } c, d \in \mathbf{V}. \tag{2.110}$$

The standard example of an element of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ is provided by the *wedge product* $a \wedge b \wedge c$ and dually $\alpha \wedge \beta \wedge \gamma$ with $a, b, c \in \mathbf{V}$ and $\alpha, \beta, \gamma \in \mathbf{V}^*$. These trilinear forms are defined to satisfy

$$(a \wedge b \wedge c)\alpha\beta\gamma = \begin{vmatrix} a\alpha & a\beta & a\gamma \\ b\alpha & b\beta & b\gamma \\ c\alpha & c\beta & c\gamma \end{vmatrix} = abc(\alpha \wedge \beta \wedge \gamma). \tag{2.111}$$

It follows that interchanging two of the three operands of $a \wedge b \wedge c$ or $\alpha \wedge \beta \wedge \gamma$ simply changes the sign of the result.

It is easily proved that the elements $e_i \wedge e_j \wedge e_k$ with $i, j, k \in \mathcal{P}, i \neq j \neq k \neq i$ and every triple $\{i, j, k\}$ counted only once, form a basis for $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ and hence $\dim \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V} = 2925$. A similar basis of elements of the form $\eta_i \wedge \eta_j \wedge \eta_k$ can be established for $\mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$.

As with $\mathbf{V} \wedge \mathbf{V}$, there are other interesting ways to construct elements of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$.

For every $A, B \in \mathbf{L}$ we define the trilinear form $A \times B$ as follows :

$$(A \times B)\gamma\delta\varepsilon \stackrel{\text{def}}{=} (A\gamma \times B)\delta\varepsilon = \langle A\gamma, B\delta, \varepsilon \rangle, \quad \text{for every } \gamma, \delta, \varepsilon \in \mathbf{V}^* \tag{2.112}$$

and dually $(A \times B)^*$ as follows :

$$cde(A \times B)^* \stackrel{\text{def}}{=} cd(A \times eB) = \langle c, dA, eB \rangle, \quad \text{for all } c, d, e \in \mathbf{V}. \tag{2.113}$$

Lemma 2.49 *Let $A, B \in \mathbf{L}$ be such that $AB = BA = 0$. Then $A \times B = -B \times A$ and $A \times B$ belongs to $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$.*

Proof : Let $\gamma, \delta, \varepsilon \in \mathbf{V}^*$. By repeatedly applying (2.56) we find

$$\begin{array}{rclcl} \langle A\gamma, B\delta, \varepsilon\rangle &=& -\langle \gamma, AB\delta, \varepsilon\rangle - \langle \gamma, B\delta, A\varepsilon\rangle &=& -\langle \gamma, B\delta, A\varepsilon\rangle \\ &=& \langle B\gamma, \delta, A\varepsilon\rangle + \langle \gamma, \delta, BA\varepsilon\rangle &=& \langle B\gamma, \delta, A\varepsilon\rangle \\ &=& -\langle AB\gamma, \delta, \varepsilon\rangle - \langle B\gamma, A\delta, \varepsilon\rangle &=& -\langle B\gamma, A\delta, \varepsilon\rangle, \end{array}$$

proving that $A \times B = -B \times A$.

Also

$$\begin{array}{rcll} (A \times B)\gamma\gamma\delta &=& \langle A\gamma, B\gamma, \delta\rangle & \\ &=& -\langle \gamma, AB\gamma, \delta\rangle - \langle \gamma, B\gamma, A\delta\rangle, & \text{by (2.56)}, \\ &=& \gamma^{\#} BA\delta = 0, & \text{by (2.54)}. \\ (A \times B)\gamma\delta\gamma &=& \langle A\gamma, B\delta, \gamma\rangle = -\gamma^{\#} AB\delta = 0. & \\ (A \times B)\delta\gamma\gamma &=& \langle A\delta, B\gamma, \gamma\rangle = -\gamma^{\#} AB\delta = 0. & \end{array}$$

These identities prove that $A \times B$ satisfies (2.109). ∎

A third way to construct elements of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ is provided by the following lemma :

Lemma 2.50 *Let $\alpha, \beta, \gamma \in \mathbf{V}^*$ be such that $\alpha^{\#} = \beta^{\#} = \gamma^{\#} = \alpha \times \beta = \beta \times \gamma = \gamma \times \alpha = 0$. Consider the unique trilinear form $X(\alpha, \beta, \gamma)$ satisfying*

$$X(\alpha, \beta, \gamma)\delta\varepsilon\varphi \stackrel{\text{def}}{=} \langle \alpha \times \delta, \beta \times \varepsilon, \gamma \times \varphi\rangle. \tag{2.114}$$

Then $X(\alpha, \beta, \gamma) \in \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$.

Proof : Note that

$$X(\alpha, \beta, \gamma)\delta\varepsilon = [(\alpha \times \delta) \times (\beta \times \varepsilon)] \times \gamma.$$

Setting $\delta = \varepsilon$ and applying (2.26) we find

$$\begin{aligned} & X(\alpha,\beta,\gamma)\delta\delta \\ = \quad & \langle \alpha,\beta,\delta\rangle\delta\times\gamma + (\delta^{\#}\alpha)\beta\times\gamma + (\delta^{\#}\beta)\alpha\times\gamma - (\delta^{\#}\times(\alpha\times\beta))\times\gamma = 0 \end{aligned}$$

hence $X(\alpha,\beta,\gamma)\delta\delta\varphi = 0$. By symmetry also the other two conditions of (2.109) are satisfied. ∎

If g is an automorphism of $\mathbf{V}$ then its action can be extended in a unique way to the trilinear forms on $\mathbf{V}^*$ by requiring

$$P^g\,\alpha^g\beta^g\gamma^g = P\,\alpha\beta\gamma, \qquad \text{for all } \alpha,\beta,\gamma\in\mathbf{V}^*. \tag{2.115}$$

(And similarly in the dual case.)

It follows that

$$\begin{array}{lcllcl} (a\wedge b\wedge c)^g &=& a^g\wedge b^g\wedge c^g, & (\alpha\wedge\beta\wedge\gamma)^g &=& \alpha^g\wedge\beta^g\wedge\gamma^g \\ (A\times B)^g &=& (A^g\times B^g), & (A\times B)^{*g} &=& (A^g\times B^g)^*, \\ X(\alpha,\beta,\gamma)^g &=& X(\alpha^g,\beta^g,\gamma^g), & X(a,b,c)^g &=& X(a^g,b^g,c^g), \end{array}$$

for $a,b,c\in\mathbf{V}$, $\alpha,\beta,\gamma\in\mathbf{V}^*$, $A,B\in\mathbf{L}$.

Lemma 2.51 *Let $a,b,c\in\mathbf{V}$ and let E be an isotropic element of $\mathbf{L}$. Then*

$$aE\wedge bE\wedge cE = (cE*(bE\times a))\times E. \tag{2.116}$$

Proof: Write $F = (cE*(bE\times a))$. Let $\alpha\in\mathbf{V}^*$. Then

$$\begin{aligned} (F\times E)\alpha &= F\alpha\times E \\ &= (cE\alpha)(bE\times a)\times E - [cE\times((bE\times a)\times\alpha)]\times E \\ &= (cE\alpha)(aE\wedge bE) + cE\wedge((bE\times a)\times\alpha)E, \end{aligned}$$

applying (2.107) to both terms. Also

$$\begin{aligned} ((bE\times a)\times\alpha)E &= -E(bE\times a)\times\alpha - (bE\times a)\times E\alpha \\ &= -(bE\times a)\times E\alpha = -(aE\wedge bE)\alpha, \end{aligned}$$

by Proposition 2.28–10 and again by (2.107). Hence

$$\begin{aligned}(F \times E)\alpha &= (cE\alpha)(aE \wedge bE) - cE \wedge (aE \wedge bE)\alpha \\ &= (cE\alpha)(aE \wedge bE) + (aE\alpha)(bE \wedge cE) + (bE\alpha)(cE \wedge aE) \\ &= (aE \wedge bE \wedge cE)\alpha.\end{aligned}$$

■

Lemma 2.52 *Let $\alpha, \beta, \gamma \in \mathbf{V}^*$ and let E be an isotropic element of $\mathbf{L}$. Then*

$$X(E\alpha, E\beta, E\gamma) = E \times ((\alpha \times E\beta) * E\gamma). \tag{2.117}$$

Proof : Write $F = ((\alpha \times E\beta) * E\gamma)$ and let $\delta, \varepsilon, \varphi \in \mathbf{V}^*$. We have

$$F\varepsilon = \langle \alpha, E\beta, \varepsilon \rangle E\gamma - (\alpha \times E\beta) \times (\varepsilon \times E\gamma),$$

and hence

$$\begin{aligned}(E \times F)\delta\varepsilon\varphi &= \langle E\delta, F\varepsilon, \varphi \rangle \\ &= -\langle \alpha \times E\beta, \varepsilon \times E\gamma, \varphi \times E\delta \rangle \\ &= -X(E\beta, E\gamma, E\delta)\alpha\varepsilon\varphi = -X(E\delta, E\beta, E\gamma)\alpha\varepsilon\varphi,\end{aligned}$$

because X is antisymmetric.

This is equal to

$$\begin{aligned}&-\langle \alpha \times E\delta, \varepsilon \times E\beta, \varphi \times E\gamma \rangle \\ &\quad = \langle E\alpha \times \delta, \varepsilon \times E\beta, \varphi \times E\gamma \rangle + \langle (\alpha \times \delta)E, \varepsilon \times E\beta, \varphi \times E\gamma \rangle \\ &\quad = \langle E\alpha \times \delta, \varepsilon \times E\beta, \varphi \times E\gamma \rangle - \langle \alpha \times \delta, (\varepsilon \times E\beta)E, \varphi \times E\gamma \rangle \\ &\qquad\quad - \langle \alpha \times \delta, \varepsilon \times E\beta, (\varphi \times E\gamma)E \rangle,\end{aligned}$$

applying (2.43) twice. By Proposition 2.28–10 the last two terms are zero, and hence $(E \times F)\delta\varepsilon\varphi = \langle E\alpha \times \delta, \varepsilon \times E\beta, \varphi \times E\gamma \rangle = X(E\alpha, E\beta, E\gamma)\delta\varepsilon\varphi$. ■

Proposition 2.53 *Let* $P \in \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$, $P \neq 0$. *Then the following are equivalent :*

1. $P = a \wedge b \wedge c$ *for some* a, b, c *that generate a* 3*-dimensional isotropic subspace of* $\mathbf{V}$.

2. $P = E \times F$ *for some isotropic elements* E, F *such that* $EF = FE = 0$ *but* $kE \neq KF$.

3. $P = X(\alpha, \beta, \gamma)$ *for some* α, β, γ *that generate a* 3*-dimensional isotropic subspace of* $\mathbf{V}^*$.

Proof : By Theorem 2.38 the pairs (E, F) that satisfy he given conditions form a single orbit of $\widehat{\mathsf{E}}_6(K)$. Likewise by Theorem 2.40 the triples (a, b, c) that satisfy the given conditions form a single orbit and so do the triples (α, β, γ). To prove the theorem it is therefore sufficient to find at least one example for which $a \wedge b \wedge c = E \times F$ and at least one for which $X(\alpha, \beta, \gamma) = E \times F$. Again we leave it to the reader to apply (2.116) and (2.117) to obtain the necessary examples. ∎

An element $P \in \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ which satisfies one of the statements of this proposition will be called *isotropicisotropic*. Isotropic elements of $\mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$ are defined in a dual way. Note that $\widehat{\mathsf{E}}_6(K)$ maps isotropic elements of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ (or $\mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$) onto isotropic elements, and it is a consequence of the proof of this proposition that the non-zero isotropic elements of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ form a single orbit.

If $P \in \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ is isotropic then $P\mathbf{V}^*\mathbf{V}^*$ is an isotropic subspace of $\mathbf{V}$ of dimension 3.

• A typical isotropic element of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ is given by $e_p \wedge e_{p'} \wedge e_{p''}$ where p, p', p'' form a coclique of size 3 in $\mathcal{Q}$. These elements are related to the elements of the orbit $\mathcal{P}_3$ of which the fundamental weight π'_3 is an example.

• The isotropic space generated by α, β, γ of Proposition 2.53 is the companion space in $\mathbf{V}^*$ of the isotropic subspace V generated by a, b, c. Likewise E and F span the companion space of V in $\mathbf{L}$.

If $\alpha \in \mathbf{V}^*$, $a \in \mathbf{V}$, define the *tensor product* $\alpha \otimes a$ to be the following linear transformation of $\mathrm{Hom}(\mathbf{V}, \mathbf{V})$:

$$\alpha \otimes a : \mathbf{V} \to \mathbf{V} : x \mapsto x(\alpha \otimes a) \stackrel{\text{def}}{=} (x\alpha)a \tag{2.118}$$

As a matrix, $\alpha \otimes a$ is obtained by multiplying the column vector α with the row vector a.

• This means that we could have used the notation αa instead of $\alpha \otimes a$, but we have chosen not to do this to avoid confusion with the product $a\alpha$.

Lemma 2.54 *Let e, f be isotropic elements of $\mathbf{V}$. Let η be an isotropic element of $\mathbf{V}^*$ such that $e\eta = f\eta = 0$. Set $E = e * \eta$, $F = f * \eta$. Then*

$$EF = FE = \eta \otimes eF = \eta \otimes fE = -\eta \otimes [(e \times f) \times \eta]. \tag{2.119}$$

Proof : Let $a \in \mathbf{V}$. We have $aEF = (a\eta)eF - ((a \times e) \times \eta)F = (a\eta)eF$, by Proposition 2.28–2. Also $eF = (e\eta) - (e \times f) \times \eta$. This proves $aEF = -(a\eta)(e \times f) \times \eta$ for all $a \in \mathbf{V}$. The right hand side of this expression is symmetric in e and f, hence $EF = FE$. ∎

Proposition 2.55 *Let $M \in \mathrm{Hom}(\mathbf{V}, \mathbf{V})$, $M \neq 0$. Then the following are equivalent*

1. *$M = \alpha \otimes a$ for some $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ such that $a^\# = 0$, $\alpha^\# = 0$, $a\alpha = 0$ and $a * \alpha = 0$.*

2. *$M = EF$ for some isotropic elements $E, F \in \mathbf{L}$ such that $EF = FE \neq 0$.*

Proof : By Theorem 2.36 the pairs (a, α) satisfying the given conditions form a single orbit of $\widehat{\mathsf{E}}_6(K)$. Likewise, by Theorem 2.38 the pairs (E, F) that satisfy he given conditions form a single orbit. It is therefore sufficient to find at least one example for which $\alpha \otimes a = EF$.

Such examples are provided by Lemma 2.54 for $e \times f \neq 0$, $\eta \neq 0$. Note that $a = (e \times f) \times \eta$ is isotropic by Proposition 2.27–2 and that $a * \alpha = [(e \times f) \times \eta] * \eta = 0$ by Proposition 2.27–5. ∎

• Calling the elements $M \in \mathrm{Hom}(\mathbf{V},\mathbf{V})$ that satisfy one of the conditions of this proposition isotropic would be ambiguous because also the isotropic elements of **L** belong to $\mathrm{Hom}(\mathbf{V},\mathbf{V})$. A typical element that satisfies these conditions is of the form $\eta_p \otimes e_q = \pm E_r E_s$ with $p \sim q$, $r \cdot s = 0$ and $p - q = r + s$. Not surprisingly, all elements $r + s$ of this type form a single orbit of $W(\mathsf{E}_6)$. This time however this orbit is not associated with a fundamental weight.

• The subspace of $\mathrm{Hom}(\mathbf{V},\mathbf{V})$ generated by the elements M of Proposition 2.55 is again a module for **L**. $\mathrm{Hom}(\mathbf{V},\mathbf{V})$ can be written as a direct product of this module, the module **L** and a trivial module of dimension 1.

2.6 The geometry $\mathcal{E}$

As was announced in Chapter 1, the isotropic elements of the various E_6-modules will be used to define a *geometry $\mathcal{E}$ of type* E_6. This geometry consists of six different *types*, numbered like the nodes of the Dynkin diagram :

1. The *points* of $\mathcal{E}$ are the isotropic 1-spaces of $\mathbf{V}$, i.e., the sets Ke where e is an isotropic element of $\mathbf{V} - \{0\}$.

2. The *lines* of $\mathcal{E}$ are the isotropic 1-spaces of $\mathbf{V} \wedge \mathbf{V}$, i.e., the sets $K(e \wedge f)$ where e, f generate an isotropic 2-space of $\mathbf{V}$.

3. The *planes* of $\mathcal{E}$ are the isotropic 1-spaces of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$, i.e., the sets $K(d \wedge e \wedge f)$ where d, e, f generate an isotropic 3-space of $\mathbf{V}$.

4. The *dual lines* of $\mathcal{E}$ are the isotropic 1-spaces of $\mathbf{V}^* \wedge \mathbf{V}^*$, i.e., the sets $K(\eta \wedge \varphi)$ where η, φ generate an isotropic 2-space of $\mathbf{V}^*$.

5. The *dual points* of $\mathcal{E}$ are the isotropic 1-spaces of $\mathbf{V}^*$, i.e., the sets $K\eta$ where η is an isotropic element of $\mathbf{V}^* - \{0\}$.

0. The *simplices* of $\mathcal{E}$ are the isotropic 1-spaces of $\mathbf{L}$, i.e., the sets KE where E is an isotropic element of $\mathbf{L} - \{0\}$.

The following table defines the incidence relations between elements of different types.

	$e \in \mathbf{V}$	$L \in \mathbf{V} \wedge \mathbf{V}$	$P \in \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$	$\Lambda \in \mathbf{V}^* \wedge \mathbf{V}^*$	$\eta \in \mathbf{V}^*$
$E \in \mathbf{L}$	$e \in \mathbf{V}E$ $E \times e = 0$	$L\mathbf{V}^* \leq \mathbf{V}E$ $E \times L\mathbf{V}^* = 0$	$P\mathbf{V}^*\mathbf{V}^* \leq \mathbf{V}E$ $E \times P\mathbf{V}^*\mathbf{V}^* = 0$	$\mathbf{V}\Lambda \leq E\mathbf{V}^*$ $\mathbf{V}\Lambda \times E = 0$	$\eta \in E\mathbf{V}^*$ $\eta \times E = 0$
e		$e \in L\mathbf{V}^*$	$e \in P\mathbf{V}^*\mathbf{V}^*$	$\mathbf{V}\Lambda \leq e \times \mathbf{V}$ $e * \mathbf{V}\Lambda = 0$	$e \in \eta \times \mathbf{V}^*$ $\eta \in e \times \mathbf{V}$ $e * \eta = 0$
L			$L\mathbf{V}^* \leq P\mathbf{V}^*$	$L\mathbf{V}^* * \mathbf{V}\Lambda = 0$	$L\mathbf{V}^* \leq \eta \times \mathbf{V}^*$ $L\mathbf{V}^* * \eta = 0$
P				$P\mathbf{V}^*\mathbf{V}^* * \mathbf{V}\Lambda = 0$	$P\mathbf{V}^*\mathbf{V}^* \leq \eta \times \mathbf{V}^*$ $P\mathbf{V}^*\mathbf{V}^* * \eta = 0$
Λ					$\eta \in \mathbf{V}\Lambda$

Incidence is symmetric and elements of the same type are never incident (so we only list the upper diagonal half of the table). If a table entry lists several relations, then they are equivalent.

Incidence between points Ke, lines KL, planes KP and simplices KE is equivalent to containment of the corresponding 1-, 2-, 3- and 6-dimensional subspaces Ke, $L\mathbf{V}^*$, $P\mathbf{V}^*\mathbf{V}^*$ and $\mathbf{V}E$ of $\mathbf{V}$. Incidence between dual points $K\eta$, dual lines $K\Lambda$ and simplices KE is equivalent to containment of the 1-, 2- and 6-dimensional subspaces $K\eta$, $\mathbf{V}\Lambda$ and $E\mathbf{V}^*$ of $\mathbf{V}^*$.

Note that a line $K(e \wedge e')$ is incident with a dual line or dual point if and only if both points Ke and Kf are incident with these elements, and similarly for planes $K(e \wedge e' \wedge e'')$. Also, a dual line $K(\eta \wedge \eta')$ is incident with a point, line or plane if and only if both dual points $K\eta$ and $K\eta'$ are incident with that point, line or plane.

A dual point $K\Lambda$ can also be identified with the 5-dimensional companion space of $\mathbf{V}\mathbf{V}\Lambda$, and then incidence with points, lines and planes is again equivalent to containment.

Proposition 2.56 *Let KE be a simplex of $\mathcal{E}$. Then the elements incident with KE form a projective 5-space $\mathcal{A}$ (i.e., a geometry of type A_5) where points, lines and planes, dual lines and dual points correspond to subspaces of $\mathcal{A}$ of (projective) dimension 0, 1, 2, 3 and 4 respectively.*

Proof : From the definitions of incidence in $\mathcal{E}$ it easily follows that points, lines and planes incident with KE are of the form $K(aE)$, $K(aE \wedge bE)$ and $K(aE \wedge bE \wedge cE)$ with $a, b, c \in \mathbf{V}$. Likewise dual points and dual lines incident with KE are of the form $K(E\alpha)$ and $K(E\alpha \wedge E\beta)$ with $\alpha, \beta \in \mathbf{V}^*$.

From the 6-dimensional space $\mathbf{V}E$ we construct a 5-dimensional projective space $\mathcal{A}$ in a natural way. By the above, there is an immediate one-to-one correspondence between points, lines and planes of $\mathcal{A}$ (i.e., 1-, 2- and 3-dimensional subspaces of $\mathbf{V}E$) and points, lines and planes of $\mathcal{E}$ incident with KE.

By Proposition 2.28–8 we have $aE * E\alpha = (aE\alpha)E$ and hence the bilinear form $F : \mathbf{V}E \times E\mathbf{V}^* \to K$ with $F(b, \beta) \stackrel{\text{def}}{=} (b * \beta)/E$ is well defined. Note that F is non-degenerate : if $F(\mathbf{V}E, \beta) = 0$, with $b = E\alpha$, then $\mathbf{V}E\alpha = 0$ and hence $\beta = E\alpha = 0$. As a consequence F defines a correspondence between any k-dimensional subspace S of $\mathbf{V}E$ and the subspace S^* of $E\mathbf{V}^*$ of dimension $6 - k$ of all elements $\beta \in E\mathbf{V}^*$ for which $F(\beta) = 0$. Moreover, by definition of F, if $b \in S$ then $\beta \in S^*$ if and only if $b * \beta = 0$, i.e., if and only if the points Kb and the dual point $K\beta$ are incident in $\mathcal{E}$. This proves that projective 3-spaces and 4-spaces of $\mathcal{A}$ are in one-to-one correspondence with the dual lines and dual planes of $\mathcal{E}$ that are incident with KE.

We leave it to the reader to verify that all incidence relations in $\mathcal{E}$ do indeed reduce to the corresponding incidence relations in $\mathcal{A}$. ∎

Proposition 2.57 *Let $K\eta$ be a dual point of $\mathcal{E}$. Then the elements incident with $K\eta$ form a geometry $\mathcal{D}$ of type* D_5 *where point, lines and planes correspond to isotropic subspaces of dimension* 0, 1 *and* 2, *and dual lines and simplexes correspond to the two different types of generators (maximal isotropic subspaces) of $\mathcal{D}$.*

Proof : From the definitions of incidence in $\mathcal{E}$ we see that points incident with $K\eta$ are of the form $K(\eta \times \alpha)$ with $\alpha \in \mathbf{V}^*$. For $\eta \times \alpha$ to be isotropic, we must have $\alpha^{\#}\eta = 0$ by the dual of Proposition 2.27–2. Similarly, two points $K(\eta \times \alpha)$ and $K(\eta \times \beta)$ will generate a space of isotropic elements if and only if $\langle \eta, \alpha, \beta \rangle = 0$, by Proposition 2.27–1.

By specializing to $\eta = \eta_p$ for some $p \in \mathcal{P}$, we see that the points incident with $K\eta$ are the points of a projective 9-space that lie on a hyperbolic quadric, i.e., points of a geometry $\mathcal{D}$ of type D_5.

Isotropic subspaces of $\mathbf{V}^*$ of elements incident with $K\eta$ correspond to subspaces of $\mathcal{D}$ all of whose points lie on the quadric. This observation provides a one-to-one correspondence between lines of planes of $\mathcal{E}$ that are incident with $K\eta$ and lines and planes of $\mathcal{D}$.

Now consider a dual point $K(\eta \wedge \beta)$. By Proposition 2.43 the companion space in $\mathbf{V}$ of $K\eta + K\beta$ is a 5-dimensional isotropic subspace, i.e., a generator of $\mathcal{D}$.

Conversely, if U is a 5-dimensional subspace of $\eta \times \mathbf{V}$ which has a companion space V in $\mathbf{V}^*$, then V is a two-dimensional isotropic subspace of $\mathbf{V}^*$ that contains η. This provides a one-to-one correspondence between generators of $\mathcal{D}$ of this type and dual points of $\mathcal{E}$ incident with $K\eta$.

The second type of maximal isotropic subspace in $\mathcal{D}$ has no companion space in $\mathbf{V}^*$ but a 1-dimensional companion space KE in $\mathbf{L}$ (cf. Proposition 2.44). Note that $\dim \mathbf{V}E \cap (\eta \times \mathbf{V}^*)$ is at least 5, and hence $\eta \in E\mathbf{V}^*$ by Theorem 2.37, i.e., KE and $K\eta$ are incident. Conversely, if $K\eta$ is incident with KE then $\mathbf{V}E$ intersects $\eta \times \mathbf{V}^*$ in an isotropic 5-space, by Proposition 2.56. This provides a one-to-one correspondence between maximal isotropic subspaces of $\mathcal{D}$ of the second type and simplices of $\mathcal{E}$ incident with $K\eta$.

Also by Proposition 2.56 the intersection of maximal isotropic spaces of different types is a subspace of $\mathbf{V}$ of dimension 4 proving that they indeed correspond to generators of different type in $\mathcal{D}$.

We again leave it to the reader to verify that all incidence relations in $\mathcal{E}$ do indeed reduce to the corresponding incidence relations in $\mathcal{D}$. ∎

As was explained in Section 1.8 of the introduction, Propositions 2.56, 2.57 and its dual, prove the following

Theorem 2.58 *$\mathcal{E}$ is the geometry arising from the exceptional building of type E_6 over the field K.*

3 The Lie algebra of type F_4 and related structures

3.1 A root system of type F_4

Define a linear transformation $\bar{\cdot}$ on $\mathbf{P}$ by its action on the basis elements, as follows :

$$\begin{aligned} p_0 &\mapsto \overline{p}_0 \stackrel{\text{def}}{=} -q_5, & p_1 &\mapsto \overline{p}_1 \stackrel{\text{def}}{=} -q_4, & p_2 &\mapsto \overline{p}_2 \stackrel{\text{def}}{=} -q_3, \\ p_3 &\mapsto \overline{p}_3 \stackrel{\text{def}}{=} -q_2, & p_4 &\mapsto \overline{p}_4 \stackrel{\text{def}}{=} -q_1, & p_5 &\mapsto \overline{p}_5 \stackrel{\text{def}}{=} -q_0, \end{aligned} \tag{3.1}$$

where we have used the standard notation for points of $\mathcal{P}$. Because $p_i \cdot p_j = q_i \cdot q_j$ equals $1/3$ whenever $i \neq j$ and $4/3$ otherwise, this transformation preserves the dot product on $\mathbf{P}$.

Note that $-3r = \sum p_i$ is mapped to $-\sum q_i = -6r - \sum p_i = -3r$, and hence $\overline{r} = r$. Then $-\overline{q}_i = -\overline{p}_i - \overline{r} = q_{5-i} - r = p_{5-i}$ and therefore $\bar{\cdot}$ is an involution. Also $\overline{a}_{ij} = -\overline{p}_i - \overline{q}_j = q_{5-i} + p_{5-j} = -a_{5-i,5-j}$. This proves that $\overline{\mathcal{P}} = -\mathcal{P}$ and hence, because every root can be written as a difference of points with inner product $1/3$, that $\overline{\Phi} = \Phi$.

Let $p \in \mathcal{P}$ and consider the point $-\overline{p}$. It is easily computed that $p = -\overline{p}$ exactly when p is either a_{01}, a_{23} or a_{45}. These points form a line, which we will denote by L_∞ and call the 'line at infinity'. We write $\mathcal{P}^* \stackrel{\text{def}}{=} \mathcal{P} - L_\infty$.

If $r \in R$ then of the six base points of r there can be at most one on L_∞, and at most one base point p such that $p + r \in L_\infty$. Because r has 6 base points, we can at least find 4 pairs $(p, q) \in \mathcal{P}^* \times \mathcal{P}^*$ such that $r = q - p$.

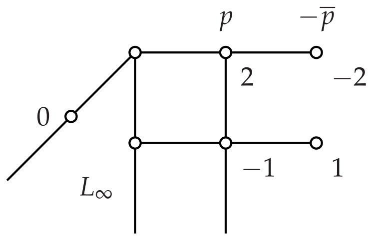
For every $p \in \mathcal{P}^*$, p and $-\overline{p}$ are collinear, and the third point $\overline{p} - p$ on the line joining them, belongs to L_∞. We will denote this point by p_∞. With this notation we have $\overline{p} = p + p_\infty$ when $p \in \mathcal{P}^*$.

The operation $\bar{\cdot}$ can now be expressed as an operation on the generalized quadrangle $\mathcal{Q}$ as follows : a point p on the line at infinity is mapped to $-p$, any other point is mapped to $-q$ where q is the third point on the line pp_∞ which projects p onto the line of infinity. The map $p \mapsto -\overline{p}$ is an automorphism of $\mathcal{Q}$ (more specifically, an *axial collineation*).

Lemma 3.1 *Let $p, q \in \mathcal{P}^*$. Then $(p+\overline{p}) \cdot q = p \cdot (q + \overline{q}) = \frac{1}{2}(p+\overline{p}) \cdot (q+\overline{q})$ and the value of this expression uniquely determines the configuration of the points p, q, $\overline{p}$ and $\overline{q}$ as summarized by the following table and the illustration below :*

$(p+\overline{p}) \cdot q$	$p \cdot q$	$-\overline{p} \cdot q$			
2	$4/3$	$-2/3$	$p = q$	$-\overline{p} \sim q$	$p_\infty = q_\infty$
1	$1/3$	$-2/3$	$p \not\sim q$	$-\overline{p} \sim q$	$p_\infty \neq q_\infty$
0	$1/3$	$1/3$	$p \not\sim q$	$-\overline{p} \not\sim q$	$p_\infty = q_\infty$
-1	$-2/3$	$1/3$	$p \sim q$	$-\overline{p} \not\sim q$	$p_\infty \neq q_\infty$
-2	$-2/3$	$4/3$	$p \sim q$	$-\overline{p} = q$	$p_\infty = q_\infty$

(3.2)



Proof : Because $\bar{\cdot}$ preserves the dot product on $\mathcal{P}$ and is an involution, we find

$$
\begin{aligned}
(p+\overline{p}) \cdot (q+\overline{q}) &= p \cdot q + p \cdot \overline{q} + \overline{p} \cdot q + \overline{p} \cdot \overline{q} \\
&= p \cdot q + p \cdot \overline{q} + p \cdot \overline{q} + p \cdot q = 2p \cdot (q + \overline{q}),
\end{aligned}
$$

and interchanging the roles of p and q this is also equal to $2(p+\overline{p}) \cdot q$.

Also

$$p_\infty \cdot q_\infty = (\overline{p} - p) \cdot (\overline{q} - q) = 2(p \cdot q) + 2(-\overline{p} \cdot q).$$

Because p, q and $-\overline{p}$ are points, the expressions $p \cdot q$ and $-\overline{p} \cdot q$ must have values $4/3$,$1/3$, or $-2/3$ and hence $(p + \overline{p}) \cdot q \in \{-2, -1, 0, 1, 2\}$. Because $p \sim -\overline{p}$ we find that $p = q$ if and only if $-\overline{p} \sim q$, and $-\overline{p} = q$ if and only if $p \sim q$. It follows that $p \cdot q$ and $-\overline{p} \cdot q$ are uniquely determined, and have values as indicated in (3.2), whenever $(p + \overline{p}) \cdot q = \pm 1$ or ± 2. This leaves the case $(p + \overline{p}) \cdot q = 0$, i.e., $p \cdot q = -\overline{p} \cdot q = 1/3$ or $-2/3$. In the second case however, we would obtain $p_\infty \cdot q_\infty = -8/3$, which is impossible. ∎

Lemma 3.2 *Let $r \in \Phi$ and $p, q \in \mathcal{P}^*$ be such that $r = q - p$. Then only the following two possibilities occur :*

1. *$r \cdot \overline{r} = 0$ if and only if $p_\infty \neq q_\infty$ if and only if q_∞ is a base point of r and p_∞ is a base point of $-r$. There are exactly 48 roots r of this type.*

2. *$r = \overline{r}$ if and only if $p_\infty = q_\infty$ if and only if r has no base point at infinity. There are exactly 24 roots r of this type.*

Proof : First note that $\overline{r} = \overline{q} - \overline{p} = (q_\infty + q) - (p_\infty + p) = r + (q_\infty - p_\infty)$, and hence $r = \overline{r}$ if and only if $p_\infty = q_\infty$.

Also $r \cdot \overline{r} = (q - p) \cdot (\overline{q} - \overline{p}) = q \cdot \overline{q} + p \cdot \overline{p} - p \cdot \overline{q} - q \cdot \overline{p} = 4/3 - p \cdot \overline{q} - \overline{p} \cdot q$ because p and $-\overline{p}$ are collinear (and so are q and $-\overline{q}$). Because $\bar{\cdot}$ preserves the dot product on $\mathbf{P}$ and is an involution, we have $p \cdot \overline{q} = \overline{p} \cdot q$ and hence $r \cdot \overline{r} = 4/3 - 2p \cdot \overline{q}$. As $\overline{q} \in -\mathcal{P}$, there are three possibilities :

Case 1. $p \cdot \overline{q} = 2/3$ and $r \cdot \overline{r} = 0$. We have $0 = r \cdot \overline{r} = r \cdot (r + (q_\infty - p_\infty))$ and hence $r \cdot q_\infty = r \cdot p_\infty - 2$. By (2.5) this is only possible when $r \cdot p_\infty = 1$ and $r \cdot q_\infty = -1$, hence p_∞ is a base point of $-r$ and q is a base point of r.

Case 2. $p \cdot \overline{q} = -1/3$ and then $r \cdot \overline{r} = 2$, i.e., $r = \overline{r}$. Let $z \in L_\infty$ (and hence $z = -\overline{z}$).

We have $z \cdot r = z \cdot \overline{r} = -\overline{z} \cdot \overline{r} = -z \cdot r$ and hence $z \cdot r = 0$. It follows that z is never a base point of r in this case.

Case 3. $p \cdot \overline{q} = -4/3$ and then $r \cdot \overline{r} = 4$, which is impossible by (2.5).

To count the number of roots of each type, we use the fact that there are exactly 16 roots with a given base point z. As base points of the same root are never collinear, there must be $3 \cdot 16 = 48$ roots of the first type, and hence the remaining 24 belong to the second type. ∎

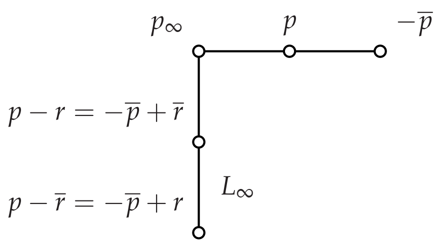
Lemma 3.3 *Let $r, s \in \Phi$ be such that $s \cdot \overline{s} = 0$ and $r \cdot s = \overline{r} \cdot s$. Then $r = \overline{r}$.*

Proof: Let $p, q \in \mathcal{P}^*$ be such that $r = q - p$. As in the proof of Lemma 3.2 we have $\overline{r} = r + (q_\infty - p_\infty)$ and hence $r = \overline{r}$ if and only if $p_\infty = q_\infty$.

From $r \cdot s = \overline{r} \cdot s$ we find $q_\infty \cdot s = p_\infty \cdot s$. By Lemma 3.2 s has a base point at infinity, say z. If z', z'' are the remaining points of L_∞, we have $s \cdot z + s \cdot z' + s \cdot z'' = 0$, and from $s \cdot z = -1$ we may conclude that all three terms must have a different value. Because $p_\infty, q_\infty \in L_\infty$, it follows that $p_\infty = q_\infty$. ∎

Lemma 3.4 *Let $r \in \Phi$ be such that $r \cdot \overline{r} = 0$. Then there exists a unique point $p \in \mathcal{P}^*$ such that $r + \overline{r} = p + \overline{p}$. Conversely, if $p \in \mathcal{P}^*$, then there is a unique pair $\{r, \overline{r}\} \subseteq \Phi$ such that $r + \overline{r} = p + \overline{p}$, and this pair satisfies $r \cdot \overline{r} = 0$.*

The points p ($-\overline{p}$, $p - r$, $p - \overline{r}$, respectively) are the (unique) common base point of $-r$ and $-\overline{r}$ (r and $\overline{r}$, r and $-\overline{r}$, $-r$ and $\overline{r}$, respectively). Both $p - r$ and $p - \overline{r}$ belong to L_∞. (See also the figure below.)



Proof: Let $r \in \Phi$ be such that $r \cdot \overline{r}$. By Lemma 3.2 r has a base point at infinity. Denote this base point by z and write $p \stackrel{\text{def}}{=} z + r$. Then

$$p + \overline{p} = z + r + \overline{z} + \overline{r} = z + r - z + \overline{r} = r + \overline{r}.$$

If $r + \overline{r} = p + \overline{p}$ we have

$$4 = (p + \overline{p})^2 = (r + \overline{r})^2 = r^2 + 2r \cdot \overline{r} + \overline{r}^2 = 4 + 2r \cdot \overline{r},$$

hence $r \cdot \overline{r} = 0$. Also

$$4 = (r + \overline{r})^2 = (r + \overline{r}) \cdot (p + \overline{p}) = p \cdot r + \overline{p} \cdot r + p \cdot \overline{r} + \overline{p} \cdot \overline{r},$$

and this can only be attained when each of the terms in the right hand side is equal to 1. Hence p is a common base point of $-r$ and $-\overline{r}$. Because $r \cdot \overline{r} = 0$ the roots $-r$ and $\overline{r}$ have exactly one base point in common (cf. Table 2.2, pg. 35), so p is unique.

There are exactly 24 points in $\mathcal{P}^*$, and exactly 24 pairs $\{r, \overline{r}\}$ with $r \cdot \overline{r} = 0$. This proves that every p corresponds to a unique pair $\{r, \overline{r}\}$. ∎

Lemma 3.5 *Let $r \in \Phi$, $x \in \mathbf{P}$. Then*

$$\overline{w_r(x)} = w_{\overline{r}}(\overline{x}). \tag{3.3}$$

If $r \cdot \overline{r} = 0$, then $w_r w_{\overline{r}} = w_{\overline{r}} w_r$ and

$$w_{r+\overline{r}}(x + \overline{x}) = w_r(w_{\overline{r}}(x + \overline{x})). \tag{3.4}$$

Proof: By definition

$$\overline{w_r(x)} = \overline{r - (r \cdot x)x} = \overline{r} - (r \cdot x)\overline{x} = \overline{r} - (\overline{r} \cdot \overline{x})\overline{x} = w_{\overline{r}}(\overline{x}),$$

using the fact that $\bar{\cdot}$ preserves the dot product on $\mathbf{P}$.

We have $(r + \overline{r})^2 = r^2 + \overline{r}^2 = 4$ and hence

$$w_{r+\overline{r}}(x + \overline{x}) = x + \overline{x} - \frac{1}{2}((r + \overline{r}) \cdot (x + \overline{x}))(r + \overline{r}). \tag{3.5}$$

Also

$$w_r(w_{\bar{r}}(x)) = w_r(x - (\bar{r}\cdot x)\bar{r}) = w_r(x) - (\bar{r}\cdot x)w_r(\bar{r}) = x - (r\cdot x)r - (\bar{r}\cdot x)\bar{r}.$$

Note that the right hand side of this expression remains the same when we interchange r and $\bar{r}$, hence $w_r w_{\bar{r}} = w_{\bar{r}} w_r$.

Likewise $w_r(w_{\bar{r}}(\bar{x})) = \bar{x} - (r\cdot\bar{x})r - (\bar{r}\cdot\bar{x})\bar{r}$, and hence

$$w_r(w_{\bar{r}}(x+\bar{x})) = x + \bar{x} - (r\cdot x)r - (\bar{r}\cdot x)\bar{r} - (r\cdot\bar{x})r - (\bar{r}\cdot\bar{x})\bar{r}. \tag{3.6}$$

Now, $r\cdot(x+\bar{x}) = \bar{r}\cdot(\overline{x+\bar{x}}) = \bar{r}\cdot(x+\bar{x})$. This proves that (3.5) and (3.6) are the same. ∎

From (3.1) the images of the fundamental roots of Φ are easily computed :

$$\overline{\pi}_0 = \pi_0, \quad \overline{\pi}_1 = \pi_5, \quad \overline{\pi}_2 = \pi_4, \quad \overline{\pi}_3 = \pi_3, \quad \overline{\pi}_4 = \pi_2, \quad \overline{\pi}_5 = \pi_1, \tag{3.7}$$

and hence the subspace $\mathbf{P}_F$ of $\mathbf{P}$ which is left invariant by $\bar{\cdot}$ is generated by the following vectors

$$\psi_1 \stackrel{\text{def}}{=} \pi_0, \quad \psi_2 \stackrel{\text{def}}{=} \pi_3, \quad \psi_3 \stackrel{\text{def}}{=} \frac{1}{2}(\pi_2+\pi_4), \quad \psi_4 \stackrel{\text{def}}{=} \frac{1}{2}(\pi_1+\pi_5), \tag{3.8}$$

and has dimension 4. The map $x \mapsto \frac{1}{2}(x+\bar{x})$ may serve as a projection of $\mathbf{P}$ onto $\mathbf{P}_F$.

Proposition 3.6 *The set*

$$\Phi_F \stackrel{\text{def}}{=} \{\frac{1}{2}(r+\bar{r}) \mid r \in \Phi\}$$

is a root system of type F_4 *with* $\Pi_F \stackrel{\text{def}}{=} \{\psi_1,\ldots,\psi_4\}$ *as a fundamental system of roots.*

Proof : By Lemma 3.2 either $r = \bar{r}$ or $r\cdot\bar{r} = 0$. In the first case $(r+\bar{r})/2 = r$, $w_{(r+\bar{r})/2} = w_r$ and then $w_r(\Phi_F) = \Phi_F$ by (3.3). If on the other hand $r\cdot\bar{r} = 0$, then $w_{(r+\bar{r})/2}(\Phi_F) = \Phi_F$, by (3.4).

Let $r, s \in \Phi$. We have

$$\langle (r+\overline{r})/2, (s+\overline{s})/2\rangle = 2\frac{\frac{1}{2}(r+\overline{r})\cdot\frac{1}{2}(s+\overline{s})}{\frac{1}{2}(r+\overline{r})\cdot\frac{1}{2}(r+\overline{r})} = 2\frac{(r+\overline{r})\cdot(s+\overline{s})}{(r+\overline{r})^2} = 4\frac{(r+\overline{r})\cdot s}{(r+\overline{r})^2}.$$

When $r = \overline{r}$ this is equal to $r \cdot s$ which is integral. When $r \cdot \overline{r} = 0$ we have $(r+\overline{r})^2 = 4$ and again this expression is an integer.

To prove that Φ_F is a root system, it remains to be proved that $(r+\overline{r})/2 = \lambda(s+\overline{s})/2$ for $r, s \in \Phi$ implies $\lambda = \pm 1$. By computing the length of both sides of this equation, we see that $\lambda = \pm 1$, $\lambda = \pm\sqrt{2}$ or $\lambda = \pm 1/\sqrt{2}$, and because both r and s can be expressed as linear combinations of the fundamental roots with coefficients that are integral, only the first two possibilities remain.

Now let $r \in \Phi$. We may express r as a linear combination $r = \lambda_0\pi_0 + \cdots + \lambda_5\pi_5$ of fundamental roots of Π. Then, by (3.7–3.8),

$$\begin{aligned}\frac{1}{2}(r+\overline{r}) &= \lambda_0\pi_0 + \frac{1}{2}(\lambda_1+\lambda_5)\pi_1 + \frac{1}{2}(\lambda_2+\lambda_4)\pi_2 + \lambda_3\pi_3 \\ &\qquad + \lambda_3\pi_3 + \frac{1}{2}(\lambda_2+\lambda_4)\pi_4 + \frac{1}{2}(\lambda_1+\lambda_5)\pi_5 \\ &= \lambda_0\psi_1 + \lambda_3\psi_2 + (\lambda_2+\lambda_4)\psi_3 + (\lambda_1+\lambda_5)\psi_4.\end{aligned}$$

This proves that every element of Φ_F can be written as a linear combination of elements of Π_F in which all coefficients are either all non-negative or all non-positive (and integral). This proves that Π_F is a fundamental system of roots for Φ_F.

We leave it to the reader to verify that the Cartan matrix, i.e., the matrix with entries $\langle\psi_i, \psi_j\rangle$, is the following :

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{3.9}$$

(Note that $\psi_3^2 = \psi_4^2 = 1$ and hence $\langle\psi_3, x\rangle = 2\psi_3 \cdot x$ and $\langle\psi_4, x\rangle = 2\psi_4 \cdot x$.) ■

The root system Φ_F can be partitioned into two sets of size 24 : the set of *short* roots Φ_S of elements of length 1 which correspond to the first case of Lemma

3.2, and the set of *long* roots Φ_L of elements of length $\sqrt{2}$, corresponding to the second case of that lemma. Lemma 3.4 provides a one-to-one correspondence between points p of $\mathcal{P}^*$ and short roots $(p+\overline{p})/2$ of Φ_S.

Note that Φ_F is not a subset of Φ. So, from now on, when ambiguity may arise, we will use the term *F-root* for elements of Φ_F and *E-root* for elements of Φ. The long (F-)roots are exactly those F-roots that are at the same time E-roots.

• When we use the qualifier 'long' or 'short' we will implicitely assume that we are talking about F-roots.

• Because not all F-roots have the same length, we must carefully distinguish between $\langle r,s\rangle$ and $r\cdot s$ when $r,s\in\Phi_F$. We have $\langle r,s\rangle = r\cdot s$ when $r\in\Phi_L$ and $\langle r,s\rangle = 2r\cdot s$ when $r\in\Phi_S$. The expression $\langle r,s\rangle$ is no longer symmetric in r and s.

Let s denote a short (F-)root. Let $u\in\Phi$ be such that $s=(u+\overline{u})/2$. Then we define

$$w'_s \overset{\text{def}}{=} w_u\cdot w_{\overline{u}} = w_{\overline{u}}\cdot w_u. \tag{3.10}$$

By (3.4) on $\mathbf{P}_F$ the element w'_s coincides with the reflection w_s in the hyperplane orthogonal to s. However, this is no longer true when we extend its action to the entire space $\mathbf{P}$.

• For example, let $u=\mathtt{3\overline{3}0000}$, $\overline{u}=\mathtt{00003\overline{3}}$ and $p=\mathtt{000003}$. We have $w_u(w_{\overline{u}}(p)) = \mathtt{000030}$ while $w_{u+\overline{u}}(p)=\frac{3}{2}\overline{\tfrac{3}{2}}\mathtt{00}\frac{3}{2}\frac{3}{2}$.

Define $W(\mathsf{F}_4)$ to be the subgroup of $W(\mathsf{E}_6)$ generated by all elements w_r with $r\in\Phi_L$ and all elements w'_s with $s\in\Phi_S$. When restricted to $\mathbf{P}_F$ the group $W(\mathsf{F}_4)$ acts like the *Weyl group* of the root system Φ_F. As an immediate consequence of Lemma 3.5 we have

$$\overline{w(x)} = w(\overline{x}), \qquad \text{for all } w\in W(\mathsf{F}_4),\, x\in\mathbf{P}. \tag{3.11}$$

It will be convenient to coordinatize $\mathbf{P}_F$, the space generated by Φ_F, by means of the following basis :

$$\psi_1+\psi_2+\psi_3,\quad \psi_2+\psi_3,\quad \psi_3,\quad \psi_1+2\psi_2+3\psi_3+2\psi_4. \tag{3.12}$$

We leave it to the reader to verify that this is an orthonormal basis and that with respect to this basis the F-roots have the following coordinates :

1. The coordinates of the 24 long roots are permutations of $(\pm 1, \pm 1, 0, 0)$.
2. There are 16 short roots with coordinates of the form $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$.
3. The 8 remaining short roots have coordinates which are permutations of $(\pm 1, 0, 0, 0)$.

We will denote coordinate quadruples by a shorthand notation of the form `1000`, $\mathtt{01\bar{1}0}$, `+--+`, where $\bar{1}$ stands for -1 (as before), `+` for $+1/2$ and `-` for $-1/2$.

• This is the representation of a root system of type F_4 which is most commonly used in the literature.

The fundamental F-roots have the following coordinates :

$$\psi_1 = \mathtt{1\bar{1}00}, \quad \psi_2 = \mathtt{01\bar{1}0}, \quad \psi_3 = \mathtt{0010}, \quad \psi_4 = \mathtt{---+}. \tag{3.13}$$

Let $r, s \in \Phi_F$. Using the coordinate representation above it is easily computed that the value of $\langle r, s\rangle$ is closely related to whether $r+s, r-s$ are F-roots and of what kind. We obtain the following results :

When $r, s \in \Phi_S$

$\langle r,s\rangle =$	if and only if
2	$r = s$
1	$r - s \in \Phi_S$
0	both $r-s, r+s \in \Phi_L$
-1	$r + s \in \Phi_S$
-2	$r = -s$

When $r, s \in \Phi_L$

$\langle r,s\rangle =$	if and only if
2	$r = s$
1	$r - s \in \Phi_L$
0	$r-s, r+s \in 2\Phi_S - \Phi$
-1	$r + s \in \Phi_L$
-2	$r = -s$

When $r \in \Phi_S, s \in \Phi_L$

$\langle r,s\rangle =$	if and only if
2	$r - s \in \Phi_S, s - 2r \in \Phi_L$
0	$r-s, r+s \notin \Phi$
-2	$r + s \in \Phi_S, s + 2r \in \Phi_L$

When $r \in \Phi_L, s \in \Phi_S$

$\langle r,s\rangle =$	if and only if
1	$r - s \in \Phi_S, r - 2s \in \Phi_L$
0	$r-s, r+s \notin \Phi$
-1	$r + s \in \Phi_S, r + 2s \in \Phi_L$

(3.14)

Lemma 3.7 *Let $r \in \Phi_L$, $s \in \Phi_S$, $p \in \mathcal{P}^*$, $u \in \Phi$ be such that $s = (p+\overline{p})/2 = (u+\overline{u})/2$. Then*

$$\langle r,s\rangle = p\cdot r = \overline{p}\cdot r = u\cdot r = \overline{u}\cdot r. \tag{3.15}$$

Let $t \in \Phi_S$, $q \in \mathcal{P}^*$, $v \in \Phi$ be such that $t = (q+\overline{q})/2 = (v+\overline{v})/2$. Then the following properties hold, depending on the value of and $\langle s,t\rangle = \frac{1}{2}(p+\overline{p})\cdot(q+\overline{q})$:

$\langle s,t\rangle =$		
2	$u\cdot q = \overline{u}\cdot q = 1$	$\{u\cdot v, \overline{u}\cdot v\} = \{0,2\}$
1	$\{u\cdot q, \overline{u}\cdot q\} = \{0,1\}$	$\{u\cdot v, \overline{u}\cdot v\} = \{0,1\}$
0	$u\cdot q = \overline{u}\cdot q = 0$	$\{u\cdot v, \overline{u}\cdot v\} = \{-1,1\}$
-1	$\{u\cdot q, \overline{u}\cdot q\} = \{-1,0\}$	$\{u\cdot v, \overline{u}\cdot v\} = \{-1,0\}$
-2	$u\cdot q = \overline{u}\cdot q = -1$	$\{u\cdot v, \overline{u}\cdot v\} = \{-2,0\}$

(3.16)

Proof : Because $r = \overline{r}$ we find $p\cdot r = \overline{p}\cdot\overline{r} = \overline{p}\cdot r$, and similar $u\cdot r = \overline{u}\cdot r$. Now $2\langle r,s\rangle = r\cdot(p+\overline{p}) = r\cdot p + r\cdot\overline{p} = 2r\cdot p$ and hence $\langle r,s\rangle = r\cdot p$. Similarly $\langle r,s\rangle = r\cdot u$.

Clearly $\langle s,t\rangle = 2s\cdot t = \frac{1}{2}(p+\overline{p})\cdot(q+\overline{q})$ and we may distinguish between 5 different cases according to (3.2). Similarly we have $\langle s,t\rangle = \frac{1}{2}(u+\overline{u})\cdot(q+\overline{q}) = u\cdot q + u\cdot\overline{q}$ where $u\cdot q, u\cdot\overline{q} \in \{-1,0,1\}$. This determines $\{u\cdot q, u\cdot\overline{q}\}$ uniquely, except when $\langle s,t\rangle = 0$. Note however that $\{u\cdot q,\overline{u}\cdot q\} = \{-1,1\}$ would imply that q is a common base point of either u and $-\overline{u}$, or of $\overline{u}$ and $-u$. Such common base points belong to L_∞ by Lemma 3.4, contradicting $q \in \mathcal{P}^*$.

Finally, $\langle s,t\rangle = \frac{1}{2}(u+\overline{u})\cdot(v+\overline{v}) = u\cdot v + u\cdot\overline{v}$, with $u\cdot v, u\cdot\overline{v} \in \{-2,-1,0,1,2\}$. Note that $u\cdot v \neq \overline{u}\cdot v$ by Lemma 3.3 and this determines $\{u\cdot v,\overline{u}\cdot v\}$ uniquely in each of the five cases. ∎

With the root system Φ_F we may associate the *dual* root system Φ_F^* of roots r^* of the form

$$r^* \stackrel{\text{def}}{=} \frac{2r}{r\cdot r} \qquad \text{with } r \in \Phi_F.$$

The element r^* is called the *co-root* corresponding to r. Note that $\langle r,s\rangle = r^*\cdot s$ and $\langle r,s^*\rangle = \langle s,r^*\rangle$.

• The notion of 'co-root' is a valid concept for every root system, but is trivial when all roots have the same size, e.g., in a root system of type E_6.

The *fundamental (F-)weights* for this root system are the following :

$$\begin{array}{lllllllll} \psi_1' & = & \texttt{1001} & = & 2\psi_1+ & 3\psi_2+ & 4\psi_3+ & 2\psi_4 \\ \psi_2' & = & \texttt{1102} & = & 3\psi_1+ & 6\psi_2+ & 8\psi_3+ & 4\psi_4 \\ \psi_3' & = & \texttt{+++}\frac{3}{2} & = & 2\psi_1+ & 4\psi_2+ & 6\psi_3+ & 3\psi_4 \\ \psi_4' & = & \texttt{0001} & = & \psi_1+ & 2\psi_2+ & 3\psi_3+ & 2\psi_4 \end{array} \tag{3.17}$$

We again leave it to the reader to verify that these elements of $\mathbf{P}_F$ satisfy $\psi_i^* \cdot \psi_j' = \delta_{ij}$ for $1 \leq i, j \leq 4$: the fundamental weights form a dual basis for the fundamental co-roots. Note that ψ_1' is a long root, ψ_4' is a short root, ψ_2' is the sum of two long roots with inner product 1 (`1001` and `0101`) and ψ_3' is the sum of two short roots with inner product $1/2$ (`++++` and `0001`).

Proposition 3.8 *The following are orbits of the group $W(\mathsf{F}_4)$:*

1. *the set of short roots Φ_S,*
2. *the set of long roots Φ_L,*
3. *five orbits of ordered pairs $(r, s) \in \Phi_S \times \Phi_S$, one for each possible value of $\langle r, s\rangle$,*
4. *three orbits of ordered pairs $(r, s) \in \Phi_S \times \Phi_L$, one for each possible value of $\langle r, s\rangle$,*
5. *five orbits of ordered pairs $(r, s) \in \Phi_L \times \Phi_L$, one for each possible value of $\langle r, s\rangle$,*
6. *the set of points of L_∞,*
7. *the set of points $\mathcal{P}^*$,*
8. *five orbits of ordered pairs $(p, q) \in \mathcal{P}^* \times \mathcal{P}^*$, one for each possible value of $(p + \overline{p}) \cdot (q + \overline{q})$, i.e., one for each row in (3.2).*
9. *an orbit of triples (p_1, p_2, p_3) with $p_i \in \mathcal{P}^*$ and $(p_i + \overline{p}_i) \cdot p_j = 1$ when $i \neq j$.*

Proof : 1. It is easily seen that the action of $w_{\texttt{1000}}$ on any root changes the sign of its first coordinate, and similar effects can be obtained on the other coordinates by the group elements $w_{\texttt{0100}}, \ldots, w_{\texttt{0001}}$. This proves that the 16 short roots with coordinates $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ belong to the same orbit. Now $w_{\texttt{++++}}(\texttt{+---}) = \texttt{1000}$, proving that also $\texttt{1000}$ belongs to this orbit, and a similar reasoning holds for the other short roots. The orbit cannot be larger, for every reflection must map roots onto roots and preserve their lengths.

2. It is easily computed that the action of $w_{\texttt{1}\bar{\texttt{1}}\texttt{00}}$ permutes the first two coordinates of a root. Using similar reflections, we see that every permutation of the coordinates is possible, and by the above, also every sign change. As a consequence, all long roots belong to the same orbit.

3–4. By the above it is sufficient to prove this for any fixed short root r. Choose $r = \texttt{++++}$. Also by the above, the subgroup of $W(\mathsf{F}_4)$ that fixes r contains all coordinate permutations. The group element $w_{\texttt{+-+-}}$ also fixes $\texttt{++++}$ and maps $\texttt{++-+}$ onto $\texttt{1000}$ and $\texttt{1}\bar{\texttt{1}}\texttt{00}$ onto $\bar{\texttt{1}}\texttt{100}$. Using this information the orbits can now easily be computed.

5. We leave this case to the reader. It can be proved in a similar way as above.

6. Because L_∞ consists of those points p for which $p = -\overline{p}$, it follows from (3.11) that L_∞ is left invariant by $W(\mathsf{F}_4)$. We therefore only need to prove that every point $z \in L_\infty$ can be mapped to any other point $z' \in L_\infty$. Now, let q denote any point of $\mathcal{P}^*$ not adjacent to z or z', then w'_{q-z} maps z onto z'.

7–8. These follow from the one-to-one correspondence between a short root s and the point p such that $s = (p + \overline{p})/2$.

9. Let $s_i = (p_i + \overline{p}_i)/2$ denote the short root corresponding to p_i. Note that $\langle s_i, s_j \rangle = 1$ when $i \neq j$. By the above, without loss of generality we may set $s_1 = \texttt{++++}$ and $s_2 = \texttt{1000}$. The subgroup of $W(\mathsf{F}_4)$ that fixes s_1 and s_2 contains all permutations of the last 3 coordinates. It is also easily computed that the only short roots s_3 such that $\langle s_1, s_3 \rangle = \langle s_2, s_3 \rangle = 1$ are $\texttt{+-++}$, $\texttt{++-+}$ and $\texttt{+++-}$. ∎

3.2 The Lie algebra of type F_4 over K

In what follows we fix a special element ∞ of $\mathbf{V}$ with the property $D(\infty) = 1$, and hence $\infty\infty^\# = 3$ and $D(\infty^\#) = 1$.

Define the linear operator $\bar{\cdot}$ as follows :

$$\begin{aligned} \overline{a} &\stackrel{\text{def}}{=} (a\infty^\#)\infty^\# - \infty \times a, &&\text{for } a \in \mathbf{V}, \\ \overline{\alpha} &\stackrel{\text{def}}{=} (\infty\alpha)\infty - \infty^\# \times \alpha, &&\text{for } \alpha \in \mathbf{V}^*, \end{aligned} \tag{3.18}$$

This operator can be regarded both as an element of $\mathrm{Hom}(\mathbf{V}, \mathbf{V}^*)$ and as an element of $\mathrm{Hom}(\mathbf{V}^*, \mathbf{V})$. The following proposition proves that $\bar{\cdot}$ is a *polarity* with respect to several algebraic operators.

Proposition 3.9 *Let $a, b \in \mathbf{V}$, $\alpha, \beta \in \mathbf{V}^*$. Then*

1. $\overline{\infty} = \infty^\#$ *and* $\overline{\infty^\#} = \infty$,
2. $a\overline{b} = b\overline{a}$ *and (dually)* $\overline{\alpha}\beta = \overline{\beta}\alpha$,
3. $\overline{\overline{a}} = a$ *and* $\overline{\overline{\alpha}} = \alpha$.
4. $\overline{a^\#} = \overline{a}^\#$ *and* $\overline{\alpha^\#} = \overline{\alpha}^\#$,
5. $\overline{a} \times \overline{b} = \overline{a \times b}$ *and (dually)* $\overline{\alpha \times \beta} = \overline{\alpha} \times \overline{\beta}$,

Proof : 1. We compute $\overline{\infty} = (\infty\infty^\#)\infty^\# - \infty \times \infty = 3\infty^\# - 2\infty^\# = \infty^\#$. The dual property follows in a similar way.

2. We compute $a\overline{b} = (b\infty^\#)(a\infty^\#) - \langle a, \infty, b\rangle$, an expression which is symmetric in a and b.

3. We have $\overline{\overline{a}} = (\infty\overline{a})\infty - \infty^\# \times \overline{a}$. Also $\infty^\# \times \overline{a} = (a\infty^\#)\infty^\# \times \infty^\# - (\infty \times a) \times \infty^\# = 2(a\infty^\#)\infty - a - (a\infty^\#)\infty$, using (2.28). From the above we know that $\infty\overline{a} = a\infty^\#$ and hence $\overline{\overline{a}} = a$.

4. By definition, we have

$$\overline{a}^{\#} = [(a\overline{\infty}^{\#})\infty^{\#} - \infty \times a]^{\#} = (a\infty^{\#})^2\infty - (a\infty^{\#})\infty^{\#} \times (\infty \times a) + (\infty \times a)^{\#}. \tag{3.19}$$

Now, by (2.27–2.28) we have

$$\begin{aligned} (\infty \times a)^{\#} &= (\infty a^{\#})\infty + (a\infty^{\#})a - a^{\#} \times \infty^{\#}, \\ (a\infty^{\#})\infty^{\#} \times (\infty \times a) &= (a\infty^{\#})a + (a\infty^{\#})(a\infty^{\#})\infty, \end{aligned}$$

and substituting these results in (3.19), we obtain $(\infty a^{\#})\infty - a^{\#} \times \infty^{\#}$ which is equal to $\overline{a^{\#}}$.

5. This is an immediate consequence of the previous property. ∎

In what follows we will prefer to write $\overline{\infty}$ instead of $\infty^{\#}$.

For any $A \in \mathrm{Hom}(\mathbf{V}, \mathbf{V})$ we define the matrix $\overline{A}$ to be the unique element of $\mathrm{Hom}(\mathbf{V}, \mathbf{V})$ satisfying

$$a\overline{A}\alpha = \overline{\alpha}A\overline{a}, \quad \text{for all } a \in \mathbf{V}, \alpha \in \mathbf{V}^*. \tag{3.20}$$

As an immediate consequence of this definition, we have the following properties :

Lemma 3.10 *Let $a \in \mathbf{V}$, $A, B \in \mathrm{Hom}(\mathbf{V}, \mathbf{V})$. Then*

$$\overline{aA} = \overline{A}\overline{a}, \quad \overline{AB} = \overline{B}\,\overline{A}, \quad \overline{[A, B]} = [\overline{B}, \overline{A}]. \tag{3.21}$$

Also, applying the definitions and using Proposition 3.9, we find

$$\overline{a * \alpha} = \overline{\alpha} * \overline{a}, \quad \text{for every } a \in \mathbf{V}, \alpha \in \mathbf{V}^*, \tag{3.22}$$

and, by Lemma 2.23,

$$\overline{A} \cdot \overline{B} = A \cdot B, \quad \text{for every } A, B \in \mathbf{L}. \tag{3.23}$$

Finally

$$\overline{\alpha \otimes a} = \overline{a} \otimes \overline{\alpha}, \quad \text{for every } a \in \mathbf{V}, \alpha \in \mathbf{V}^*. \tag{3.24}$$

We may also extend this definition to $\mathbf{V} \wedge \mathbf{V}$ (and dually to $\mathbf{V}^* \wedge \mathbf{V}^*$) : for $L \in \mathbf{V} \wedge \mathbf{V}$, $\overline{L} \in \mathbf{V}^* \wedge \mathbf{V}^*$ is the unique symmetric bilinear form satisfying $ab\overline{L} = L\overline{b}\overline{a}$ for all $a, b \in \mathbf{V}$. Likewise, if $P \in \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ then $\overline{P}$ is the unique element of $\mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$ satisfying $abc\overline{P} = P\,\overline{c}\overline{b}\overline{a}$. We may now easily prove

$$\begin{array}{lll} \overline{a \wedge b} = \overline{b} \wedge \overline{a}, & & \text{for } a, b \in \mathbf{V}, \\ \overline{B \times b} = \overline{b} \times \overline{B}, & \overline{A \times B} = (\overline{B} \times \overline{A})^*, & \text{for } b \in \mathbf{V},\ A, B \in \mathbf{L}, \\ \overline{a \wedge b \wedge c} = \overline{c} \wedge \overline{b} \wedge \overline{a}, & \overline{X(a,b,c)} = X(\overline{c}, \overline{b}, \overline{a}), & \text{for } a, b, c \in \mathbf{V}. \end{array} \tag{3.25}$$

Note that isotropic elements are always mapped to isotropic elements by $\bar{\cdot}$.

Proposition 3.11 *Let $A \in \mathbf{L}$. Then the following statements are equivalent*

1. $\infty A = 0$.

2. $A\overline{\infty} = 0$.

3. $aA\overline{a} = 0$ *for every* $a \in \mathbf{V}$.

If these are satisfied then also

$$aA\overline{b} + bA\overline{a} = 0, \quad \textit{for every } a, b \in \mathbf{V}, \tag{3.26}$$

or equivalently

$$\overline{A} = -A. \tag{3.27}$$

Proof : Substituting $a = \infty$ in (2.41) we obtain $\infty \times \infty A = -A\overline{\infty}$, hence $A\overline{\infty} = 0$ when $\infty A = 0$, and by duality also the converse holds.

We have $aA\overline{a} = (aA\overline{\infty})(a\overline{\infty}) - \langle aA, \infty, a \rangle$ and hence by (2.41) :

$$aA\overline{a} = (aA\overline{\infty})(a\overline{\infty}) + \infty A a^{\#}. \tag{3.28}$$

Hence $\infty A = 0$ and $A\overline{\infty} = 0$ imply $aA\overline{a} = 0$.

Conversely, assume $aA\overline{a} = 0$ for all a. Then also $(a+b)A(\overline{a+b}) - aA\overline{a} - bA\overline{b} = 0$ which implies (3.26). Substituting e_p for a in (3.28), with $p \in \mathcal{P}$, yields $(e_pA\overline{\infty})(e_p\overline{\infty}) = 0$ and hence $e_pA\overline{\infty} = 0$ for any p such that $e_p\overline{\infty} \neq 0$.

Now consider $p \in \mathcal{P}$ such that $e_p\overline{\infty} = 0$. As $D(\overline{\infty}) = 1$, there must be at least one line ijk of $\mathcal{Q}$ such that the i-th, j-th and k-th coordinate of $\overline{\infty}$ are nonzero, i.e., such that $e_i\overline{\infty}, e_j\overline{\infty}, e_k\overline{\infty} \neq 0$. We may also find at least one point on ijk, say i, which is not collinear to p. Then $e_i\overline{\infty} \neq 0$ and $e_i \times e_p = 0$. Now apply (3.26) to $a = e_i, b = e_p$.We find

$$\begin{aligned}
0 &= e_iA\overline{e_p} + e_pA\overline{e_i} \\
&= (e_iA\overline{\infty})(e_p\overline{\infty}) - e_iA(\infty \times e_p) + (e_pA\overline{\infty})(e_i\overline{\infty}) - e_pA(\infty \times e_i) \\
&= (e_pA\overline{\infty})(e_i\overline{\infty}) + \infty A(e_i \times e_p) = (e_pA\overline{\infty})(e_i\overline{\infty}).
\end{aligned}$$

It follows that $e_pA\overline{\infty} = 0$ also in this case.

This proves that $e_pA\overline{\infty} = 0$ for every basis element e_p of $\mathbf{V}$, and hence $A\overline{\infty} = 0$. ∎

We denote the set of all $A \in \mathbf{L}$ that satisfy the above proposition by $\mathbf{J}$. It is easily seen that $\mathbf{J}$ is a Lie algebra which is a subalgebra of $\mathbf{L}$.

Lemma 3.12 *Let* $A \in \mathbf{L}$. *Then* $\infty A = \infty\overline{A}$ *and hence* $A - \overline{A} \in \mathbf{J}$.

Proof : It is sufficient to prove this when A is of the form $a * \alpha$ with $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$, for $\mathbf{L}$ is generated as a vector space by elements of this form. We have

$$\begin{aligned}
\infty A = \infty(a * \alpha) &= (\infty\alpha)a - (a \times \infty) \times \alpha \\
&= (\infty\alpha)a + \overline{a} \times \alpha - (a\overline{\infty})\overline{\infty} \times \alpha \\
&= (\infty\alpha)a + \overline{a} \times \alpha + (a\overline{\infty})\overline{\alpha} - (a\overline{\infty})(\infty\alpha)\infty
\end{aligned}$$

Interchanging a with $\overline{\alpha}$, α with $\overline{a}$ and hence $a\overline{\infty}$ with $\infty\alpha$, leaves this expression invariant. Hence $\infty(a * \alpha) = \infty(\overline{\alpha} * \overline{a}) = \infty(\overline{a * \alpha}) = \infty\overline{A}$. ∎

Denote by $\mathbf{W}$ the subspace of elements $a \in \mathbf{V}$ such that $a\overline{\infty} = 0$. By Proposition 3.11 it follows that $\mathbf{V}A \leq \mathbf{W}$ whenever $A \in \mathbf{J}$. Hence the action of $\mathbf{J}$ on $\mathbf{V}$ leaves $\mathbf{W}$ invariant. In other words : $\mathbf{W}$ is a 26-dimensional module for $\mathbf{J}$.

Dually, the subspace of elements $\alpha \in \mathbf{V}^*$ such that $\infty\alpha = 0$ is also left invariant by $\mathbf{J}$. We denote this space by $\mathbf{W}^*$.

• The notation $\mathbf{W}^*$ is a little misleading, because it suggests that $\mathbf{W}^*$ can be obtained from $\mathbf{W}$ using the map $e_p \mapsto \eta_p$, which is not correct in general. It is however true for the particular choice of ∞ which we will introduce later.

• It is not sufficient that $a, b \in \mathbf{W}$ to have $a \times b \in \mathbf{W}^*$ or $a^\# \in \mathbf{W}^*$ or $a * \overline{b} \in \mathbf{J}$. However, $a \in \mathbf{W}$ always implies $\overline{a} \in \mathbf{W}^*$ (and conversely).

On $\mathbf{W}$ and $\mathbf{W}^*$ the definitions of $\overline{\cdot}$ simplify to

$$\begin{aligned} \overline{a} &\stackrel{\text{def}}{=} -\infty \times a, && \text{for } a \in \mathbf{W}, \\ \overline{\alpha} &\stackrel{\text{def}}{=} -\overline{\infty} \times \alpha, && \text{for } \alpha \in \mathbf{W}^*. \end{aligned} \tag{3.29}$$

• In [1] the space $\mathbf{W}$ is introduced in essentially the same way as ours, but the polarity $\overline{\cdot}$ is defined only on $\mathbf{W}$, as in (3.29) but without the minus sign.

Note that $a\overline{b} = b\overline{a} = -\langle a, b, \infty\rangle$ when $a, b \in \mathbf{W}$. Substituting $d = \infty$ in (2.24) yields

$$\begin{aligned} (a \times b) \times \overline{c} + (b \times c) \times \overline{a} + (c \times a) \times \overline{b} = \\ (a\overline{b})c + (b\overline{c})a + (c\overline{a})b - \langle a, b, c\rangle\infty, \end{aligned} \tag{3.30}$$

for all $a, b, c \in \mathbf{W}$. Similarly, setting $c = \infty$ in (2.26) yields

$$(a \times b) \times \overline{a} + a^\# \times \overline{b} = (a\overline{b})a - (\infty a^\#)b - (ba^\#)\infty. \tag{3.31}$$

Applying the same technique to (2.39) and (2.40), we obtain

$$a * \overline{a} = \infty * a^\# + (\infty a^\#)\mathbf{1}, \tag{3.32}$$

$$a * \overline{b} + b * \overline{a} = \infty * (a \times b) - (a\overline{b})\mathbf{1}, \tag{3.33}$$

for all $a, b \in \mathbf{W}$.

The following proposition lists some properties of isotropic elements that belong to $\mathbf{W}$.

Proposition 3.13 *Let e be an isotropic element that belongs to $\mathbf{W}$. Let $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$, $A \in \mathbf{L}$. Then*

1. $e\overline{e} = 0$.
2. $\overline{e} \times (e \times a) = (a\overline{e})e$, *and dually* $e \times (\overline{e} \times \alpha) = (e\alpha)\overline{e}$.
3. $e * \overline{e} = 0$.
4. $(e \times a) \wedge \overline{e} = (a * \overline{e}) \times e$ *and dually* $e \wedge (\overline{e} \times \alpha) = \overline{e} \times (e * \alpha)$.
5. $eA\overline{e} = 0$.
6. $eA * \overline{e} = e * A\overline{e}$.

Proof : Most properties are easy consequences of (3.29–3.33), Proposition 2.27 and Lemma 2.47 where ∞ is substituted for the appropriate variable. Note that $e * \overline{e} = 0$ implies $[e * \overline{e}, A] = 0$ yielding the last statement, by (2.61). ∎

Proposition 3.14 *Let E be an isotropic element of $\mathbf{L} - \{0\}$. Then we may find isotropic elements $e, f \in \mathbf{W}$ such that $e\overline{f} = 0$ and $E = e * \overline{f}$ (and then $\overline{E} = f * \overline{e}$). Let $A \in \mathbf{L}$. Then*

1. $e, f \in \mathbf{V}E$.
2. $\infty E = \infty\overline{E} = \overline{e} \times \overline{f}$ *and dually* $E\overline{\infty} = \overline{E}\overline{\infty} = e \times f$.
3. $eE = fE = e\overline{E} = f\overline{E} = 0$ *and* $E \cdot \overline{E} = 0$.
4. $\overline{e} \times E = e \wedge \infty E$, $\overline{f} \times E = 0$ *and dually* $E \times f = E\overline{\infty} \wedge \overline{f}$, $E \times e = 0$.
5. $E\overline{E} = \overline{E}E = E\overline{\infty} \otimes \infty E$. *In particular* $[E, \overline{E}] = 0$.
6. $[\overline{E}, [E, A]] = [E, [\overline{E}, A]] = -(E \cdot A)\overline{E} - (\overline{E} \cdot A)E + \infty EA * E\overline{\infty}$.
7. $\infty EA * E\overline{\infty} = \infty E * AE\overline{\infty}$.

Proof : Because $\dim \mathbf{V}E = 6$ the intersection of this isotropic subspace of $\mathbf{V}$ with the hyperplane $\mathbf{W}$ has dimension at least 5. Hence, we may always find $c \in \mathbf{V}$ such that $cE \in \mathbf{W}$ and with the extra condition that cE is not a multiple of ∞E. Because of this we can find $\gamma \in \mathbf{V}^*$ such that $\infty E\gamma = 0$ and $cE\gamma \neq 0$, and even $cE\gamma = 1$. The elements $e = cE$ and $f = \overline{E\gamma}$ satisfy the stated conditions.

1. By the above $e = cE \in \mathbf{V}E$ and $f = \overline{\gamma}\overline{E} = -\overline{\gamma}E \in \mathbf{V}E$.

2. We have $\infty E = \infty(e * \overline{f}) = (\infty\overline{f})e - (e \times \infty) \times \overline{f} = \overline{e} \times \overline{f}$. Interchanging e and f yields $\infty\overline{E} = \overline{e} \times \overline{f}$.

3. $eE = 0$ by Proposition 2.28–1 and dually $E\overline{f} = 0$ and hence $f\overline{E} = 0$. Applying Proposition 2.28–2 yields $E\overline{e} = -E(e \times \infty) = 0$, and dually $e\overline{E} = 0$. Also $E \cdot \overline{E} = (e * \overline{f}) \cdot \overline{E} = e\overline{E}\overline{f} = 0$.

4. The first equality is immediate from Proposition 3.13–4 with $a = f$ and $\alpha = \overline{f}$, the second follows from Proposition 2.28–3.

5. Let $x \in \mathbf{V}$. We have

$$\begin{aligned}
x E\overline{E} = xE(f * \overline{e}) &= (xE\overline{e})f - (xE \times f) \times \overline{e} \\
&= -x(E \times f) \times \overline{e} \\
&= -x(E\overline{\infty} \wedge \overline{f}) \times \overline{e} \\
&= -(x\overline{f})(E\overline{\infty} \times \overline{e} + (xE\overline{\infty})\overline{f} \times \overline{e} \\
&= -(x\overline{f})((e \times f) \times \overline{e}) + (xE\overline{\infty})(\infty E) \\
&= -(x\overline{f})(f\overline{e})e + (xE\overline{\infty})(\infty E) = (xE\overline{\infty})(\infty E)
\end{aligned}$$

Interchanging E and $\overline{E}$ leaves this expression unchanged, and hence $E\overline{E} = \overline{E}E$.

6. By (2.61) we have $[E, [\overline{E}, A]] = [E, [f * \overline{e}, A]] = [E, fA * \overline{e}] - [E, f * A\overline{e}]$. Now, by the above,

$$\begin{aligned}
&[E, fA * \overline{e}] \\
&\quad = fA * E\overline{e} - fAE * \overline{e} = -fAE * \overline{e}, \\
&\quad = -fA(e * \overline{f}) * \overline{e} \\
&\quad = -(fA\overline{e})e * \overline{e} + ((fA \times e) \times \overline{f}) * \overline{e}
\end{aligned}$$

$$\begin{aligned} &= -((fA\times e)\times\overline{e})*\overline{f} - (\overline{e}\times\overline{f})*(fA\times e) - \langle fA\times e,\overline{e},\overline{f}\rangle\mathbf{1}, \text{ by (2.40)} \\ &= -(fA\overline{e})e*\overline{f} - \infty E*(fA\times e) - (fA\overline{e})(e\overline{f})\mathbf{1}, \text{ by Proposition 3.13–2} \\ &= -(fA\overline{e})E - \infty E*(fA\times e) = -(\overline{E}\cdot A)E - \infty E*(fA\times e) \end{aligned}$$

Also $-[E, f*A\overline{e}] = -f*EA\overline{e} + fE*A\overline{e} = -f*EA\overline{e}$, and then

$$\begin{aligned} -f*EA\overline{e} &= -f*(e*\overline{f})A\overline{e} \\ &= -f*(e*A\overline{e})\overline{f} && \text{by (2.35),} \\ &= -f(e*A\overline{e})*\overline{f} \\ &= -f(eA*\overline{e})*\overline{f} && \text{by Proposition 3.13–6,} \\ &= -eA(f*\overline{e})*\overline{f} = -eA\overline{E}*\overline{f}. \end{aligned}$$

The value of this last expression can be computed by interchanging e and f in the expression for $-fAE*\overline{e}$ above, hence

$$-eA\overline{E}*\overline{f} = -(E\cdot A)\overline{E} - \infty E*(eA\times f).$$

Adding both expressions and using the fact that $-fA\times e - eA\times f = A(e\times f) = AE\overline{\infty}$ we obtain the stated value for $[E,[\overline{E},A]]$ Note that interchanging E and $\overline{E}$ does not alter the value of this expression.

7. Applying statement 6 of this proposition to $\overline{A}$ we find

$$[\overline{E},[E,A]] = \overline{[E,[\overline{E},\overline{A}]]} = -(\overline{E}\cdot A)E - (E\cdot A)\overline{E} + \infty\overline{E}*A\overline{E}\overline{\infty}$$

comparing this to the statement 6 and using statement 2 we obtain the listed result. ∎

In what follows we will choose a specific vector ∞ and hence a specific vector space $\mathbf{W}$ and algebra $\mathbf{J}$. Fix a line $\{z_1, z_2, z_3\}$ of Σ. Define

$$\infty \stackrel{\text{def}}{=} e_{z_1} + e_{z_2} + e_{z_3} \tag{3.34}$$

Note that $D(\infty) = 1$ and that $\overline{\infty} = \eta_{z_1} + \eta_{z_2} + \eta_{z_3}$.

• By Theorem 2.34 all elements $a\in\mathbf{V}$ with $D(a) = 1$ are known to lie in the same orbit of $\widehat{\mathsf{E}}_6(K)$. Our choice of ∞ is therefore not so special from an algebraic point of view. However, this particular choice will enable us to easily establish a connection with the root system of type F_4 which we have constructed in Section 3.1.

We will choose the line $\{z_1, z_2, z_3\}$ to coincide with the line L_∞ as defined in Section 3.1. This is possible because the definition of the involution $\bar{\cdot}$ depends on the root which is used in the coordinatization of $\mathbf{P}$. By an appropriate choice of this root we can make L_∞ coincide with any given line of $\mathcal{L}$ and of the spread Σ in particular.

With this choice of ∞, the polarity $\bar{\cdot}$ can now be expressed in terms of the involution with the same notation. It is easily computed that the following holds :

$$\overline{e_p} = \eta_{-\overline{p}}, \quad \overline{\eta_p} = e_{-\overline{p}}, \quad \text{for every } p \in \mathcal{P}. \tag{3.35}$$

Note that $e_p \in \mathbf{W}$ if and only if $p \in \mathcal{P}^*$. These elements serve as typical examples of isotropic elements of $\mathbf{W}$.

We will occasionally be interested in the 3-dimensional subspace $\mathbf{V}_\infty$ of $\mathbf{V}$ generated by the elements $e_{z_1}, e_{z_2}, e_{z_3}$, and the 2-dimensional space $\mathbf{W}_\infty = \mathbf{V}_\infty \cap \mathbf{W}$, generated by $e_{z_1} - e_{z_2}$ and $e_{z_1} - e_{z_3}$. We leave it to the reader to verify that the isotropic elements of $\mathbf{V}_\infty$ are those of the form ke_{z_i} with $k \in K$, $i = 1, 2, 3$, and hence that $\mathbf{W}_\infty$ does not contain any isotropic element, except 0.

Lemma 3.15 *Let ω denote the defining automorphism for the spread Σ. Then $\overline{x\omega} = \overline{x}\omega$ for all $x \in \mathbf{P}$.*

Let $p \in \mathcal{P}^*$ be a positive base point of $r \in \Phi$. Then $-\overline{p}$ is a positive base point of $-\overline{r}$. If moreover $r = \overline{r}$ then $p + \overline{p} + r \in \Phi_L$ and $-\overline{p}$ is a negative base point of $p + \overline{p} + r$.

Proof : First note that the map $\mathcal{P} \to \mathcal{P} : q \mapsto -\overline{q}$ maps lines of Σ to lines of Σ. Indeed L_∞ is left invariant, and any other line L is mapped onto the third line M of the grid determined by L because q and $-\overline{q}$ lie on a line that intersects L_∞ when $q \in \mathcal{P}^*$. Because Σ is a regular spread, M also belongs to Σ.

Hence, for $q \in \mathcal{P}^*$, the unique line of Σ through q must be mapped onto the unique line of Σ through $-\overline{q}$, i.e., $\{q, q\omega, q\omega^2\}$ is mapped onto $\{-\overline{q}, -\overline{q}\omega, -\overline{q}\omega^2\}$. Because the image of $q\omega$ must be collinear to it, this image can only be $-\overline{q}\omega$.

We have just proved that $\overline{q\omega} = \overline{q}\omega$ for all $q \in \mathcal{P}^*$, and the same property holds trivially for $q \in L_\infty$. Because $\mathcal{P}$ generates $\mathbf{P}$ as a vector space (and ω is a linear transformation) we find that $\overline{x\omega} = \overline{x}\omega$ for all $x \in \mathbf{P}$.

Now, let $p \in \mathcal{P}^*$ be a base point of $r \in \Phi$. Applying the involution $\bar{\cdot}$ to (2.13) and using the first part of this lemma easily proves that $-\overline{p}$ is a positive base point of $-\overline{r}$. If moreover $r = \overline{r}$, then $-\overline{p} \cdot (p + r) = -2/3 - \overline{p} \cdot r = -2/3 - p \cdot r = 1/3$, hence $-\overline{p} \not\perp (p + r)$ and then $p + \overline{p} + r$ is a root with base point $-\overline{p}$. Also $-\overline{p} \cdot (p + \overline{p} + r)\omega = -\overline{p} \cdot p\omega - \overline{p} \cdot \overline{p}\omega - \overline{p} \cdot r\omega = 1/3 + 2/3 - \overline{p} \cdot r\omega = 1 - p \cdot r\omega$. Hence $-\overline{p}$ is a negative base point of $p + \overline{p} + r$. ∎

Let $p \in \mathcal{P}$ be a positive base point of $r \in \Phi$. Then $E_r = e_{p+r} * \eta_p$. By (3.22) we have $\overline{E_r} = \overline{\eta_p} * \overline{e_{p+r}} = e_{-\overline{p}} * \eta_{-\overline{p}-\overline{r}} = \pm E_{\overline{r}}$. By Lemma 3.15 $-\overline{p}$ is a positive base point of $-\overline{r}$, hence $-\overline{p} - \overline{r}$ is a negative base point of $\overline{r}$. We find

$$\overline{E_r} \quad = \quad -E_{\overline{r}}, \qquad \text{for every } r \in \Phi. \tag{3.36}$$

A similar argument can be used to prove

$$\begin{array}{rclll} \overline{H_p} & = & H_{-\overline{p}}, & \text{for every } p \in \mathcal{P} \\ \overline{H_r} & = & H_{-\overline{r}}, & \text{for every } r \in \Phi \end{array} \tag{3.37}$$

Let $r \in \Phi$. Then $E_r \in \mathbf{J}$ if and only if $\infty E_r = 0$ if and only if no base point of r belongs to L_∞ if and only if $r \in \Phi_L$. (An element E_r of this type may serve as a typical example of an isotropic element of $\mathbf{J}$.) Likewise, $H_r \in \mathbf{J}$ if and only if $r \in \Phi_L$.

Let $s \in \Phi_S$ and write $s = (u + \overline{u})/2$ with $u \in \Phi_L$. Define

$$E_s \stackrel{\text{def}}{=} E_u + E_{\overline{u}} = E_u - \overline{E_u}, \tag{3.38}$$

then $E_s \in \mathbf{J}$ by Lemma 3.12. Note that s uniquely determines the pair $\{u, \overline{u}\}$ (by Lemma 3.4) and hence the definition of E_s is sound.

It is a consequence of Lemma 3.4 that $\infty E_u = \pm e_p$ and hence, by Proposition 3.14,

$$E_u E_{\overline{u}} = -\overline{e_p} \otimes e_p = -\eta_{-\overline{p}} \otimes e_p \tag{3.39}$$

and

$$\begin{aligned}[E_u,[E_{\overline{u}},A]] &= (E_u\cdot A)E_{\overline{u}} + (E_{\overline{u}}\cdot A)E_u + e_p A * \eta_{-\overline{p}} \\ &= (E_u\cdot A)E_{\overline{u}} + (E_{\overline{u}}\cdot A)E_u + e_p * A\eta_{-\overline{p}}, \quad \text{for } A\in \mathbf{L}\end{aligned} \tag{3.40}$$

Similarly, define $H_s \in \mathbf{J}$ as follows :

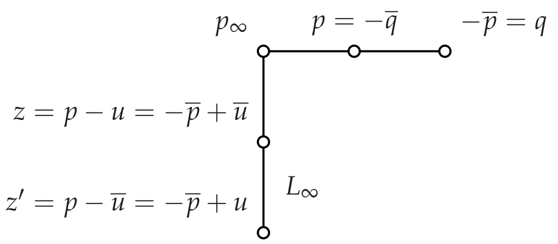
$$H_s \stackrel{\text{def}}{=} H_u + H_{\overline{u}} = H_u - \overline{H_u}. \tag{3.41}$$

If $s = (p+\overline{p})/2$, then $p+\overline{p} = u+\overline{u}$ implies $H_s = H_p - \overline{H_p} = H_p - H_{-\overline{p}}$.

Lemma 3.16 *Let $s\in\Phi_s$, $p\in\mathcal{P}^*$ such that $s=\frac{1}{2}(p+\overline{p})$. Let z,z' denote the points of $L_\infty - \{p_\infty\}$. Let $q\in\mathcal{P}$. Then*

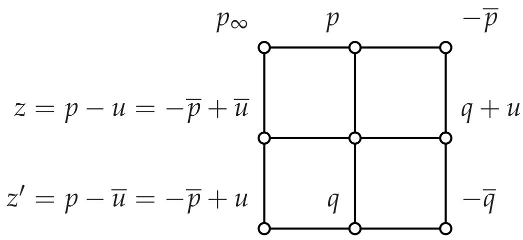
$$\begin{array}{llll} e_{-\overline{p}}E_s & = & \pm(e_z - e_{z'}), & \\ e_zE_s = -e_{z'}E_s & = & \pm e_p, & \\ e_qE_s & = & \pm e_{\overline{p}+\overline{q}}, & \textit{when } q\in\mathcal{P}^* \textit{ such that } \langle s,q\rangle = -1 \\ e_qE_s & = & 0, & \textit{otherwise.} \\ & & & \\ e_qH_s & = & \langle s,q\rangle e_q = 2(s\cdot q)e_q. & \end{array} \tag{3.42}$$

Proof : Let $u\in\Phi$ such that $s=\frac{1}{2}(u+\overline{u})$. We have $e_qE_s = e_qE_u + e_qE_{\overline{u}}$. By (2.45) $e_qE_u = \pm e_{q+u}$ when $q\cdot u = -1$, and $e_qE_u = 0$ when $q\cdot u \geq 0$. The value of $e_qE_{\overline{u}}$ is analogous. From Lemma 3.4 we know that the line at infinity consists of the points p_∞, $p-u$ and $p-\overline{u}$. We set $z = p-u$ and $z = p-\overline{u}$.



Firstly, when $q \in \mathcal{P}^*$ we may use (3.16) to distinguish between various cases. It follows that $e_q E_s = 0$ when $p = q$ or $p \sim q$. If $q = -\overline{p}$, then $e_q E_u = \pm e_{p-u}$ and $e_q E_{\overline{u}} = \pm e_{p-\overline{u}}$. (Refer to the illustration above.) Because $E_s \in \mathbf{J}$ we have $e_q E_s \overline{\infty} = 0$ and hence both terms must have opposite sign.

If on the other hand $p \sim q$ but $q \neq -\overline{p}$, then without loss of generality (we may always interchange u and $\overline{u}$) we may choose $q \cdot u = -1$ and then $q + u = \overline{p} + \overline{q}$ and hence $e_q E_s = \pm e_{\overline{p}+\overline{q}}$. (Refer to the illustration below.)



Secondly, consider $q \in L_\infty$. The base points of u and $\overline{u}$ on L_∞ are z and z', and in both cases $e_q E_s = \pm e_p$. It also follows that $e_{p_\infty} E_s = 0$. We have $0 = \infty E_s = (e_{p_\infty} + e_z + e_{z'})E_s = e_z E_s + e_{z'} E_s$ and hence $e_z E_s$ and $e_{z'} E_s$ have opposite sign.

Finally $e_q H_s = e_q H_u + e_q H_{\overline{u}} = (q \cdot u)e_q + (q \cdot \overline{u})e_q = (q \cdot 2s)e_q$. ∎

The last result of (3.42) can be generalized to all of Φ_F :

$$e_q H_r = \langle r, q \rangle e_q = (r^* \cdot q)e_q, \quad \text{for all } r \in \Phi_F, q \in \mathcal{P}. \tag{3.43}$$

• Because the operation $\langle \cdot, \cdot \rangle$ is not linear in the first argument, it is not always true that $H_r + H_s = H_{r+s}$ when $r, s, r+s \in \Phi_F$, as in the case of E_6. The elements H_r behave like co-roots of the root system, and not like ordinary roots.

Let us now consider the subalgebra $\mathbf{G}$ of diagonal elements of $\mathbf{J}$, i.e., $\mathbf{G} \stackrel{\text{def}}{=} \mathbf{H} \cap \mathbf{J}$.

Lemma 3.17 *$\mathbf{G}$ consists of those diagonal matrices A of $\mathbf{H}$ satisfying $A[z_1, z_1] = A[z_2, z_2] = A[z_3, z_3] = 0$. The dimension of $\mathbf{G}$ is 4 and $\{H_{\psi_1}, H_{\psi_2}, H_{\psi_3}, H_{\psi_4}\}$ is a basis for $\mathbf{G}$.*

Proof : Consider the linear transformation τ_∞ which maps $A \in \mathbf{H}$ to $\infty A \in \mathbf{V}$. The kernel of τ_∞ is $\mathbf{G}$. Because A is a diagonal matrix, we find $\tau_\infty(A) = \infty A = A[z_1, z_1]e_{z_1} + A[z_2, z_2]e_{z_2} + A[z_3, z_3]e_{z_3}$. Hence $A \in \mathbf{G}$ if and only if $A[z_1, z_1] = A[z_2, z_2] = A[z_3, z_3] = 0$.

By Proposition 2.17 we also have $A[z_1, z_1] + A[z_2, z_2] + A[z_3, z_3] = 0$, and hence the image of τ_∞ has dimension at most 2. It is not so difficult to find roots r, s such that

$$\begin{array}{lll} r \cdot z_1 = 1, & r \cdot z_2 = -1, & r \cdot z_3 = 0, \\ s \cdot z_1 = 1, & s \cdot z_2 = 0, & s \cdot z_3 = -1, \end{array}$$

and then $\infty H_r = e_{z_1} - e_{z_2}$ and $\infty H_s = e_{z_1} - e_{z_3}$ are clearly linearly independent. Hence $\dim \tau_\infty(\mathbf{H}) = 2$. As a consequence we find $\dim \mathbf{G} = 4$.

It is easily seen that $H_{\psi_i} \in \mathbf{G}$ for $i = 1, \ldots, 4$. It remains to be proved that they are linearly independent. The following matrix contains the values of $H_{\psi_i} \cdot H_{\pi_j}$ with $i = 1, \ldots, 4$ (row numbers) and $j = 0, \ldots, 3$ (column numbers).

$$\begin{pmatrix} 2 & 0 & 0 & -1 \\ -1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -2 \\ 0 & 2 & -1 & 0 \end{pmatrix}$$

This matrix can easily be computed from the Cartan matrix (2.10). Its determinant is -1, hence the $H_{\psi_i} \in \mathbf{G}$ are linearly independent. ∎

Theorem 3.18 *The algebra $\mathbf{J}$ has dimension 52 and can be written as a direct sum as follows :*

$$\mathbf{J} = \mathbf{G} \oplus \bigoplus_{r \in \Phi_L} KE_r \oplus \bigoplus_{s \in \Phi_S} KE_s. \tag{3.44}$$

Proof : It is easily seen that the sum in (3.44) is a direct sum and, using Lemma 3.17, that it has dimension 52. It is also a subspace of $\mathbf{J}$.

Now let τ_∞ be the linear transformation which maps $A \in \mathbf{L}$ to $\infty A \in \mathbf{V}$. Then $\mathbf{J}$ is the kernel of τ_∞. Let $p \in \mathcal{P}^*$, then there exists $r \in \Phi$ such that $\infty E_r = \pm e_p$. From the proof of Lemma 3.17 we also known that $\tau_\infty(\mathbf{H})$ is a 2-dimensional subspace of the space generated by the e_{zi}. Hence $\tau_\infty(\mathbf{L})$ has dimension 26 (and is in fact equal to $\mathbf{W}$). Therefore $\dim \mathbf{J} = \dim \mathbf{L} - \dim \mathbf{W} = 78 - 26 = 52$. ∎

Proposition 3.19 *Let $r, s \in \Phi_F$. Then*

1. $[E_r, E_{-r}] = H_r$.
2. $[E_r, E_s] = 2E_{r+s}$ *or* $-2E_{r+s}$*, when both* $r - s, r + s \in \Phi_F$.
3. $[E_r, E_s] = E_{r+s}$ *or* $-E_{r+s}$*, when* $r + s \in \Phi_F$ *but* $r - s \notin \Phi_F$.
4. $[E_r, E_s] = 0$ *when* $\langle r, s\rangle \geq 0$.
5. $[E_r, H_s] = \langle s, r\rangle E_r$.
6. $[H_r, H_s] = 0$.

Proof : We distinguish four cases, according to whether r, s are short or long roots. The last statement of this proposition is a consequence of the fact that H_r, H_s are diagonal matrices.

Case 1. If both $r, s \in \Phi_L$ nothing remains to be proved, for in that case the proposition is equivalent to Proposition 2.22.

Case 2. Assume $r \in \Phi_L$, $s \in \Phi_S$ and consider $v \in \Phi$ such that $s = (v + \overline{v})/2$. Then

$$[E_r, E_s] = [E_r, E_v] + [E_r, E_{\overline{v}}] = [E_r, E_v] + [\overline{E_r}, \overline{E_v}] = [E_r, E_v] - \overline{[E_r, E_v]},$$

because $r = \overline{r}$. By Proposition 2.22, $[E_r, E_v] = 0$ unless $r \cdot v = -1$ in which case it is equal to $\pm E_{r+v}$. Because $r + v \neq \overline{r} + \overline{v}$, it follows that $[E_r, E_s] = 0$ or

$[E_r, E_s] = \pm E_{r+s}$ (with $r + s \in \Phi_S$) when $r \cdot v = -1$. By (3.15) $r \cdot s = r \cdot v$, hence $[E_r, E_s] = \pm E_{r+s}$ if and only if $r \cdot s = -1$, which is equivalent to $r + s \in \Phi_F$, by (3.14).

Similarly, $[E_r, H_s] = [E_r, H_v + H_{\overline{v}}] = ((v + \overline{v}) \cdot r)E_r = (2s \cdot r)E_r = \langle s, r\rangle E_r$.

Case 3. If $r \in \Phi_S$ and $s \in \Phi_L$ then we may apply the previous case with the roles of r and s interchanged, except for statement 5. Now, let $u \in \Phi$ such that $r = (u + \overline{u})/2$. Then $[E_r, H_s] = [E_u + E_{\overline{u}}, H_s] = (s \cdot u)E_u + (s \cdot \overline{u})E_{\overline{u}} = (s \cdot u)E_r$, because $s \cdot u = \overline{s} \cdot \overline{u} = s \cdot u$.

Case 4. Assume $r, s \in \Phi_S$ and define $u, v \in \Phi$ as before. We have

$$[E_r, E_s] = [E_u, E_v] - \overline{[E_u, E_v]} + [E_{\overline{u}}, E_v] - \overline{[E_{\overline{u}}, E_v]}. \tag{3.45}$$

Note that this expression is non-zero if and only if $u \cdot v < 0$ or $\overline{u} \cdot v < 0$. By (3.16) this is the case if and only if $\langle r, s\rangle \leq 0$.

If $\langle r, s\rangle = -2$, then we may without loss of generality assume that $u = -v$ (otherwise we interchange the roles of u and $\overline{u}$) and then $[E_u, E_v] = H_u - \overline{H_u} = H_r$ by (3.41).

If $\langle r, s\rangle = 0$ or -1 we may without loss of generality assume $u \cdot v = -1$ and $\overline{u} \cdot v \geq 0$. We obtain $[E_u, E_v] = \pm E_{u+v}$, $[E_{\overline{u}}, E_v] = 0$ and hence $[E_r, E_s] = \pm(E_{u+v} + E_{\overline{u+v}})$.

When $\langle r, s\rangle = 0$, $r + s$ is a long root, and therefore $u + v = \overline{u + v}$. Indeed $8 = 4(r+s)^2 = (u + v + \overline{u + v})^2 = (u + v)^2 + 2(u + v) \cdot (\overline{u + v}) + (\overline{u + v})^2$ can only be satisfied when $(u + v) \cdot (\overline{u + v}) = 2$. Then also $u + v = \overline{u + v} = r + s$ and $[E_r, E_s] = \pm 2E_{r+s}$. When $\langle r, s\rangle = -1$, $r + s$ is a short root and $[E_r, E_s] = \pm E_{r+s}$.

Finally, $[E_r, H_s] = [E_u + E_{\overline{u}}, H_v + H_{\overline{v}}] = (u \cdot v + u \cdot \overline{v})E_u + (\overline{u} \cdot v + \overline{u} \cdot \overline{v})E_{\overline{u}} = (2r \cdot s)E_r = \langle s, r\rangle E_r$. ∎

• This proposition 'almost' provides us with a Chevalley basis for $\mathbf{J}$. To obtain a true Chevalley basis we use the same technique as was explained in the note after Proposition 2.22.

• Compared to Proposition 2.22 we need to make an extra distinction when $r + s \in$

Φ_F. The resulting coefficient in the right hand side of $[E_r, E_s]$ is now either ± 2 or ± 1 depending on whether $r - s$ is also a root, or not. By (3.14), if $r - s, r + s$ both belong to Φ_F, then both r, s must be short roots.

Lemma 3.20 *Let $p, q \in \mathcal{P}^*$ such that $p \not\perp q$ and $p_\infty = q_\infty$. Then*

$$e_p * \overline{e_q} = -e_q * \overline{e_p} = \pm E_{r+s}, \tag{3.46}$$

with $r = (p + \overline{p})/2$, $s = (q + \overline{q})/2$.

Proof : We have $e_p \times e_q = 0$ and $e_p \overline{e_q} = 0$, hence $e_p * \overline{e_q} = -e_q * \overline{e_p}$ by (3.33). Also $p \not\perp -\overline{q}$ and therefore $u = p + \overline{q}$ is a root. By definition $E_u = E_{p+\overline{q}} = \pm e_p * \eta_{-\overline{q}} = \pm e_p * \overline{e_q}$. Also $\overline{p} - p = p + \infty = q_\infty = \overline{q} - q$ and therefore $q + \overline{p} = \overline{p} + q$. Hence $r + s = (p + q + \overline{p} + \overline{q})/2 = p + \overline{q}$. ∎

The following proposition establishes the behaviour of the dot product on $\mathbf{J}$.

Proposition 3.21 *Let $r, s \in \Phi_F$, $H \in \mathbf{G}$. Then*

$$\begin{aligned} E_r \cdot E_s &= \begin{cases} -2 & \text{when } r = -s \in \Phi_S \\ -1 & \text{when } r = -s \in \Phi_L \\ 0 & \text{otherwise.} \end{cases} \\ E_r \cdot H = H \cdot E_r &= 0 \\ H_r \cdot H_s &= r^* \cdot s^* \end{aligned} \tag{3.47}$$

Proof : We will only consider the special case where both r, s are short roots and leave the other cases to the reader. Let $u, v \in \Phi$ be such that $r = (u + \overline{u})/2$, $s = (v + \overline{v})/2$. We have $E_r \cdot E_s = E_u \cdot E_v + E_{\overline{u}} \cdot E_v + E_u \cdot E_{\overline{v}} + E_{\overline{u}} \cdot E_{\overline{v}} = 2E_u \cdot E_v + 2E_{\overline{u}} \cdot E_v$, by (3.23). This is nonzero only when either $u = -v$ or $u = -\overline{v}$, in which case its value is -2. This happens exactly when $r = -s$.

Similarly $H_r \cdot H_s$ evaluates to $(u + \overline{u}) \cdot (v + \overline{v}) = 4r \cdot s$ and $E_r \cdot H = E_u \cdot H + E_{\overline{u}} \cdot H = 0$. ∎

• When char $K = 2$ the dot product on $\mathbf{J}$ is no longer nondegenerate as it was on $\mathbf{L}$. In fact, in that case $E_s \cdot \mathbf{J} = 0$ and $H_s \cdot \mathbf{J} = 0$ whenever s is a short root.

3.3 Automorphisms of W

An automorphism g of $\mathbf{V}$ will be called *an automorphism of* $\mathbf{W}$ if and only if it satisfies $\infty^g = \infty$. As an immediate consequence g will then satisfy

$$\overline{a}^g = \overline{a^g}, \quad \overline{A}^g = \overline{A^g}, \quad \overline{\alpha}^g = \overline{\alpha^g}, \tag{3.48}$$

and similar equalities for all other operations on the various $\mathbf{L}$-modules which were introduced. Also note that $\mathbf{W}^g = \mathbf{W}$ and $\mathbf{J}^g = \mathbf{J}$.

• For our particular choice of ∞, all field automorphisms of $\mathbf{V}$ are also (field) automorphisms of $\mathbf{W}$.

If E is an isotropic element of $\mathbf{J}$, then also $x(E)$ is an automorphism of $\mathbf{W}$. This is however not true when $E \in \mathbf{L} - \mathbf{J}$. The following lemma shows how an isotropic element of this type can still be used to construct an automorphism of $\mathbf{W}$:

Lemma 3.22 *Let E be an isotropic element of* $\mathbf{L}$. *Define*

$$x'(E) = x'(-\overline{E}) \stackrel{\text{def}}{=} x(E)x(-\overline{E}) = x(-\overline{E})x(E). \tag{3.49}$$

Then $x'(E)$ is an automorphism of $\mathbf{W}$ *and the following properties hold (with $S = E - \overline{E}$) :*

$$\begin{aligned} a^{x'(E)} &= a - aS - aE\overline{E} = a - aS - (aE\overline{\infty})\infty E, \\ \alpha^{x'(E)} &= \alpha + S\alpha - \overline{E}E\alpha = \alpha + S\alpha - (\infty E\alpha)E\overline{\infty}, \\ A^{x'(E)} &= A + [S, A] - (E \cdot A)E - (\overline{E} \cdot A)\overline{E} - [\overline{E}, [E, A]] \\ &= A + [S, A] - (S \cdot A)S - \infty E A * E\overline{\infty}, \end{aligned} \tag{3.50}$$

for all $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$, $A \in \mathbf{L}$.

Proof : Note that $[E, -\overline{E}] = 0$ by Proposition 3.14–5, and hence $x(E)$ and $x(-\overline{E})$ commute, by (2.97). This proves that (3.49) is sound.

We have $\infty^{x'(E)} = \infty^{x(E)x(-\overline{E})} = \infty - \infty E + \infty\overline{E} - \infty E\overline{E}$. Because $\infty E = \infty\overline{E}$ we find $\infty E\overline{E} = \infty(\overline{E})^2$ and this is zero because $\overline{E}$ is isotropic. This proves $\infty^{x'(E)} = \infty$ and therefore $x'(E)$ is an automorphism of $\mathbf{W}$.

Let $A \in \mathbf{L}$. By (2.84) we find

$$\begin{aligned}
A^{x'(E)} &= A^{x(E)x(-\overline{E})} \\
&= [A + [E, A] - (E \cdot A)E]^{x(-\overline{E})} \\
&= A + [E, A] - (E \cdot A)E - [\overline{E}, A] - [\overline{E}, [E, A]] + (E \cdot A)[\overline{E}, E] \\
&\qquad - (\overline{E} \cdot A)\overline{E} - (\overline{E} \cdot [E, A])\overline{E} + (E \cdot A)(\overline{E} \cdot E)\overline{E} \\
&= A + [E, A] - [\overline{E}, A] - [\overline{E}, [E, A]] - (E \cdot A)E - (\overline{E} \cdot A)\overline{E} \\
&= A + [E, A] - [\overline{E}, A] + (E \cdot A)\overline{E} + (\overline{E} \cdot A)E - \infty E A * E\overline{\infty}
\end{aligned}$$

using Proposition 3.14. It can easily be verified that this is the result listed in (3.50).

We leave it to the reader to compute the images of $a \in \mathbf{V}$ and $\alpha \in \mathbf{V}^*$ in a similar way. ∎

The elements $x'(E)$ are only of interest when $E \neq \overline{E}$, for otherwise $x'(E)$ is the identity.

- When char $K \neq 2$ then (3.50) can be further simplified to

$$\begin{aligned}
a^{x'(E)} &= a - aS + \tfrac{1}{2}aS^2, \\
\alpha^{x'(E)} &= \alpha + S\alpha + \tfrac{1}{2}S^2\alpha, \\
A^{x'(E)} &= A + [S, A] + \tfrac{1}{2}[S, [S, A]].
\end{aligned}$$

The group generated by all automorphisms of the form $x(E)$ where E is an isotropic element of $\mathbf{L}$ such that $E = \overline{E}$, together with all elements $x'(E)$ as defined in Lemma 3.22, will be denoted by $\widehat{\mathsf{F}}_4(K)$ and is called a *Chevalley group* of type F_4. It is a subgroup of $\widehat{\mathsf{E}}_6(K)$ and all its elements are automorphisms of $\mathbf{W}$.

- In [1] $\widehat{\mathsf{F}}_4(K)$ is defined as the subgroup of $\widehat{\mathsf{E}}_6(K)$ of isometries of the bilinear form $[a, b] \mapsto \langle \infty, a, b \rangle$, which is essentially the same as our notion of automorphism of $\mathbf{W}$.

When $r \in \Phi_L$, $k \in K$, then $x_r(k) \in \widehat{\mathsf{F}}_4(K)$. If s is a short root, then write $s = (u + \overline{u})/2 = (p + \overline{p})/2$ with $u \in \Phi$, $p \in \mathcal{P}^*$ and define

$$x_s(k) \stackrel{\text{def}}{=} x'(kE_u) = x_u(k)x_{\overline{u}}(k) \tag{3.51}$$

By Lemma 3.22 $x_s(k) \in \widehat{\mathsf{F}}_4(K)$. Also $\infty E_u = \pm e_p$, so (3.50) translates to

$$\begin{array}{rcl} a^{x_s(k)} & = & a - kaE_s - k^2(a\eta_{-\overline{p}})e_p, \\ \alpha^{x_s(k)} & = & \alpha + kE_s\alpha - k^2(e_p\alpha)\eta_{-\overline{p}}, \\ A^{x_s(k)} & = & A + k[E_s, A] - k^2(E_s \cdot A)E_s - k^2 e_p A * \eta_{-\overline{p}}, \end{array} \tag{3.52}$$

for all $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$, $A \in \mathbf{L}$. This proves that the definition of $x_s(k)$ is independent of the choice of u.

• The elements $x_s(k)$ correspond to those given in [8], except that as before, for certain s we should use $x_s(-k)$ instead of $x_s(k)$.

When r is a long root, the action of $x_r(k)$ on $\mathbf{V}$ is as in (2.85). When s is a short root, say $s = (p + \overline{p})/2$ with $p \in \mathcal{P}^*$, then from (3.42) and (3.52) we obtain

$$\begin{array}{rcll} e_{-\overline{p}}^{x_s(k)} & = & e_{-\overline{p}} \pm k(e_z - e_{z'}) - k^2 e_p, & \\ e_z^{x_s(k)} & = & e_z \pm k e_p, & \\ e_{z'}^{x_s(k)} & = & e_{z'} \mp k e_p, & \text{(sign opposite to the previous case)}, \\ e_q^{x_s(k)} & = & e_q \pm k e_{\overline{p}+\overline{q}}, & \text{for } q \in \mathcal{P}^* \text{ such that } \langle s, q\rangle = -1, \\ e_q^{x_s(k)} & = & e_q, & \text{for all other } q \in \mathcal{P}, \end{array} \tag{3.53}$$

where z, z' are the points of L_∞ different from p_∞.

Lemma 3.23 *Let $r \in \Phi_L$, $s \in \Phi_S$. Then*

$$E_r^{x_s(k)} = \begin{cases} E_r \pm kE_{r+s} + k^2 E_{r+2s}, & \textit{when } \langle r, s\rangle = -1, \\ E_r, & \textit{otherwise.} \end{cases} \tag{3.54}$$

Proof : The action of $x_s(k)$ on E_r with $r \in \Phi_L$ can be computed from Proposition 3.19 and (3.52). Consider $p \in \mathcal{P}^*$ such that $s = (p + \overline{p})/2$. We compute $e_p E_r * \eta_{-\overline{p}}$. Note that $e_p E_r \neq 0$ only if $p \cdot r = -1$, i.e., when $\langle r, s\rangle = -1$, and then $e_p E_r = \pm e_{p+r}$. Hence $e_p E_r * \eta_{-\overline{p}} = \pm E_{r+p+\overline{p}} = \pm E_{r+2s}$.

To see that the sign of this result must be negative, note that p is a positive base point of r if and only if $-\overline{p}$ is a negative base point of $p + \overline{p} + r$ by Lemma 3.15. ∎

Lemma 3.24 *Let* E, F *be isotropic elements of* $\mathbf{L}$ *such that* $E \cdot F = -1$ *and* $[E, \overline{F}] = 0$. *Define*

$$n'(E,F) \stackrel{\text{def}}{=} x'(E)x'(F)x'(E) \tag{3.55}$$

Then $n'(E,F) = n'(F,E) = n'(-\overline{E}, -\overline{F}) = n'(-\overline{F}, -\overline{E})$. *Also*

$$n'(E,F) = n(E,F)n(-\overline{E},-\overline{F}) = n(-\overline{E},-\overline{F})n(E,F) \tag{3.56}$$

and

$$E^{n'(E,F)} = F, \quad \overline{E}^{n'(E,F)} = \overline{F}, \quad F^{n'(E,F)} = E, \quad \overline{F}^{n'(E,F)} = \overline{E} \tag{3.57}$$

Proof : Note that $[E, -\overline{E}] = [E, -\overline{F}] = [F, -\overline{E}]$ and hence by (2.97) both $x(E)$ and $x(F)$ commute with both $x(-\overline{E})$ and $x(-\overline{F})$. Hence

$$\begin{aligned} n'(E,F) = x'(E)x'(F)x'(E) &= x(E)x(-\overline{E})x(F)x(-\overline{F})x(E)x(-\overline{E}) \\ &= x(E)x(F)x(-\overline{E})x(E)x(-\overline{F})x(-\overline{E}) \\ &= x(E)x(F)x(E)x(-\overline{E})x(-\overline{F})x(-\overline{E}) \\ &= n(E,F)n(-\overline{E},-\overline{F}). \end{aligned}$$

Because $x'(E) = x'(-\overline{E})$ and $x'(F) = x'(-\overline{F})$, we have $n'(E,F) = n'(-\overline{E},-\overline{F})$ and therefore $n(E,F)n(-\overline{E},-\overline{F}), = n(-\overline{E},-\overline{F})n(E,F)$. This expression is left unchanged when E and F are interchanged, because $n(E,F) = n(F,E)$ and $n(-\overline{E},-\overline{F}) = n(-\overline{F},-\overline{E})$. It follows that $n'(E,F) = n'(F,E)$.

By (2.89) we find

$$E^{n'(E,F)} = F^{n(-\overline{F},-\overline{E})} = F^{x(-\overline{F})x(-\overline{E})x(-\overline{F})}$$

and this is equal to F because $[F, -\overline{F}] = [F, -\overline{E}] = 0$. This yields the first identity of (3.57). The other identities can be obtained from the first by interchanging E and F, or substituting $-\overline{E}$ for E and $-\overline{F}$ for F. ∎

Consider a short root s with $s = (u + \overline{u})/2$ as usual. Note that $E = kE_u$ and $F = k^{-1}E_{-u}$ satisfy the conditions of Lemma 3.24, and then

$$n_s(k) \stackrel{\text{def}}{=} n(kE_u, k^{-1}E_{-u}) = x_s(k)x_{-s}(k^{-1})x_s(k) = n_u(k)n_{\overline{u}}(k) \tag{3.58}$$

is well defined (and belongs to $\widehat{\mathsf{F}}_4(K)$). Also of interest are the following group elements, for $k \in K - \{0\}$, $s \in \Phi_S$:

$$h_s(k) \stackrel{\text{def}}{=} n_s(-1)n_s(k). \tag{3.59}$$

Using (2.33) we may now easily compute the following identities

$$\begin{array}{lllllll} e_p^{n_s(k)} & = & \pm k^{-\langle s,p\rangle} e_{w_s'(p)}, & \quad & e_p^{h_s(k)} & = & k^{\langle s,p\rangle} e_p, \\ \eta_p^{n_s(k)} & = & \pm k^{\langle s,p\rangle} \eta_{w_s'(p)}, & & \eta_p^{h_s(k)} & = & k^{-\langle s,p\rangle} \eta_p, \\ E_r^{n_s(k)} & = & k^{-\langle s,r\rangle} E_{w_s'(r)}, & & E_r^{h_s(k)} & = & k^{\langle s,r\rangle} E_r, \\ H_p^{n_s(k)} & = & H_{w_s'(p)}, & & H_p^{h_s(k)} & = & H_p, \\ H_r^{n_s(k)} & = & H_{w_s'(r)}, & & H_r^{h_s(k)} & = & H_r. \end{array} \tag{3.60}$$

for all $k \in K$, $r \in \Phi_F \cup \Phi$, $s \in \Phi_S$, $p \in \mathcal{P}$.

Again this shows that the element $n_s(k)$ acts on the 1-dimensional subspace Ke_p of **V** (respectively $KE_r \leq$ **L**) in the same way as the Weyl group element w_s' acts on the points of $\mathcal{P}$ (respectively the roots of Φ). Hence $W(\mathsf{F}_4)$ is a quotient group of the subgroup N' of $\widehat{\mathsf{E}}_6(K)$ generated by all $n_r(k)$ with $r \in \Phi_L$, and all $n_s(k)$ with $s \in \Phi_S$. The group elements $h_r(k)$ and $h_s(k)$ belong to the kernel of the corresponding homomorphism.

In the remainder of this section we will establish various transitivity properties of the group $\widehat{\mathsf{F}}_4(K)$ using a similar reduction process as in Section 2.4. This time we will use elements of the form $x_r(k)$ and $x_s(k)$ with $r \in \Phi_L$, $s \in \Phi_S$ and $k \in K$.

Theorem 3.25 *The orbits of $\widehat{\mathsf{F}}_4(K)$ on isotropic elements of* **V** *are as follows :*

1. *The trivial orbit $\{0\}$.*

2. *The set of all isotropic elements that belong to* **W**. *A typical representative of this orbit is given by e_p with $p \in \mathcal{P}^*$.*

3. *The set of all isotropic elements that belong to* **V** $-$ **W**. *A typical representative of this orbit is given by e_z with $z \in L_\infty$.*

Proof : Let e denote an isotropic element of $\mathbf{V}$. We will prove that e can be mapped by $\widehat{\mathsf{F}}_4(K)$ onto an element of the form ke_p with $p \in \mathcal{P}$ and $k \in K$. In fact, the scalar k can be removed by an appropriate application of $h_r(k)$ (for some $r \in \Phi_F$) and the point p can be chosen freely in one of the two orbits of $W(\mathsf{F}_4)$ on $\mathcal{P}$ by a suitable $n_r(1)$.

First assume that $e \in \mathbf{V}_\infty$. Then e is isotropic if and only if $e = ke_p$ with $k \in K$ and $p \in L_\infty$ and therefore e belongs to one of the stated orbits.

If $e \in \mathbf{V} - \mathbf{V}_\infty$ then there is at least one $p \in \mathcal{P}^*$ for which the coordinate $e[p] \neq 0$. For this p we partition the set $\mathcal{P}^*$ into 5 parts which we will denote by $\mathcal{P}^*_2$, $\mathcal{P}^*_1$, $\mathcal{P}^*_0$, $\mathcal{P}^*_{-1}$ and $\mathcal{P}^*_{-2}$. The set $\mathcal{P}^*_i$ contains those elements q of $\mathcal{P}^*$ for which $(p + \overline{p}) \cdot q = i$. Note that these sets correspond to the five cases of (3.16). We have $\mathcal{P}^*_2 = \{p\}$ and $\mathcal{P}^*_{-2} = \{-\overline{p}\}$.

Consider $s \in \Phi_S$, $p' \in \mathcal{P}^*$ such that $s = (p' + \overline{p}')/2$ and $\langle s, p \rangle = -1$. If $q \in \mathcal{P}^*$ such that $e[q] \neq 0$ then (3.16) shows us that applying $x_s(k)$ may affect coordinates at positions $\overline{p}' + \overline{q}$ (provided this is a point of $\mathcal{P}^*$), p' and at infinity. Now $(p + \overline{p}) \cdot (\overline{p}' + \overline{q}) = -1 + (p + \overline{p}) \cdot q$, and hence $\overline{p}' + \overline{q} \in \mathcal{P}^*_{i-1}$ when $q \in \mathcal{P}^*_i$. It follows that $x_s(k)$ will only affect coordinate positions in $\mathcal{P}^*_1, \ldots,$ $\mathcal{P}^*_{-2}$ and L_∞, and of those only a single coordinate position (i.e., $\overline{p} + \overline{p}'$) will belong to $\mathcal{P}^*_1$. This means that we may apply consecutive reductions of this type to annihilate every single coordinate with an index in $\mathcal{P}^*_1$.

Similarly, let $r \in \Phi_L$ be such that $p \cdot r = -1$, and then $(p + \overline{p}) \cdot r = -2$. If $q \in \mathcal{P}^*$ such that $e[q] \neq 0$, then applying $x_r(k)$ may affect coordinates at positions $q + r$ (when $q + r \in \mathcal{P}$). Because $q + r \in \mathcal{P}^*_{i-2}$ when $q \in \mathcal{P}^*_i$ we see that $x_r(k)$ will only affect coordinate positions in $\mathcal{P}^*_0$, $\mathcal{P}^*_{-1}$ and $\mathcal{P}^*_{-2}$, and of those only a single coordinate position (i.e., $p + r$) will belong to $\mathcal{P}^*_0$. This means that we may apply consecutive reductions of this type to annihilate every single coordinate with an index in $\mathcal{P}^*_0$ while leaving the coordinates with index in $\mathcal{P}^*_1$ intact.

After applying the necessary reductions of this type we end up with an element for which all non-zero coordinates have index in $\mathcal{P}^*_0$, $\mathcal{P}^*_{-1}$, $\mathcal{P}^*_{-2}$ or L_∞, i.e., at positions q such that either $p \sim q$ or $q \in L_\infty$. As in the proof of Theorem 2.35 we may compute $e^\#$ (which should be 0) to show that e must then be of the form $e = k_1 e_p + k_2 e_{-\overline{p}} + k_3 e_z + k_4 e_{z'}$, with $k_1 k_2 = k_3 k_4$, where z, z' denote the points of L_∞ different from p_∞.

We may apply $x_s(k)$ with $s = (-\overline{p} - p)/2$ to reduce k_4 to zero, and then also $k_2 = 0$ (because $k_1 \neq 0$). Finally, when $k_3 \neq 0$ we use $x_s(k)$ with $s = (p + \overline{p})/2$ to reduce e to a multiple of e_z, otherwise e is a multiple of e_p. ∎

• Theorem 8.6 in [1] states that $\widehat{\mathsf{F}}_4(K)$ acts transitively on the isotropic 1-spaces of **W**, which is essentially a 'projective version' of the second statement of this theorem.

Theorem 3.26 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{F}}_4(K)$ on ordered pairs (e, f) where both $e, f \in \mathbf{W} - \{0\}$ are isotropic. Let $x, p \in \mathcal{P}^*$.*

1. *For each $\ell \in K - \{0\}$ an orbit with representative $(e_x, \ell e_{-\overline{x}})$.*
2. *An orbit with representative (e_x, e_p) where $(p + \overline{p}) \cdot x = -1$.*
3. *An orbit with representative (e_x, e_p) where $(p + \overline{p}) \cdot x = 0$.*
4. *An orbit with representative (e_x, e_p) where $(p + \overline{p}) \cdot x = +1$.*
5. *For each $\ell \in K - \{0\}$ an orbit with representative $(e_x, \ell e_x)$.*

The following table list several properties of the corresponding pairs (e, f).

	Case 1	*Case 2*	*Case 3*	*Case 4*	*Case 5*
$e\overline{f}$	ℓ	0	0	0	0
$e \times f$	*isotr.*, $\notin \mathbf{W}$	*isotr.*, $\in \mathbf{W}$, $\neq 0$	0	0	0
$e * \overline{f}$	*not isotr.*	*isotr.*, $\notin \mathbf{J}$	*isotr.*, $\in \mathbf{J}$, $\neq 0$	0	0
$f \in Ke$	*no*	*no*	*no*	*no*	$f = \ell e$

Proof : Because $\widehat{\mathsf{F}}_4(K)$ is transitive on isotropic points of **W** we may choose f of the form $f = e_x$ without loss of generality.

We will apply the same reduction process as in the proof of Theorem 3.25, but this time we must make sure that the elements $x_r(k)$ and $x_s(k)$ which we use, leave e_x invariant. This means that we will only use elements $r \in \Phi_L$ such that $r \cdot x \geq 0$, i.e., we can only reduce coordinate positions $q = r + p$ of $\mathcal{P}_0^*$ which satisfy the following property :

$$q \cdot x \geq p \cdot x. \tag{3.61}$$

Similarly, we can only use elements $s = (p' + \overline{p}')/2 \in \Phi_S$ such that $s \cdot x \geq 0$, i.e., we can only reduce coordinate positions $q = \overline{p} + \overline{p}'$ of $\mathcal{P}_1^*$ for which

$$(q + \overline{q}) \cdot x \geq (p + \overline{p}) \cdot x, \tag{3.62}$$

for we have $q + \overline{q} = (\overline{p} + \overline{p}') + (p + p') = p + \overline{p} + 2s$.

In the final reduction stage we need $x_s(k)$ with $s = (-\overline{p} - p)/2$ to obtain an element of the form $k_1 e_p + k_3 e_z$ with $p \in \mathcal{P}^*$ and $z \in L_\infty$. Again this is only allowed when $x_s(k)$ leaves e_x invariant, i.e., when

$$(p + \overline{p}) \cdot x \leq 0. \tag{3.63}$$

We consider five different cases.

Case 1. Assume $e[-\overline{x}] \neq 0$. We apply the reductions with $p = -\overline{x}$. Note that $p \cdot x = -\overline{x} \cdot x = -2/3$, $(p + \overline{p}) \cdot x = -2$ and hence (3.61–3.63) are trivially satisfied.

Case 2. Assume $e[-x] = 0$ but $e[p] \neq 0$ for some p such that $(p + \overline{p}) \cdot x = -1$ (and then $p \cdot x = -2/3$). Again (3.61–3.63) are satisfied, except for $q \in \mathcal{P}_1^*$ such that $(q + \overline{q}) \cdot x = -2$. But then $q = -x$ and the coordinate at this position is already known to be zero and hence need not be reduced.

Case 3. Otherwise, assume $e[p] \neq 0$ for some p such that $(p + \overline{p}) \cdot x = 0$, and then $p \cdot x = 1/3$. Again (3.61–3.63) are satisfied, except for those coordinate positions q which are already known to be zero.

Case 4. Otherwise, assume $e[p] \neq 0$ for some p such that $(p + \overline{p}) \cdot x = -1$, and then $p \cdot x = 1/3$. Again (3.61–3.62) are satisfied, except for those coordinate positions q which are already known to be zero. However, now (3.63) is not, hence we cannot execute the final reduction step.

In other words, we end up with an element of the form $e = k_1e_p + k_2e_{-\overline{p}} + k_3e_z + k_4e_{z'}$ with $k_1k_2 = k_3k_4$ where z, z' denote the points of L_∞ different from p_∞. If $k_2 \neq 0$, then interchanging the roles of $-\overline{p}$ and p this element can be further reduced using case 2 of this proof. Otherwise, the fact that $a \in \mathbf{W}$ implies $k_3 = k_4$, so $k_3 = k_4 = 0$ and a is a multiple of e_p.

Case 5. Otherwise, e must be of the form $e = \ell e_x + \ell_1 e_{x_\infty} + \ell_2 e_z + \ell_3 e_{z'}$ with $L_\infty = \{x_\infty, z, z'\}$. Then $e^\# = 0$ and $e \in \mathbf{W}$ implies $\ell_1 = \ell_2 = \ell_3 = 0$.

In each of these five cases we see that e is equivalent to an element of the form ke_p with $k \in K$. To get rid of the factor k we must find a suitable element $h_r(k)$ or $h_s(k)$ which leaves e_x invariant. We leave it to the reader to prove that this is always possible, except in the case $p = x$ or $p = -\overline{x}$. ∎

- A 'projective version' of this theorem can be found as Theorem 9.5 in [1].

It is an immediate consequence of this theorem that $\widehat{\mathsf{F}}_4(K)$ has three orbits on the isotropic elements of $\mathbf{L}$: the trivial orbit $\{0\}$, the set of all isotropic elements of $\mathbf{J}$ and the set of all isotropic elements of $\mathbf{L} - \mathbf{J}$.

Theorem 3.27 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{F}}_4(K)$ on ordered pairs (E, e) where F is a non-zero isotropic element of* $\mathbf{J}$ *and e is a non-zero isotropic element of* $\mathbf{W}$. *Let $t \in \Phi_L$, $p \in \mathcal{P}^*$.*

1. *An orbit with representative (E_t, e_p) where $\langle t, p\rangle = -1$.*

2. *An orbit with representative (E_t, e_p) where $\langle t, p\rangle = 0$.*

3. *An orbit with representative (E_t, e_p) where $\langle t, p\rangle = +1$.*

These orbits are the intersections with $\mathbf{J} \times \mathbf{W}$ of the corresponding orbits of $\widehat{\mathsf{E}}_6(K)$. If $e \in \mathbf{V}E$, then there exist isotropic elements $d, f \in \mathbf{W}$ such that $e = dE$ and $E = e * \overline{f}$.

Proof : We set $E = E_t$ and apply the reduction process to e, using only group elements $x_r(k)$ and $x_s(k)$ that leave E_t invariant.

For $r \in \Phi_L$ this means that we can only reduce coordinate positions q of $\mathcal{P}_0^*$ which satisfy the following property :

$$q \cdot t \geq p \cdot t, \tag{3.64}$$

by (2.86). For $s \in \Phi_S$ we find that $\langle t, s \rangle \geq 0$ by (3.54) and hence

$$(q + \overline{q}) \cdot t \geq (p + \overline{p}) \cdot t. \tag{3.65}$$

Similarly, in the final stage we need

$$(p + \overline{p}) \cdot t \leq 0. \tag{3.66}$$

Case 1. Assume we can find a base point p of t such that $e[p] \neq 0$. We have $p \cdot t = -1$, $(p + \overline{p}) \cdot t = -2$ and therefore (3.64–3.66) are trivially satisfied. After reduction we end up with an element of the form ℓe_p.

Case 2. Assume $e[q] = 0$ for all base points q of t, but we may find a point p such that $e[p] \neq 0$, $p \cdot t = 0$ and $(p + \overline{p}) \cdot t = 0$. As in earlier proofs, the reduction process can again be applied to all coordinates which were not 'taken care of' by the previous case. Again we end up with an element of the form ℓe_p.

Case 3. Assume $e[q] = 0$ except when q is a base point for $-r$. This time we need to use the special technique of case 4 of the proof of Theorem 3.26 because we cannot always satisfy (3.66) to once again reduce e to an element of the form ℓe_p.

We leave it to the reader to prove that we can get rid of the factor ℓ in each of these three cases and to prove the stated properties for the case $e \in \mathbf{V}E$. ∎

Theorem 3.28 *The following is an exhaustive list of all orbits of $\widehat{\mathsf{F}}_4(K)$ on ordered pairs (E, F) of non-zero isotropic elements of $\mathbf{J}$. Let $t, t' \in \Phi_L$.*

1. *For every $\ell \in K - \{0\}$ an orbit with representative $(E_t, \ell E_{-t})$.*

2. *An orbit with representative $(E_t, E_{t'})$ where $\langle t, t' \rangle = -1$.*

3. *An orbit with representative $(E_t, E_{t'})$ where $\langle t, t' \rangle = 0$.*

4. *An orbit with representative $(E_t, E_{t'})$ where $\langle t, t' \rangle = +1$.*

5. *For every $\ell \in K - \{0\}$ an orbit with representative $(E_t, \ell E_t)$.*

These orbits are the intersections with $\mathbf{J} \times \mathbf{J}$ of the corresponding orbits of $\widehat{\mathsf{E}}_6(K)$.

Proof : (The proof uses a setting similar to that of Theorem 2.38.)

Write $F = a * \overline{b}$ with $a, b \in \mathbf{W}$ and $a\overline{b} = 0$. By Theorem 3.27 we may without loss of generality assume that $E = E_t$ and $b = e_{-\overline{x}}$ (i.e., $\overline{b} = \eta_x$) for some $t \in \Phi_L$ and $x \in \mathcal{P}^*$. Consider $e \stackrel{\text{def}}{=} e_x F$, then $e \in \mathbf{W}$, $F = e * \eta_x$, $e^\# = 0$ and $e\eta_x = 0$. Also, because $F \in \Phi_L$, we have $F\infty^\# = e \times e_{-\overline{x}} = 0$, by Proposition 3.14–3.

As before we will apply the reduction process to e using elements $x_r(k)$ and $x_s(k)$ that leave both E_t and $e_{-\overline{x}}$ invariant. This yields the following restrictions on the coordinate indices p and q that can be used in the reduction process :

$$\begin{array}{lcl} q \cdot t \geq p \cdot t & \text{and} & q \cdot \overline{x} \leq p \cdot \overline{x} \\ (q + \overline{q}) \cdot t \geq (p + \overline{p}) \cdot t & \text{and} & (q + \overline{q}) \cdot x \leq (p + \overline{p}) \cdot x \\ (p + \overline{p}) \cdot t \leq 0 & \text{and} & (p + \overline{p}) \cdot x \geq 0 \end{array} \tag{3.67}$$

Additionally, many coordinates $e[q]$ of e are already zero, in particular, $e = e_x F$ implies $e[q] = 0$ whenever $q \sim x$ and $\overline{e} \times \eta_x = 0$ implies $e[q] = 0$ whenever $q \sim -\overline{x}$. In other words, by (3.2), $e[q] = 0$ whenever $(q + \overline{q}) \cdot x \neq 0$.

Because the reduction process only involves coordinates $e[p]$, $e[q]$ which are non-zero, and in those cases, $p \cdot \overline{x} = q \cdot \overline{x} = -1/3$ and $(p + \overline{p}) \cdot x = (q + \overline{q}) \cdot x = 0$, by (3.2), this means that we essentially only have to worry about the inequalities in the left hand column of (3.67).

We need to consider three different cases :

Case 1. Assume there exists $p \in \mathcal{P}^*$ such that $p \cdot t = -1$ and $e[p] \neq 0$. Then (3.67) is trivially satisfied.

Case 2. Otherwise, assume we find $p \in \mathcal{P}^*$ such that $p \cdot t = 0$ and $e[p] \neq 0$. As before, the coordinates $e[q]$ for which the reduction process would not work because of (3.67) are ruled out by the fact that the previous case did not apply.

Case 3. Otherwise, we must find $p \in \mathcal{P}^*$ such that $p \cdot t = +1$. Again we may apply similar techniques as before to reduce e to a single non-zero coordinate.

This proves that F is equivalent to some element of the form $\ell e_p * \eta_x$ and hence to some $\pm \ell E_{t'}$ with $t' \in \Phi_L$. We leave it to the reader to verify that the scalar factor can always be removed when $t \neq t'$ and $t \neq -t'$. ∎

• In [19, Theorem A] it is stated that there are six different orbits on pairs of long root subgroups in $\widehat{\mathsf{F}}_4(K)$, with two orbits instead of one for the configuration that corresponds to case 3 of the theorem above. (A *long root subgroup* consists of all elements $x(kE)$ with $k \in K$, where E is a fixed isotropic element.)

This is probably a misprint, for in the same paper the author writes "when char $K = 2$, the short root subgroups behave exactly as the long root subgroups", while Theorem 2 of that paper, which deals exactly with the case quoted, lists a result which differs from that of Theorem A.

Theorem 3.29 *The group $\widehat{\mathsf{F}}_4(K)$ acts transitively on all linearly independent triples $(a, b, c) \in \mathbf{W}^3$ such that $a, b, c \neq 0$, $a^\# = b^\# = c^\# = 0$ and $a * \overline{b} = b * \overline{c} = c * \overline{a} = 0$.*

Proof : By Theorem 3.26 we may without loss of generality choose $a = e_x$, $b = e_y$ with $x, y \in \mathcal{P}^*$ such that $(x + \overline{x}) \cdot y = 1$. The conditions of the theorem then force all coordinates of c zero except possibly $c[x]$, $c[y]$ or $c[q]$ with $q \in L_\infty$ or $q \in \mathcal{P}^*$ such that $(x + \overline{x}) \cdot q = (y + \overline{y}) \cdot q = 1$ (and then $q \cdot x = q \cdot y = 1/3$ by (3.2).

The fact that c is isotropic, belongs to $\mathbf{W}$ and is linearly independent of a and b, shows that $c[p] = 0$ for at least one coordinate $p \in \mathcal{P}^*$ different from x and y. We will apply the standard reduction process to reduce all other coordinates of c to zero. The fact that we may only use group elements that leave both

e_x and e_y invariant, implies the following conditions in the first stage of the reduction process :

$$q \cdot x \geq p \cdot x, \quad q \cdot y \geq p \cdot y,$$

and in the second stage

$$(q + \overline{q}) \cdot x \geq (p + \overline{p}) \cdot x, \quad (q + \overline{q}) \cdot y \geq (p + \overline{p}) \cdot y.$$

By the above, both of these are satisfied for all coordinate positions q such that $c[q] \neq 0$.

This means that we can reduce c to the form

$$c = k_1 e_p + k_2 e_{-\overline{p}} + k_3 e_z + k_4 e_{z'},$$

where $\{z, z'\} = L_\infty - \{p_\infty\}$, $k_1 k_2 = k_3 k_4$ and $k_3 + k_4 = 0$. Note however that $k_2 = 0$ because $-\overline{p} \cdot (x + \overline{x}) = -1$. This implies $k_3 = k_4 = 0$ and therefore c can be reduced to a multiple of e_p. By Proposition 3.8 we may choose p freely (provided $(p + \overline{p}) \cdot x = (p + \overline{p}) \cdot y = 1$). We leave it to the reader to show that that we may get rid of the scalar factor k_1 by an appropriate group element $h_r(k_1)$. ∎

The following is the equivalent of the technical Lemma 2.41 for **J** instead of **L**.

Lemma 3.30 *Let $p \in \mathcal{P}^*$, $r \in \Phi_L$. Let $A \in \mathbf{J}$ and write $A = A_G + \sum_{s \in \Phi_F} A[s] E_s$ with $A_G \in \mathbf{G}$. Then*

1. *$e_p A = 0$ if and only if $A[p, p] = 0$ and $A[s] = 0$ for all $s \in \Phi_F$ such that $p \cdot s < 0$.*

2. *$A(e_p \times \mathbf{V}) = 0$ if and only if $A_G = 0$ and $A[s] = 0$ for all $s \in \Phi_F$ such that $p \cdot s = 1$.*

3. *$E_r A = 0$ if and only if $A[s] = 0$ for all $s \in \Phi_F$ such that $r \cdot s \leq 0$ and $A_G = 0$ (except when* char $K = 3$ *in which case this latter condition is relaxed to $A_H \in K(H_r + \mathbf{1})$).*

Proof : Note that the 'coordinates' in $\mathbf{L}$ of A are $A_H = A_G$, $A[s]$ for $s \in \Phi_L$ and $A[u] = A[\overline{u}] = A[s]$ for $u \in \Phi - \Phi_L$ such that $s = (u + \overline{u})/2$. We now apply Lemma 2.41 to these coordinates.

If $s \in \Phi_S$ such that $s = (u + \overline{u})/2$, then Lemma 3.7 shows that $p \cdot u$ or $p \cdot \overline{u} = -1$ if and only if $p \cdot s = -1/2$ or -1. This proves the first statement of this lemma.

Likewise $p \cdot u \geq 0$ or $p \cdot \overline{u} \geq 0$ if and only if $p \cdot s = 1$. Also $H_p + \mathbf{1} \notin \mathbf{J}$ when $p \in \mathcal{P}^*$. This proves the second statement.

Similarly, $\langle r, s \rangle = r \cdot u = r \cdot \overline{u}$ and hence $r \cdot s \leq 0$ if and only if both $r \cdot u = r \cdot \overline{u} \leq 0$. (Note that $H_r + \mathbf{1} \in \mathbf{J}$ when r is a long root.) ∎

Theorem 3.31 *For $n = 1, 2$ or 3 the group $\widehat{\mathsf{F}}_4(K)$ acts transitively on the n-tuples $(E_1, \ldots, E_n)$ of linearly independent isotropic elements of $\mathbf{J}$ that satisfy $E_i E_j = 0$ for all i, j, $1 \leq i, j \leq n$. A representative of each orbit is given by $(E_{r_1}, \ldots, E_{r_n})$ with $r_1 = \mathtt{1100}$, $r_2 = \mathtt{1010}$ and $r_3 = \mathtt{1001}$.*

There are no such n-tuples in $\mathbf{J}$ with $n > 3$.

Proof : By Theorem 3.28 without loss of generality we may set $E_1 = E_{\mathtt{1100}}$ and $E_2 = E_{\mathtt{1010}}$. From Lemma 3.30 it follows that any $E_3 \in \mathbf{J}$ which satisfies $E_1 E_3 = E_2 E_3 = 0$ must be of the form $E_3 = k_1 E_1 + k_2 E_2 + k_3 E_{\mathtt{1001}} + k_3' E_{\mathtt{100\overline{1}}}$.

If E_1, E_2, E_3 are linearly independent, then either $k_3 \neq 0$ or $k_3' = 0$ and because of the properties of the Weyl group we may assume $k_3 \neq 0$ without loss of generality. Applying $x_r(k) \in \widehat{\mathsf{F}}_4(K)$ to E_3 with $r = \mathtt{010\overline{1}}$ and an appropriate $k \in K$ annihilates k_1. Similarly $r = \mathtt{001\overline{1}}$ can be used to annihilate k_2. We leave it to the reader to prove that the resulting linear combination can be isotropic only when also $k_3' = 0$. Finally, we get rid of the factor k_3 by applying $h_s(k_3)$ with $s = \mathtt{-++-}$.

A similar application of Lemma 3.30 shows that any $E_4 \in \mathbf{J}$ such that $E_1 E_4 = E_2 E_4 = E_3 E_4 = 0$ must necessarily be a linear combination of E_1, E_2 and E_3. ∎

3.4 F-isotropic subspaces

It turns our that the notion of isotropic subspace of $\mathbf{V}$ needs to be strenghtened when we are working with F_4-modules.

An isotropic subspace U of $\mathbf{V}$ will be called *F-isotropic* if and only if $e * \overline{f} = 0$ for every two elements $e, f \in U$. It is a consequence of Theorem 3.25 that isotropic elements $e \in \mathbf{V}$ satisfy $e * \overline{e} = 0$ if and only if they belong to $\mathbf{W}$, hence all F-isotropic subspaces are subspaces of $\mathbf{W}$.

Note that $\widehat{\mathsf{F}}_4(K)$ maps F-isotropic subspaces onto F-isotropic subspaces.

Proposition 3.32 *Let U denote an isotropic subspace of $\mathbf{V}$. Let V denote the companion subspace in $\mathbf{V}^*$ of U, as in Proposition 2.43. Then U is F-isotropic if and only if $\overline{U} \leq V$. If U is F-isotropic then* $\dim U \leq 3$.

Proof : Let e, f denote isotropic elements. By Theorem 2.35 $e * \overline{f} = 0$ if and only if $\overline{f} \in e \times \mathbf{V}$. Hence U is F-isotropic if and only every element of $\overline{U}$ belongs to the intersection $\bigcap_{e \in U} e \times \mathbf{V}$, i.e., if and only if $\overline{U}$ is a subspace of V.

Because the dimension of a companion space cannot be larger than the dimension of the original space, and $\dim V < 3$ when $\dim U > 3$, we must have $\dim U \leq 3$. ∎

• [1] uses a very colorful terminology: F-isotropic spaces are called *amber* while the spaces $\mathbf{V}E$ with $E \in \mathbf{J}$ are *tangerine*. Isotropic 2-spaces of $\mathbf{W}$ which are not F-isotropic are called *scarlet*.

Proposition 3.33 *Let $L \in \mathbf{V} \wedge \mathbf{V}, L \neq 0$. Then the following are equivalent :*

1. *$L = e \wedge f$ with $e, f \in \mathbf{W}$ such that $e^\# = f^\# = 0$ and $e * \overline{f} = 0$.*

2. *$L = \overline{d} \times E$ for some isotropic $E \in \mathbf{J}$ and isotropic $d \in \mathbf{W}$ such that $dE = 0$ but $d \notin \mathbf{V}E$.*

Proof : By Theorem 3.26 the pairs (e, f) satisfying the given conditions form a single orbit of $\widehat{\mathsf{F}}_4(K)$, and so do the pairs (d, E). To prove the theorem it is therefore sufficient to provide a single example for which $e \wedge f = \overline{d} \times E$.

Choose $a, b \in \mathbf{W}$ such that $aE\overline{b} = 0$. Set $\overline{d} = bE \times a$, $e = aE$, $f = -bE$. Then $e * \overline{f} = aE * E\overline{b} = (aE\overline{b})E = 0$ and $e \wedge f = \overline{d} \times E$ by (2.107). Also $\infty\overline{d} = \infty(bE \times a) = \langle \infty, bE, a \rangle = bE(a \times \infty) = -bE\overline{a} = 0$, and therefore $\overline{d} \in \mathbf{W}$. ∎

An element $L \in \mathbf{V} \wedge \mathbf{V}$ which satisfies one of the statements of this proposition will be called *F-isotropic*. An isotropic element L of $\mathbf{V} \wedge \mathbf{V}$ is F-isotropic if and only if $L\mathbf{V}^*$ is an F-isotropic subspace of $\mathbf{V}$.

• It is not sufficient for an isotropic element of $\mathbf{V} \wedge \mathbf{V}$ to belong to $\mathbf{W} \wedge \mathbf{W}$ to be F-isotropic. From Theorem 3.26 it follows that there are two different orbits of $\widehat{\mathsf{F}}_4(K)$ on the isotropic elements of $\mathbf{W} \times \mathbf{W}$ and of these the F-isotropic elements only consititute a single one.

Proposition 3.34 *Let $P \in \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$, $P \neq 0$. Then the following are equivalent :*

1. *$P = e_1 \wedge e_2 \wedge e_3$ with $e_i \in \mathbf{W}$ such that $e_i^{\#} = 0$ and $e_i * \overline{e}_j = 0$, for all i, j, $1 \leq i, j \leq 3$.*

2. *$P = E \times F$ for some isotropic $E, F \in \mathbf{J}$ such that $EF = FE = 0$ but $KE \neq KF$.*

Proof : Theorems 3.28 and 3.29 show that both statements correspond to a single orbit of $\widehat{\mathsf{F}}_4(K)$. It is therefore sufficient to provide a single element that belongs to both orbits.

Let E be an isotropic element of $\mathbf{J}$ and choose $a, b, c \in \mathbf{V}$ such that $aE\overline{b} = bE\overline{c} = cE\overline{a} = 0$. (To see that this is always possible, specialise to the case $E = E_r$ with $r \in \Phi_L$.) Setting $e_1 = aE$, $e_2 = bE$ and $e_3 = cE$, it follows from (2.116) that $e_1 \wedge e_2 \wedge e_3 = F \times E$, with $F = cE * (bE \times a)$. Note that $e_1 * \overline{e}_2 = -aE * E\overline{b} = -(aE\overline{b})E = 0$, and likewise $e_i * \overline{e}_j = 0$ for all pairs i, j. It is therefore sufficient to prove that $F \in \mathbf{J}$.

Now $cE\overline{\infty} = 0$ and $\langle bE, a, \infty \rangle = 0$, hence $cE \in \mathbf{W}$ and $bE \times a \in \mathbf{W}^*$. By Proposition 3.14–2 we need only prove that $(bE \times a) \times E\bar{c} = 0$ and because $a * E\bar{c} = 0$ we find $0 = bE(a * E\bar{c}) = -(bE \times a) \times E\bar{c}$. ∎

An element P of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ that satisfies one of the statements of this proposition will be called *F-isotropic*. An isotropic element P of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ is F-isotropic if and only if $P\mathbf{V}^*\mathbf{V}^*$ is an F-isotropic subspace of $\mathbf{V}$.

Proposition 3.35 *Let $E, F \in \mathbf{J}$ be isotropic such that $KE \neq KF$. Then $\mathbf{V}E \cap \mathbf{V}F$ is an F-isotropic subspace of $\mathbf{W}$.*

Proof : Theorems 2.38 and 3.28 show that there are only a few cases to consider. If $\mathbf{V}E \cap \mathbf{V}F$ has dimension 0 or 1, then the statement is trivial, otherwise $\mathbf{V}E \cap \mathbf{V}F = E\mathbf{V}^* \times F\mathbf{V}^* = (E \times F)\mathbf{V}^*\mathbf{V}^*$, which is an F-isotropic 3-space by Proposition 3.34. ∎

Proposition 3.36 *Let U denote an F-isotropic subspace of $\mathbf{W}$ of dimension n. Let J_n denote the intersection in $\mathbf{J}$ of the companion subspace in $\mathbf{L}$ of U. Then $\dim J_1 = 7$, $\dim J_2 = 3$ and $\dim J_3 = 2$. and the isotropic elements of J_1 are the absolute elements of a non-degenerate (parabolic) quadratic form on J_n.*

Conversely, if J is an isotropic subspace of $\mathbf{J}$ with $\dim J \geq 2$, then the companion subspace in $\mathbf{V}$ of J is F-isotropic.

Proof : Consider an element $e_1 \in U$. Without loss of generality we may choose e_1 to be e_{p_0} with $p_0 = \mathtt{300000}$. Using Lemma 3.30 (or the information in the proof of Proposition 2.44) we may compute that any $E \in \mathbf{J}$ such that $e_1 \in \mathbf{V}E$ must be of the following form :

$$\begin{aligned} E \quad = \quad & k_1 E_{\mathtt{30000\bar{3}}} + k_2 E_{\mathtt{111111}} + k_3 E_{\mathtt{2\bar{1}\bar{1}22\bar{1}}} + k_4 E_{\mathtt{222\bar{1}\bar{1}\bar{1}}} \\ & + k_5 E_{\mathtt{2\bar{1}2\bar{1}2\bar{1}}} + k_6 E_{\mathtt{22\bar{1}2\bar{1}\bar{1}}} + k_0 (E_{\mathtt{22\bar{1}\bar{1}2\bar{1}}} + E_{\mathtt{2\bar{1}22\bar{1}2}}), \end{aligned} \qquad (3.68)$$

with $k_0, \ldots, k_6 \in K$. This proves that E belongs to a 7-dimensional subspace of $\mathbf{J}$.

To determine which E of this form are isotropic, we may derive from (3.68) that E can be written as $e_{p_0} * \alpha$ with

$$\begin{aligned}\alpha &= \pm k_1\eta_{\mathtt{000003}} \pm k_2\eta_{\mathtt{2}\overline{\mathtt{11111}}} \pm k_3\eta_{\mathtt{111}\overline{\mathtt{22}}\mathtt{1}} \pm k_4\eta_{\mathtt{1}\overline{\mathtt{22}}\mathtt{111}} \\ &\quad \pm k_5\eta_{\mathtt{11}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{2}}\mathtt{1}} \pm k_6\eta_{\mathtt{1}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{2}}\mathtt{11}} \pm k_0\eta_{\mathtt{1}\overline{\mathtt{2}}\mathtt{11}\overline{\mathtt{2}}\mathtt{1}} \pm k_0\eta_{\mathtt{11}\overline{\mathtt{22}}\mathtt{1}\overline{\mathtt{2}}}\end{aligned}$$

and in turn α can be written as $e_{a_{05}} \times a$ (with $a_{05} = \overline{\mathtt{2}}\mathtt{1111}\overline{\mathtt{2}}$) with

$$\begin{aligned}a &= \pm k_2e_{\mathtt{000003}} \pm k_1e_{\mathtt{2}\overline{\mathtt{11111}}} \pm k_4e_{\mathtt{111}\overline{\mathtt{22}}\mathtt{1}} \pm k_3e_{\mathtt{1}\overline{\mathtt{22}}\mathtt{111}} \\ &\quad \pm k_6e_{\mathtt{11}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{2}}\mathtt{1}} \pm k_5e_{\mathtt{1}\overline{\mathtt{2}}\mathtt{1}\overline{\mathtt{2}}\mathtt{11}} \pm k_0e_{\mathtt{1}\overline{\mathtt{2}}\mathtt{11}\overline{\mathtt{2}}\mathtt{1}} \pm k_0e_{\mathtt{11}\overline{\mathtt{22}}\mathtt{1}\overline{\mathtt{2}}}.\end{aligned}$$

By Proposition 2.29, E is isotropic if and only if $e_{p_0}\alpha = 0$ (which is trivially satisfied) and $e_{p_0} \times \alpha^\# = 0$. Note that $\alpha^\# = (e_{a_{05}}a^\#)e_{a_{05}}$ and hence $e_{p_0} \times \alpha^\# = \pm(e_{a_{05}}a^\#)e_{q_0}$ with $q_0 = \overline{\mathtt{211111}}$. Therefore E is isotropic if and only if $e_{a_{05}}a^\#$ which is equivalent to $\pm k_1k_2 \pm k_3k_4 \pm k_5k_6 \pm k_0^2 = 0$. This proves the statements about J_1.

Now, let $\dim U \geq 2$ and consider $e_2 \in U$ such that e_1, e_2 are linearly independent. Without loss of generality we may choose $e_2 = e_{p_1}$ with $p_1 = \mathtt{030000}$. From the above we easily compute that the $E \in \mathbf{J}$ for which $e_{p_0}E = e_{p_1}E = 0$ is of the form (3.68) with $k_1 = k_3 = k_5 = k_0 = 0$. Hence $\dim L_2 = 3$. We may prove $\dim L_3 = 2$ in a similar way (or obtain it as a consequence of Proposition 3.34).

Finally, it is a consequence of Proposition 3.34 that any 2-dimensional isotropic subspace $KE + KF$ of $\mathbf{J}$ has a companion subspace in $\mathbf{V}$ which is F-isotropic. The same holds for isotropic subspaces $KE + KF + KG$ of larger dimension, because their companions are subspaces of the companions of $KE + KF$. ∎

We will call the intersection with $\mathbf{J}$ of the companion in $\mathbf{L}$ of an F-isotropic subspace U, its *companion in* $\mathbf{J}$.

3.5 The geometry $\mathcal{F}$

The *geometry* $\mathcal{F}$ *of type* F_4 has elements and incidence defined as follows. Elements are of four different *types* numbered like the nodes of the Dynkin diagram :

1. The *hyperlines* of $\mathcal{F}$ are the isotropic 1-spaces of $\mathbf{J}$, i.e., the sets KE where E is an isotropic element of $\mathbf{J} - \{0\}$.

2. The *planes* of $\mathcal{F}$ are the 1-spaces KP of $\mathbf{W} \wedge \mathbf{W} \wedge \mathbf{W}$ where P is F-isotropic.

3. The *lines* of $\mathcal{F}$ are the 1-spaces KL of $\mathbf{W} \wedge \mathbf{W}$ where L is F-isotropic.

4. The *points* of $\mathcal{F}$ are the isotropic 1-spaces of $\mathbf{W}$, i.e., the sets Ke where e is an isotropic element of $\mathbf{W} - \{0\}$.

Incidence between points Ke, lines KL, planes KP and hyperlines KE is (symmetrized) containment of the corresponding 1-, 2-, 3- and 6-dimensional subspaces Ke, $L\mathbf{V}^*$, $P\mathbf{V}^*\mathbf{V}^*$ and $\mathbf{V}E$ of $\mathbf{W}$. As with incidence in $\mathcal{E}$, containment can often be expressed in a different way using one of the many operators defined on $\mathbf{V}$ and $\mathbf{L}$.

• Every point, line, plane and hyperline of $\mathcal{F}$ is a point, line, plane or simplex of $\mathcal{E}$ with the same incidence relation. A point Ke of $\mathcal{E}$ is a point of $\mathcal{F}$ if and only if $e \in \mathbf{W}$ and a simplex KE of $\mathcal{E}$ is a hyperline of $\mathcal{F}$ if and only if $E \in \mathbf{J}$. It is however dangerous to take this analogy too far. For example, two points that are on the same line in $\mathcal{F}$ are always on the same line in $\mathcal{E}$ but the converse is not always true. The same care should be taken with planes.

Proposition 3.37 *Let KE be a hyperline of $\mathcal{F}$. Then the points, lines and planes incident with KE form a symplectic* 5*-space $\mathcal{C}$ (i.e., a geometry of type* C_3*).*

Proof : The points, lines and planes incident with KE are points, lines and planes of the 5-dimensional projective space $\mathcal{A}$ of Proposition 2.56. Because $E \in \mathbf{J}$ and then $\mathbf{V}E \in \mathbf{W}$, every point of $\mathcal{A}$ is also a point of $\mathcal{C}$. However, a line $K(aE \wedge bE)$ of $\mathcal{A}$ is a line of $\mathcal{C}$ only if $aE \wedge bE$ is F-isotropic, i.e., if and only if $aE * \overline{bE} = 0$. Now $aE * \overline{bE} = -aE * E\overline{b} = -(aE\overline{b})E$, hence this is equivalent to requiring $aE\overline{b} = 0$. From $aE\overline{b} = -bE\overline{a}$ it follows that the map $(aE, bE) \mapsto -(aE * \overline{bE})/E = aE\overline{b}$ is an antisymmetric bilinear form on $\mathbf{V}E$ which is easily proved to be non-degenerate. Therefore the lines of $\mathcal{C}$ are the absolute lines in $\mathcal{A}$ of this symplectic polarity. And similarly, the planes of $\mathcal{C}$ are those planes of $\mathcal{A}$ all of whose lines are absolute. ∎

Proposition 3.38 *Let Ke be a point of $\mathcal{F}$. Then the hyperlines, planes and lines incident with Ke are the points, lines and planes of a 7-dimensional polar space $\mathcal{B}$ of type B_3.*

Proof : Let $\mathcal{B}$ denote the projective space associated with the companion subspace of KE in $\mathbf{J}$. From Proposition 3.36 it follows that the projective dimension of $\mathcal{B}$ is 6 and that the hyperlines of $\mathcal{E}$ that belong to $\mathcal{B}$ exactly correspond to the points of a non-degenerate parabolic quadric in $\mathcal{B}$. In a similar way, with every F-isotropic 2- and 3-space that contains e we may associate its companion 3- and 2-space in $\mathbf{J}$, each of which are absolute for the same quadratic form.

We leave it to the reader to verify that this correspondence between lines (planes, hyperlines, respectively) of $\mathcal{E}$ that are incident with Ke and planes (lines, points, respectively) on the parabolic quadric, preserves incidence. ∎

Proposition 3.39 *Let KL and KP denote an incident line/plane-pair of $\mathcal{F}$. Then the points and hyperlines incident with both KL and KP form a generalized digon : a 2-space in $\mathbf{W}$ and a 2-space of $\mathbf{J}$ such that every 1-space contained in the first 2-space is incident with every 1-space contained in the second.*

Proof : The element $e \in \mathbf{V}$ for which Ke is incident with KL form an F-isotropic 2-space $L\mathbf{V}^*$ which is a subspace of the F-isotropic 3-space $P\mathbf{V}^*\mathbf{V}^*$. By Proposition 3.36 the elements $E \in \mathbf{J}$ for which KE is incident with KP form the companion 2-space of $P\mathbf{V}^*\mathbf{V}^*$ in $\mathbf{J}$, a subspace of the companion 3-space of $L\mathbf{V}^*$. Clearly every such e belongs to every such $\mathbf{V}E$. ∎

Note that $\mathcal{F}$ satisfies the intersection property (cf. Section 1.8). Indeed, because the intersection of two F-isotropic spaces is always F-isotropic, and because every F-isotropic subspace of $\mathbf{W}$ is in one-to-one correspondence to a single element of $\mathcal{F}$, it only remains to be proved that $\mathbf{V}E \cap \mathbf{V}F$ is F-isotropic for every isotropic $E, F \in \mathbf{J}$ such that $KE \neq KF$. And this is Proposition 3.35.

As was pointed out in the introduction, the intersection property together with Propositions 3.37–3.39 prove the following

Theorem 3.40 *$\mathcal{F}$ is the geometry arising from the exceptional split building of type F_4 over the field K.*

• This proves that $\mathcal{F}$ is a so-called *metasymplectic space* according to the definitions given in [22] and [26]. Although these definitions are essentially self-dual, often the type 'point' is (implicitely) assumed to correspond to the long roots of $\mathbf{J}$ (see for instance [9] where metasymplectic spaces are characterized axiomatically in terms of points and lines), while we have chosen to use the type 'point' for the elements of the F_4-module of smallest dimension. In the next chapter we shall see that in the case of char $K = 2$ the geometry $\mathcal{F}$ is self-dual and then this choice is immaterial.

Theorem 3.41 *Let Ke, Kf denote different points of $\mathcal{F}$.*

1. *If $e * \overline{f} = 0$ then Ke and Kf are collinear. The points collinear (or equal) to both Ke and Kf form a projective 4-space of $\mathcal{F}$ which corresponds to the 5-dimensional companion space of $K\overline{e} + K\overline{f}$ in $\mathbf{V}$. The hyperlines incident with both Ke and Kf form a bundle of projective dimension 2 which corresponds to the 3-dimensional companion space of $Ke + Kf$ in $\mathbf{J}$.*

2. *If $e \times f = 0$ but $e * \overline{f} \neq 0$ then the points Kd collinear to both Ke and Kf form a projective 3-space of $\mathcal{F}$. The hyperline $K(e * \overline{f})$ is the unique hyperline incident with both Ke and Kf.*

3. *If $e\overline{f} = 0$ but $e \times f \neq 0$ then $K(e \times f)$ is the unique point collinear to both Ke and Kf and there are no hyperlines that are incident with both points.*

4. *If $e\overline{f} \neq 0$ then there are no points or hyperlines incident with both Ke and Kf.*

These are the only possible relations between different points of $\mathcal{F}$.

Proof : By Theorem 3.26 Kd is collinear with Ke if and only if $d \in \bar{e} \times \mathbf{V}^*$. Likewise, Ke belongs to the hyperline KE if and only if $e \in \mathbf{V}E$. This proves the statements about the companion spaces, except that it still needs to be shown that the companion space in $\mathbf{V}^*$ of a 2-dimensional F-isotropic space lies entirely in $\mathbf{W}$.

The different cases of the theorem correspond to the first four cases of Theorem 3.26 (in reverse order). Without loss of generality we may choose representatives for (e, f) of the form (e_x, e_p) with $p, x \in \mathcal{P}^*$ from the respective orbits. We leave it to the reader to prove the theorem for these special cases. ∎

Point pairs satisfying statement 1 (2, 3 or 4, respectively) of this theorem will be called *collinear* (*cohyperlinear*, *almost opposite* or *opposite*, respectively).

4 The Lie algebra of type F_4 when $\operatorname{char} K = 2$

Throughout this chapter we assume that $\operatorname{char} K = 2$. In this special case the Lie algebra $\mathbf{J}$ turns out to be no longer irreducible.

4.1 Two Lie algebras isomorphic to $\mathbf{W}$.

In Lie algebras of characteristic 2 the operator $\cdot^2$ often plays a special role:

Proposition 4.1 *Let $A \in \widehat{\mathbf{L}}$. Then $A^2 \in \widehat{\mathbf{L}}$ with $\tau(A^2) = \tau(A)^2$. If $A \in \mathbf{J}$ then also $A^2 \in \mathbf{J}$.*

For every $A, B \in \widehat{\mathbf{L}}$ holds

1. $(A+B)^2 = A^2 + [A,B] + B^2$,
2. $[B, A^2] = [[B,A],A]$,

Proof : Let $a \in \mathbf{V}$. Applying (2.42) with $b = aA$ and then (2.41) twice, we obtain

$$\begin{aligned} a \times aA^2 &= aA \times aA + A(a \times aA) + \tau(A)a \times aA \\ &= 2(aA)^\# + A(Aa^\#) + \tau(A)Aa^\# + \tau(A)a \times aA \\ &= A^2 a^\# + \tau(A)^2 a^\#, \end{aligned}$$

which by definition indicates that $A^2 \in \widehat{\mathbf{L}}$ with $\tau(A^2) = \tau(A)^2$. Also $\infty A^2 = (\infty A)A$ and hence $A^2 \in \mathbf{J}$ when $A \in \mathbf{J}$.

The first stated property then follows from $[A, B] = AB - BA = AB + BA$ in even characteristic. The second from $[[B, A], A] = BA^2 - 2[A, B] + A^2B$. ∎

Proposition 4.2 *For $a, b \in \mathbf{V}$ define the following operations :*

$$\begin{aligned}
S(a) &\stackrel{\text{def}}{=} a * \overline{\infty} + (a\overline{\infty})\mathbf{1}, \\
[a,b] = [b,a] &\stackrel{\text{def}}{=} aS(b) = bS(a) = (a\overline{\infty})b + (b\overline{\infty})a + (a \times b) \times \overline{\infty}, \\
&= (a \times \infty) \times (b \times \infty) + \langle a, b, \infty \rangle \infty, \\
a^2 &\stackrel{\text{def}}{=} (a\overline{\infty})a + a^{\#} \times \overline{\infty} = (a \times \infty)^{\#} + (\infty a^{\#})\infty.
\end{aligned} \tag{4.1}$$

Then for all $a, b, c, d \in \mathbf{V}$, $A \in \mathbf{J}$ we have

1. $(a + b)^2 = a^2 + [a, b] + b^2$,
2. $\infty^2 = \infty$, $[a, a] = 0$ *and* $[a, \infty] = 0$.
3. $a^2\overline{\infty} = (a\overline{\infty})^2$ *and* $[a, b] \in \mathbf{W}$.
4. $S(a) \in \mathbf{J}$ *and* $S(\infty) = 0$,
5. $[a, b]\overline{c} = [b, c]\overline{a} = [c, a]\overline{b}$ *and* $[a, b]\overline{[c, d]} + [a, c]\overline{[b, d]} + [a, d]\overline{[b, c]} = 0$.
6. $a^2A = [aA, a]$ *and* $[a, b]A = [aA, b] + [a, bA]$,
7. $[a^2, c] = [[c, a], a]$ *and* $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$,
8. $S(a) \cdot A = 0$ *and* $S(a) \cdot S(b) = 0$.
9. $S(aA) = [S(a), A]$ *and* $S([a, b]) = [S(a), S(b)]$.
10. $S(a^2) = S(a)^2 = a * \overline{a} + (a\overline{\infty})^2\mathbf{1}$ *and* $S([a, b]) = a * \overline{b} + b * \overline{a}$.

Proof : (In most cases we will leave the proof of the second part of a statement to the reader. Often this amounts to substituting $a+b$ for a in the first part and applying statement 1, or substituting $S(b)$ for A, which is allowed by statement 4.)

1. Evaluating $(a+b)^2 + a^2 + b^2$ by means of (4.1) yields the definition of $[a,b]$.

2. By definition, we find $\infty^2 = (\infty \times \infty)^\# + (\infty\overline{\infty})\infty = (2\overline{\infty})^\# + 3D(\infty)\infty = \infty$ in characteristic 2. Substituting $a = b$ in statement 1 yields $[a,a] = 0$. Finally $[a,\infty] = (a \times \infty) \times (\infty \times \infty) + \langle a, \infty, \infty \rangle \infty = 2(a \times \infty) \times \overline{\infty} + 2(a\overline{\infty})\infty = 0$.

3. From the definition we obtain $a^2\overline{\infty} = (a\overline{\infty})^2 + \langle a^\#, \overline{\infty}, \overline{\infty} \rangle = (a\overline{\infty})^2$.

4. By definition $\tau(S(a)) = a\overline{\infty} + (a\overline{\infty})\tau(1) = 0$. Also $\infty S(a) = [\infty, a] = 0$ by the above, and hence $S(a) \in \mathbf{J}$.

5. We have $[a,b]\overline{c} = bS(a)\overline{c} = c\,\overline{bS(a)} = c\overline{S(a)}\overline{b} = c\overline{S(a)}\overline{b} = [c,a]\overline{b}$. The second part is obtained by multiplying the Jacobi identity to the right with $\overline{d}$ and applying te first part.

6, 7. We find $a^2 A = (a\overline{\infty})aA + (\overline{\infty} \times a^\#)A = (a\overline{\infty})aA + \overline{\infty} \times Aa^\#$, using (2.42). The second term of this expression evaluates to $\overline{\infty} \times Aa^\# = \overline{\infty} \times (aA \times a)$, using (2.41). On the other hand, also $[aA, a] = aA(a * \overline{\infty}) = (a\overline{\infty})aA + (a \times aA) \times \overline{\infty}$, which is the same expression. Statement 7 is obtained by substituting $S(c)$ for A.

8. By definition $S(a) \cdot A = aA\overline{\infty} + (a\overline{\infty})\tau(A) = 0$, because $A \in \mathbf{J}$.

9. Let $b \in \mathbf{V}$. Then $b[S(a), A] = bS(a)A + bAS(a) = [b,a]A + [bA, a] = [aA, b]$ by statement 6. But this is equal to $bS(aA)$.

10. By the above $cS(a^2) = [c, a^2] = [[c,a],a] = cS(a)^2$ for all $c \in \mathbf{V}$. Hence $S(a^2) = S(a)^2$. Also

$$\begin{aligned} a * \overline{a} &= a * (a\overline{\infty})\overline{\infty} + a * (a \times \infty) \\ &= (a\overline{\infty})S(a) + (a\overline{\infty})^2\mathbf{1} + \infty * a^\# + (\infty a^\#)\mathbf{1} \\ &= (a\overline{\infty})S(a) + (a\overline{\infty})^2\mathbf{1} + S(a^\#) = S(a^2) + (a\overline{\infty})^2\mathbf{1}. \end{aligned}$$

∎

Note that this proposition makes $\mathbf{W}, [\cdot,\cdot]$ into a Lie algebra. (The Lie bracket is symmetric in even characteristic and statement 7 provides the Jacobi identity.)

Proposition 4.3 *The map $a \mapsto S(a)$ is a homomorphism of $\mathbf{J}$-modules (and Lie algebras). The image $S(\mathbf{V}) = S(\mathbf{W})$ of this map is an ideal (and hence a subalgebra) of $\mathbf{J}$ of dimension 26, isomorphic to $\mathbf{W}$ as a $\mathbf{J}$-module.*

Proof : Proposition 4.2–9 shows that S is a homomorphism of $\mathbf{J}$-modules. Because $S(\infty) = 0$ we have $S(\mathbf{V}) = S(\mathbf{W})$, hence it only remains to be proved that 0 is the only element $a \in \mathbf{W}$ for which $S(a) = 0$.

Let $a \in \mathbf{W}$. We will prove that $S(a) \cdot A = 0$ for every $A \in \mathbf{L}$ only if $a = 0$.

We easily compute that $S(a) \cdot A = aA\overline{\infty}$. The map $\mathbf{L} \to \mathbf{V} : A \mapsto A\overline{\infty}$ has kernel $\mathbf{J}$, and hence the dimension of its image is $\dim \mathbf{L} - \dim \mathbf{J} = 26$. Also note that $\infty A\overline{\infty} = 0$ by (2.39) and hence this image is a subspace of $\mathbf{W}^*$, and because of the dimension, it is equal to $\mathbf{W}^*$. Therefore $a\alpha = 0$ for all $\alpha \in \mathbf{W}^*$, and hence $a = 0$. ∎

In $\mathbf{W}$ definitions (4.1) can be simplified as follows :

$$\begin{aligned} S(a) &= a * \overline{\infty} = \infty * \overline{a} && \\ [a,b] &= \overline{a} \times \overline{b} + (a\overline{b})\infty, && \text{for all } a \in \mathbf{W}, \\ a^2 &= \overline{a}^\# + (\infty a^\#)\infty, && \text{for all } a, b \in \mathbf{W}. \end{aligned} \tag{4.2}$$

When char $K = 2$ it appears to be more convenient to discard the operators # and $\times$ in favour of the new operators. Note also that

$$\langle a,b,c\rangle = [a,b]\overline{c} = [b,c]\overline{a} = [c,a]\overline{b}, \quad D(a) = a^2\overline{a} \quad \text{for all } a,b,c \in \mathbf{W} \tag{4.3}$$

Clearly, every isotropic element a of $\mathbf{W}$ satisfies $a^2 = 0$. But also the converse is true : if $a \in \mathbf{W}$ and $a^2 = 0$, then $a^\# = (\infty a^\#)\overline{\infty}$, and hence $D(a)a = (a^\#)^\# = (\infty a^\#)^2\infty$, which implies $\infty a^\# = 0$, and hence $a^2 = a^\#$.

In other words : an element a of $\mathbf{W}$ is isotropic if and only if $a^2 = 0$.

Lemma 4.4 *Let $a, b, c, d \in \mathbf{W}$. Then*

$$\begin{array}{lcl} a(b * \overline{c}) & = & (a\overline{b})c + (a\overline{c})b + [[a,b],c] + ([a,b]\overline{c})\infty \\ a(b * \overline{c})\overline{d} & = & (a\overline{b})(c\overline{d}) + (a\overline{c})(b\overline{d}) + [a,b]\,\overline{[c,d]} \\ a^2 * \overline{a} & = & (\infty a^{\#})S(a) + (a^2\overline{a})\mathbf{1}, \\ [a,b] * \overline{a} + a^2 * \overline{b} & = & (\infty a^{\#})S(b) + (a\overline{b})S(a) + (a^2\overline{b})\mathbf{1}, \\ [a,b] * \overline{c} + [b,c] * \overline{a} + [c,a] * \overline{b} & & \\ & = & (a\overline{b})S(c) + (b\overline{c})S(a) + (c\overline{a})S(b) + ([a,b]\overline{c})\mathbf{1}. \end{array} \tag{4.4}$$

Proof : We have

$$\begin{array}{lcl} [[a,b],c] & = & \overline{[a,b]} \times \overline{c} + ([a,b]\overline{c})\infty \\ & = & (a \times b) \times \overline{c} + (a\overline{b})\infty \times \overline{c} + ([a,b]\overline{c})\infty \\ & = & (a \times b) \times \overline{c} + (a\overline{b})c + ([a,b]\overline{c})\infty. \end{array}$$

Also note that $a(b * \overline{c}) = (a\overline{c})b + (a \times b) \times \overline{c}$ by definition. This yields the first equality, and multiplying with $\overline{d}$ yields the second.

We have $a^2 * \overline{a} = \overline{a^{\#}} * \overline{a} + (\infty a^{\#})\infty * \overline{a} = D(a)\mathbf{1} + (\infty a^{\#})S(a)$, by (2.38), yielding the third equality. Substituting $a + kb$ for a in this equality (where k is a formal variable) and considering the terms in k of the result, yields the next equality. From this the last result can be obtained, by substituting $a + c$ for a. ∎

Proposition 4.5 *Let e be an isotropic element of* **W**. *Let $a, b \in \mathbf{W}$, $A, B \in \mathbf{J}$. Then*

1. $[a,e]\overline{e} = 0$, $[a,e]\overline{[b,e]} = 0$ *and* $[a,e]A\overline{e} = 0$,
2. $[a,e]^2 = (a\overline{e})[a,e] + (a^2\overline{e})e$,
3. $[[[a,e],b],e] = [[a,e],[b,e]] = (a\overline{e})[b,e] + (b\overline{e})[a,e] + ([a,b]\overline{e})e$,
4. $eA * \overline{e} \in \mathbf{J}$,
5. $eS(a) * \overline{e} = [e,a] * \overline{e} = (a\overline{e})S(e)$,
6. $eB(eA * \overline{e}) = (eBA\overline{e})\,e$.

Proof :

1. By Proposition 4.2–5 we obtain $[a,e]\overline{e} = [e,e]\overline{a} = 0$ and hence $[a,e]\overline{[b,e]} = [[b,e],e]\overline{a} = [b,e^2]\overline{a} = 0$. Also $[a,e]A\overline{e} = [a,e]\overline{eA} = [e,eA]\overline{a} = e^2A\overline{a} = 0$.

2. By (4.2) we find

$$\begin{aligned} [a,e]^\# &= (\overline{a} \times \overline{e} + (a\overline{e})\infty)^\# \\ &= (\overline{a} \times \overline{e})^\# + (a\overline{e})(\overline{a} \times \overline{e}) \times \infty + (a\overline{e})^2\overline{\infty} \\ &= (\overline{a}^\#\overline{e})\overline{e} + (a\overline{e})\overline{[a,e]}, \end{aligned}$$

applying the definition (4.1) in reverse. It follows that

$$\begin{aligned} [a,e]^2 &= \overline{[a,e]}^\# + (\infty[a,e]^\#)\infty \\ &= (a\overline{e})[a,e] + (ea^\#)e + (a\overline{e})([a,e]\infty)\infty + (ea^\#)(e\infty)\infty \\ &= (a\overline{e})[a,e] + (ea^\#)e \end{aligned}$$

and also $a^2\overline{e} = \overline{a}^\#\overline{e} = ea^\#$.

3. The first equality is a consequence of the Jacobi identity and the fact that $[[a,e],e] = [a,e^2] = 0$. The second identity follows form the first statement of this proposition by substituting $a + b$ for a.

4. We have $\infty(eA * \overline{e}) = \infty\overline{e} + (\infty \times eA) \times \overline{e} = \overline{eA} \times \overline{e} = \overline{eA \times e} = \overline{Ae^\#} = 0$.

5. This is an immediate consequence of (4.4).

6. We use (4.4) to compute

$$eB(eA * \overline{e}) = (eBA\overline{e})e + (eB\overline{e})eA + [[eB,eA],e] + ([eB,eA]\overline{e})\infty.$$

Now $[[eB,eA],e] = [[eB,e],eA] + [[e,eA],eB] = 0$, because $[eA,e] = Ae^2 = 0$, and similarly, $[eB,e] = 0$. Also $eB\overline{e} = 0$ by Proposition 3.11–3 and $[eB,eA]\overline{e} = [e,Be]\overline{eA} = 0$ by Proposition 4.2–5. ∎

By Proposition 4.3 $S(\mathbf{W})$ is an ideal of $\mathbf{J}$ and hence it makes sense to consider the quotient algebra $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{J}/S(\mathbf{W})$. The following proposition proves that many of the operations defined on $\mathbf{J}$ also make sense on $\mathbf{Q}$.

Proposition 4.6 *Let* $A, B \in \mathbf{J}$. *Then*

1. $(A + S(\mathbf{W}))^2 = A^2 + S(\mathbf{W})$,
2. $[A + S(\mathbf{W}), B + S(\mathbf{W})] = [A, B] + S(\mathbf{W})$.
3. $(A + S(\mathbf{W})) \cdot (B + S(\mathbf{W})) = A \cdot B$,
4. *The coset* $A + S(\mathbf{W})$ *contains at most one isotropic element. In particular, the only isotropic element of* $S(\mathbf{W})$ *is* 0.

Proof :

1. Let $a \in \mathbf{W}$. By Proposition 4.2 we find

$$\begin{aligned}(A + S(a))^2 &= A^2 + AS(a) + S(a)A + S(a)^2 \\ &= A^2 + [A, S(a)] + S(a^2) \\ &= A^2 + S(aA) + S(a^2) \in A^2 + S(\mathbf{W}).\end{aligned}$$

2. Note that $S(\mathbf{W})$ is an ideal of $\mathbf{J}$. (This statement also follows by substituting $A + B$ for A in the above).

3. This is a consequence of Proposition 4.2–8.

4. Assume A, B are isotropic elements of $\mathbf{J}$ such that $A = B \bmod S(\mathbf{W})$. By Proposition 3.28, the pair (A, B) can be mapped by $\widehat{\mathsf{F}}_4(K)$ onto a pair of the form (E_r, kE_s) with $r, s \in \Phi_L$, $k \in K$. It is easily seen that the group $\widehat{\mathsf{F}}_4(K)$ leaves $S(\mathbf{W})$ invariant, and hence also $E_r = kE_s \bmod S(\mathbf{W})$. This is only possible when $r = s$ and $k = 1$, i.e., when $A = B$. ∎

Statement 4 of this proposition proves that the notion of isotropic element can be extended to $\mathbf{Q}$.

Note that $\mathbf{Q}$ has dimension 26. It even turns out that algebra $\mathbf{Q}$ is isomorphic to $S(\mathbf{W})$. To prove this, we will express the Lie bracket on both algebras in terms of the corresponding base elements.

The following lemma can be used to establish the structure of $S(\mathbf{W})$.

Lemma 4.7 *Let $p \in \mathcal{P}$. Then*

$$\begin{array}{rcll} S(e_p) & = & E_s, & \textit{when } p \in \mathcal{P}^*, \textit{ with } s = (p+\overline{p})/2 \in \Phi_S \\ S(e_p) & = & H_p + \mathbf{1}, & \textit{when } p \in L_\infty \end{array} \tag{4.5}$$

Proof : Consider $s \in \Phi_S$, $p \in \mathcal{P}^*$, $r \in \Phi$ such that $s = (r+\overline{r})/2 = (p+\overline{p})/2$. By Lemma 3.4, the line at infinity consists of the points p_∞, $p - r$ and $p - \overline{r}$. Hence $\overline{\infty} = \eta_{p_\infty} + \eta_{p-r} + \eta_{p-\overline{r}}$. We then easily compute that $e_p * \overline{\infty} = E_r + E_{\overline{r}} = E_s$.

When $p \in L_\infty$, a similar computation shows that $e_p * \overline{\infty} = H_p$, and because $e_p\overline{\infty} = 1$ in this case, we obtain $S(e_p) = H_p + \mathbf{1}$. ∎

It follows that $S(\mathbf{W})$ can be written as a direct sum in the following way :

$$S(\mathbf{W}) = \mathbf{G}_S \oplus \bigoplus_{s \in \Phi_S} KE_s \tag{4.6}$$

where $\mathbf{G}_S$ denotes the 2-dimensional subspace of $\mathbf{G}$ generated by $H_{z_1} + \mathbf{1}$, $H_{z_2} + \mathbf{1}$ and $H_{z_3} + \mathbf{1}$ with $L_\infty = \{z_1, z_2, z_3\}$. (Note that $\sum_i H_{z_i} + \mathbf{1} = 0$.)

The following lemma proves that $\mathbf{G}_S$ is generated by all H_s with $s \in \Phi_S$.

Lemma 4.8 *Let $s \in \Phi_S$, $p \in \mathcal{P}^*$ be such that $s = (p+\overline{p})/2$. Then $H_s = H_{p_\infty} + \mathbf{1}$.*

Proof : Let $q \in \mathcal{P}$. By $e_q(H_{p_\infty} + \mathbf{1}) = (p_\infty \cdot q + 2/3)e_q$ where $p_\infty \cdot q + 2/3 \in \{0,1,2\}$ should be interpreted as an element of K. Because char $K = 2$, we find that $e_q(H_{p_\infty} + \mathbf{1}) = e_q$ or 0 according to whether $p_\infty \not\perp q$ or not.

Similarly, $e_q H_s = (s^* \cdot q)e_q$ by (3.43) and $s^* \cdot q = 2s \cdot q = p \cdot q + \overline{p} \cdot q$. For $q \in \mathcal{P}^*$, by (3.2), this result is odd if and only if $q_\infty \neq p_\infty$, or equivalently $q \not\perp p$. For $q \in L_\infty$ we have $q = -\overline{q}$ and $\overline{p} \cdot q = p \cdot \overline{q}$, hence $s \cdot q = 0$.

This proves that $e_q H_s = e_q(H_{p_\infty} + \mathbf{1})$ for all $q \in \mathcal{P}$. Hence $H_s = H_p + \mathbf{1}$. ∎

For $p \in \mathcal{P}^*, s \in \Phi_S$ with $s = (p + \overline{p})/2$ it turns out to be convenient to use the notation $e_s \stackrel{\text{def}}{=} e_p$ when working in even characteristic. With this notation we have $S(e_s) = E_s$. We will also write $h_s \stackrel{\text{def}}{=} [e_s, e_{-s}] = e_{p_\infty} + \infty$ and then $S(h_s) = H_s$.

From Proposition 3.19 we obtain the following expressions for the action of the Lie bracket on $S(\mathbf{W})$. Let $r, s \in \Phi_S$. Then

$$\begin{array}{lcll} [E_r, E_{-r}] & = & H_r & \\ [E_r, E_s] & = & E_{r+s}, & \text{when } \langle r, s\rangle = -1, \\ [E_r, E_s] & = & 0, & \text{when } \langle r, s\rangle \leq 0, \\ [E_r, H_s] & = & \langle s, r\rangle E_r, & \\ [H_r, H_s] & = & 0 & \end{array} \tag{4.7}$$

To establish a base of $\mathbf{Q}$ first note that $E_r - E_s \notin S(\mathbf{W})$ whenever $r, s \in \Phi_R$ are different. This proves that the elements $E_r + S(\mathbf{W})$ can be used as the first 24 base elements of $\mathbf{Q}$ and that we may write

$$\mathbf{Q} = \mathbf{G}_L \oplus \bigoplus_{r \in \Phi_F} K(E_r + S(\mathbf{W})) \tag{4.8}$$

where $\mathbf{G}_L \stackrel{\text{def}}{=} \mathbf{G}/S(\mathbf{W})$ is isomorphic to $\mathbf{G}/\mathbf{G}_S$ and has dimension 2.

Lemma 4.9 *Let $r, r' \in \Phi_L$. Then $H_r - H_{r'} \in S(\mathbf{W})$ if and only if $r \cdot r'$ is even (or equivalently $H_r \cdot H_{r'} = 0$). The elements $H_{\psi_1} + S(\mathbf{W})$ and $H_{\psi_2} + S(\mathbf{W})$ form a basis for $\mathbf{G}_L$.*

Proof : Assume $r \cdot r'$ is even. If $r' = r$ or $r' = -r$ then clearly $H_{r'} = H_r = H_{-r}$. Otherwise, by (3.14), $r' = r + 2s$ for some short root s. And then $H_{r'} = H_{r+2s}$. We claim that $H_{r+2s} - H_r = H_s$. Indeed, for all $q \in \mathcal{P}$ we have $e_q H_{r+2s} - e_q H_r = ((r + 2s) \cdot q - r \cdot q)e_q = (2s \cdot q)e_q = \langle s, q\rangle e_q$ because $r' = r + 2s$ and r are long roots, and s is a short root. By Lemma 4.8 $H_s \in S(\mathbf{W})$.

Conversely, if $H = H_r - H_{r'} \in S(\mathbf{W})$ then $H_r \cdot H = H_r \cdot H_{r'}$ because $H_r \cdot H_r = 2 = 0$. And also, by Proposition 4.2–8, $H_r \cdot H = 0$.

Note that $H_{\psi_1} \cdot H_{\psi_2} = \psi_1 \cdot \psi_2 = -1$ and hence $H_{\psi_1} + S(\mathbf{W})$ and $H_{\psi_2} + S(\mathbf{W})$ are linearly independent, by the above. ∎

We may use this information (and that of Proposition 3.19) to derive the following formulas for the Lie bracket on $\mathbf{Q}$. Let $r, s \in \Phi_L$.

$$
\begin{aligned}
{[E_r + S(\mathbf{W}), E_{-r} + S(\mathbf{W})]} &= H_r + S(\mathbf{W}) && \\
{[E_r + S(\mathbf{W}), E_s + S(\mathbf{W})]} &= E_{r+s} + S(\mathbf{W}), && \text{when } \langle r, s \rangle = -1, \\
{[E_r + S(\mathbf{W}), E_s + S(\mathbf{W})]} &= 0, && \text{when } \langle r, s \rangle \leq 0, \\
{[E_r + S(\mathbf{W}), H_s + S(\mathbf{W})]} &= \langle s, r \rangle E_r + S(\mathbf{W}), && \\
{[H_r + S(\mathbf{W}), H_s + S(\mathbf{W})]} &= 0 &&
\end{aligned}
\tag{4.9}
$$

The similarity between (4.7) and (4.9) can be used to show that $S(\mathbf{W})$ and $\mathbf{Q}$ are isomorphic. To establish this isomorphism, we consider the linear transformation † of $\mathbf{P}_F$ defined on the fundamental roots in the following way :

$$\psi_1^\dagger \stackrel{\text{def}}{=} \psi_4^* = 2\psi_4,\ \psi_2^\dagger \stackrel{\text{def}}{=} \psi_3^* = 2\psi_3,\ \psi_3^\dagger \stackrel{\text{def}}{=} \psi_2^* = \psi_2,\ \psi_4^\dagger \stackrel{\text{def}}{=} \psi_1^* = \psi_1. \tag{4.10}$$

This map satisfies $r^\dagger \cdot s^\dagger = 2r \cdot s$ and $r^{\dagger\dagger} = 2r$. It follows that $\langle r^\dagger, s^\dagger \rangle = \langle r, s \rangle$.

The following table lists the 24 short roots $r \in \Phi_S$, the corresponding images $r^\dagger$ and the value of $r + r^\dagger$ when it belongs to Φ_S. Note that $r^\dagger$ is always a long root when r is a short root. Because $r^{\dagger\dagger} = 2r$ the same table can also be used to find the image of a long root (in the second column). Clearly $\Phi_S^\dagger = \Phi_L$ and $\Phi_L^\dagger = 2\Phi_S$.

$r \cdot r^\dagger = 1$

r	$r^\dagger$
$0\underline{1}00$	$0\underline{11}0$
0100	0110
$000\underline{1}$	$\underline{1}00\underline{1}$
0001	1001
++++	$0\underline{1}01$
+--+	$010\underline{1}$
-++-	$0\underline{1}0\underline{1}$
----	0101

$r \cdot r^\dagger = 0$

r	$r^\dagger$
+++-	1100
++-+	$00\underline{1}1$
+-++	$\underline{00}11$
+---	1100
-+++	$110\underline{0}$
-+--	$00\underline{11}$
--+-	$0\underline{0}11$
---+	1100

$r \cdot r^\dagger = -1$

r	$r^\dagger$	$r + r^\dagger$
$\underline{1}000$	$100\underline{1}$	$000\underline{1}$
1000	$10\underline{0}1$	0001
$00\underline{1}0$	$0\underline{1}10$	$0\underline{1}00$
0010	$\underline{0}110$	0100
++--	$\underline{1}0\underline{1}0$	-++-
+-+-	1010	----
-+-+	$10\underline{1}0$	++++
--++	1010	+--+

(4.11)

Note that both Φ_S and Φ_L can be regarded as root systems of type D_4 that are embedded in the larger root system Φ_F of type F_4. The map † serves as an isomorphism between those two root systems.

Theorem 4.10 *Define the linear transformation* $\mu : \mathbf{V} \to \mathbf{Q}$ *as follows :*

$$\begin{array}{rcl} \mu(e_s) & \stackrel{\text{def}}{=} & E_{s^\dagger} + S(\mathbf{W}), \\ \mu(h_s) & \stackrel{\text{def}}{=} & H_{s^\dagger} + S(\mathbf{W}). \end{array} \tag{4.12}$$

Then μ *is well-defined and satisfies*

$$\begin{array}{lcl} \mu(\infty) & = & 0, \\ \mu(a^2) & = & \mu(a)^2, \\ \mu([a,b]) & = & [\mu(a),\mu(b)], \\ \mu(a)\cdot\mu(b) & = & a\overline{b} \end{array} \tag{4.13}$$

for all $a, b \in \mathbf{V}$.

The restriction of μ *to* $\mathbf{W}$ *provides an isomorphism between* $\mathbf{W}$ *and* $\mathbf{Q}$.

Proof : To prove that μ is well-defined we need to check that the second definition in (4.12) is independent of the choice of s. In other words, when $s = (p+\overline{p})/2$ and $t = (q+\overline{q})/2$ with $p, q \in \mathcal{P}^*$ such that $p_\infty = q_\infty$, we must have $H_{s^\dagger} = H_{t^\dagger}$. From (3.1) we know that $p_\infty = q_\infty$ implies that $(p+\overline{p})\cdot q$ is even, and hence $\langle s, t\rangle$ is even. But then also $\langle s^\dagger, t^\dagger\rangle$ is even and therefore $H_{s^\dagger} = H_{t^\dagger} \bmod S(\mathbf{W})$ by Lemma 4.9.

By (4.5) and Lemma 4.8 we see that $\mu(\infty) = \mu(e_{z_1} + e_{z_2} + e_{z_3}) = H_{z_1} + H_{z_2} + H_{z_3} + \mathbf{1} = 0$, with $L_\infty = \{z_1, z_2, z_3\}$. The similarity between (4.7) and (4.9) and the fact that $\langle r^\dagger, s^\dagger\rangle = \langle r, s\rangle$ proves $[\mu(a), \mu(b)] = \mu([a,b])$. This proves that μ is an epimorphism of $\mathbf{V}$ onto $\mathbf{Q}$. Because $\dim \mathbf{V} = \dim \mathbf{Q} + 1$, the kernel of μ must have dimension 1, and is therefore equal to $K\infty$. Therefore μ is an isomorphism when restricted to $\mathbf{W}$.

Because $(a+b)^2 = a^2 + b^2 + [a,b]$ for all $a, b \in \mathbf{W}$ and also $(A+B)^2 = A^2 + B^2 + [A,B]$ for all $A, B \in \mathbf{J}$, by the above it is sufficient to prove $\mu(a^2) = \mu(a)^2$

only for the base elements of $\mathbf{V}$. Firstly, $e_p^2 = 0$ for all $p \in \mathcal{P}^*$ and $E_r^2 = 0$ for all $r \in \Phi$. Secondly, by (4.1), we see that $e_p^2 = e_p$ when $p \in L_\infty$, and also $H_r^2 = H_r$ for all $r \in \Phi$, as H_r is a diagonal matrix with entries 0 and 1.

Finally, for $p, q \in \mathcal{P}$, $e_p\bar{e}_q = e_p\eta_{-q} = 1$ or 0 according to whether $p = -\overline{q}$ or not. Also, for $r, s \in \Phi$, $E_r \cdot E_s = 1$ or 0 according to whether $r = -s$ or not, $E_r \cdot H_s = 0$ and $H_r \cdot H_s = \langle r, s\rangle$. We leave it to the reader to verify that this implies that $\mu(e_p) \cdot \mu(e_q) = 0$ or 1 according to whether $p = -\overline{q}$ or not. ∎

• To simplify notations in the proofs and results below we will often silently use $\mu(a)$ to denote the element $A \in \mathbf{J}$ such that $A = \mu(a) \bmod S(\mathbf{W})$ instead of the coset $A + S(\mathbf{W}) \in \mathbf{Q}$ itself. (Of course this will only be done in those situations where the choice of coset representative is not important.)

4.2 Duality

Let e be an isotropic element of $\mathbf{W}$, let $A \in \mathbf{J}$. The map $A \mapsto eA * \overline{e}$ is a well-defined linear transformation of $\mathbf{Q}$, because by Proposition 4.5–5, $eS(a) * \overline{e} = (a\overline{e})S(e)$ belongs to $S(\mathbf{W})$, for all $a \in \mathbf{W}$. Also note that $(eA * \overline{e}) \cdot B = eAB\overline{e}$ for $B \in \mathbf{J}$, and therefore also the value of $eAB\overline{e}$ is well-defined for $A, B \in \mathbf{Q}$.

Now, define $Q(e)$ to be the unique element of $\mathrm{Hom}(\mathbf{V}, \mathbf{V})$ satisfying

$$aQ(e)\overline{b} \stackrel{\text{def}}{=} e\mu(a)\mu(b)\overline{e} = e\mu(b)\mu(a)\overline{e} = bQ(e)\overline{a}, \quad \text{for every } a, b \in \mathbf{V}. \tag{4.14}$$

By the above, this definition makes sense. Note that $Q(\cdot)$ is a quadratic operator, i.e., $Q(ke) = k^2Q(e)$ for all $k \in K$.

Proposition 4.11 *Let e be an isotropic element of $\mathbf{W}$. Then $Q(e)$ belongs to $\mathbf{J}$ and for all $a \in \mathbf{V}$ we have $\mu(aQ(e)) = e\mu(a) * \overline{e}$.*

Proof: Let $a, b \in \mathbf{V}$. Choose $A, B \in \mathbf{J}$ such that $A = \mu(a), B = \mu(b) \bmod S(\mathbf{V})$. Note that $e\mu(b)\mu(a)\overline{e} = eBA\overline{e} = e\,\overline{eBA} = e\,\overline{A}\,\overline{B}\,\overline{e} = eAB\overline{e}$ by Proposition 3.9–2 and because $A, B \in \mathbf{J}$.

Let $a, b \in \mathbf{V}$. We have $aQ(e)\overline{\infty} = eA\mu(\infty)\overline{e} = 0$ and hence $Q(e)\overline{\infty} = 0$. Similarly $\infty Q(e) = 0$. We also have $aQ(e)\overline{a} = eA^2\overline{e} = 0$, by Proposition 3.11–3 (and because $A^2 \in \mathbf{J}$).

Now, consider $\langle a, aQ(e), b\rangle = aQ(e)(a \times b)$. By (4.1) we have $\overline{a \times b} = [a, b] + (a\overline{b})\infty + (a\overline{\infty})b + (b\overline{\infty})a$, and hence

$$\begin{aligned}
\langle a, aQ(e), b\rangle &= aQ(e)(a \times b) \\
&= aQ(e)\overline{[a,b]} + (a\overline{b})aQ(e)\overline{\infty} + (a\overline{\infty})aQ(e)\overline{b} + (b\overline{\infty})aQ(e)\overline{a}. \\
&= aQ(e)\overline{[a,b]} + (a\overline{\infty})aQ(e)\overline{b} \\
&= eA\mu([a,b])\overline{e} \\
&= eA(AB + BA)\overline{e} + (a\overline{\infty})aQ(e)\overline{b} \\
&= eA^2B\overline{e} + eABA\overline{e} + (a\overline{\infty})aQ(e)\overline{b}.
\end{aligned}$$

The middle term in this last expression is zero by Proposition 3.11–3, with eA substituted for a and B for A.

Similarly $\overline{a^{\#}} = a^2 + (\infty a^{\#})\infty + (a\overline{\infty})a$ and hence

$$\begin{aligned} bQ(e)a^{\#} &= bQ(e)\overline{a^2} + (\infty a^{\#})bQ(e)\overline{\infty} + (a\overline{\infty})bQ(e)\overline{a} \\ &= e\mu(a^2)\mu(b)\overline{e} + (a\overline{\infty})aQ(e)\overline{b} \\ &= eA^2B\overline{e} + (a\overline{\infty})aQ(e)\overline{b} \end{aligned}$$

This proves that $\langle a, aQ(e), b\rangle = bQ(e)a^{\#}$ for all $b \in \mathbf{V}$, and hence $a \times aQ(e) = Q(e)a^{\#}$ for all $a \in \mathbf{V}$. Therefore $Q(e) \in \mathbf{L}$. We have already proved that $\infty Q(e) = 0$ for all $a \in \mathbf{V}$, and hence $Q(e) \in \mathbf{J}$ by Proposition 3.11.

Finally, $aQ(e)\overline{b} = \mu(aQ(e)) \cdot \mu(b)$ by applying the last identity in (4.13). On the other hand, by definition $aQ(e)\overline{b} = (e\mu(a) * \overline{e}) \cdot \mu(b)$. Because the inner product $a\overline{b}$ is non-degenerate on $\mathbf{W}$, also the dot-product on $\mathbf{Q}$ is non-degenerate, and hence $\mu(aQ(e)) = e\mu(a) * \overline{e}$, as $\mu(b)$ ranges over $\mathbf{Q}$ when b ranges over $\mathbf{W}$. ∎

Proposition 4.12 *Let $e, f \in \mathbf{W}$ be such that $e^2 = f^2 = 0$. Then*

1. *$Q(e)$ is isotropic (or zero).*
2. *$Q(e + [e, f] + (e\overline{f})f) = Q(e) + [Q(e), Q(f)] + (e\overline{f})^2 Q(f)$.*
3. *If $[e, f] = 0$ then $Q(e + f) = Q(e) + Q(f) + S(\mu^{-1}(e * \overline{f}))$.*
4. *If $e\overline{f} = 0$, then $Q([e, f]) = [Q(e), Q(f]$.*
5. *If $[e, f] = 0$ then $Q(e)Q(f) = 0$ if and only if $e * \overline{f} = 0$.*
6. *If $A \in \mathbf{J}$ is isotropic and $a \in \mathbf{W}$ such that $A = \mu(a)$, then $Q(eA) = aQ(e) * \overline{a}$.*
7. *$Q(e) \cdot Q(f) = (e\overline{f})^2$*

Proof : Let $a, b, c, d \in \mathbf{V}$. For ease of notation we introduce the following abbreviations :

$$E \stackrel{\text{def}}{=} Q(e),\ F \stackrel{\text{def}}{=} Q(f),\ x \stackrel{\text{def}}{=} aQ(e),\ y \stackrel{\text{def}}{=} bQ(e).$$

Note that $x, y \in \mathbf{W}$. We choose $A, B, C, D, X, Y \in \mathbf{J}$ such that

$$A = \mu(a),\ B = \mu(b),\ C = \mu(c),\ D = \mu(d) \bmod S(\mathbf{W}),$$

and

$$X = \mu(x) = eA * \overline{e},\ Y = \mu(y) = eB * \overline{e} \bmod S(\mathbf{W}).$$

We will also abbreviate $uQ(e)\overline{v} = uE\overline{v}$ to E_{uv} for general $u, v \in \mathbf{W}$. Note that $eX = eY = 0$ and $eCX = eC(eA * \overline{e}) = (eCA\overline{e})e = (aQ(e)\overline{c})e$ by Proposition 4.5–6. Hence

$$eCX = E_{ac}e,\ eDX = E_{ad}e,\ eCY = E_{bc}e,\ eDY = E_{bd}e.$$

Also $eAY = eBX = E_{ab}e$.

1. We compute

$$\begin{aligned}
[[c,x],y]\overline{d} &= [c,x]\overline{[d,y]} = [c,[d,y]]\overline{x} \\
&= \mu([c,[d,y]]) \cdot \mu(x) = [C,[D,Y]] \cdot X \\
&= e[C,[D,Y]]A\overline{e} \\
&= eCDYA\overline{e} + eCYDA\overline{e} + eDYCA\overline{e} + eYDCA\overline{e} \\
&= eCD(E_{ab}\overline{e}) + (E_{bc}e)DA\overline{e} + (E_{bd}e)CA\overline{e} \\
&= E_{ab}(eCD\overline{e}) + E_{bc}(eDA\overline{e}) + E_{bd}(eCA\overline{e})
\end{aligned}$$

This proves

$$[[c,x],y]\overline{d} = E_{ac}E_{bd} + E_{bc}E_{ad} + E_{ab}E_{cd}. \tag{4.15}$$

Setting $a = b$ in this equality yields $[c, x^2]\overline{d} = 0$, for all $c, d \in \mathbf{V}$. Therefore $x^2 = 0$ and by symmetry $y^2 = 0$.

Also

$$x\overline{y} = aQ(e)\overline{y} = eAY\overline{e} = E_{ab}e\overline{e} = 0.$$

This proves that $x * \overline{y}$ is an isotropic element of $\mathbf{L}$. Applying (4.4) we obtain

$$\begin{aligned} c(x*\overline{y})\overline{d} &= (c\overline{x})(y\overline{d}) + (c\overline{y})(x\overline{d}) + [c,x]\,\overline{[d,y]} \\ &= (C\cdot X)(Y\cdot D) + (C\cdot Y)(X\cdot D) + [[c,x],y]\overline{d} \\ &= (eAC\overline{e})(eBD\overline{e}) + (eAD\overline{e})(eBC\overline{e}) + E_{ac}E_{bd} + E_{bc}E_{ad} + E_{ab}E_{cd} \\ &= E_{ab}E_{cd} = E_{ab}cQ(e)\overline{d}. \end{aligned}$$

This is true for all $c, d \in \mathbf{V}$ and therefore $x * \overline{y} = (aQ(e)\overline{b})Q(e)$. If $Q(e) \neq 0$ we may always find a, b such that $aQ(e)\overline{b} = 1$, and then $Q(e)$ is equal to $x * \overline{y}$ and hence isotropic.

2. By Proposition 4.5–2 $[e,f]^2 = (e\overline{f})[e,f]$ and hence $(e + [e,f] + (e\overline{f})f)^2 = e^2 + [e,f]^2 + (e\overline{f})^2 f^2 + [e,[e,f]] + (e\overline{f})[e,f] + (e\overline{f})[[e,f],f] = 0$, so $e + [e,f] + (e\overline{f})f$ is isotropic and $Q(e + [e,f] + (e\overline{f})f)$ is well-defined. We have

$$\begin{aligned} &aQ(e + [e,f] + (e\overline{f})f)\overline{b} \\ &\quad= (e + [e,f] + (e\overline{f})f)AB(\overline{e} + [\overline{e},\overline{f}] + (e\overline{f})\overline{f}) \\ &\quad= eAB\overline{e} + (e\overline{f})eAB\overline{f} + (e\overline{f})fAB\overline{e} + (e\overline{f})^2 fAB\overline{f} + \\ &\qquad [e,f]AB\overline{e} + (e\overline{f})[e,f]AB\overline{f} + eAB[\overline{e},\overline{f}] + (e\overline{f})fAB[\overline{e},\overline{f}] + [e,f]AB[\overline{e},\overline{f}] \end{aligned} \tag{4.16}$$

Now $eAB\overline{e} = aQ(e)\overline{b}$ and $fAB\overline{f} = aQ(f)\overline{b}$ by definition. Also,

$$[e,f]AB\overline{e} + eAB[\overline{e},\overline{f}] = [e,f]AB\overline{e} + [e,f]BA\overline{e} = [e,f][A,B]\overline{e} = 0,$$

by Proposition 4.2. Likewise $[e,f]AB\overline{f} + fAB[\overline{e},\overline{f}] = 0$.

Finally, we compute

$$\begin{aligned} aEF\overline{b} &= xQ(f)\overline{b} = fXB\overline{f} = f(eA*\overline{e})B\overline{f} \\ &= (f\overline{e})eAB\overline{f} + (fA\overline{e})(eB\overline{f}) + [f,eA]\overline{[e,fB]} \\ &= (e\overline{f})eAB\overline{f} + (eA\overline{f})(eB\overline{f}) + [fB,eA]\overline{[e,f]}, \quad \text{by Proposition 4.2–5} \\ &= (e\overline{f})eAB\overline{f} + (eA\overline{f})(eB\overline{f}) + [fB,e]A\overline{[e,f]} + [fBA,e]\overline{[e,f]} \\ &= (e\overline{f})eAB\overline{f} + (eA\overline{f})(eB\overline{f}) + [fB,e]A\overline{[e,f]}, \quad \text{by Proposition 4.5–1} \end{aligned}$$

and hence, adding the same formula with the roles of e and f interchanged,

$$\begin{aligned} a[E,F]\overline{b} &= (e\overline{f})eAB\overline{f} + (e\overline{f})fAB\overline{e} + [fB,e]A\overline{[e,f]} + [eB,f]A\overline{[e,f]} \\ &= (e\overline{f})eAB\overline{f} + (e\overline{f})fAB\overline{e} + [e,f]BA\overline{[e,f]}. \end{aligned}$$

Substituting these results into (4.16) we obtain

$$aQ(e+[e,f]+(e\overline{f})f)\overline{b} = aQ(e)\overline{b} + a[Q(e),Q(f)]\overline{b} + (e\overline{f})^2 aQ(f)\overline{b}.$$

This is true for all $a,b \in \mathbf{V}$, hence $Q(e+[e,f]+(e\overline{f})f) = Q(e)+[Q(e),Q(f)]+(e\overline{f})^2 Q(f)$.

3. Note that $e+f$ is isotropic and hence $Q(e+f)$ is well-defined. Applying the definition of $Q(\cdot)$, we find

$$\begin{aligned}
&aQ(e+f)\overline{b} - aQ(e)\overline{b} - aQ(f)\overline{b} \\
&\quad= (e+f)AB(\overline{e+f}) - eAB\overline{e} - fAB\overline{f} \\
&\quad= eAB\overline{f} + fAB\overline{e} \\
&\quad= eAB\overline{f} + eBA\overline{f} \\
&\quad= e[A,B]\overline{f} \\
&\quad= (e*\overline{f})\cdot[A,B] \\
&\quad= [a,b]\overline{\mu^{-1}(e*\overline{f})} = [a,\mu^{-1}(e*\overline{f})]\overline{b} = aS(\mu^{-1}(e*\overline{f}))\overline{b}
\end{aligned}$$

This is true for all $a,b \in \mathbf{V}$, hence $Q(e+f)-Q(e)-Q(f) = S(\mu^{-1}(e*\overline{f}))$.

4. Applying statement 2 of this proposition to the case $e\overline{f} = 0$ yields $[Q(e), Q(f)] = Q(e)+Q(e+[e,f])$. Because $[e,[e,f]] = 0$ and $[e,f]*\overline{e} = 0$, we may apply statement 3 to find $Q(e+[e,f]) = Q(e)+Q([e,f])$. Hence $[Q(e),Q(f)] = Q([e,f])$.

5. Let $[e,f] = 0$. From the proof of statement 2 above we know that in this case $aEF\overline{b} = (eA\overline{f})(eB\overline{f})$. Clearly, if $e*\overline{f} = 0$ then also $eA\overline{f} = (e*\overline{f})\cdot A = 0$, and then $EF = 0$. Conversely, if $EF = 0$ then $(e*\overline{f})\cdot A = 0$ for all $A \in \mathbf{J}$ and then $e*\overline{f} = 0 \bmod S(\mathbf{W})$, because the dot-product is non-degenerate on $\mathbf{Q}$. Because $e*\overline{f}$ is isotropic, we must have $e*\overline{f} = 0$ by Proposition 4.6–4.

6. (In this part of the proof A is isotropic and a belongs to $\mathbf{W}$.)

Note that eA is isotropic because A is isotropic and hence $Q(eA)$ is well-defined. We compute

$$\begin{aligned} c(aQ(e) * \overline{a})\overline{d} &= c(x * \overline{a})\overline{d} = (x\overline{c})(a\overline{d}) + (a\overline{c})(x\overline{d}) + [c,x]\overline{[a,d]} \\ &= E_{ac}a\overline{d} + E_{ad}a\overline{c} + [[a,c],d]\overline{x} \end{aligned}$$

Now

$$\begin{aligned} [[a,c],d]\overline{x} &= aQ(e)\overline{[[a,c],d]} \\ &= eA[[A,C],D]\overline{e} \\ &= eA^2CD\overline{e} + eACAD\overline{e} + eADAC\overline{e} + eADCA \\ &= (A \cdot C)eAD\overline{e} + (A \cdot D)eAC\overline{e} + eADCA\overline{e} \\ &= (a\overline{c})E_{ad} + (a\overline{d})E_{ac} + cQ(eA)\overline{d}. \end{aligned}$$

Combining this with the previous result yields $c(aQ(e) * \overline{a})\overline{d} = cQ(eA)\overline{d}$ for all $c, d \in \mathbf{V}$, and then $aQ(e) * \overline{a} = Q(eA)$.

7. Because E, F are isotropic by statement 1 of this proposition, we have $EFE = (E \cdot F)E$. Now

$$\begin{aligned} aEFE\overline{b} &= f\mu(aE)\mu(bE)\overline{f} = fXY\overline{f} = f(eA * \overline{e})Y\overline{f} \\ &= (eAY\overline{f})(e\overline{f}) + (fA\overline{e})(eY\overline{f}) + [eA,f]\overline{[eY,\overline{f}]} \\ &= E_{ab}(e\overline{f})^2 = (aE\overline{b})(e\overline{f})^2, \end{aligned}$$

and this is true for every $a, b \in \mathbf{V}$. ∎

• Up to now we have only used the properties of μ listed in (4.13) to prove the various properties of $Q(\cdot)$. Proposition 4.13 below is the first to make use of the actual definition given in (4.12).

In what follows we will use the Frobenius endomorphism $k \mapsto k^{\mathrm{frob}} \stackrel{\mathrm{def}}{=} k^2$ over K to $\mathbf{V}$ by applying it to the coordinates with respect to the Chevalley basis, i.e.,

$$a^{\mathrm{frob}} \stackrel{\mathrm{def}}{=} \sum_{p \in \mathcal{P}} a[p]^2 e_p.$$

Note that the map $\cdot^{\text{frob}}$ satisfies $[a,b]^{\text{frob}} = [a^{\text{frob}}, b^{\text{frob}}]$, $(a^2)^{\text{frob}} = (a^{\text{frob}})^2$, $\overline{a^{\text{frob}}} = \overline{a}^{\text{frob}}$ and $a^{\text{frob}}\overline{b}^{\text{frob}} = (a\overline{b})^2$. The action of $\cdot^{\text{frob}}$ can be extended in a natural way to $\text{Hom}(\mathbf{V},\mathbf{V})$ (and hence on $\widehat{\mathbf{L}}$) by the identity $a^{\text{frob}}A^{\text{frob}}\alpha^{\text{frob}} = (aA\alpha)^{\text{frob}}$.

• In general, the operation $\cdot^{\text{frob}}$ is not an automorphism of $\mathbf{V}$, because it is semi-linear but not linear (except when $K = \text{GF}(2)$ in which case it is the identity). However, it does satisfy (2.80), i.e., $(a^{\text{frob}})^{\#} = (a^{\#})^{\text{frob}}$, for all $a \in \mathbf{V}$.

The operation $\cdot^{\text{frob}}$ is a bijection on $\mathbf{V}$ if and only it is a bijection on the field K, i.e., if and only if $K^{\text{frob}} = K$. In that case the field is called *perfectperfect field*. In general, K is called perfect when $K^p = K$, with $p = \text{char}\, K \neq 0$. Every finite field is perfect because the Frobenius morphism is an injection and hence K^p must be a subfield of K of the same size.

Proposition 4.13 *Let e be an isotropic element of $\mathbf{W}$. Then*

$$Q(e) = \mu(e^{\text{frob}}) \bmod S(\mathbf{W}). \tag{4.17}$$

Proof : We will first prove this proposition for $e = e_p$ with $p \in \mathcal{P}^*$. Let $q \in \mathcal{P}$. We will compute $e_q Q(e_p)$. Write $s = (p + \overline{p})/2$.

First assume $q \in \mathcal{P}^*$ and write $t = (q + \overline{q})/2$. By Proposition 4.11 we have $\mu(e_q Q(e_p)) = e_p \mu(e_q) * \eta_{-\overline{p}} = e_p E_{t^\dagger} * \eta_{-\overline{p}}$. When $p \cdot t^\dagger \geq 0$ we find that $\mu(e_q Q(e_p)) = 0$ and then $e_q Q(e_p) = 0$. Otherwise, we have $p + t^\dagger \in \mathcal{P}$ and then $(p + t^\dagger) \cdot (-\overline{p}) = 1/3$ so $r \stackrel{\text{def}}{=} p + t^\dagger + \overline{p} \in \Phi$ and $\mu(e_q Q(e_p)) = E_r$. We have $r = 2s + t^\dagger = (s^\dagger + t)^\dagger$ and $s^\dagger + t = (q + s^\dagger + \overline{q} + s^\dagger)/2$, hence $E_r = \mu(e_{q+s^\dagger})$. Because $q \cdot s^\dagger = t \cdot s^\dagger = s \cdot t^\dagger = p \cdot t^\dagger$ we see that $\mu(e_q Q(e_p)) = \mu(e_q E_{s^\dagger})$ in both cases, and hence $e_q Q(e_p) = e_q E_{s^\dagger}$ because both sides belong to $\mathbf{W}$.

Similarly, for $z \in L_\infty$ we have $\mu(e_z Q(e_p)) = e_p H * \eta_{-\overline{p}}$ for some diagonal element $H \in \mathbf{G}_L$. The value of this expression is a multiple of $e_p * \eta_{-\overline{p}}$ and this is zero. Note that also $e_z E_{s^\dagger} = 0$ because $s^\dagger \in \Phi_L$.

This proves that $e_q Q(e_p) = e_q E_{s^\dagger}$ for all $q \in \mathcal{P}$ and hence that $Q(e_p) = E_{s^\dagger} = \mu(e_p) \bmod S(\mathbf{W})$.

Finally, consider a general isotropic element e. Let e' be the unique element in $\mathbf{W}$ such that $\mu(e') = Q(e)$. We need to prove that $e'[p] = e[p]^2$ for all $p \in \mathcal{P}$. For $p \in \mathcal{P}^*$ we may write $e'[p] = e'\eta_p = e'\overline{e_{-\overline{p}}} = \mu(e) \cdot \mu(e_{-\overline{p}}) = Q(e) \cdot Q(e_{-\overline{p}}) = (e \cdot \overline{e_{-\overline{p}}})^2$ by Proposition 4.12–7. And this is equal to $e[p]^2$.

We still need to determine the coordinates of e' 'at infinity'. For this purpose we will deduce a special property of coordinates of isotropic elements of $\mathbf{W}$. Note that $Q(e)$ is isotropic and therefore $Q(e)^2 = 0$ and $e'^2 = 0$, hence e' is isotropic. Also e is isotropic.

Let $L_\infty = \{z, z', z''\}$. Expressing the fact that $e^\# \eta_z = e^\# \eta_{z'} = e^\# \eta_{z''} = 0$ in terms of coordinates, yields

$$\begin{array}{rcl} e[z']e[z''] & = & Z \stackrel{\text{def}}{=} \displaystyle\sum_{\{p,-\overline{p}\}, p_\infty = z} e[p]e[-\overline{p}], \\ e[z'']e[z] & = & Z' \stackrel{\text{def}}{=} \displaystyle\sum_{\{p,-\overline{p}\}, p_\infty = z'} e[p]e[-\overline{p}], \\ e[z]e[z'] & = & Z'' \stackrel{\text{def}}{=} \displaystyle\sum_{\{p,-\overline{p}\}, p_\infty = z''} e[p]e[-\overline{p}], \end{array}$$

and the fact that $e\overline{\infty} = 0$ yields $e[z] + e[z'] + e[z''] = 0$. Hence, adding these identities two by two, we find

$$e[z]^2 = Z' + Z'', \quad e[z']^2 = Z + Z'', \quad e[z'']^2 = Z + Z''.$$

A similar identity holds for e' and using the fact that $e'[p] = e[p]^2$ when $p \in \mathcal{P}^*$, we obtain

$$e'[z]^2 = Z'^2 + Z''^2, \quad e'[z']^2 = Z^2 + Z''^2, \quad e'[z'']^2 = Z^2 + Z''^2.$$

It follows that $e'[z] = e[z]^2$, $e'[z'] = e[z']^2$ and $e'[z''] = e'[z'']^2$. ∎

It is a consequence of this proposition that $Q(e) = 0$ if and only if $e = 0$. Also, applying Proposition 4.12–6 to $a = f^{\text{frob}}$ and $A = Q(f)$, now yields

$$Q(eQ(f)) = f^{\text{frob}} Q(e) * \overline{f}^{\text{frob}}, \quad \text{for all } e, f \in \mathbf{W} \text{ such that } e^2 = f^2 = 0. \quad (4.18)$$

The following proposition provides some explicit example values of $Q(e)$.

Proposition 4.14 *Let $s, t \in \Phi_S$. Let $k, \ell \in K$. Then*

$$
\begin{array}{lcll}
Q(ke_s) & = & k^2 E_{s^\dagger}, & \\
Q(ke_s + \ell e_{-s} + \sqrt{k\ell} h_s) & = & k^2 E_{s^\dagger} + \ell^2 E_{-s^\dagger} + k\ell H_{s^\dagger}, & \\
Q(ke_s + \ell e_t) & = & k^2 E_{s^\dagger} + \ell^2 E_{t^\dagger}, & \text{when } \langle s, t\rangle = 1, \\
Q(ke_s + \ell e_t) & = & k^2 E_{s^\dagger} + \ell^2 E_{t^\dagger} + k\ell E_{(s+t)^{\dagger *}}, & \text{when } \langle s, t\rangle = 0.
\end{array}
\tag{4.19}
$$

Proof : The first identity can be found in the proof of Proposition 4.13 for the special case $k = 1$. The case for general k then follows from the fact that $Q(\cdot)$ is quadratic.

The second identity follows from the first, using Proposition 4.12–2 with $e = ke_s$ and $f = \sqrt{\ell/k}e_{-s}$. The third and fourth identities are a consequence of Proposition 4.12–3. Note that $\langle s, t\rangle = 1$ implies $e_s * \overline{e_t} = 0$. Likewise, if $\langle s, t\rangle = 0$ then $e_s * \overline{e_t} = E_{s+t}$ by Proposition 3.46, and then $S(\mu^{-1}(E_{s+t})) = E_{(s+t)^{\dagger *}}$. ∎

4.3 The graph endomorphism of $\widehat{\mathsf{F}}_4(K)$

Let e be an isotropic element of $\mathbf{W}$. Define the linear map

$$x(e) : \mathbf{V} \to \mathbf{V} : a \mapsto a^{x(e)} \stackrel{\text{def}}{=} a + [e, a] + (a\bar{e})e. \tag{4.20}$$

In the special case of $e = ke_p$ with $k \in K$, $p \in \mathcal{P}^*$ it easily follows from (3.52) that $x(ke_p)$ is equal to $x_s(k)$ with $s = (p + \overline{p})/2$, using $kE_S = S(ke_p)$ by (4.5). Because $\widehat{\mathsf{F}}_4(K)$ is transitive on all isotropic elements of $\mathbf{W}$, we see that $x(e)$ must belong to $\widehat{\mathsf{F}}_4(K)$ also in the general case.

Generalizing the last statement of (3.52) using a similar argument and applying Proposition 4.2, yields

$$A^{x(e)} \stackrel{\text{def}}{=} A + S(eA) + eA * \bar{e}. \tag{4.21}$$

Note that $x(ke)x(k'e) = x((k + k')e)$ for all $k, k' \in K$ and that in particular $x(e)^2 = 1$. It also follows that $\widehat{\mathsf{F}}_4(K)$ is generated by all elements $x(e)$ and $x(E)$ where e is an isotropic element of $\mathbf{W}$ and E is an isotropic element of $\mathbf{J}$.

Lemma 4.15 *Let e, f be isotropic elements of* $\mathbf{W}$ *such that $e\overline{f} = 1$. Define*

$$n(e, f) \stackrel{\text{def}}{=} x(e)x(f)x(e). \tag{4.22}$$

Then $n(e, f) = n(f, e)$ and

$$e^{n(e,f)} = f, \quad f^{n(e,f)} = e. \tag{4.23}$$

Proof : Note that $e^{x(e)} = e$, $f^{x(f)} = f$ and $e^{x(f)} = e + f + [e, f] = f^{x(e)}$. From $x(e)^2 = 1$, we then obtain $(e + f + [e, f])^{x(e)} = f$. It follows that $e^{n(e,f)} = e^{x(e)x(f)x(e)} = f$ and $f^{n(f,e)} = e$, by symmetry. Applying (4.24) we obtain $n(e, f) = x(e)x(f)x(e) = x(f^{x(e)}) = x(e + f + [e, f])$ which is symmetric in e and f. This implies $n(e, f) = n(f, e)$. ∎

• As with the elements $n(E, F)$ of Chapter 2 the elements $n(e, f)$ can be used in a short proof of the fact that the non-zero isotropic elements of $\mathbf{W}$ form a single orbit of $\widehat{\mathsf{F}}_4(K)$.

Consider $p \in \mathcal{P}^*$, $k \in K - \{0\}$ then $n(ke_p, k^{-1}e_{-\overline{p}}) = x(kE_s)x(k^{-1}E_{-s})x(kE_s)$ with $s = (p + \overline{p})/2$, and this is equal to $n_s(k)$ as defined in (3.58).

Let e be an isotropic element of $\mathbf{W}$. Applying an automorphism g of $\mathbf{W}$ to (4.20) we obtain $(a^{x(e)})^g = (a^g)^{x(e^g)}$ and hence

$$x(e)^g = x(e^g), \tag{4.24}$$

for all isotropic elements $e \in \mathbf{W}$ and all automorphisms g of $\mathbf{J}$. This result is similar to (2.87).

Formula (4.24) can be extended to endomorphisms like $\cdot^{\text{frob}}$. Let g be an automorphism of $\mathbf{W}$ and define g^{frob} to be the transformation that maps a^{frob} onto $(a^g)^{\text{frob}}$ for all $a \in \mathbf{V}$. Then g^{frob} is an automorphism of $\mathbf{W}^{\text{frob}}$. Applying the Frobenius map to (4.20) we find that $(a^{\text{frob}})^{x(e^{\text{frob}})} = (a^{x(e)})^{\text{frob}}$ for all $a \in \mathbf{V}$, and hence

$$x(e)^{\text{frob}} = x(e^{\text{frob}}), \tag{4.25}$$

for all isotropic elements $e \in \mathbf{W}$. Because $\widehat{\mathsf{F}}_4(K)$ is generated by elements of the form $x(e)$, we find that $g^{\text{frob}} \in \widehat{\mathsf{F}}_4(K^2)$ whenever $g \in \widehat{\mathsf{F}}_4(K)$.

We are now in the position to extend the commutation relations of Proposition 2.39 with some new identities involving elements of the form $x(e)$ with $e \in \mathbf{W}$.

Proposition 4.16 *Let e, f be isotropic elements of $\mathbf{W}$ such that $e\overline{f} = 0$. Let E be an isotropic element of $\mathbf{J}$. Then*

$$\begin{array}{rcl} x(e)x(f) & = & x(f)x(e)x([e,f]), \\ x(e)x(E) & = & x(E)x(e)x(eE)x(eE * \overline{e}), \end{array} \tag{4.26}$$

If also $[e, f] = 0$ then

$$x(f * \overline{e}) = x(e * \overline{f}) = x(e)x(f)x(e + f). \tag{4.27}$$

Proof : First consider the case $[e, f] = 0$. Note that in that case $f^{x(e)} = f +$

$[e, f] + (e\overline{f})e = f$. For $a \in \mathbf{W}$ we find

$$\begin{aligned} a^{x(f)x(e)} &= (a + [f, a] + (a\overline{f})f)^{x(e)} \\ &= a^{x(e)} + [f^{x(e)}, a^{x(e)}] + (a\overline{f})f^{x(e)} \\ &= a + [e, a] + (a\overline{e})e + [f, a] + [f, [e, a]] + (a\overline{e})[e, f] + (a\overline{f})f \\ &= a + [e, a] + [f, a] + [[a, e], f] + (a\overline{e})e + (a\overline{f})f, \end{aligned}$$

and then

$$a^{x(f)x(e)x(e*f)} = a^{x(f)x(e)} + [a^{x(f)x(e)}, e * \overline{f}].$$

Using Lemma 4.4 to discard most of the resulting terms, we find $[a^{x(f)x(e)}, e * \overline{f}] = [a, e * \overline{f}]$ and we finally obtain

$$\begin{aligned} a^{x(f)x(e)x(e*f)} &= a + [e, a] + [f, a] + (a\overline{e})f + (a\overline{f})e + (a\overline{e})e + (a\overline{f})f \\ &= a + [e + f, a] + ((\overline{e} + \overline{f})a)(e + f) = a^{x(e+f)}. \end{aligned}$$

Rearranging terms yields (4.27).

Now, for general values of $[e, f]$, by (4.24) we have

$$x(f)x(e)x(f) = x(e^{x(f)}) = x(e + [e, f] + (e\overline{f})f) = x(e + [e, f]).$$

Because $[e, f]\overline{e} = 0$ by Proposition 4.5–1 and also $[[e, f], e] = [f, e^2] = 0$, we may apply (4.27) to find that $x(e + [e, f]) = x(e)x([e, f])x([e, f] * \overline{e}) = x(e)x([e, f])$, because $[e, f] * \overline{e} = 0$ by Proposition 4.5–6. This proves the first identity of (4.26).

Similarly,

$$x(E)x(e)x(E) = x(e + eE) = x(e)x(eE)x(eE * \overline{e}),$$

because $eE\overline{e} = 0$ and $[e, eE] = 0$. This yields the second identity. ∎

Because every isotropic element E of $\mathbf{J}$ can be written as $E = e * \overline{f}$ with $e, f \in \mathbf{W}$ isotropic such that $[e, f] = 0$ it follows from (4.27) that $\widehat{\mathsf{F}}_4(K)$ can be generated by elements of the form $x(e)$ with $e \in \mathbf{W}$ isotropic.

Because $\widehat{\mathsf{F}}_4(K)$ leaves $S(\mathbf{W})$ invariant, the action of $\widehat{\mathsf{F}}_4(K)$ on $\mathbf{L}$ can be extended to $\mathbf{Q}$ by setting $(A + S(\mathbf{W}))^g = A^g + S(\mathbf{W})$ for all $A \in \mathbf{J}$, $g \in \widehat{\mathsf{F}}_4(K)$. Hence the operation † from the following proposition is well-defined.

Proposition 4.17 *Let $g \in \widehat{\mathsf{F}}_4(K)$. Define*

$$g^\dagger \stackrel{\text{def}}{=} \mu g \mu^{-1}. \tag{4.28}$$

Then $g^\dagger \in \widehat{\mathsf{F}}_4(K)$, and

$$x(e)^\dagger = x(Q(e)), \qquad x(E)^\dagger = x(e'), \tag{4.29}$$

for every isotropic element $e \in \mathbf{W}$ and $E \in \mathbf{J}$, where e' is the unique element of $\mathbf{W}$ such that $E = \mu(e')$.

Proof : It is sufficient to prove (4.29).

Let $a \in \mathbf{W}$ and write $A = \mu(a) \bmod S(\mathbf{W})$. By definition, $x(e)^\dagger$ maps a onto $\mu^{-1}(A + S(eA) + eA * \bar{e})$ which is equal to $a + aQ(e)$ by Proposition 4.11. This proves the first part of (4.29). Similarly, $x(E)$ maps a onto $\mu^{-1}(A + [A, E] + (A \cdot E)E) = a + [a, e'] + (a\bar{e}')e'$. ∎

Definition (4.28) also implies

$$\mu(a)^g = \mu(a^{g^\dagger}), \qquad \text{for all } a \in \mathbf{V}. \tag{4.30}$$

As a particular case of (4.29) we find

$$x_s(k)^\dagger = x_{s^\dagger}(k^2), \qquad x_{s^\dagger}(k)^\dagger = x_s(k), \tag{4.31}$$

for any $k \in K$ and $s \in \Phi_S$.

As a further consequence of Proposition 4.17 we have

$$n(e, f)^\dagger = n(Q(e), Q(f)), \tag{4.32}$$

for isotropic elements $e, f \in \mathbf{W}$ satisfying $e\overline{f} = 1$ (and hence $Q(e) \cdot Q(f) = 1$).

Note that by definition $(gh)^\dagger = g^\dagger h^\dagger$ for any $g, h \in \widehat{\mathsf{F}}_4(K)$. This proves that $\dagger$ is an endomorphism of the group $\widehat{\mathsf{F}}_4(K)$ (called a *graph* endomorphism of $\widehat{\mathsf{F}}_4(K)$). If e is an isotropic element of $\mathbf{W}$, we may use (4.17) and (4.29) to obtain

$x(e)^{\dagger\dagger} = x(e^{\text{frob}}) = x(e)^{\text{frob}}$, and in general, because $\widehat{\mathsf{F}}_4(K)$ can be generated by elements of the form $x(e)$, we have $g^{\dagger\dagger} = g^{\text{frob}}$, or

$$g\,\text{frob} = \text{frob}\,g^{\dagger\dagger}, \quad \text{for } g \in \widehat{\mathsf{F}}_4(K). \tag{4.33}$$

• The endomorphism $\dagger$ is not necessarily an automorphism of $\widehat{\mathsf{F}}_4(K)$ for its image might be a proper subgroup of $\widehat{\mathsf{F}}_4(K)$. From (4.33) it follows that $\dagger$ will be an automorphism of $\widehat{\mathsf{F}}_4(K)$ if and only if the field K is perfect.

Proposition 4.18 *Let e be an isotropic element of* $\mathbf{W}$. *Let $g \in \widehat{\mathsf{F}}_4(K)$. Then*

$$Q(e^g) = Q(e)^{g^\dagger}. \tag{4.34}$$

Proof: Let $a, b \in \mathbf{W}$, $g \in \widehat{\mathsf{F}}_4(K)$. We have

$$\begin{aligned} a^{g^\dagger} Q(e)^{g^\dagger} \overline{b}^{g^\dagger} &= aQ(e)\overline{b} \\ &= e\mu(a)\mu(b)\overline{e} \\ &= e^g \mu(a)^g \mu(b)^g \overline{e}^g \\ &= e^g \mu(a^{g^\dagger})\mu(b^{g^\dagger})^g \overline{e}^g = a^{g^\dagger} Q(e^g)\overline{b^{g^\dagger}} = a^{g^\dagger} Q(e^g)\overline{b}^{g^\dagger}, \end{aligned}$$

and this is true for all a, b. ∎

• Let e, f denote isotropic elements of $\mathbf{W}$ and set $g = x(f)$ (and $g^\dagger = x(Q(f))$) in (4.34). We obtain $Q(e + [e, f] + (e\overline{f})f) = Q(e) + [Q(e), Q(f)] + (Q(e) \cdot Q(f))Q(f)$, which is equivalent to Proposition 4.12–2.

5 The perfect Ree-Tits generalized octagon

Apart from requiring that K is a field of characteristic 2, as in the previous chapter, we will additionally assume throughout this chapter that the field K is perfect (i.e., $K^2 = K$) and that K has a *Tits endomorphism*, i.e., a field endomorphism σ with the property

$$(k^\sigma)^\sigma = k^2, \quad \text{for every } k \in K. \tag{5.1}$$

Because K is assumed to be perfect the Tits endomorphism is an automorphism, i.e., $K = K^\sigma$ and $k^{\sigma^{-1}}$ is defined for every $k \in K$. For finite fields a Tits automorphism exists if and only if the order of the field is an odd power 2^{2m+1} of 2, and then $k^\sigma = k^{2^{m+1}}$.

• Note that $k^{\sigma^{-1}} = \sqrt{k^\sigma} = (k^\sigma)^{1/2}$ and we will always write this as $k^{\sigma/2}$. (Note that the square root operator is well-defined. Because $\operatorname{char} K = 2$ every element K has at most one square root — we need not worry about the sign — and because K is perfect, every element has at least one square root.) Formally the symbol σ behaves very much like an exponent of value $\sqrt{2}$.

As with the Frobenius endomorphism we will extend σ to $\mathbf{V}$ by applying it to the coordinates with respect to the Chevalley basis, i.e.,

$$a^\sigma \stackrel{\text{def}}{=} \sum_{p\in\mathcal{P}} a[p]^\sigma e_p.$$

We have $a^{\text{frob}} = (a^\sigma)^\sigma$ for every $a \in \mathbf{W}$ and $A^{\text{frob}} = (A^\sigma)^\sigma$ for every $A \in \widehat{\mathbf{L}}$. Also $[a,b]^\sigma = [a^\sigma, b^\sigma]$, $(a^2)^\sigma = (a^\sigma)^2$, $a^\sigma \overline{b^\sigma} = (a\overline{b})^\sigma$, ..., as expected. When e (and hence e^σ) is an isotropic element of $\mathbf{W}$, we find that $Q(e^\sigma) = Q(e)^\sigma$.

• We regret that we have to restrict ourselves to the case of a field K that is *perfect*, for a lot of the results of this chapter also hold in the non-perfect case. Unfortunately, the proofs in this chapter frequently use the fact that σ has an inverse σ^{-1} and we also occasionally have to extract a square root of an element of K. At this moment we do not see how to avoid this.

5.1 A root system of type ${}^2\mathsf{F}_4$

Lemma 5.1 *Let $r \in \Phi_S$. Let $x \in \mathbf{P}_F$.*

If $r \cdot r^\dagger = 0$, then

$$w_{\sqrt{2}r+r^\dagger}(\sqrt{2}x + x^\dagger) = w_r(w_{r^\dagger}(\sqrt{2}x + x^\dagger)) \tag{5.2}$$

Otherwise, $r \cdot r^\dagger = \pm 1$, and then

$$w_{\sqrt{2}r+r^\dagger}(\sqrt{2}x + x^\dagger) = w_r(w_{r^\dagger}(w_r(w_{r^\dagger}(\sqrt{2}x + x^\dagger)))). \tag{5.3}$$

Proof : Let $x \in \mathbf{P}_F$. For ease of notation we write $s = \sqrt{2}r + r^\dagger$, $y = \sqrt{2}x + x^\dagger$. Note that $s^\dagger = \sqrt{2}s$ and $y^\dagger = \sqrt{2}y$. From $r \cdot x^\dagger = r^\dagger \cdot x$ and $r^\dagger \cdot x^\dagger = 2r \cdot x$ we derive $r \cdot y = s \cdot x$, $r^\dagger \cdot y = \sqrt{2}r \cdot y$ and $s \cdot y = 2\sqrt{2}r \cdot y$. Also $w_{r^\dagger}(y) = y - (r^\dagger \cdot y)r^\dagger$, $w_r(y) = y - 2(r \cdot y)r$ and $w_r(r^\dagger) = r^\dagger - 2(r \cdot r^\dagger)r$.

Assume $r \cdot r^\dagger = 0$. Then

$$\begin{aligned} w_r(w_{r^\dagger}(y)) &= w_r(y - (r^\dagger \cdot y)r^\dagger) \\ &= y - 2(r \cdot y)r - (r^\dagger \cdot y)r^\dagger \\ &= y - (s \cdot y)r/\sqrt{2} - (s \cdot y)r^\dagger/2 \quad = \quad y - \tfrac{1}{2}(s \cdot y)s. \end{aligned} \tag{5.4}$$

This is equal to $w_s(y)$ because $s \cdot s = 4$ when $r \cdot r^\dagger = 0$.

Secondly, let $r \cdot r^\dagger = \pm 1$. Note that $w_{r^\dagger} w_r w_{r^\dagger} = w_{w_{r^\dagger}(r)} = w_{r \mp r^\dagger}$, with $r \mp r^\dagger \in \Phi_S$, and hence

$$\begin{aligned} w_{r^\dagger}(w_r(w_{r^\dagger}(y))) &= w_{r \mp r^\dagger}(y) \\ &= y - 2((r \mp r^\dagger) \cdot y)(r \mp r^\dagger) = y - 2(1 \mp \sqrt{2})(r \cdot y)(r \mp r^\dagger). \end{aligned}$$

Also $r \cdot (r \mp r^\dagger) = 0$ and therefore $w_r(r \mp r^\dagger) = r \mp r^\dagger$. This yields

$$\begin{aligned} w_r(w_{r^\dagger}(w_r(w_{r^\dagger}(y)))) &= w_r(y) - 2(1 \mp \sqrt{2})(r \cdot y) w_r(r \mp r^\dagger) \\ &= y - 2(r \cdot y) - 2(1 \mp \sqrt{2})(r \cdot y)(r \mp r^\dagger) \\ &= y - 2(r \cdot y)[(2 \mp \sqrt{2})r + (\sqrt{2} \mp 1)r^\dagger] \\ &= y - 2(r \cdot y)(\sqrt{2} \mp 1)(\sqrt{2}r + r^\dagger) \\ &= y - 2(r \cdot y)(\sqrt{2} \mp 1)s. \end{aligned} \tag{5.5}$$

To complete the proof of (5.3) we compute

$$2\frac{s \cdot y}{s \cdot s}\frac{2(s \cdot y)}{4 \pm 2\sqrt{2}} = \frac{2(\sqrt{2} \mp 1)(s \cdot y)}{(\sqrt{2} \mp 1)(4 \pm 2\sqrt{2})} = \frac{2(\sqrt{2} \mp 1)(s \cdot y)}{2\sqrt{2}} = 2(\sqrt{2} \mp 1)(r \cdot y),$$

which proves that the right hand side of (5.5) is equal to $w_s(y)$. ∎

The lemma above shows that it is natural to consider the subspace $\mathbf{P}_O = \{\sqrt{2}x + x^\dagger\}$ of $\mathbf{P}_F$.

If we define

$$\Phi_O \stackrel{\text{def}}{=} \{\sqrt{2}s + s^\dagger | s \in \Phi_S\},$$

then it follows from Lemma 5.1 that $w_s(t) \in \Phi_O$ whenever $s, t \in \Phi_O$. We will call Φ_O a *root system of type* $^2\mathsf{F}_4$. Note that Φ_O is not a subset of Φ_F, hence when ambiguity may arise, we will use the term *O-root* for elements of Φ_O.

• Unlike Φ and Φ_F, the set Φ_O is not really a root system in the proper sense, because it contains elements r, s for which $\langle r, s\rangle$ is not integral. There also exist $r, s \in \Phi_O$, $r \neq \pm s$ such that r is a scalar multiple of s.

We depict this root system in Figure 5.1 on the next page. Each O-root $s = \sqrt{2}r + r^\dagger$ has been labelled with the short root r to which it corresponds. Note that O-roots have three different lengths according to whether $r \cdot r^\dagger = 1$, 0 or -1 (those of intermediate length have been printed in black). Angles between O-roots are always a multiple of $\pi/8$.

Let $s = \sqrt{2}r + r^\dagger$. Define $w''_s \stackrel{\text{def}}{=} w'_r w_{r^\dagger}$ when $r \cdot r^\dagger = 0$ (i.e., s is of intermediate length) and $w''_s \stackrel{\text{def}}{=} w'_r w_{r^\dagger} w'_r w_{r^\dagger}$ when $r \cdot r^\dagger = \pm 1$. By (5.2–5.3) the element w''_s coincides on $\mathbf{P}_O$ with the reflection w_s in the line orthogonal to s.

Define $W(^2\mathsf{F}_4)$ to be the subgroup of $W(\mathsf{F}_4)$ generated by all elements w_s'' of this form. When restricted to $\mathbf{P}_O$ the group $W(^2\mathsf{F}_4)$ acts like a kind of *Weyl group* of Φ_O, isomorphic to the dihedral group of order 16 (i.e., the group of symmetries of the regular octagon).

The group $W(^2\mathsf{F}_4)$ has three orbits on Φ_O (corresponding to the three different lengths of O-roots). Two ordered pairs of O-roots are equivalent under the action of $W(^2\mathsf{F}_4)$ if and only if the corresponding lengths and the inner product are the same.

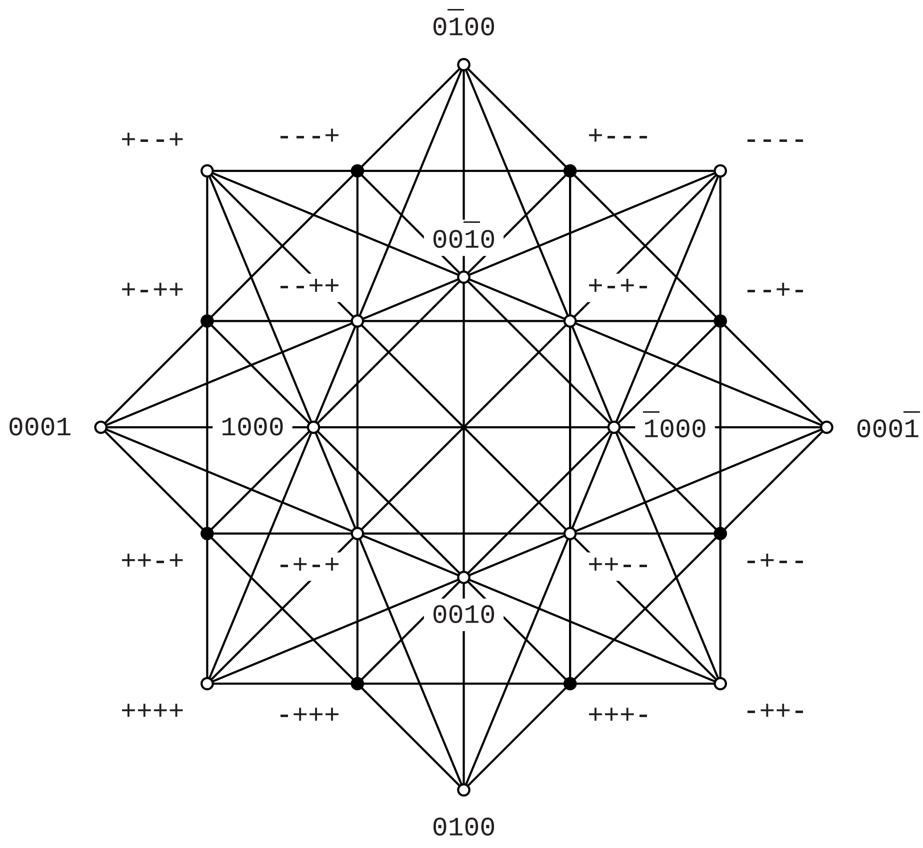


Figure 5.1: The root system Φ_O of type $^2\mathsf{F}_4$

5.2 Octagonality and the Ree group $^2\widehat{\mathsf{F}}_4(K)$

For $a, e \in \mathbf{W}$ such that e is isotropic we define the operator $q(\cdot,\cdot)$ as follows :

$$q(a,e) \stackrel{\text{def}}{=} aQ(e)^{\sigma/2} = aQ(e^{\sigma/2}). \tag{5.6}$$

We write $q(\mathbf{V}, e)$ for the set of elements $q(a,e)$ with $a \in \mathbf{V}$, i.e., $q(\mathbf{V},e) = \mathbf{V}Q(e)^{\sigma/2}$.

• When the field K is not perfect, the definition of the operator $q(\cdot,\cdot)$ must be restricted to those isotropic elements e for which $\mathbf{V}Q(e) \leq \mathbf{V}^\sigma$. Note that $e = ke_s$ is of this type for every $s \in \Phi_S$ and every $k \in K$, even when $k \notin K^\sigma$.

Proposition 5.2 *Let e, f denote isotropic elements of* $\mathbf{W}$. *Then*

1. $Q(q(f,e))^{\sigma/2} = q(e,f) * \overline{e}$,
2. $q(e,f)\overline{e} = 0$, $[q(e,f),e] = 0$, $q(e,q(f,e)) = 0$ *and* $q(q(e,f),q(f,e)) = 0$.
3. $q(e,f) = 0$ *if and only if* $q(f,e) = 0$,
4. $e \in q(\mathbf{V},f)$ *if and only if* $f \in q(\mathbf{V},e)$,
5. e *and* $q(e,e)$ *both belong to* $q(\mathbf{V},q(e,e))$.

Proof :

1. Applying (4.18) to e and $f^{\sigma/2}$ yields $Q(q(e,f)) = q(f,e)^\sigma * \overline{f}^\sigma$.

2. The first two equalities follow from the definition and the fact that e is isotropic. The last two are a consequence of the previous statement.

3. Setting $q(e,f) = 0$ in statement 1 of this proposition proves $Q(q(f,e))^{\sigma/2} = 0$ and hence $q(f,e) = 0$.

4. Every $f \in q(\mathbf{V}, e) = \mathbf{V}Q(e)^{\sigma/2}$ can be written as $f = q(a, e)$ for some $a \in \mathbf{V}$. We may choose a to be isotropic and belong to $\mathbf{W}$ by Theorem 3.27. Then $Q(f)^{\sigma/2} = q(e, a) * \overline{e}$ and therefore $e \in \mathbf{V}Q(f)^{\sigma/2}$ by Proposition 3.14–1.

5. Setting $e = f$ in statement 1 proves $Q(q(e, e))^{\sigma/2} = q(e, e) * \overline{e}$, and hence both e and $q(e, e)$ belong to $\mathbf{V}Q(q(e, e))^{\sigma/2}$ by Proposition 3.14–1. ∎

An element $e \in \mathbf{W}$ will be called *semi-octagonal* if and only if it is isotropic and $q(e, e) = 0$. An element $e \in \mathbf{W}$ will be called *octagonal* if and only if it is isotropic and $e \in q(\mathbf{W}, e)$. An element $e \in \mathbf{W}$ will be called *para-octagonal* if it is isotropic but not semi-octagonal. By Theorem 2.37 $e \in \mathbf{W}$ is octagonal if and only if it is semi-octagonal and $e \times Q(e)^{\sigma/2} = 0$. All octagonal elements are semi-octagonal, but not conversely. By Proposition 5.2–5 $q(e, e)$ is always octagonal (and different from zero) when e is para-octagonal. We will call $q(e, e)$ the *kernel* of e.

• To prove that an element e is not octagonal we will often try to establish an element f such that $[e, q(f, e)] \neq 0$. Indeed, when e is octagonal, both e and $q(f, e)$ belong to $\mathbf{V}Q(e)^{\sigma/2}$, and this subspace of $\mathbf{W}$ is isotropic.

Typical octagonal (semi-octagonal but not octagonal, para-octagonal, respectively) elements are the elements $e_r \in \mathbf{W}$ with $r \in \Phi_S$ such that $r \cdot r^\dagger = -1$ (0, 1, respectively). These correspond to long (intermediate, short, respectively) O-roots.

Before we proceed we need to introduce one more operator :

Lemma 5.3 *Let $e, f \in \mathbf{W}$ be isotropic elements satisfying $[e, f] = 0$. Define $c(e, f)$ to be the unique element of $\mathbf{W}$ such that $\mu(c(e, f)) = (e * \overline{f})^{\sigma/2}$. Then*

1. $c(e, f) = c(f, e)$,
2. $Q(c(e, f))^{\sigma/2} = e * \overline{f}$,
3. $q(e, c(e, f)) = q(f, c(e, f)) = 0$ *and hence* $q(c(e, f), e) = q(c(e, f), f) = 0$,
4. *For all $a \in \mathbf{W}$ we have* $q(a, e + f) = q(a, e) + q(a, f) + [a, c(e, f)]$.

Proof :

1. Immediate by Proposition 4.2–10.

2. By Proposition 4.13, $Q(c(e,f))$ and $(e * \overline{f})^\sigma$ are equal, modulo $S(\mathbf{W})$. Because both are isotropic elements of $\mathbf{J}$, Proposition 4.6–4) proves that they are also equal in $\mathbf{J}$.

3. By the above $q(e,c(e,f)) = eQ(c(e,f))^{\sigma/2} = e(e * \overline{f}) = 0$.

4. This is a simple rephrasing of Proposition 4.12–3. ■

Let g be an automorphism of $\mathbf{W}$ and define g^σ to be the transformation that maps a^σ onto $(a^g)^\sigma$ for all $a \in \mathbf{V}$. Then g^σ is an automorphism of $\mathbf{W}^\sigma$. As with the Frobenius map, we have

$$x(e)^\sigma = x(e^\sigma), \tag{5.7}$$

for all isotropic elements $e \in \mathbf{W}$. It follows that $g^\sigma \in \widehat{\mathsf{F}}_4(K)$ whenever $g \in \widehat{\mathsf{F}}_4(K)$. Note that $(g^\sigma)^\sigma = g^{\text{frob}}$ also in this context.

Proposition 5.4 *Let g be an automorphism of $\mathbf{W}$ satisfying $g^\dagger = g^\sigma$, i.e.,*

$$(a^g)^\sigma = (a^\sigma)^{g^\dagger}, \qquad \textit{for all } a \in \mathbf{W}. \tag{5.8}$$

Let e, f be isotropic elements of $\mathbf{W}$. Then

1. $q(a^g, e^g) = q(a,e)^g$, *for all* $a \in \mathbf{V}$,
2. $q(\mathbf{V}, e^g) = q(\mathbf{V}, e)^g$,
3. $c(e^g, f^g) = c(e,f)^g$.

Proof : Let $a \in \mathbf{V}$. We have $q(a^g, e^g)^\sigma = (a^g)^\sigma Q(e^g) = (a^g)^\sigma Q(e)^{g^\dagger}$ by (4.34) and this is equal to $(a^\sigma)^{g^\dagger} Q(e)^{g^\dagger} = (a^\sigma Q(e))^{g^\dagger} = q(a,e)^{\sigma g^\dagger} = q(a,e)^{g\sigma}$ by (5.8). Because g is an automorphism, we also have $\mathbf{V}^g = \mathbf{V}$.

Finally, $c(e^g, f^g)$ is the unique element of $\mathbf{W}$ satisfying $\mu(c(e^g, f^g)^\sigma) = e^g * \overline{f}^g = (e * \overline{f})^g$. Also $(e * \overline{f})^g = \mu(c(e,f)^\sigma)^g = \mu(c(e,f)^{\sigma g^\dagger})$ by 4.30, and this is equal to $\mu(c(e,f)^{g\sigma})$ by (5.8). ∎

Proposition 5.5 *Let e be an isotropic element of* $\mathbf{W}$. *Define*

$$y(e) \stackrel{\text{def}}{=} x(Q(e)^{\sigma/2})x(e)x(q(e,e)). \tag{5.9}$$

Then $y(e)$ *satisfies* (5.8).

Proof : Write $d = q(e,e)$. Note that $Q(d)^{\sigma/2} = d * \overline{e}$ by Proposition 5.2–1. Applying the commutation relations (4.26) we obtain

$$\begin{aligned} x(e^\sigma)x(Q(e)) &= x(Q(e))x(e^\sigma)x(e^\sigma Q(e))x(e^\sigma Q(e) * \overline{e}^\sigma) \\ &= x(Q(e))x(e^\sigma)x(d^\sigma)x(d^\sigma * \overline{e}^\sigma) \\ &= x(Q(e))x(e^\sigma)x(d^\sigma)x(Q(d)). \end{aligned} \tag{5.10}$$

Hence,

$$\begin{aligned} y(e)^\dagger &= x(Q(e)^{\sigma/2})^\dagger x(e)^\dagger x(d)^\dagger \\ &= x(e^\sigma)x(Q(e))x(Q(d)), && \text{by (4.29)} \\ &= x(Q(e))x(e^\sigma)x(d^\sigma)x(Q(d))x(Q(d)), && \text{by (5.10)} \\ &= x(Q(e))x(e^\sigma)x(d^\sigma) = y(e^\sigma) = y(e)^\sigma. \end{aligned}$$

∎

Note that $[e, q(e,e)] = 0$ and $q(e,e)Q(e)^{\sigma/2} = 0$, and hence we may apply the commutation relations (4.26) to obtain

$$\begin{aligned} y(e) &= x(Q(e)^{\sigma/2})x(e)x(q(e,e)) \\ &= x(Q(e)^{\sigma/2})x(q(e,e))x(e) \quad = \quad x(q(e,e))x(Q(e)^{\sigma/2})x(e). \end{aligned} \tag{5.11}$$

Because every $x(\cdot)$ is an involution, we obtain the following formulas for the three remaining permutations of the factors

$$\begin{aligned} y(e)^{-1} &= x(q(e,e))x(e)x(Q(e)^{\sigma/2}) \\ &= x(e)x(q(e,e)x(Q(e)^{\sigma/2}) \quad = \quad x(e)x(Q(e)^{\sigma/2})x(q(e,e)). \end{aligned} \tag{5.12}$$

Similarly,

$$y(e) = x(Q(e)^{\sigma/2})x(e) = x(e)x(Q(e)^{\sigma/2}), \quad \text{when } q(e,e) = 0. \tag{5.13}$$

The group generated by all elements $y(e)$ as defined in (5.9) will be denoted by ${}^2\widehat{\mathsf{F}}_4(K)$ and is called a *Ree* group or *twisted Chevalley group* of type ${}^2\mathsf{F}_4$. It follows from Proposition 5.4 that this group preserves octagonality, semi-octagonality and para-octagonality.

If e is an isotropic element of $\mathbf{W}$ and g an automorphism of $\mathbf{W}$ that satisfies (5.8), in particular, if $g \in {}^2\widehat{\mathsf{F}}_4(K)$, then applying Proposition 5.4 and (4.24), we obtain

$$y(e^g) = y(e)^g. \tag{5.14}$$

Let $s \in \Phi_S$, $k \in K$. Define the following elements of ${}^2\widehat{\mathsf{F}}_4(K)$:

$$y_s(k) \overset{\text{def}}{=} y(ke_s) = \begin{cases} x_{s^\dagger}(k^\sigma)x_s(k), & \text{when } s \cdot s^\dagger \neq -1, \\ x_{s^\dagger}(k^\sigma)x_s(k)x_{s+s^\dagger}(k^{1+\sigma}), & \text{when } s \cdot s^\dagger = -1. \end{cases} \tag{5.15}$$

We may permute the factors in these definitions in similar ways as in (5.11) and (5.13).

• It can be proved that the set of all elements $y_s(k)$ with $k \in K$, $s \in \Phi_S$ generate the group ${}^2\widehat{\mathsf{F}}_4(K)$. This is often taken as the definition of ${}^2\widehat{\mathsf{F}}_4(K)$ (cf. [8]).

Before we investigate some properties of the full group ${}^2\widehat{\mathsf{F}}_4(K)$ we will first consider some of its smaller subgroups.

Proposition 5.6 *Let e be an isotropic element of* $\mathbf{W}$ *with kernel $d = q(e,e)$. Let $k_0, \ell_0, k_1, \ell_1 \in K$. Then*

$$y(k_0e)y(k_1d)\,y(\ell_0e)y(\ell_1d) = y((k_0+\ell_0)e)y((k_1+\ell_1+k_0^\sigma\ell_0)d). \tag{5.16}$$

In particular $y(e)^2 = 1$ when e is semi-octagonal and $y(e)^4 = 1$ in general.

Proof : Write $E = Q(e)^{\sigma/2}$ and $D = Q(d)^{\sigma/2} = d * \bar{e}$. Note that $d = eE$ by definition and hence $dE = [d,e] = 0$. Likewise, $eD = 0$ and $[D,E] = 0$.

It follows that both $x(kd)$ and $x(k'D)$ commute with both $x(\ell e)$ and $x(\ell' E)$, for any $k, k', \ell, \ell' \in K$. Also note that $dD = 0$ and so also $x(kd)$ and $x(k'D)$ commute with each other. From

$$y(ke) = x(k^\sigma E)x(ke)x(k^{1+\sigma}d), \qquad y(\ell d) = x(\ell^\sigma D)x(\ell d)$$

it then follows that $y(\ell d)$ commutes with $y(ke)$ and $y(\ell' d)$, for any $k, \ell, \ell' \in k$. However, $y(k_0 e)$ and $y(\ell_0 e)$ do not necessarily commute.

We have

$$\begin{aligned} y(k_0e)y(\ell_0e) &= x(k_0^\sigma E)x(k_0e)x(k_0^{1+\sigma}d)x(\ell_0^\sigma E)x(\ell_0e)x(\ell_0^{1+\sigma}d) \\ &= x(k_0^\sigma E)x(k_0e)x(\ell_0^\sigma E)x(\ell_0e)x(k_0^{1+\sigma}d)x(\ell_0^{1+\sigma}d). \end{aligned}$$

Applying (4.26) yields

$$\begin{aligned} x(k_0e)x(\ell_0^\sigma E) &= x(\ell_0^\sigma E)x(k_0e)x(k_0\ell_0^\sigma eE)x(k_0^2\ell_0^\sigma eE * \bar{e}) \\ &= x(\ell_0^\sigma E)x(k_0e)x(k_0\ell_0^\sigma d)x(k_0^2\ell_0^\sigma D), \end{aligned}$$

and then

$$\begin{aligned} &y(k_0e)y(\ell_0e) \\ &\quad= x(k_0^\sigma E)x(\ell_0^\sigma E)x(k_0e)x(k_0\ell_0^\sigma d)x(k_0^2\ell_0^\sigma D)x(\ell_0e)x(k_0^{1+\sigma}d)x(\ell_0\ell_0^\sigma d) \\ &\quad= x(k_0^\sigma E)x(\ell_0^\sigma E)x(k_0e)x(\ell_0e)x(k_0\ell_0^\sigma d)x(k_0^{1+\sigma}d)x(\ell_0^{1+\sigma}d)x(k_0^2\ell_0^\sigma D), \end{aligned}$$

using the commutation properties given above. This further simplies to

$$\begin{aligned} &y(k_0e)y(\ell_0e) \\ &\quad= x((k_0+\ell_0)^\sigma E)x((k_0+\ell_0)e)x(k_0\ell_0^\sigma d + k_0^{1+\sigma}d + \ell_0^{1+\sigma}d)x(k_0^2\ell_0^\sigma D) \\ &\quad= y((k_0+\ell_0)e)x((k_0+\ell_0)^{1+\sigma}d + k_0\ell_0^\sigma d + k_0^{1+\sigma}d + \ell_0^{1+\sigma}d)x(k_0^2\ell_0^\sigma D) \\ &\quad= y((k_0+\ell_0)e)x(k_0^\sigma\ell_0 d)x(k_0^2\ell_0^\sigma D) \\ &\quad= y((k_0+\ell_0)e)y(k_0^\sigma\ell_0 d) \end{aligned}$$

Note that in particular this implies $y(k_0e)y(\ell_0e) = y((k_0+\ell_0)e)$ whenever e is semi-octagonal, because d is octagonal,

$$y(kd)y(k'd) = y((k+k')d), \qquad \text{for all } k, k' \in K. \tag{5.17}$$

Finally,

$$\begin{aligned}
&y(k_0e)y(k_1d)\,y(\ell_0e)y(\ell_1d)\\
&\quad = \; y(k_0e)y(\ell_0e)\,y(k_1d)y(\ell_1d)\\
&\quad = \; y((k_0+\ell_0)e)y(k_0^\sigma\ell_0d)y(k_1d)y(\ell_1d)\\
&\quad = \; y((k_0+\ell_0)e)y((k_1+\ell_1+k_0^\sigma\ell_0)d), \qquad \text{by (5.17).}
\end{aligned}$$

∎

In order to study the group elements introduced by this proposition it is convenient to treat the pairs (k_0, k_1), (ℓ_0, ℓ_1) as elements $\boldsymbol{k}, \boldsymbol{\ell}$ of an algebraic structure $K_\sigma^{(2)}$ endowed with the following addition operation, trace and norm :

$$\begin{aligned}
\boldsymbol{k} \oplus \boldsymbol{\ell} &\stackrel{\text{def}}{=} (k_0+\ell_0, k_1+\ell_1+k_0^\sigma\ell_0)\\
T(\boldsymbol{k}) &\stackrel{\text{def}}{=} k_0^{1+\sigma}+k_1,\\
N(\boldsymbol{k}) &\stackrel{\text{def}}{=} k_0^{2+\sigma}+k_0k_1+k_1^\sigma = k_0T(k)+k_1^\sigma = k_0k_1+T(k)^\sigma.
\end{aligned} \tag{5.18}$$

Note that $2\boldsymbol{k}(=\boldsymbol{k}\oplus\boldsymbol{k}) = (0, k_0^{1+\sigma})$, $3\boldsymbol{k} = (k_0, T(\boldsymbol{k}))$ and $4\boldsymbol{k} = (0,0)$. We will write $3\boldsymbol{k}$ as $-\boldsymbol{k}$ and $(0,0)$ as 0, giving $\boldsymbol{k}\oplus(-\boldsymbol{k}) = (-\boldsymbol{k})\oplus\boldsymbol{k} = 0$. We have

$$T(-\boldsymbol{k}) = k_1, \qquad N(-\boldsymbol{k}) = N(\boldsymbol{k}). \tag{5.19}$$

For an isotropic element $e \in \mathbf{W}$ we will write

$$y(\boldsymbol{k}e) \stackrel{\text{def}}{=} y(k_0e)y(k_1q(e,e)), \tag{5.20}$$

and then (5.16) simplifies to

$$y(\boldsymbol{k}e)y(\boldsymbol{\ell}e) = y((\boldsymbol{k}\oplus\boldsymbol{\ell})e), \qquad \text{for all } \boldsymbol{k},\boldsymbol{\ell} \in K_\sigma^{(2)}, \tag{5.21}$$

proving that the set $y(K_\sigma^{(2)}e)$ of all elements of the form $y(\boldsymbol{k}e)$ with $\boldsymbol{k} \in K_\sigma^{(2)}$ is a subgroup of ${}^2\widehat{\mathsf{F}}_4(K)$.

- $(K_\sigma^{(2)}, \oplus)$ is a group of which $((0,K),\oplus)$ is a subgroup isomorphic to $(K,+)$. The group $y(K_\sigma^{(2)}e)$ is a quotient of $(K_\sigma^{(2)},\oplus)$ which will turn out to be isomorphic to $(K,+)$ when e is semi-octagonal and non-zero, and to $(K_\sigma^{(2)},\oplus)$ when e is para-octagonal.

We leave it to the reader to verify the following identities :

$$e^{y(\boldsymbol{k}e)} = e + k_0^\sigma q(e,e), \qquad q(e,e)^{y(\boldsymbol{k}e)} = q(e,e). \tag{5.22}$$

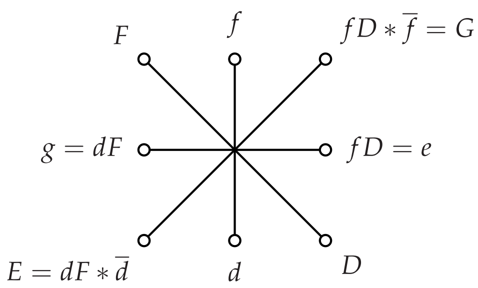
5.3 The Suzuki-Tits ovoid and the Suzuki group

Let d, f be octagonal elements of **W** satisfying $d\overline{f} = 1$.

Write $D = Q(d)^{\sigma/2}$, $F = Q(f)^{\sigma/2}$ and define $e = q(f,d) = fD$, $g = q(d,f) = dF$, $h = [d,f]$, $E = Q(e)^{\sigma/2} = dF * \overline{d}$ and $G = Q(g)^{\sigma/2} = fD * \overline{f}$. We will investigate the structure of the 5-dimensional subspace $\mathbf{B}(d,f)$ of **W** generated by d, e, f, g and h. (We will write $\mathbf{B} = \mathbf{B}(d,f)$ when d and f are clear from context.)

Because $d\overline{f} = 1$ we have $D \cdot F = 1$. From $d \in \mathbf{V}D$ and $D \cdot F = 1$ we obtain $eF = fDF = f$, $e * \overline{d} = fD * \overline{d} = D$ and $[d,e] = [d,fD] = 0$. By symmetry $gD = d$, $g * \overline{f} = F$ and $[f,g] = 0$. Also $[e,f] = [fD,f] = 0$, $[d,g] = [d,dF] = 0$ and $e\overline{g} = fDF\overline{d} = f\overline{d} = 1$. Finally $[e,g] = [fD,g] = [f,g]D + [f,gD] = [f,d] = h$. This proves that **B** is a subalgebra of **W**.

It is also easily verified that $dD = eD = eG = fG = fF = gF = gE = dE = 0$. In other words, any two elements that are at an angle $\pi/4$ in the picture below, have a trivial product. Likewise, we invite the reader to verify that the Lie bracket applied to any two elements at a right angle also is zero, i.e., $[d,e] = 0$, $\ldots$, $[D,E] = 0$.



Note that $dD = fF = 0$, $q(e,e) = eE = e(e * \overline{d}) = d$ and $q(g,g) = f$. Also $hD = [d,f]D = [dD,f] + [d,fD] = 0$, and similarly $hE = hF = hG = 0$. Also $h\overline{d} = h\overline{e} = h\overline{f} = h\overline{g} = h\overline{h} = 0$.

For $k \in K$ we have

$$\begin{array}{rclrcl} y(kd) & = & x(k^\sigma D)x(kd) & y(kf) & = & x(k^\sigma F)x(kf) \\ y(ke) & = & x(k^\sigma E)x(ke)x(k^{1+\sigma}d) & y(kg) & = & x(k^\sigma G)x(kg)x(k^{1+\sigma}f), \end{array}$$

and hence the following are easily computed :

$$\begin{array}{rclrcl} d^{y(kd)} & = & d & d^{y(kf)} & = & d + k^2 f + k^\sigma g + k^2 f + kh \\ e^{y(kd)} & = & e & e^{y(kf)} & = & e + k^\sigma f \\ f^{y(kd)} & = & k^2 d + k^\sigma e + f + kh & f^{y(kf)} & = & f \\ g^{y(kd)} & = & k^\sigma d + g & g^{y(kf)} & = & g \\ h^{y(kd)} & = & h & h^{y(kf)} & = & h \end{array}$$

$$\begin{array}{rclrcl} d^{y(ke)} & = & d & d^{y(kg)} & = & d + k^\sigma e + k^{2+2\sigma} f + k^{2+\sigma} g \\ e^{y(ke)} & = & k^\sigma d + e & e^{y(kg)} & = & e + k^2 g + kh \\ f^{y(ke)} & = & k^{2+2\sigma} d + k^{2+\sigma} e + f + k^\sigma g & f^{y(kg)} & = & f \\ g^{y(ke)} & = & k^2 e + g + kh & g^{y(kg)} & = & k^\sigma f + g \\ h^{y(ke)} & = & h & h^{y(kg)} & = & h. \end{array}$$

In terms of $\boldsymbol{k} \in K_\sigma^{(2)}$ we find

$$\begin{array}{rclrcl} d^{y(\boldsymbol{k}e)} & = & d & d^{y(\boldsymbol{k}g)} & = & d + k_0^\sigma e + N(\boldsymbol{k})^\sigma f + T(\boldsymbol{k})^\sigma g + k_1 h \\ e^{y(\boldsymbol{k}e)} & = & k_0^\sigma d + e & e^{y(\boldsymbol{k}g)} & = & e + k_1^\sigma f + k_0^2 g + k_0 h \\ f^{y(\boldsymbol{k}e)} & = & N(\boldsymbol{k})^\sigma d + T(\boldsymbol{k})^\sigma e + f + k_0^\sigma g + k_1 h & & & \\ & & & f^{y(\boldsymbol{k}g)} & = & f \\ g^{y(\boldsymbol{k}e)} & = & k_1^\sigma d + k_0^2 e + g + k_0 h & & & \\ & & & g^{y(\boldsymbol{k}g)} & = & k_0^\sigma f + g \\ h^{y(\boldsymbol{k}e)} & = & h & h^{y(\boldsymbol{k}g)} & = & h \end{array} \tag{5.23}$$

• From the formula for $d^{y(\boldsymbol{k}g)}$ we see that $y(K_\sigma^{(2)} g)$ must be isomorphic to $(K_\sigma^{(2)}, \oplus)$ and not a proper quotient.

We will denote the subgroup of ${}^2\widehat{\mathsf{F}}_4(K)$ generated by $y(K_\sigma^{(2)} e)$ and $y(K_\sigma^{(2)} g)$

by $\mathrm{Suz}(d,f)$ (or simply Suz when d,f are clear from context). Suz is called a *Suzuki group* or *twisted Chevalley group of type* ${}^2\mathsf{B}_2$.

Theorem 5.7 *Let d,f denote octagonal elements of $\mathbf{W}$ such that $d\overline{f}=1$. Let $\mathbf{B}(d,f)$ and $\mathrm{Suz}(d,f)$ be defined as above.*

Then every non-zero isotropic element of $\mathbf{B}(d,f)$ can be mapped by $\mathrm{Suz}(d,f)$ onto an element of the form $d+k[d,f]+k^2f$ with $k\in K$. All semi-octagonal elements of $\mathbf{B}(d,f)$ are octagonal and the set of non-zero (semi-)octagonal elements of $\mathbf{B}(d,f)$ constitutes a single orbit of Suz*.*

The subgroup $y(K_\sigma^{(2)}q(d,f))$ of Suz *leaves the (semi-)octagonal element f invariant and acts transitively on all (semi-)octagonal elements b such that $b\overline{f}=1$.*

Proof : Let d,e,f,g,h,D,E,F,G be as above. A general element b of $\mathbf{B}$ can be written as

$$b=k_dd+k_ee+k_ff+k_gg+k_hh,$$

with $k_d,\ldots,k_h\in K$. We distinguish between three cases, according to the value of k_d and k_f.

We have $b^2=(k_dk_f+k_ek_g+k_h^2)h$, because $h^2=[d,f]^2=[d,f]=h$ by Proposition 4.5–2. Hence b is isotropic if and only if $k_h^2=k_dk_f+k_ek_g$.

Case 1. Assume $k_d\neq 0$. Using (5.23) we compute $b^{y(\boldsymbol{\ell}g)}$ with $\boldsymbol{\ell}\in K_\sigma^{(2)}$:

$$\begin{aligned} b^{y(\boldsymbol{\ell}g)} \;=\;& k_dd+(k_d\ell_0^\sigma+k_e)e+(k_dN(\boldsymbol{\ell})^\sigma+k_e\ell_1^\sigma+k_f+k_g\ell_0^\sigma)f+\\ &(k_dT(\boldsymbol{\ell})^\sigma+k_e\ell_0^2+k_g)g+(k_d\ell_1+k_e\ell_0+k_h)h. \end{aligned}\tag{5.24}$$

Because $k_d\neq 1$ (and $K^\sigma=K$) we may always find $\boldsymbol{\ell}=(\ell_0,\ell_1)$ such that $k_d\ell_0^\sigma=k_e$ and $k_d\ell_1^\sigma=k_g$. It is easily verified that this choice of $\boldsymbol{\ell}$ makes the coefficients of e and g in (5.24) disappear. We obtain

$$b^{y(\boldsymbol{\ell}g)}=k_dd+(k_dN(\boldsymbol{\ell})^\sigma+k_f)f+(k_dN(\boldsymbol{\ell})^{\sigma/2}+\sqrt{k_dk_f}+k_h)h.$$

This element is of the form $b' = k_d d + k'_f f + k'_h h$. If we assume that b and hence b' is isotropic, we find ${k'_h}^2 = k_d k'_f$, i.e., b' is of the form $b' = k_d(d + k^2 f + kh)$ with $k = \sqrt{k'_f/k_d}$. Applying Proposition 4.12–2 to the pair $k_d d, kf$ we obtain $Q(b') = k_d^2(Q(d) + k^4 Q(f) + k^2[Q(d), Q(f)])$, and hence

$$\begin{aligned} q(b', b') = b' Q(b')^{\sigma/2} &= k_d^{1+\sigma}(d + k^2 f + kh)(D + k^{2\sigma} F + k^\sigma [D, F]) \\ &= k_d^{1+\sigma}(k^{2\sigma} g + k^\sigma d + k^2 e + k^{2+\sigma} f) \\ &= k_d^{1+\sigma} k^\sigma (d + k^{2-\sigma} e + k^2 f + k^\sigma g) \end{aligned}$$

Because $k_d \neq 0$ it follows that $q(b', b') = 0$ if and only if $k'_f = 0$, i.e., if and only if $k_f = k_d N(\boldsymbol{\ell})$. In that case $b' = k_d d$ and hence b' (and therefore b) is octagonal.

This last result can also be summarized as follows : b is semi-octagonal if and only if it is octagonal if and only if it is of the form

$$b = k_d d + k_d \ell_0^\sigma e + k_d N(\boldsymbol{\ell})^\sigma f + k_d \ell_1^\sigma g + k_d \ell_1 h = k_d d^{y(-\boldsymbol{\ell} g)} \tag{5.25}$$

for some $\boldsymbol{\ell} \in K_\sigma^{(2)}$.

Case 2. Assume $k_f \neq 0$. Using the same techniques as in the previous case, we see that b can be mapped by Suz onto an element of the form $b'' = k_f(f + \ell^2 d + \ell h)$ for some $\ell \in K$. Now, let $k \in K$ and consider $y(kd)$. We have

$$b''^{y(kd)} = k_f(f + \ell^2 d + \ell h)^{y(kd)} = k_f(k^2 + \ell^2)d + k_f k^\sigma e + f + k_f(k + \ell)h,$$

and hence, choosing $k = \ell k_f^{-1/2}$, we may map b'' onto an element that belongs to case 1 above, with $k_d = 1$.

By symmetry, we can also map any element of case 1 to an element that satisfies case 2 (with $k_f = 1$) and then back again to case 1, with $k_d = 1$.

Case 3. Assume $k_d = k_f = 0$,i.e., $b = k_e e + k_g g + k_h h$. Applying either $y(kd)$ or $y(kg)$ with suitably chosen k, brings this case back to one of the previous cases, unless of course $k_e = k_g = 0$, and then $b = 0$ when b is isotropic.

These three cases prove that any non-zero isotropic element b can be mapped to an element of the form $d + k^2 f + kh$ with $k \in K$. As part of the proof of case 1 it was shown that an element of this form is semi-octagonal only if $k = 0$ and hence it is octagonal. It also follows that all non-zero (semi-)octagonal elements belong to the same orbit.

Finally, it is a consequence of (5.25) that all octagonal elements b with $k_d = 1$, i.e., with $d\overline{f} = 1$, belong to the same orbit of $y(K_\sigma^{(2)} e)$. ∎

As an immediate consequence we see that $\mathrm{Suz}(d, f) = \mathrm{Suz}(d', f')$ for any two octagonal elements $d', f' \in \mathbf{B}(d, f)$ such that $d'\overline{f}' = 1$. This means Suz only depends on $\mathbf{B}$.

The set S of 1-dimensional subspaces Kb of octagonal elements is called a *Suzuki-Tits ovoid* in the 4-dimensional projective space associated with $\mathbf{B}$. Theorem 5.7 proves that Suz acts doubly transitive on the points of this ovoid. A typical pair of points of this ovoid is given by (Kd, Kf). From the proof of Theorem 5.7 it follows that no other nontrivial linear combination of d and f is semi-octagonal, hence that no three points of S lie on the same line, justifying the term 'ovoid' for this set.

• We may project $\mathbf{B}$ onto a subspace $\mathbf{C}$ of co-dimension 1 by leaving out the coordinate associated with h. Because Suz leaves h invariant, the group also acts naturally on $\mathbf{C}$. The projection of the set S is again an ovoid in the corresponding 3-dimensional projective space.

Proposition 5.8 *Let d, f be octagonal elements of $\mathbf{W}$ satisfying $d\overline{f} = 1$. Define*

$$N(d, f) \stackrel{\text{def}}{=} y(d)y(f)y(d)y(f)y(d). \tag{5.26}$$

Then $N(d, f) = N(f, d)$, also

$$N(d, f) = (n(d, f)n(Q(d)^{\sigma/2}, Q(f)^{\sigma/2}))^2 = (n(Q(d)^{\sigma/2}, Q(f)^{\sigma/2})n(d, f))^2 \tag{5.27}$$

and

$$d^{N(d,f)} = f, \quad f^{N(d,f)} = d \tag{5.28}$$

Proof: Let $d, \ldots, G$ be as above. From properties like $dD = 0$ we may derive commutation relations like $x(d)x(D) = x(D)x(d)$ and $x(d)x(e) = x(e)x(d)$. For elements at an angle of $3\pi/4$ in the diagram printed in the beginning in this section, we may derive the following identities from (4.26) :

$$\begin{array}{rclrcl} (x(d)x(F))^2 & = & x(g)x(E), & (x(e)x(F))^2 & = & x(f)x(G) \\ (x(e)x(E))^2 & = & x(d)x(D), & (x(f)x(E))^2 & = & x(g)x(F) \\ (x(f)x(D))^2 & = & x(e)x(G), & (x(g)x(D))^2 & = & x(d)x(E) \\ (x(g)x(G))^2 & = & x(f)x(F), & (x(d)x(G))^2 & = & x(e)x(D) \end{array} \tag{5.29}$$

Together with the other relations, these can be used in different ways. For example, we have $x(d)x(F) = x(F)x(d)x(g)x(E)$ but also $x(F)x(d) = x(d)x(F)$ $x(g)x(E)$. Also note that the right hand sides of (5.29) commute.

Now

$$\begin{array}{rl} n(d,f)n(D,F) & \\ = & x(d)x(f)\ x(d)x(D)\ x(F)x(D) \\ = & x(d)\ x(f)x(D)\ x(d)x(F)\ x(D) \\ = & x(d)x(D)x(f)x(e)x(G)\ x(g)x(E)x(F)x(d)x(D) \\ = & x(d)x(D)x(f)\ x(e)x(G)x(f)x(F)\ x(F)x(f)x(g)x(E)\ x(F)x(d)x(D) \\ = & x(d)x(D)x(f)x(F)x(e)x(E)x(f)x(F)x(d)x(D) \\ = & y(d)y(f)x(e)x(E)y(f)y(d) \end{array}$$

Note that $y(d)^2 = y(f)^2 = 1$ because d and f are (semi-)octagonal. Hence, squaring both sides of the identity above, we obtain

$$\begin{array}{rcl} (n(d,f)n(D,F))^2 & = & y(d)y(f)\ (x(e)x(E))^2\ y(f)y(d) \\ & = & y(d)y(f)\ x(d)x(D)\ y(f)y(d) \end{array}$$

and this is equal to $y(d)y(f)y(d)y(f)y(d)$, proving (5.27).

Note that $n(d,f)$ interchanges d and f. Also

$$\begin{array}{rcl} d^{n(D,F)} & = & d^{x(D)x(F)x(D)} = d^{x(F)x(D)} = (d+g)^{x(D)} = g \\ f^{n(D,F)} & = & f^{x(D)x(F)x(D)} = (f+e)^{x(F)x(D)} = e^{x(D)} = e \\ e^{n(d,f)} & = & e^{x(d)x(f)x(d)} = e \end{array}$$

$$\begin{array}{rcl} g^{n(d,f)} & = & e^{x(d)x(f)x(d)} = g \\ e^{n(D,F)} & = & e^{x(D)x(F)x(D)} = e^{x(F)x(D)} = (e+f)^{x(D)} = f \\ g^{n(D,F)} & = & g^{x(D)x(F)x(D)} = (g+d)^{x(F)x(D)} = d^{x(D)} = d \end{array}$$

From this it may be easily seen that $n(d,f)n(D,F)$ permutes d,e,f,g cyclicly, and hence $N(d,f)$ interchanges d and f (and e and g). ∎

Because $N(d,f)$ is an element of ${}^2\widehat{\mathsf{F}}_4(K)$ it follows that it also interchanges D and F. The action of $N(d,f)$ corresponds to a symmetry of the diagram at the start of this section.

• As with $n(E,F)$ in Chapter 2 and $n(e,f)$ in Chapter 4, the group elements $N(d,f)$ can be used in a short proof of the fact that the non-zero octagonal elements of **W** form a single orbit of ${}^2\widehat{\mathsf{F}}_4(K)$. We will give a different proof of this fact in section 5.4.

It is interesting to apply this Proposition 5.8 to the special case $d = e_s$, $f = e_{-s}$, where $s \in \Phi_S$ such that $s \cdot s^\dagger = 1$. We obtain $N(e_s, e_{-s}) = (n_s(1)n_{-s}(1))^2$ and hence

$$e_r^{N(e_s,e_{-s})} = e_{w''_s(r)}, \qquad h_r^{N(e_s,e_{-s})} = h_{w''_s(r)} \tag{5.30}$$

for all $r \in \Phi_S$. In other words, the elements $N(e_s, e_{-s})$ are strongly related to the elements w''_s of $W({}^2\mathsf{F}_4)$. The following propositions provide similar companions to the elements w'_s of $W({}^2\mathsf{F}_4)$, where $s \cdot s^\dagger = 0$.

Lemma 5.9 *Let e, f be semi-octagonal elements of* **W** *satisfying $q(e,f) = q(f,e) = 0$ and $e\overline{f} = 1$. Define*

$$N'(e,f) \stackrel{\text{def}}{=} y(e)y(f)y(e). \tag{5.31}$$

Then $N(e,f) = N(f,e)$. Also

$$N'(e,f) = n(e,f)n(Q(e)^{\sigma/2}, Q(f)^{\sigma/2}) = n(Q(e)^{\sigma/2}, Q(f)^{\sigma/2})n(e,f) \tag{5.32}$$

and

$$e^{N'(e,f)} = f, \quad f^{N'(e,f)} = e, \quad E^{N'(e,f)} = F, \quad F^{N'(e,f)} = E. \tag{5.33}$$

Proof : We will use the abbreviations $E = Q(e)^{\sigma/2}$, $F = Q(f)^{\sigma/2}$. We have $E \cdot F = 1$ and $eE = eF = fE = fF = 0$. Hence $x(e)$ and $x(f)$ commute with both $x(E)$ and $x(F)$.

$$\begin{aligned} y(e)y(f)y(e) &= x(e)x(E)x(f)x(F)x(e)x(E) \\ &= x(e)x(f)x(e)x(E)x(F)x(E) = n(e,f)n(E,F), \end{aligned}$$

and likewise $y(e)y(f)y(e) = n(E,F)n(e,f)$. Also note that $n(e,f) = n(f,e)$ and $n(E,F) = n(F,E)$.

Finally, it is easily proved that $n(E,F)$ leaves both e and f invariant, while it interchanges E and F, and $n(e,f)$ leaves both E and F invariant, interchanging e and f. Hence $N'(e,f)$ interchanges E with F and e with f. ∎

We leave it to the reader to verify that $e = e_s$, $f = e_{-s}$ with $s \in \Phi_S$ such that $s \cdot \bar{s} = 0$ do indeed satisfy the conditions of this proposition, and as a consequence, that

$$e_r^{N'(e_s,e_{-s})} = e_{w_s''(r)}, \qquad h_r^{N'(e_s,e_{-s})} = h_{w_s''(r)}, \tag{5.34}$$

for all $r \in \Phi_S$, also in this case.

5.4 Transitivity properties of ${}^2\widehat{\mathsf{F}}_4(K)$

Consider the spaces $\mathbf{B}_1 = \mathbf{B}(e_{\mathtt{0100}}, e_{\mathtt{0\bar{1}00}})$ and $\mathbf{B}_2 = \mathbf{B}(e_{\mathtt{0001}}, e_{\mathtt{000\bar{1}}})$. Let the corresponding Suzuki groups be denoted by $\mathrm{Suz}_i = \mathrm{Suz}(\mathbf{B}_i)$, $i = 1,2$. The following table lists the corresponding values of $d, \dots, G$.

d	e	f	g	D	E	F	G
$e_{\mathtt{0100}}$	$e_{\mathtt{0010}}$	$e_{\mathtt{0\bar{1}00}}$	$e_{\mathtt{00\bar{1}0}}$	$E_{\mathtt{0110}}$	$E_{\mathtt{01\bar{1}0}}$	$E_{\mathtt{0\bar{1}\bar{1}0}}$	$E_{\mathtt{0\bar{1}10}}$
$e_{\mathtt{0001}}$	$e_{\mathtt{1000}}$	$e_{\mathtt{000\bar{1}}}$	$e_{\mathtt{\bar{1}000}}$	$E_{\mathtt{1001}}$	$E_{\mathtt{\bar{1}001}}$	$E_{\mathtt{\bar{1}00\bar{1}}}$	$E_{\mathtt{100\bar{1}}}$

The element h is the same in both cases, and equal to $h = h_{\mathtt{0100}} = h_{\mathtt{0010}} = h_{\mathtt{0001}} = h_{\mathtt{1000}}$. Note that h can be written as $h = e_z + \infty$ with $z \in L_\infty$. It follows that $\mathbf{B}_1 \cap \mathbf{B}_2 = Kh$ and hence the space $\mathbf{B}_1 + \mathbf{B}_2$ has dimension 9.

• The corresponding Lie algebra is of type B_4 and consists of those elements of $\mathbf{W}$ that belong to $\eta_z \times \mathbf{V}$.

Observe that $\mathbf{B}_1$ involves only roots for which the first and last coordinate are 0, while $\mathbf{B}_2$ uses roots for which the middle coordinates are 0. As a consequence we see that Suz_1 must leave every element of $\mathbf{B}_2$ invariant, and conversely, that Suz_2 leaves every element of $\mathbf{B}_1$ invariant.

The element $N \stackrel{\text{def}}{=} N(e_{\mathtt{-++-}}, e_{\mathtt{+--+}})$, which can be interpreted as a symmetry along the line connecting ++++ and - - - - in figure 5.1, interchanges the spaces $\mathbf{B}_1$ and $\mathbf{B}_2$. We will therefore be interested in the group generated by $\mathrm{Suz}_1, \mathrm{Suz}_2$ and N, which we denote by $2\mathrm{Suz}^2(\mathbf{B}_1, \mathbf{B}_2)$. This group leaves $\mathbf{B}_1 + \mathbf{B}_2$ invariant.

• The group $2\mathrm{Suz}^2(\mathbf{B}_1, \mathbf{B}_2)$ is a wreath product of two copies of Suz.

Theorem 5.10 *The group* $2\mathrm{Suz}^2(\mathbf{B}_1, \mathbf{B}_2)$ *has exactly the following orbits on the semi-octagonal elements of* $\mathbf{B}_1 + \mathbf{B}_2$ *:*

1. *The trivial orbit* $\{0\}$.

2. *A single orbit of non-zero octagonal elements. Every octagonal element of* $\mathbf{B}_1 + \mathbf{B}_2$ *belongs either to* $\mathbf{B}_1$ *or to* $\mathbf{B}_2$.

3. *A single orbit of semi-octagonal elements that are not octagonal. This is exactly the set of elements that can be written as the sum of a non-zero octagonal element of* $\mathbf{B}_1$ *and a non-zero octagonal element of* $\mathbf{B}_2$.

Proof : Any element c of $\mathbf{B}_1 + \mathbf{B}_2$ can be written as

$$\begin{aligned} c \quad = \quad & k_1 e_{\mathtt{1000}} + k_2 e_{\mathtt{0100}} + k_3 e_{\mathtt{0010}} + k_4 e_{\mathtt{0001}} + \\ & \ell_1 e_{\bar{\mathtt{1}}\mathtt{000}} + \ell_2 e_{\mathtt{0}\bar{\mathtt{1}}\mathtt{00}} + \ell_3 e_{\mathtt{00}\bar{\mathtt{1}}\mathtt{0}} + \ell_4 e_{\mathtt{000}\bar{\mathtt{1}}} + mh \end{aligned} \tag{5.35}$$

with $k_1, \ell_1, \ldots, m \in K$. It is easily verified that c is isotropic if and only if $m^2 = k_1\ell_1 + \cdots + k_4\ell_4$. In that case we may write $c = b_1 + b_2$ with

$$\begin{aligned} b_1 &= k_2 e_{\mathtt{0100}} + \ell_2 e_{\mathtt{0}\bar{\mathtt{1}}\mathtt{00}} + k_3 e_{\mathtt{0010}} + \ell_3 e_{\mathtt{00}\bar{\mathtt{1}}\mathtt{0}} + (k_2\ell_2 + k_3\ell_3)^{1/2} h, \\ b_2 &= k_1 e_{\mathtt{1000}} + \ell_1 e_{\bar{\mathtt{1}}\mathtt{000}} + k_4 e_{\mathtt{0001}} + \ell_4 e_{\mathtt{000}\bar{\mathtt{1}}} + (k_1\ell_1 + k_4\ell_4)^{1/2} h. \end{aligned}$$

Note that b_i is an isotropic element of $\mathbf{B}_i$ (for $i = 1, 2$).

Applying Theorem 5.7 independently on b_1 and b_2, we see that c can always be mapped by ${}^2\widehat{\mathsf{F}}_4(K)$ onto an element of the form $c' = b_1' + b_2'$ with

$$\begin{aligned} b_1' = 0 \quad \text{or} \quad b_1' = e_{\mathtt{0100}} + \ell_2' e_{\mathtt{0}\bar{\mathtt{1}}\mathtt{00}} + {\ell_2'}^{1/2} h, \\ b_2' = 0 \quad \text{or} \quad b_2' = e_{\mathtt{0001}} + \ell_4' e_{\mathtt{000}\bar{\mathtt{1}}} + {\ell_4'}^{1/2} h. \end{aligned}$$

Assume that both $b_1', b_2' \neq 0$ (and hence $b_1', b_2' \neq 0$). Applying Proposition 4.12–3 we find

$$\begin{aligned} & Q(c') = Q(b_1') + Q(b_2') + S(\mu^{-1}(b_1' * \overline{b}_2')) = \\ & \quad E_{\mathtt{0110}} + {\ell_2'}^2 E_{\mathtt{0}\overline{\mathtt{11}}\mathtt{0}} + \ell_2' H_{\mathtt{0110}} + E_{\mathtt{1001}} + {\ell_4'}^2 E_{\bar{\mathtt{1}}\mathtt{00}\bar{\mathtt{1}}} + \ell_4' H_{\mathtt{1001}} + \\ & \quad E_{\mathtt{++++}} + \ell_2' E_{\mathtt{+--+}} + \ell_4' E_{\mathtt{-++-}} + \ell_2' \ell_4' E_{\mathtt{----}}, \end{aligned} \tag{5.36}$$

and after some computation we obtain

$$
\begin{aligned}
q(c',c')^{\sigma} = c'^{\sigma}Q(c') = & \\
& \ell_2'^{\,2} e_{\mathtt{00\bar{1}0}} + \ell_2' e_{\mathtt{0100}} + \ell_2' e_{\mathtt{++-+}} + \ell_2'\ell_4' e_{\mathtt{-+--}} + \\
& \ell_2'^{\,\sigma} e_{\mathtt{0010}} + \ell_2'^{\,1+\sigma} e_{\mathtt{0\bar{1}00}} + \ell_2'^{\,\sigma} e_{\mathtt{+-++}} + \ell_2'^{\,\sigma}\ell_4' e_{\mathtt{--+-}} + \\
& \ell_4'^{\,2} e_{\mathtt{\bar{1}000}} + \ell_4' e_{\mathtt{0001}} + \ell_4' e_{\mathtt{-+++}} + \ell_2'\ell_4' e_{\mathtt{---+}} + \\
& \ell_4'^{\,\sigma} e_{\mathtt{1000}} + \ell_4'^{\,1+\sigma} e_{\mathtt{000\bar{1}}} + \ell_4'^{\,\sigma} e_{\mathtt{+++-}} + \ell_2'\ell_4'^{\,\sigma} e_{\mathtt{+---}} + \\
& (\ell_2' + \ell_4')^{\sigma/2}(e_{\mathtt{++++}} + \ell_2' e_{\mathtt{+--+}} + \ell_4' e_{\mathtt{-++-}} + \ell_2'\ell_4' e_{\mathtt{----}}).
\end{aligned}
\tag{5.37}
$$

This proves that $q(c',c') = 0$ if and only if $\ell_2' = \ell_4' = 0$. Extending this result with the cases $b_1' = 0$ or $b_2' = 0$ we obtain four typical examples of semi-octagonal elements : 0, $e_{\mathtt{0100}}$, $e_{\mathtt{0001}}$ and $e_{\mathtt{0100}} + e_{\mathtt{0001}}$. The middle two are in the same orbit because of N, the last one is semi-octagonal but not octagonal. Indeed, by (5.36), we find

$$Q(e_{\mathtt{0100}} + e_{\mathtt{0001}}) = E_{\mathtt{0110}} + E_{\mathtt{1001}} + E_{\mathtt{++++}}.$$

and hence in this case $[q(e_{\mathtt{--+-}}, c'), c'] = e_{\mathtt{++++}} \neq 0$. ∎

Theorem 5.10 can be extended to the action of the full group ${}^2\widehat{\mathsf{F}}_4(K)$ on the full space $\mathbf{W}$. To prove this, we will apply a reduction process similar to that used in Chapter 2 and 3 for the groups $\widehat{\mathsf{E}}_6(K)$ and $\widehat{\mathsf{F}}_4(K)$. We will apply successive group elements of the form $y_s(k)$, with $s \in \Phi_S$, $k \in K$ to reduce the number of non-zero coordinates of an isotropic element $a \in \mathbf{W}$, while staying in the same orbit of ${}^2\widehat{\mathsf{F}}_4(K)$.

Let a denote a non-zero octagonal element of $\mathbf{W}$. Because a is isotropic and non-zero it cannot belong to $\mathbf{W}_\infty$ and hence must have at least one coordinate $a[r]$ different from zero such that $r \in \Phi_S$. We will first prove that we can choose $r = \mathtt{0100}$.

If a has coordinate $a[r] \neq 0$ for some r such that $r \cdot r^\dagger = 1$, then subsequent operations of the form $N(e_s, e_{-s})$ or $N'(e_s, e_{-s})$ will bring a in the required form. Indeed, using figure 5.1 and referring to (5.30) and (5.34), it is easily seen that r can be mapped onto `0100` using appropriate Weyl group elements w''_s.

If $a[r] = 0$ for all r satisfying $r \cdot r^\dagger = 1$ but a has a non-zero coordinate for some $r \in \Phi_S$ with $r \cdot r^\dagger = 0$, then we may easily find an element $y_s(1)$ which reduces a to the previous case. For example, assume $a[\texttt{-+--}] \neq 0$, then applying $y_{\texttt{++++}}(1) = x_{\texttt{0101}}(1)x_{\texttt{++++}}(1)$ will create a non-zero coordinate at position `0100`. By symmetry, other cases can be resolved in a similar way. The same technique can also be used when $r \cdot r^\dagger = -1$. For example, we may apply $y_{\texttt{-+++}}(1) = x_{\texttt{1100}}(1)x_{\texttt{-+++}}(1)$ when $a[\texttt{++--}] \neq 0$.

Henceforth we will assume that $a[r] = a[\texttt{0100}] \neq 0$. We now apply the reductions $y_t(k)$ of Table 5.1 (in the order given) to annihilate subsequent coordinates $a[s]$.

s	$t^\dagger$	t	$t + t^\dagger$
`++++`	$00\bar{1}1$	`+-++`	
`-+++`	$10\bar{1}0$	`--++`	`+--+`
`+++-`	$\bar{1}0\bar{1}0$	`+-+-`	`----`
`-++-`	$00\bar{1}\bar{1}$	`--+-`	
`++-+`	$0\bar{1}01$	`+--+`	
`-+-+`	$0\bar{1}00$	`---+`	
`++--`	$\bar{1}\bar{1}00$	`+---`	
`-+--`	$0\bar{1}0\bar{1}$	`----`	

Table 5.1: Reductions used in the proof of Theorems 5.11 and 5.16.

We have listed s, t, $t^\dagger$ and $t + t^\dagger$ (when it belongs to Φ_S) for each step (note that $s = r + t$). The table is ordered in such a way that each group element $y_t(k)$ does not affect the value of coordinate $a[r]$ or the values of $a[s]$ for each s from the previous rows. (We leave it to the reader to verify that this is indeed the case.) In each step the scalar k is equal to $a[s]/a[r]$.

After applying the stated reductions, we end up with an image b of a for which $b[r] \neq 0$ and $b[s] = 0$ whenever $r \cdot s = 1$. Now, for such s we may compute that $b^2[s] = b[r]b[s-r]$, and because b is isotropic and $b[r] \neq 0$, we find that $b[s-r] = 0$. When s runs through all short roots that satisfy $r \cdot s = 1$, $v = s - r$ runs through all short roots that satisfy $r \cdot v = -1$.

In other words, the eight remaining short roots s for which a coordinate is (possibly) non-zero, are the 8 short roots s satisfying $r \cdot s = -2, 0$ or 2, which for our particular choice of r correspond to the roots with integral coordinates, i.e., the cyclic permutations of $\mathtt{1000}$ and $\bar{\mathtt{1}}\mathtt{000}$. The fact that $b^2 = 0$ also allows us to easily compute the coordinates $b[z]$ with $z \in L_\infty$ (refer to the proof of proposition 4.13 for details). It follows that b is of the following form:

$$\begin{aligned} b &= k_1 e_{\mathtt{1000}} + k_2 e_{\mathtt{0100}} + k_3 e_{\mathtt{0010}} + k_4 e_{\mathtt{0001}} + \\ &\quad \ell_1 e_{\bar{\mathtt{1}}\mathtt{000}} + \ell_2 e_{\mathtt{0}\bar{\mathtt{1}}\mathtt{00}} + \ell_3 e_{\mathtt{00}\bar{\mathtt{1}}\mathtt{0}} + \ell_4 e_{\mathtt{000}\bar{\mathtt{1}}} + mh \end{aligned} \tag{5.38}$$

with $k_1, \ell_1, \ldots, m \in K$ such that $m^2 = k_1\ell_1 + \cdots + k_4\ell_4$ and $h = h_{\mathtt{1000}} = \cdots = h_{\mathtt{000}\bar{\mathtt{1}}}$. In other words, b belongs to the space $\mathbf{B}_1 + \mathbf{B}_2$. We may now apply Theorem 5.10 to complete the proof of

Theorem 5.11 *The group ${}^2\widehat{\mathsf{F}}_4(K)$ has exactly the following orbits on the semi-octagonal elements of* $\mathbf{W}$ *:*

1. *The trivial orbit* $\{0\}$.
2. *An orbit of non-zero octagonal elements.*
3. *An orbit of semi-octagonal elements that are not octagonal.*

5.5 Pairs of octagonal elements

In this section we will provide more information on the orbit structure of ${}^2\widehat{\mathsf{F}}_4(K)$ on pairs of (non-zero) octagonal elements. The main result will be stated in Theorem 5.16. The proof of that theorem is subdivided into several lemmas.

Lemma 5.12 *Let $k \in K - \{0\}$. Then ${}^2\widehat{\mathsf{F}}_4(K)$ acts transtively on all ordered pairs (b, c) of octagonal elements $b, c \in \mathbf{W}$ such that $b\overline{c} = k$.*

Proof : By Theorem 5.11 we may choose $c = e_{\mathtt{0\bar{1}00}}$ without loss of generality. Note that $b\overline{c} = b[\mathtt{0100}] = k$ is non-zero. We will again apply the reductions $y_t(k)$ of Table 5.1 to map b onto an element for which the number of non-zero coordinates is little. We must however take care only to use such group elements $y_t(k)$ that leave $c = e_{\mathtt{0\bar{1}00}}$ invariant. A quick inspection proves that this is the case for all $y_t(k)$ in the table. Hence, using the same argument as in the proof of Theorem 5.11, when b is isotropic it can always be mapped onto an element that belongs to $\mathbf{B}_1 + \mathbf{B}_2$.

By Theorem 5.10–2 b, which is octagonal, must either belong to $\mathbf{B}_1$ or $\mathbf{B}_2$. The condition $b\overline{c} \neq 0$ disposes of the second case, hence $b \in \mathbf{B}_1$. Then, by (5.25), we may write

$$b = ke_{\mathtt{0100}}^{\,y(-\boldsymbol{\ell} e_{\mathtt{00\bar{1}0}})},$$

for some $\boldsymbol{\ell} \in K_\sigma^{(2)}$. Note that $y(-\boldsymbol{\ell} e_{\mathtt{00\bar{1}0}})$ leaves c invariant and hence b is equivalent to $ke_{\mathtt{0100}}$ under the action of the stabilizer of $\widehat{\mathsf{F}}_4(K)$ of the element f. ∎

Lemma 5.13 *The group ${}^2\widehat{\mathsf{F}}_4(K)$ acts transtively on all ordered pairs (b, c) of octagonal elements $b, c \in \mathbf{W}$ such that $b\overline{c} = 0$ and $[b, c] \neq 0$.*

Proof : This proof will use the same techniques as the proofs of Theorem 5.11 and Lemma 5.12. By Theorem 5.11 we may choose $c = e_{\texttt{----}}$ without loss of generality. The main argument of this proof will rely on the assumption that the coordinate $b[\texttt{0100}]$ is non-zero. We will therefore first prove that it is always possible to map b onto b' such that $b'[\texttt{0100}] \neq 0$, by means of an element of ${}^2\widehat{\mathsf{F}}_4(K)$ that leaves $c = e_{\texttt{----}}$ invariant.

Because b is isotropic, $b \notin \mathbf{W}_\infty$ and then from $[b, c] = [b, e_{\texttt{----}}] \neq 0$ we easily deduce that at least one of the coordinates $b[r]$ is non-zero with $r = \texttt{1000}$, `0100`, `0010`, `0001`, `-+++`, `+-++`, `++-+` or `+++-`.

Below we give a list of group elements $y_t(1)$ which can be used to map b onto b' with the required properties, whenever $b[r] \neq 0$ for one of the given r.

r	s	$t^\dagger$	t	$t + t^\dagger$
`1000`	`+++-`	$\mathtt{010\bar{1}}$	`-++-`	
`0100`		none needed		
`0010`	`+++-`	$\mathtt{\bar{1}010}$	`++--`	`-++-`
`0001`	`0100`	$\mathtt{010\bar{1}}$	`-++-`	
`-+++`	`0100`	$\mathtt{\bar{1}010}$	`++--`	`-++-`
`+-++`	`+++-`	$\mathtt{010\bar{1}}$	`-++-`	
`++-+`	`0100`	$\mathtt{010\bar{1}}$	`-++-`	
`+++-`	`0100`	$\mathtt{10\bar{1}0}$	`--++`	

For example, the last row of the table tells us that, when $b[\texttt{0100}] = 0$ but $b[\texttt{+++-}] \neq 0$, we may apply $y_{\texttt{--++}}(1)$ to create a non-zero coordinate at position `0100`. Similarly, the first row of the table indicates that when $b[\texttt{0100}] = b[\texttt{+++-}] = 0$ but $b[\texttt{1000}] \neq 0$ we may first apply $y_{\texttt{-++-}}(1)$ to obtain an image with a non-zero coordinate at position `+++-`, bringing us back to the first example. We leave it to the reader to verify that each of the listed group elements indeed leaves $c = e_{\texttt{----}}$ invariant.

Henceforth we may assume that the coordinate $b[\texttt{0100}]$ is non-zero. We now apply the reductions $y_t(k)$ of Table 5.1 to b, *except* the first reduction (which corresponds to $s = \texttt{++++}$). Luckily we have no need for this reduction because $b\bar{c} = 0$ implies that $b[\texttt{++++}]$ is zero already. It is easily verified that all other reductions in the table leave $c = e_{\texttt{----}}$ invariant.

Hence, using the same argument as in the proof of Lemma 5.12, when b is octagonal it can always be mapped onto an element that belongs to either $\mathbf{B}_1$ or $\mathbf{B}_2$, and because $b[\texttt{0100}] \neq 0$ it must belong to $\mathbf{B}_1$ and be of the form

$$b = ke_{\texttt{0100}}^{\;y(-\boldsymbol{\ell} e_{\texttt{00}\bar{\texttt{1}}\texttt{0}})},$$

for some $k \in K, \boldsymbol{\ell} \in K_\sigma^{(2)}$. Note that $y(-\boldsymbol{\ell} e_{\texttt{00}\bar{\texttt{1}}\texttt{0}})$ leaves c invariant.

Finally, we apply an element $N = N(k^{\sigma/2}e_{\texttt{-++-}}, k^{-\sigma/2}e_{\texttt{+--+}})$ to prove that the orbit of (b,c) does not depend on k. This group element leaves both $e_{\texttt{----}}$ and $e_{\texttt{++++}}$ invariant and interchanges $k^{\sigma/2}e_{\texttt{-++-}}$ and $k^{-\sigma/2}e_{\texttt{+--+}}$. Hence it also interchanges the following elements :

$$c(k^{\sigma/2}e_{\texttt{-++-}}, e_{\texttt{++++}}) = k^{1/2}e_{\texttt{0100}} \quad \text{and} \quad c(k^{-\sigma/2}e_{\texttt{+--+}}, e_{\texttt{++++}}) = k^{-1/2}e_{\texttt{0001}},$$

and hence it interchanges $ke_{\texttt{0100}}$ and $e_{\texttt{0001}}$ while it leaves $e_{\texttt{----}}$ untouched. This proves that every (b,c) satisfying the conditions of this lemma is equivalent to $(e_{\texttt{----}}, e_{\texttt{0001}})$ under the action of ${}^2\widehat{\mathsf{F}}_4(K)$. ∎

Lemma 5.14 *The group ${}^2\widehat{\mathsf{F}}_4(K)$ acts transtively on all ordered pairs (b,c) of octagonal elements $b,c \in \mathbf{W}$ such that $b\bar{c} = 0$, $[b,c] = 0$ and $b * \bar{c} \neq 0$.*

Proof : This proof will use the same techniques as the proofs of the previous lemmas in this section. Without loss of generality we may choose $c = e_{\texttt{0001}}$. As in the proof of Lemma 5.12 we first need to prove that b can be mapped onto an element b' for which $b'[\texttt{0100}]$ is non-zero, using only group elements from ${}^2\widehat{\mathsf{F}}_4(K)$ that leave $c = e_{\texttt{0001}}$ invariant. (In this case this turns out to be slightly more complicated.)

Because b is isotropic, $b \notin \mathbf{W}_\infty$, and then from $[b,c] = 0$, $b * \bar{c} \neq 0$ we deduce that $b[r] \neq 0$ for at least one root r from the list $\mathtt{1000}$, $\mathtt{\bar{1}000}$, $\mathtt{0100}$, $\mathtt{0\bar{1}00}$, $\mathtt{0010}$ or $\mathtt{00\bar{1}0}$. We list these roots below, together with the group elements $y_t(1)$ which can be used to map b onto b' with the required properties, whenever $f[r] \neq 0$. (Refer to the proof of Lemma 5.13 for an explanation of how this table should be interpreted.) All of these group elements leave c invariant.

r	s	$t^\dagger$	t	$t + t^\dagger$
$\mathtt{0100}$			none needed	
$\mathtt{0010}$	$\mathtt{0100}$	$\mathtt{01\bar{1}0}$	$\mathtt{0010}$	$\mathtt{0100}$
$\mathtt{\bar{1}000}$	$\mathtt{0100}$	$\mathtt{1100}$	-+++	
$\mathtt{0\bar{1}00}$	$\mathtt{0100}$	$\mathtt{0110}$	$\mathtt{0100}$	
$\mathtt{00\bar{1}0}$	$\mathtt{0100}$	$\mathtt{0110}$	$\mathtt{0100}$	

Unfortunately, the case $r = \mathtt{1000}$ cannot be handled in the same way and will be postponed until the end of this proof. For now let us assume that we have $b[\mathtt{0100}] \neq 0$. We again apply the reductions of Table 5.1 except those that do not leave $c = e_{\mathtt{0001}}$ invariant, i.e., those corresponding to $s =$ +++-, -++-, ++-- and -+--. Luckily, the corresponding coordinates $b[s]$ are already zero, because $[b,c] = 0$. As in Lemma 5.13, this means that we have mapped b onto an octagonal element which must belong to $\mathbf{B}_1$ and because $\mathrm{Suz}(\mathbf{B}_1)$ leaves c invariant, we may apply Theorem 5.7 to see that all such b are in the same orbit of the stabilizer c in $\widehat{\mathsf{F}}_4(K)$.

This still leaves the special case $b[\mathtt{0100}] = b[\mathtt{0010}] = b[\mathtt{\bar{1}000}] = b[\mathtt{0\bar{1}00}] = b[\mathtt{00\bar{1}0}] = 0$, but $b[\mathtt{1000}] \neq 0$. In this case we may further annihilate 4 coordinates $b[s]$ using the elements $y_t(1)$ listed in the following table :

s	$t^\dagger$	t	$t + t^\dagger$
++++	$\mathtt{1100}$	-+++	
++-+	$\mathtt{1010}$	-+-+	++++
+-++	$\mathtt{10\bar{1}0}$	--++	+--+
+--+	$\mathtt{1\bar{1}00}$	---+	

Note that all of these elements leave $c = e_{\mathtt{0001}}$ invariant.

Together with $[b, c] = 0$ this implies that we end up with an element b that belongs to the space generated by $\mathbf{W}_\infty$ and the 6 elements $e_{\mathtt{0001}}$, $e_{\mathtt{1000}}$, $e_{\mathtt{---+}}$, $e_{\mathtt{--++}}$, $e_{\mathtt{-+-+}}$ and $e_{\mathtt{-+++}}$. The fact that b is isotropic induces the following further simplification : b must be of the form

$$b = k^2 e_{\mathtt{1000}} + \ell^2 e_{\mathtt{0001}} + k\ell h_{\mathtt{0001}},$$

with $k, \ell \in K$. Finally, Theorem 5.7 (applied to $\mathbf{B}_2$) proves that b can only be (semi-)octagonal when $k = 0$, contradicting $b[\mathtt{1000}] \neq 0$. ∎

Lemma 5.15 *The group ${}^2\widehat{\mathsf{F}}_4(K)$ acts transtively on all ordered pairs (b, c) of octagonal elements b, c such that $b\bar{c} = 0$, $[b, c] = 0$, $b * \bar{c} = 0$, but $b \notin Kc$.*

Proof : This proof will use the same techniques as the proofs of the other lemmas in this section. Without loss of generality we may choose $c = e_{\mathtt{++++}}$.

Because b is isotropic, $b \notin \mathbf{W}_\infty$, and then from $[b, c] = 0$, $b * \bar{c} \neq 0$ we deduce that $b[r] \neq 0$ for at least one root r from the list $r = \mathtt{1000}, \mathtt{0100}, \mathtt{0010}, \mathtt{0001}$, `-+++`, `+-++`, `++-+` or `+++-`. This is exactly the same list of roots as in the proof of Lemma 5.13, and in fact we can use the same transformations as in that lemma to ensure that we may choose $r = \mathtt{0100}$. (The transformations of Lemma 5.13 happen to also leave `++++` invariant, not entirely by coincidence.)

We may now apply the following reductions to b (taken from Table 5.1) in order to annihilate three of its coordinates :

s	$t^\dagger$	t	$t + t^\dagger$
`++++`	$\mathtt{00\bar{1}1}$	`+-++`	
`-+++`	$\mathtt{10\bar{1}0}$	`--++`	`+--+`
`++-+`	$\mathtt{0\bar{1}01}$	`+--+`	

This reduces b to the form

$$b = k_1 e_{\mathtt{1000}} + k_2 e_{\mathtt{0100}} + k_3 e_{\mathtt{0010}} + k_4 e_{\mathtt{0001}} + k_5 e_{\mathtt{+-++}} + k_6 e_{\mathtt{+++-}},$$

with $k_1, \ldots, k_6 \in K$ such that $k_2 \neq 0$. Note that $b^2 = (k_2k_5 + k_4k_6)e_{{++++}}$ and hence $k_2k_5 + k_4k_6 = 0$.

We will now compute $Q(b)$. We write $b = b_1 + b_2$ with

$$\begin{aligned} b_1 &= k_2e_{0100} + k_2k_5e_{++++} + k_3e_{0010} + k_6e_{+++-} \\ b_2 &= k_1e_{1000} + k_4k_6e_{++++} + k_4e_{0001} + k_5e_{+-++}. \end{aligned}$$

Note that $[b_1, b_2] = 0$, and hence $Q(b) = Q(b_1) + Q(b_2) + S(\mu^{-1}(b_1 * \overline{b}_2)$ by Proposition 4.12–3.

To compute $Q(b_1)$ we apply Proposition 4.12–2 to $e = k_2e_{0100}$ and $f = k_5e_{+-++}$ to obtain

$$\begin{aligned} Q(k_2e_{0100} + k_2k_5e_{++++}) &= k_2^2E_{0110} + k_2^2k_5^2[E_{0110}, E_{00\bar{1}1}] \\ &= k_2^2E_{0110} + k_2^2k_5^2E_{0101}. \end{aligned}$$

Repeated application of Proposition 4.12–3 then yields

$$Q(b_1) = k_2^2E_{0110} + k_2^2k_5^2E_{0101} + k_3^2E_{01\bar{1}0} + k_6^2E_{\bar{1}100} + k_2k_3E_{0100}.$$

Likewise

$$Q(b_2) = k_4^2E_{1001} + k_4^2k_6^2E_{0101} + k_1^2E_{\bar{1}001} + k_5^2E_{00\bar{1}1} + k_1k_4E_{0001},$$

and hence

$$\begin{aligned} Q(b) = {}& k_1^2E_{\bar{1}001} + k_2^2E_{0110} + k_3^2E_{01\bar{1}0} + k_4^2E_{1001} + k_5^2E_{00\bar{1}1} + k_6^2E_{\bar{1}100} \\ &+ k_1k_2E_{-+++} + k_1k_4E_{0001} + k_2k_3E_{0100} + k_2k_4E_{++++} + k_3k_4E_{++-+} \\ &+ (k_1k_3 + k_5k_6)E_{-+-+}. \end{aligned}$$

The fact that b is octagonal will allow us to establish further relations between the coefficients $k_1, \ldots, k_6$. Recall that $[b, q(\mathbf{V}, b)] = 0$ when b is octagonal, or equivalently, that $[b^\sigma, aQ(b)] = 0$ for all $a \in \mathbf{V}$.

Choosing $a = e_{+--+}$ we obtain

$$\begin{aligned} e_{\texttt{+--+}}Q(b) &= k_2^2 e_{\texttt{++++}} + k_6^2 e_{\texttt{-+-+}} + k_1 k_2 e_{\texttt{0001}} + k_2 k_3 e_{\texttt{++-+}} \\ [b^\sigma, e_{\texttt{+--+}}Q(b)] &= k_1^\sigma k_6^2 e_{\texttt{++-+}} + k_3^\sigma k_6^2 e_{\texttt{-+++}} + k_5^\sigma k_6^2 e_{\texttt{0001}} + k_6^{2+\sigma} e_{\texttt{0100}} \\ &\quad + (k_1 k_2 k_6^\sigma + k_2 k_3^{1+\sigma}) e_{\texttt{++++}} \end{aligned}$$

and hence $[b^\sigma, e_{\texttt{++--}}Q(b)] = 0$ implies $k_6 = 0$ and $k_3 = 0$ (as $k_2 \neq 0$). Because also $k_2k_5 + k_3k_6 = 0$ this implies $k_5 = 0$, and hence b belongs to $\mathbf{B}_1 + \mathbf{B}_2$. Because of Theorem 5.10 and because $k_2 \neq 0$, we must even have $b \in \mathbf{B}_1$ and hence $k_1 = k_4 = 0$. Hence $b = k_2 e_{\texttt{0100}}$.

Finally we map b to $e_{\texttt{0001}}$ using the same $N = N(k_2^{\sigma/2} e_{\texttt{-++-}}, k_2^{-\sigma/2} e_{\texttt{+--+}})$ as in the proof of Lemma 5.13. It was already indicated there that N leaves $e_{\texttt{++++}}$ invariant. ∎

We may now combine Lemmas 5.12–5.15 into the following

Theorem 5.16 *The following is an exhaustive list of all orbits of* ${}^2\widehat{\mathsf{F}}_4(K)$ *on ordered pairs* (e, f) *of non-zero octagonal (isotropic) elements* $e, f \in \mathbf{W}$.

1. *For each* $k \in K - \{0\}$ *an orbit with representative* $(e_{\texttt{0100}}, k e_{\texttt{0}\bar{\texttt{1}}\texttt{00}})$.
2. *An orbit with representative* $(e_{\texttt{0100}}, e_{\texttt{----}})$.
3. *An orbit with representative* $(e_{\texttt{0100}}, e_{\texttt{0001}})$.
4. *An orbit with representative* $(e_{\texttt{0100}}, e_{\texttt{++++}})$.
5. *For each* $k \in K - \{0\}$ *an orbit with representative* $(e_{\texttt{0100}}, k e_{\texttt{0100}})$.

These orbits are the intersections of the corresponding orbits of $\widehat{\mathsf{F}}_4(K)$ *with the set of pairs of octagonal elements.* ../..

The following table list several properties of the corresponding pairs (e, f).

	Case 1	*Case 2*	*Case 3*	*Case 4*	*Case 5*
$[e, f]$	*not isotr.*	*semi-oct.*‡	0	0	0
$q(e, f)$	*para-oct.*	$\neq 0$, *oct.*	0	0	0
$e + f$	*not isotr.*	*not isotr.*	*semi-oct.*‡	$\neq 0$, *oct.*	*octag.*
$c(e, f)$	*undef.*	*undef.*	$\neq 0$, *oct.*	0	0
$f \in q(\mathbf{V}, e)$	*no*	*no*	*no*	*yes*	*yes*
$q(\mathbf{V}, e) \cap q(\mathbf{V}, f)$	$\{0\}$	$\{0\}$	*K c(e,f)*	dim = 3	$q(\mathbf{V}, e) = q(\mathbf{V}, f)$

‡ *semi-octagonal, but not octagonal*

Proof : Only the table of properties remains to be checked. Note that all properties are 'invariant' for ${}^2\widehat{\mathsf{F}}_4(K)$, for example, when $g \in {}^2\widehat{\mathsf{F}}_4(K)$, then $q(e, f)^g = q(e^g, f^g)$, $c(e, f)^g = c(e^g, f^g)$, etc. As a consequence it is sufficient to check all listed properties only for a single (representative) pair in each orbit. We will discuss the most difficult cases here and leave the rest to be verified by the reader.

When $e = e_{\texttt{0100}}$ and $f = e_{\texttt{0001}}$ we have $Q(e)^{\sigma/2} = E_{\texttt{0110}}$ and $Q(f)^{\sigma/2} = E_{\texttt{1001}}$, and then $\mathbf{V}Q(e)^{\sigma/2} \cap \mathbf{V}Q(f)^{\sigma/2} = \mathbf{V}Q(e)^{\sigma/2}Q(f)^{\sigma/2}$ by Theorem 2.38. It is easily seen that $\mathbf{V}E_{\texttt{0110}}E_{\texttt{1001}} = Ke_{\texttt{++++}}$, and indeed $e_{\texttt{++++}} = c(e_{\texttt{0100}}, e_{\texttt{0001}})$.

Consider the value of $e + f$. When $e = e_{\texttt{0100}}$ and $f = e_{\texttt{0001}}$ then $e + f$ is semi-octagonal but not octagonal by Theorem 5.10–3. When $f = e_{\texttt{++++}}$ we obtain $Q(e + f) = Q(e_{\texttt{0100}} + e_{\texttt{++++}}) = e_{\texttt{0110}} + e_{\texttt{0101}}$, and then $(e_{\texttt{000}\bar{\texttt{1}}} + e_{\texttt{+--+}})Q(e + f) = (e + f)^\sigma$, proving that $e + f$ is octagonal. ∎

A subspace of $\mathbf{W}$ all of whose elements are octagonal, will be called an *octagonal* subspace of $\mathbf{W}$. Clearly, when d is octagonal, then Kd is octagonal. It follows from Theorem 5.16 that $e + f$ is octagonal if and only if $e\overline{f} = 0$,

$[e, f] = 0$ and $e * \overline{f} = 0$, hence every octagonal space must be F-isotropic.

For such e, f and for $k, \ell \in K$ we have $Q(ke + \ell f)^{\sigma/2} = k^\sigma Q(e)^{\sigma/2} + \ell^\sigma Q(f)^{\sigma/2}$, hence for any octagonal subspace S of $\mathbf{W}$ the set $Q(S)^{\sigma/2} \stackrel{\text{def}}{=} \{Q(e)^{\sigma/2} \mid e \in S\}$ is a subspace of $\mathbf{J}$. Because S is octagonal, $Q(S)^{\sigma/2}$ must be a subspace of the companion subspace of S in $\mathbf{J}$. As a consequence $\dim S \leq 2$.

Proposition 5.17 *Let $L \in \mathbf{W} \wedge \mathbf{W}$ be F-isotropic. Then the following are equivalent :*

1. *$L = e \wedge f$ with $e, f \in \mathbf{W}$ such that $Ke \neq Kf$ and $Ke + Kf$ an octagonal subspace of $\mathbf{W}$.*

2. *$L = \overline{d} \times Q(d)^{\sigma/2}$ for some $d \in \mathbf{W}$ which is semi-octagonal but not octagonal.*

Proof : By Theorem 5.11 the elements d which are semi-octagonal but not octagonal form a single orbit of ${}^2\widehat{\mathsf{F}}_4(K)$. Likewise, by Theorem 5.16, also the pairs (e, f) in the first statement form a single orbit of ${}^2\widehat{\mathsf{F}}_4(K)$. To prove the theorem it is therefore sufficient to provide a single example of each orbit such that $e \wedge f = \overline{d} \times Q(d)^{\sigma/2}$. Such an example is given by $e = e_{\texttt{0100}}$, $f = e_{\texttt{++++}}$, $d = e_{\texttt{-+++}}$ and then $Q(d)^{\sigma/2} = e_{\texttt{1100}}$. ∎

Proposition 5.18 *Let S denote a two-dimensional octagonal subspace of $\mathbf{W}$. Let T denote the companion subspace of $Q(S)$ in $\mathbf{W}$. Then $\dim T = 3$, $S \leq T$, all elements of T are semi-octagonal and S contains all octagonal elements of T. If $d \in T - S$, then $S = (\overline{d} \times Q(d)^{\sigma/2})\mathbf{V}^*$.*

Proof : By Theorem 5.16 without loss of generality we may take $S = Ke + Kf$ with $e = e_{\texttt{0100}}$, $f = e_{\texttt{++++}}$ and then $Q(e)^{\sigma/2} = E_{\texttt{0110}}$ and $Q(f)^{\sigma/2} = E_{\texttt{0101}}$. The companion subspace of $Q(S)$ in $\mathbf{W}$ is equal to $\mathbf{V}E_{\texttt{0110}} \cap \mathbf{V}E_{\texttt{0101}}$, which is the 3-dimensional space generated by $e = e_{\texttt{0100}}$, $f = e_{\texttt{++++}}$ and $e_{\texttt{-+++}}$. Consider an element $d = k_1 e_{\texttt{0100}} + k_2 e_{\texttt{++++}} + k_3 e_{\texttt{-+++}}$ of this space. We have $Q(d)^{\sigma/2} =$

$k_1^\sigma e_{0110} + k_2^\sigma e_{0101} + k_3^\sigma e_{1100}$, and hence $q(d,d) = dQ(d)^{\sigma/2} = 0$, proving that d is semi-octagonal. Also note that $[d, e_{0\bar{1}00} Q(d)^{\sigma/2}] = k_3^{1+\sigma} e_{++++}$ and hence d is octagonal if and only if $k_3 = 0$.

Finally, note that $e, f \in \mathbf{V}Q(d)^{\sigma/2}$ and that $d \in \mathbf{V}Q(e)^{\sigma/2}$ and $d \in \mathbf{V}Q(f)^{\sigma/2}$. Hence $\overline{e} \times \mathbf{V}Q(d)^{\sigma/2} = \overline{f} \times \mathbf{V}Q(d)^{\sigma/2} = \overline{d} \times \mathbf{V}Q(e)^{\sigma/2} = \overline{d} \times \mathbf{V}Q(f)^{\sigma/2} = 0$ and therefore $\overline{d} \times Q(d)^{\sigma/2} = k_3^{1+\sigma} \overline{e}_{-+++} \times E_{1100}$ which was shown to be equal to $k_3^{1+\sigma} e \wedge f$ in the proof of Proposition 5.17. ∎

5.6 The Ree-Tits octagon

Define a new geometry $\mathcal{O}$ whose *points* are the one-dimensional octagonal subspaces of **W** and whose *lines* are the two-dimensional octagonal subspaces of **W**. Points of $\mathcal{O}$ are points of $\mathcal{F}$, and lines of $\mathcal{O}$ are exactly those lines of $\mathcal{F}$ all of whose points belong to $\mathcal{O}$.

• Because an octagonal subspace of **W** has dimension at most 2, there are no planes or hyperlines of $\mathcal{F}$ all of whose points belong to $\mathcal{O}$.

As an immediate consequence of properties defined in earlier sections, we find that there are many equivalent ways of stating that two points Ke and Kf belong to the same line of $\mathcal{O}$. For example :

- Every 1-dimensional subspace $K(k_1e + k_2f)$ of $Ke + Kf$ is a point of $\mathcal{O}$.
- $f \in q(\mathbf{V}, e)$.
- $f \in [e, \mathbf{V}]$.
- $[e, \mathbf{J}]\overline{f} = 0$.
- $\dim q(\mathbf{V}, e) \cap q(\mathbf{V}, f) = 3$.
- $e\overline{f} = 0$, $[e, f] = 0$ and $e * \overline{f} = 0$.

The theorem on the next page (which is in some sense the final 'thesis' of this text) proves that the point-line geometry $\mathcal{O}$ satisfies the axioms of a generalized octagon, where 'collinear', 'cohyperlinear', 'almost opposite' and 'opposite' correspond to distance 2, 4, 6 and 8 in the incidence graph of $\mathcal{O}$.

Theorem 5.19 *Let Ke, Kf be different points of $\mathcal{O}$.*

1. *If Ke and Kf are collinear then every point of the line $Ke + Kf$ belongs to $\mathcal{O}$. Apart from the points of $Ke + Kf$ there are no other points of $\mathcal{O}$ collinear to both Ke and Kf.*

2. *If Ke and Kf are cohyperlinear, then there is a unique point of $\mathcal{O}$ that is collinear to both Ke and Kf. This point is $Kc(e,f)$. In the incidence graph of $\mathcal{O}$, Ke and Kf are at mutual distance 4.*

3. *If Ke and Kf are almost opposite then there are no points of $\mathcal{O}$ that are collinear to both Ke and Kf. There is a unique point of $\mathcal{O}$ that is collinear to Ke and cohyperlinear to Kf. This point is $Kq(f,e)$. In the incidence graph of $\mathcal{O}$, Ke and Kf are at mutual distance 6.*

4. *If Ke and Kf are opposite, then there are no points of $\mathcal{O}$ that are collinear to both Ke and Kf, or that are collinear to Ke and cohyperlinear to Kf (or vice versa). Every line of $\mathcal{O}$ through Kf contains exactly one point of $\mathcal{O}$ that is almost opposite to Ke. In the incidence graph of $\mathcal{O}$, Ke and Kf are at mutual distance 8.*

These are the only possible relations between different points of $\mathcal{O}$.

Proof : The different cases of this theorem correspond to the 4 different cases of Theorem 3.41. They also correspond to 4 different cases of Theorem 5.16, but numbered in the opposite order. (The fifth case of Theorem 5.16 occurs when $Ke = Kf$.)

1. (Case 4 of Theorem 5.16.) Any point Kd collinear to both Ke and Kf must belong to the companion space in $\mathbf{W}$ of $KQ(e) + KQ(f)$. By Proposition 5.18, the only octagonal elements of this 3-dimensional space are exactly those belonging to $Ke + Kf$.

2. (Case 3 of Theorem 5.16.) A point collinear to both Ke and Kf must belong to the intersection $q(\mathbf{W}, e) \cap q(\mathbf{W}, f)$, which by Theorem 5.16 is exactly $Kc(e,f)$. Note also that $c(e,f)$ is octagonal by the same theorem.

3. (Case 2 of Theorem 5.16.) We have $q(\mathbf{W}, e) \cap q(\mathbf{W}, f) = \{0\}$, hence there are no points collinear to both Ke and Kf.

Without loss of generality we may set $e = e_{\texttt{0100}}$ and $f = e_{\texttt{----}}$. If $Kd \in \mathcal{O}$ is collinear to e and cohyperlinear to f, we must have $d \in q(\mathbf{V}, e) = \mathbf{V}E_{\texttt{0110}}$ and $[d, f] = 0$. We easily compute that this implies $d \in Ke_{\texttt{-++-}}$. Also $q(f, e) = e_{\texttt{----}}E_{\texttt{0110}} = e_{\texttt{-++-}}$.

4. (Case 1 of Theorem 5.16.) We have $q(\mathbf{W}, e) \cap q(\mathbf{W}, f) = \{0\}$, hence there are no points collinear to both Ke and Kf.

Without loss of generality we may set $e = e_{\texttt{0100}}$ and $f = e_{\texttt{0}\bar{\texttt{1}}\texttt{00}}$. As in the previous case, if $Kd \in \mathcal{O}$ is collinear to Ke and cohyperlinear to Kf, we must have $d \in q(\mathbf{V}, e) = \mathbf{V}E_{\texttt{0110}}$ and $[d, f] = 0$. This implies $d \in Ke_{\texttt{0010}}$, but then d is not even semi-octagonal.

Finally, consider the hyperplane of elements $w \in \mathbf{W}$ such that $w\bar{e} = 0$. Because f does not belong to this hyperplane, every line of $\mathcal{O}$ through f intersects this hyperplane in a single point $Kw \in \mathcal{O}$. By definition $w\bar{e} = 0$, hence Kw is either almost opposite to Ke, cohyperlinear to Ke, collinear to Ke or equal to Ke. The last possibility is easily ruled out, and the second and third possibility have been disproved above. ∎

The following theorem provides some insight into the 'local structure' of $\mathcal{O}$.

Theorem 5.20 *Let Kc be a point of $\mathcal{O}$. Then we may find octagonal elements $d, f \in \mathbf{W}$ with $d\bar{f} = 1$ such that $\mathrm{Suz}(d, f)$ fixes c and acts doubly transitive on the lines of $\mathcal{O}$ through Kc. These lines form a cone with vertex Kc and a Suzuki-Tits ovoid as a base. The semi-octagonal elements of $q(\mathbf{V}, c)$ form a 5-dimensional subspace of $q(\mathbf{V}, c)$*

Let L be a line through Kc. Then d, f can be chosen in such a way that $y(K_\sigma^{(2)} q(f, d))$ fixes L and acts transitively on the remaining lines through Kc.

Proof : By Theorem 5.16 all points collinear with Kc must lie in $q(\mathbf{V}, c)$. Let Kc' be a point of L different from Kc. By Theorem 5.16 we may set $c = e_{0001}$ and $c' = e_{++++}$ without loss of generality.

Every element $a \in q(\mathbf{V}, e)$ is of the form

$$a = k_1 e_{1000} + k_2 e_{0001} + k_3 e_{++++} + k_4 e_{+--+} + k_5 e_{+-++} + k_6 e_{++-+},$$

with $k_1, \ldots, k_6 \in K$.

It is easily verified that a may be written as $a = c(c, b)$ with

$$b = k_1^\sigma e_{\bar{1}000} + k_2^\sigma e_{1000} + k_3^\sigma e_{0100} + k_4^\sigma e_{0\bar{1}00} + k_5^\sigma e_{00\bar{1}0} + k_6^\sigma e_{0010},$$

which is an element of $\mathbf{B}_1 + \mathbf{B}_2$ (cf. section 5.4). By Theorem 5.7 we can always apply an element of $\mathrm{Suz}(\mathbf{B}_1)$ to map b to b' of the form

$$b' = k_1^\sigma e_{\bar{1}000} + k_2^\sigma e_{1000} + e_{0100} + k^\sigma h_{0100} + k^{2\sigma} e_{0\bar{1}00},$$

Because the group $\mathrm{Suz}(\mathbf{B}_1)$ leaves c invariant, the same element will reduce $a = c(c, b)$ to

$$a' = c(c, b') = k_1 e_{1000} + k_2 e_{0001} + e_{++++} + k^2 e_{+--+}.$$

Let us now compute $Q(a')$. We find

$$Q(a') = k_1^2 E_{\bar{1}001} + k_2^2 E_{1001} + E_{0101} + k^2 E_{0001} + k^4 E_{0\bar{1}01}$$

and then $q(a', a') = k_1^{1+\sigma} e_{1000}$. Hence a' (and a) is semi-octagonal if and only if $k_1 = 0$. It follows that the semi-octagonal elements form a 5-dimensional subspace of $q(\mathbf{V}, c)$.

Now assume $k_1 = 0$ and compute $[a', q(e_{--+-}, a')]$. We have

$$\begin{aligned} q(e_{--+-}, a') &= k_2^\sigma e_{+-++} + e_{-+++} + k^\sigma E_{--++} \\ [a', q(e_{--+-}, a')] &= k^2 e_{0001} \end{aligned}$$

and therefore $k = 0$ when a is octagonal. Conversely, it is easily verified that $a' = k_2 e_{0001} + e_{++++}$ is octagonal for every $k_2 \in K$.

In other words, the octagonal elements of $q(\mathbf{V}, c)$ form a cone with vertex e_{0001} and with base the orbit of e_{++++} under the action of $\mathrm{Suz}(\mathbf{B}_1)$.

From the proof of Theorem 5.7 we know that every element b' in the orbit of e_{0100} is either of the form ke_{0100} with $k \in K$, or $k(e_{0\bar{1}00} + \ell_0^\sigma e_{00\bar{1}0} + N(\boldsymbol{\ell})^\sigma e_{0100} + T(\boldsymbol{\ell})^\sigma e_{0010} + \ell_1 h_{0100})$ with $k \in K, \boldsymbol{\ell} \in K_\sigma^{(2)}$ (in the notations of Theorem 5.7, $b' = kd$ or $b' = kf^{y(\boldsymbol{\ell}e)}$).

Hence, every octagonal $a \in q(\mathbf{V}, c)$ must satisfy

$$\begin{aligned} a &= k_2 e_{0001} + ke_{++++} \\ \text{or } a &= k_2 e_{0001} + k(e_{+--+} + \ell_0^\sigma e_{+-++} + N(\boldsymbol{\ell})^\sigma e_{++++} + T(\boldsymbol{\ell})^\sigma e_{++-+}). \end{aligned}$$

The first case, i.e., $a = k_2 c + kc'$, corresponds to a point Ka that belongs to the line L. The second case corresponds to the orbit of $a = k_2 c + ke_{+--+}$ under the action of $y(K_\sigma^{(2)} e_{0010})$.

Comparing this with the results of section 5.3 we see that the base of the cone does indeed form a Suzuki-Tits ovoid in the space generated by e_{++++}, e_{++-+}, e_{+--+} and e_{+-++}. ∎

We end this section by establishing the two types of *root groups* for $\mathcal{O}$.

Proposition 5.21 *Let Kd, Ke be points of $\mathcal{O}$ that are almost opposite. Then $y([d, e])$ fixes all elements (points and lines) on the shortest path joining Kd and Ke and all elements incident to (at least) one of those elements.*

Let L be any line of $\mathcal{O}$ through Kd and not on this path. Then the subgroup $y(K\,[d, e])$ of ${}^2\widehat{\mathsf{F}}_4(K)$ acts transitively on the points of L different from Kd.

Proof : By Theorem 5.16 we may choose $d = e_{0001}$ and $e = e_{----}$ without loss of generality. Then $[d, e]$ is the semi-octagonal element e_{---+} and $Q([d, e]^{\sigma/2}) = E_{1\bar{1}00}$.

The points on the shortest path joining Kd and Ke are Kd' and Ke' with $d' = q(e, d) = e_{+--+}$ and $e' = q(d, e) = e_{0\bar{1}00}$ and they are easily seen to be left invariant by $y([d, e])$. Hence also the lines on that path are left invariant, and all points on those lines.

Consider a line M through Kd'. M lies entirely inside the space $q(\mathbf{V}, d') = \mathbf{V}E_{0\bar{1}01}$ which is generated by the following vectors :

$$e_{0\bar{1}00},\ e_{0001},\ e_{+-++},\ e_{+--+},\ e_{--++},\ e_{---+}.$$

It is easily seen that each of these vectors is left invariant by $y([d, e])$, hence so is M. (By symmetry, the same holds for lines through e'.)

Finally, let L be a line through Kd, not equal to $Kd + Kd'$. From the proof of Theorem 5.20 we know that L lies on the cone with vertex Kd and a base belonging to the 4-dimensional space generated by e_{++++}, e_{+--+}, e_{+-++}, e_{++-+}. Of these four vectors, the only one not left invariant by $y(k\,[d, e])$, for $k \in K$, is e_{++++} which maps to $e_{++++} + ke_{0001} = e_{++++} + kd$.

Now consider a point Kc of L. Using the notation of the proof of Theorem 5.20 we assign 'coordinates' $k_2, \ldots, k_5$ to the element c. It is easily seen that the case $k_3 = 0$ occurs only when $c \in Kd + Kd'$. Because k_3 is the 'coordinate' that corresponds to e_{++++}, we find that c is mapped by $y(k\,[d, e])$ to $c + kk_3d$, which lies on the same line L.

Also, when k runs through K, the image of c encounters all points of the line L, except d. ∎

The elements $y([e, f])$ of the proposition above are called *point-elations* of $\mathcal{O}$ and the group $y(K\,[e, f])$ is a *root group* [26].

Proposition 5.22 *Let Kb, Kb' be opposite points of $\mathcal{O}$ and let Kd be a point of $\mathcal{O}$ at distance 4 of both Kb and Kb'. Then*

1. *The subset of elements c of $q(\mathbf{V}, d)$ that satisfy $[c, b] = [c, b'] = 0$ is a 2-dimensional subspace of the form $Kd + Ke$ where e is para-octagonal and $d = q(e, e)$.*

2. *The group $y(K_\sigma^{(2)} e)$ fixes all points and lines on the unique path of length 8 joining Kb to Kb' through Kd.*

3. *$y(K_\sigma^{(2)} e)$ fixes all points incident with any of the four lines of the path.*

4. *$y(K_\sigma^{(2)} e)$ fixes all lines incident with any of the three 'middle' points of that path (i.e. Kd, $Kc(b, d)$ and $Kc(b', d)$).*

5. *$y(K_\sigma^{(2)} e)$ acts transitively on the set of lines through Kb which are not on this path.*

Proof : By Theorem 5.16 without loss of generality we may take $b = e_{\texttt{0001}}$ and $b' = e_{\texttt{000}\bar{\texttt{1}}}$. It is easily verified that the set of all $c \in \mathbf{W}$ such that $[b, c] = [b', c] = 0$ is the 5-dimensional space $\mathbf{B}_1$ in the notations of Section 5.4. By Theorem 5.7 the group $\mathrm{Suz}(\mathbf{B}_1)$ acts transitively on all octagonal elements of this space, hence without loss of generality we may choose $d = e_{\texttt{0100}}$.

In this case we have $Q(d)^{\sigma/2} = E_{\texttt{0110}}$ and then the intersection of $q(\mathbf{V}, d)$ with $\mathbf{B}_1$ is equal to $Kd + Ke$ with $e = e_{\texttt{0010}}$, proving statement 1 of this proposition.

Note that $y(K_\sigma^{(2)} e)$ fixes d, b and b' (as elements of $\mathbf{B}_2$) and hence also $c(b, d) = e_{\texttt{++++}}$ and $c(b', d) = e_{\texttt{-++-}}$, proving statements 2 and 3.

The three 'middle' points on the unique path of length 8 joining Kb to Kb' through Kd correspond to $e_{\texttt{++++}}$, $e_{\texttt{0100}}$ and $e_{\texttt{-++-}}$. For each of these three vectors e_r, the following table lists the generators $e_r, e_{s_1}, \ldots, e_{s_4}$ of the corresponding

5-dimensional subspace of semi-octagonal elements of $q(\mathbf{V}, e_r)$. (These results were obtained from the example used in the proof of Theorem 5.20 by applying an appropriate element of the form $N'(e_s, e_{-s})$, i.e., a symmetry of Φ_O.)

r	s_1	s_2	s_3	s_4
`++++`	`0001`	`0100`	`-+++`	`++-+`
`0100`	`++++`	`-++-`	`+++-`	`-+++`
`-++-`	`0100`	$\mathtt{000\bar{1}}$	`-+--`	`+++-`

Note that each of the listed vectors e_r, e_{s_i} is left invariant by $y(K_\sigma^{(2)} e)$, except for $e_{\mathtt{++-+}}$ and $e_{\mathtt{-+--}}$. Observe that $e_{\mathtt{++-+}} = c(e, e_{\mathtt{0001}})$ and $e_{\mathtt{-+--}} = c(e, e_{\mathtt{000\bar{1}}})$. And hence, for $\boldsymbol{k} \in K_\sigma^{(2)}$, using $e^{y(\boldsymbol{k}e)} = k_0^\sigma d + e$ we obtain

$$e_{\mathtt{++-+}}{}^{y(\boldsymbol{k}e)} = e_{\mathtt{++-+}} + k_0^\sigma e_{\mathtt{++++}}, \qquad e_{\mathtt{-+--}}{}^{y(\boldsymbol{k}e)} = e_{\mathtt{-+--}} + k_0^\sigma e_{\mathtt{-++-}}.$$

This proves statement 4.

Statement 5 was already proved as part of the proof of Theorem 5.16. ∎

The group $y(K_\sigma^{(2)} e)$ is again a *root group*. The elements $y(\boldsymbol{k}e)$ are called *line-elations* of $\mathcal{O}$, and in the special case $k_0 = 0$, i.e., $y(\boldsymbol{k}e) = y(k_1 d)$ they are the *central elations* with center Kd. The central elations leave invariant all points at distance 4 of Kd. Propositions 5.21 and 5.22 imply that $\mathcal{O}$ is a *Moufang polygon* [26] and hence prove

Theorem 5.23 *$\mathcal{O}$ is the classical perfect Ree-Tits generalized octagon $\mathsf{O}(K, \sigma)$.*

And so, after a quest spanning more than two hundred pages, we finally find our Grail.

A The Suzuki-Tits ovoid

In this chapter we want to provide an illustration of how the techniques of the main text can be applied to a much simpler example. We will fold the Lie algebra of type A_3 to one of type B_2 and finally twist it to obtain a Suzuki-Tits ovoid.

To keep the length of this chapter within bounds we have omitted the descriptions of the corresponding root systems and Chevalley groups although they can easily be constructed using the techniques of the main text. We have also left out most of the proofs.

• This chapter is largely independent of Chapters 2–5. We do however occasionally refer to concepts which were introduced in Chapter 1. To illustrate the similarity with the main text, notations will often be 'inherited' : for example, in the main text the symbol $\mathbf{V}$ denotes the 27-dimensional module of type E_6, while here it will denote the 4-dimensional module of type A_3, which plays an equivalent role in the construction.

A.1 Modules for the Lie algebra of type A_3

Consider a 4-dimensional vector space $\mathbf{V}$ over a field K and denote its dual by $\mathbf{V}^*$. Choose a base $(e_1, \ldots, e_4)$ of $\mathbf{V}$ and a base $(\eta_1, \ldots, \eta_4)$ of $\mathbf{V}^*$ in such a way that

$$e_i\eta_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

We may represent an element $a \in \mathbf{V}$ by a row vector $(a[1]\ a[2]\ a[3]\ a[4])$ where $a[i] = a\eta_i$ denotes the ith coordinate of a with respect to the chosen base. Similarly, we may represent $\alpha \in \mathbf{V}^*$ by the column vector $(\alpha[1]\ \cdots\ \alpha[4])^T$, where $\alpha[i] = e_i\alpha$ denotes the ith coordinate of α with respect to the dual base. With these notations we obtain the following expressions for the inner and tensor product of $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$:

$$\begin{aligned}
a\alpha &= (a[1]\ a[2]\ a[3]\ a[4]) \begin{pmatrix} \alpha[1] \\ \alpha[2] \\ \alpha[3] \\ \alpha[4] \end{pmatrix} = a[1]\alpha[1] + \cdots + a[4]\alpha[4], \\
\alpha \otimes a &= \begin{pmatrix} \alpha[1] \\ \alpha[2] \\ \alpha[3] \\ \alpha[4] \end{pmatrix} (a[1]\ a[2]\ a[3]\ a[4]) \\
&= \begin{pmatrix} a[1]\alpha[1] & a[2]\alpha[1] & a[3]\alpha[1] & a[4]\alpha[1] \\ a[1]\alpha[2] & a[2]\alpha[2] & a[3]\alpha[2] & a[4]\alpha[2] \\ a[1]\alpha[3] & a[2]\alpha[3] & a[3]\alpha[3] & a[4]\alpha[3] \\ a[1]\alpha[4] & a[2]\alpha[4] & a[3]\alpha[4] & a[4]\alpha[4] \end{pmatrix}.
\end{aligned}$$

Denote by $\mathbf{L}$ the set of all linear transformations $A \in \mathrm{Hom}(\mathbf{V}, \mathbf{V})$ such that $\mathrm{Tr}\, A = 0$. It is easily seen that $\dim \mathbf{L} = 15$. An element A of $\mathbf{L}$ can be regarded as a 4×4 matrix over K. We write $A[i, j]$ for the element at row i and column j of this matrix, i.e., $A[i, j] = e_i A \eta_j$. If $a \in \mathbf{V}$ and $\alpha \in \mathbf{V}^*$, then $\alpha \otimes a \in \mathbf{L}$ if and only if $a\alpha = 0$.

The standard Lie product $[A, B] \stackrel{\text{def}}{=} AB - BA$ makes $\mathbf{L}$ into a Lie algebra of type A_3. The action $a^A \stackrel{\text{def}}{=} aA$ of $\mathbf{L}$ on $\mathbf{V}$ makes $\mathbf{V}$ into a 4-dimensional module of $\mathbf{L}$ and so does the action $\alpha^A \stackrel{\text{def}}{=} -A\alpha$ of A on $\mathbf{V}^*$. Note that $a^A\alpha + a\alpha^A = 0$ and $\alpha^A \otimes a + \alpha \otimes a^A = [\alpha \otimes a, A]$, hence both operations are 'compatible' with $\mathbf{L}$.

Define $\mathbf{V} \wedge \mathbf{V}$ to be the vector space of all antisymmetric bilinear forms L on $\mathbf{V}^*$. We will write $L\alpha\beta$ for $L(\alpha, \beta)$ with $\alpha, \beta \in \mathbf{V}^*$. In other words, $L \in \mathbf{V} \wedge \mathbf{V}$ if and only if $L\alpha\alpha = 0$ for all $\alpha \in \mathbf{V}^*$ (and hence $L\alpha\beta = -L\beta\alpha$). Note that we may also regard $\mathbf{V} \wedge \mathbf{V}$ as a subspace of $\mathrm{Hom}(\mathbf{V}^*, \mathbf{V})$, with $L\alpha \in \mathbf{V}$ for all

$\alpha \in \mathbf{V}^*$.

For every $a, b \in \mathbf{V}$ we define $a \wedge b \in \mathrm{Hom}(\mathbf{V}^*, \mathbf{V})$ as follows :

$$(a \wedge b)\alpha\beta \stackrel{\text{def}}{=} \begin{vmatrix} a\alpha & a\beta \\ b\alpha & b\beta \end{vmatrix} \text{ or } (a \wedge b)\alpha = (a\alpha)b - (b\alpha)a.$$

Note that $a \wedge a = 0$ and $a \wedge b = -b \wedge a$. It is easily proved that $a \wedge b \in \mathbf{V} \wedge \mathbf{V}$ and that the elements $e_i \wedge e_j$ with $1 \leq i < j \leq 4$ form a base for $\mathbf{V} \wedge \mathbf{V}$. Hence $\mathbf{V} \wedge \mathbf{V}$ has dimension 6.

For $L \in \mathbf{V} \wedge \mathbf{V}$ we write $L[i,j] \stackrel{\text{def}}{=} L\eta_i\eta_j$. Then $L[i,i] = 0$ and $L[i,j] = -L[j,i]$. Also

$$L = \sum_{i<j} L[i,j](e_i \wedge e_j) \text{ and } (a \wedge b)[i,j] = \begin{vmatrix} a[i] & a[j] \\ b[i] & b[j] \end{vmatrix}.$$

Dually, let $\mathbf{V}^* \wedge \mathbf{V}^*$ denote the vector space of all antisymmetric bilinear forms Λ on $\mathbf{V}$. Write $ab\Lambda$ for $\Lambda(a,b)$. Again we may regard $b\Lambda$ as an element of $\mathbf{V}^*$ and hence $\mathbf{V}^* \wedge \mathbf{V}^*$ as a subspace of $\mathrm{Hom}(\mathbf{V}, \mathbf{V}^*)$. For $\alpha, \beta \in \mathbf{V}^*$ we define $\alpha \wedge \beta \in \mathbf{V}^* \wedge \mathbf{V}^*$ as follows :

$$ab(\alpha \wedge \beta) \stackrel{\text{def}}{=} \begin{vmatrix} a\alpha & a\beta \\ b\alpha & b\beta \end{vmatrix} \text{ or } b(\alpha \wedge \beta) = (b\beta)\alpha - (b\alpha)\beta.$$

As before, $\alpha \wedge \alpha = 0$ and $\alpha \wedge \beta = -\beta \wedge \alpha$. $\mathbf{V}^* \wedge \mathbf{V}^*$ has dimension 6 with base elements $\eta_i \wedge \eta_j$ (for $i < j$). If we write $\Lambda[i,j] \stackrel{\text{def}}{=} e_i e_j \Lambda$, then

$$\Lambda = \sum_{i<j} \Lambda[i,j]\eta_i \wedge \eta_j \text{ and } (\alpha \wedge \beta)[i,j] = \begin{vmatrix} \alpha[i] & \alpha[j] \\ \beta[i] & \beta[j] \end{vmatrix}.$$

It is easily verified that the following action of $\mathbf{L}$ on $\mathbf{V} \wedge \mathbf{V}$ establishes $\mathbf{V} \wedge \mathbf{V}$ as a 6-dimensional module of $\mathbf{L}$:

$$(a \wedge b)^A \stackrel{\text{def}}{=} aA \wedge b + a \wedge bA, \text{ for } a, b \in \mathbf{V}, A \in \mathbf{L}.$$

When $L \in \mathbf{V} \wedge \mathbf{V}$ and $\Lambda \in \mathbf{V}^* \wedge \mathbf{V}^*$ we define the product $L \cdot \Lambda$ from the following property :

$$(a \wedge b) \cdot (\alpha \wedge \beta) \stackrel{\text{def}}{=} ab(\alpha \wedge \beta) = (a \wedge b)\alpha\beta = \begin{vmatrix} a\alpha & a\beta \\ b\alpha & b\beta \end{vmatrix}.$$

This definition is sound because the defining formula is linear and antisymmetric in the symbols a and b. We have

$$\begin{array}{rcll} L \cdot (\alpha \wedge \beta) & = & L\alpha\beta, & \text{for } \alpha, \beta \in \mathbf{V}^*, L \in \mathbf{V} \wedge \mathbf{V} \\ (a \wedge b) \cdot \Lambda & = & ab\Lambda, & \text{for } a, b \in \mathbf{V}, \Lambda \in \mathbf{V}^* \wedge \mathbf{V}^*. \end{array}$$

It is easily shown that $(e_i \wedge e_j) \cdot (\eta_k \wedge \eta_l)$ is non-zero only when $\{i, j\} = \{k, l\}$ and therefore

$$L \cdot \Lambda = \sum_{i<j} L[i, j]\Lambda[i, j].$$

Now, let $a, b, c, d \in \mathbf{V}$. Define $(a \wedge b)^*$ to be the unique element of $\mathrm{Hom}(\mathbf{V}, \mathbf{V}^*)$ satisfying

$$cd(a \wedge b)^* \stackrel{\text{def}}{=} \begin{vmatrix} a[1] & a[2] & a[3] & a[4] \\ b[1] & b[2] & b[3] & b[4] \\ c[1] & c[2] & c[3] & c[4] \\ d[1] & d[2] & d[3] & d[4] \end{vmatrix},$$

for all $c, d \in \mathbf{V}$. We find

$$\begin{array}{rclrcl} (e_1 \wedge e_2)^* & = & \eta_3 \wedge \eta_4, & (e_2 \wedge e_3)^* & = & \eta_1 \wedge \eta_4, \\ (e_1 \wedge e_3)^* & = & \eta_4 \wedge \eta_2, & (e_2 \wedge e_4)^* & = & \eta_3 \wedge \eta_1, \\ (e_1 \wedge e_4)^* & = & \eta_2 \wedge \eta_3, & (e_3 \wedge e_4)^* & = & \eta_1 \wedge \eta_2. \end{array}$$

These expressions can be extended to a linear operator $*$ on $\mathbf{V} \wedge \mathbf{V}$ (with values in $\mathbf{V}^* \wedge \mathbf{V}^*$). Using coordinates we find (for $L \in \mathbf{V} \wedge \mathbf{V}$) :

$$\begin{array}{rclrcl} L^*[1,2] & = & L[3,4] & L^*[2,3] & = & L[1,4] \\ L^*[1,3] & = & L[4,2] & L^*[2,4] & = & L[3,1] \\ L^*[1,4] & = & L[2,3] & L^*[3,4] & = & L[1,2] \end{array}$$

For $L, M \in \mathbf{V} \wedge \mathbf{V}$ the expression $L \cdot M^*$ defines a symmetric bilinear form on $\mathbf{V} \wedge \mathbf{V}$ with associated quadratic form #, given by

$$L^\# = L[1,2]L[3,4] + L[1,3]L[4,2] + L[1,4]L[2,3].$$

Note that $L \cdot L^* = 2L^\#$. Finally, composition of $L, L' \in \mathbf{V} \wedge \mathbf{V}$ and $\Lambda, \Lambda' \in \mathbf{V}^* \wedge \mathbf{V}^*$ is denoted by $\circ$:

$$\begin{array}{rclrcl} a(L \circ \Lambda) & \stackrel{\text{def}}{=} & L(a\Lambda), & \alpha(\Lambda \circ L) & \stackrel{\text{def}}{=} & (L\alpha)\Lambda, \\ a(\Lambda' \circ L \circ \Lambda) & \stackrel{\text{def}}{=} & [L(a\Lambda)]\Lambda', & \alpha(L' \circ \Lambda \circ L) & \stackrel{\text{def}}{=} & L'[(L\alpha)\Lambda], \end{array}$$

for $a \in \mathbf{V}$ and $\alpha \in \mathbf{V}^*$.

Proposition A.1 *Let $L \in \mathbf{V} \wedge \mathbf{V}$. Then $L \circ L^* = L^\#$, and dually $L^* \circ L = L^\#$, i.e.,*

$$(L\alpha)L^* = L^\#\alpha, \quad L(aL^*) = L^\# a, \quad \text{for all } a \in \mathbf{V}, \alpha \in \mathbf{V}^*.$$

Proposition A.2 *Let $L \in \mathbf{V} \wedge \mathbf{V}, L \neq 0$. Then the following are equivalent :*

1. *L is of the form $L = a \wedge b$ for some $a, b \in \mathbf{V}$.*
2. rank $L = 2$, *i.e.*, $\dim L\mathbf{V}^* = 2$.
3. $L^\# = 0$.
4. *$L\alpha \wedge L\beta = (L\alpha\beta)L$ for all $\alpha, \beta \in \mathbf{V}^*$.*
5. *$L \circ \Lambda \circ L = (L \cdot \Lambda)L$ for every $\Lambda \in \mathbf{V}^* \wedge \mathbf{V}^*$.*

An element L of $\mathbf{V} \wedge \mathbf{V}$ satisfying one of these conditions is called *isotropic* and we define the isotropic elements of $\mathbf{V}^* \wedge \mathbf{V}^*$ in a dual way. Because the expression for $L^\#$ is self-dual, $L \in \mathbf{V} \wedge \mathbf{V}$ is isotropic if and only if L^* is isotropic.

• The notion of isotropy can also be defined for other A_3-modules. Every element of $\mathbf{V}$ and of $\mathbf{V}^*$ is isotropic. An element A of $\mathbf{L}$ will be isotropic if and only if rank $A = 1$. This is the case if and only if A can be written as $\alpha \otimes a$ for some $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$.

We define the geometry $\mathcal{A}$ of points, lines and planes as follows :

- The *points* of $\mathcal{A}$ are the 1-dimensional subspaces of $\mathbf{V}$,
- the *planes* of $\mathcal{A}$ are the 1-dimensional subspaces of $\mathbf{V}^*$,
- the *lines* of $\mathcal{A}$ are the 1-dimensional subspaces generated by *isotropic* elements of $\mathbf{V} \wedge \mathbf{V}$,

with the following incidence :

- A point Ka is incident with a line KL if and only if there exists $\beta \in \mathbf{V}^*$ such that $a = L\beta$,

- a line KL is incident with a plane $K\alpha$ if and only if $L\alpha = 0$,

- a point Ka is incident with a plane $K\alpha$ if and only if $a\alpha = 0$.

Because $a = L\beta$ and $L\alpha = 0$ implies $L\alpha\beta = -a\alpha = 0$, incidence of points, lines and planes is 'transitive'.

The reader will have guessed that $\mathcal{A}$ is the geometry of points, lines and planes of a projective space of dimension 3 with the classical notion of incidence. The coordinates $L[i,j]$ are *Plücker coordinates* of the line KL (and $L^\# = 0$ is the equation of the *Klein quadric*).

The following proposition provides several links between algebraic properties of $\mathbf{V}$, $\mathbf{V}^*$ and $\mathbf{V} \wedge \mathbf{V}$, and geometrical properties of $\mathcal{A}$.

Proposition A.3 *Let $a, b \in \mathbf{V} - \{0\}$, $\alpha, \beta \in \mathbf{V}^* - \{0\}$ and $L, M \in \mathbf{V} \wedge \mathbf{V} - \{0\}$ be such that $L^\# = M^\# = 0$. Then*

1. *If Ka and Kb are different points then $K(a \wedge b)$ is the unique line joining Ka and Kb.*

2. *If Ka is not incident with KL, then $K(aL^*)$ is the unique plane containing both Ka and KL.*

3. *If the line KL is not incident with the plane $K\alpha$ then $K(L\alpha)$ is the unique point they have in common.*

4. *If $K\alpha$ and $K\beta$ are different planes, then $K(\alpha \wedge \beta)^*$ is the unique line incident with both planes.*

5. *Two different lines KL and KM intersect if and only if KL and KM lie in the same plane if and only if $L \cdot M^* = 0$. In that case the linear operator M^*L has rank 1 and can be written as αa where Ka is the intersection point and $K\alpha$ is the plane containing KL and KM.*

A.2 Modules for the Lie algebra of type B_2

Consider the special element $\infty = e_1 \wedge e_2 + e_3 \wedge e_4$. We have $\infty^* = \eta_1 \wedge \eta_2 + \eta_3 \wedge \eta_4$ and $\infty^\# = 1$. For $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ we define $\overline{a} \stackrel{\text{def}}{=} a\infty^*$, $\overline{\alpha} \stackrel{\text{def}}{=} \infty\alpha$. Note that $\overline{a} \in \mathbf{V}^*$, $\overline{\alpha} \in \mathbf{V}$. In terms of coordinates this reads

$$\overline{a} = \begin{pmatrix} a[2] \\ -a[1] \\ a[4] \\ -a[3] \end{pmatrix}, \qquad \overline{\alpha} = \begin{pmatrix} \alpha[2] & -\alpha[1] & \alpha[4] & -\alpha[3] \end{pmatrix}.$$

We have $\overline{\overline{a}} = \infty(a\infty^*) = \infty^\# a = a$, for all $a \in \mathbf{V}$, by Proposition A.1. Also $a\overline{b} = ab\infty^* = -ba\infty^* = -b\overline{a}$, for $a, b \in \mathbf{V}$, because $\infty^* \in \mathbf{V}^* \wedge \mathbf{V}^*$. Using a similar argument we find that $a\overline{a} = 0$ for all $a \in \mathbf{V}$.

Denote by $\mathbf{J}$ the subspace of $\mathbf{L}$ generated by all products of the form $\overline{a} \otimes a$ with $a \in \mathbf{V}$. We find

$$\overline{a} \otimes a = \begin{pmatrix} a[1]a[2] & a[2]^2 & a[2]a[3] & a[2]a[4] \\ -a[1]^2 & -a[1]a[2] & -a[1]a[3] & -a[1]a[4] \\ a[1]a[4] & a[2]a[4] & a[3]a[4] & a[4]^2 \\ -a[1]a[3] & -a[2]a[3] & -a[3]^2 & -a[3]a[4] \end{pmatrix}$$

Hence, if $A \in \mathbf{J}$, the elements of A must satisfy the following conditions :

$$\begin{array}{rclcrcl} A[1,1] + A[2,2] & = & 0, & \qquad & A[1,4] & = & A[3,2], \\ A[3,3] + A[4,4] & = & 0, & & A[2,3] & = & A[4,1], \\ A[1,3] + A[4,2] & = & 0, & & & & \\ A[2,4] + A[3,1] & = & 0. & & & & \end{array}$$

From this information it can easily be proved that $\dim \mathbf{J} = 10$.

Proposition A.4 *Let $A \in \mathbf{L}$. Then $A \in \mathbf{J}$ if and only if $aA\overline{b} = bA\overline{a}$ for all $a, b \in \mathbf{V}$.*

It is a consequence of this proposition that $\mathbf{J}$ is a Lie algebra and that the operation $a, b \mapsto a\overline{b}$ is compatible with $\mathbf{J}$. $\mathbf{J}$ turns out to be an algebra of type B_2.

The operator $\bar{\cdot}$ can be extended to $\mathbf{V} \wedge \mathbf{V}$ by setting $\overline{b \wedge c} \stackrel{\text{def}}{=} \overline{b} \wedge \overline{c}$. We have

$$a(\overline{b \wedge c}) = a(\overline{b} \wedge \overline{c}) = (a\overline{c})\overline{b} - (a\overline{b})\overline{c} = (b\overline{a})\overline{c} - (c\overline{a})\overline{b} = \overline{(b \wedge c)\overline{a}},$$

and hence, in general $a\overline{L} = \overline{L\overline{a}}$, for all L in $\mathbf{V} \wedge \mathbf{V}$. Expressed in terms of coordinates, this yields

$$\begin{array}{rclcrcl} \overline{L}[1,2] & = & L[1,2], & & \overline{L}[2,3] & = & L[4,1], \\ \overline{L}[1,3] & = & L[2,4], & & \overline{L}[2,4] & = & L[1,3], \\ \overline{L}[1,4] & = & L[3,2], & & \overline{L}[3,4] & = & L[3,4]. \end{array}$$

For $L \in \mathbf{V} \wedge \mathbf{V}$ consider the expression $L \cdot \infty^* = L[1,2] + L[3,4]$. (We have $(a \wedge b) \cdot \infty^* = a\overline{b}$.) Denote the subspace of all elements $L \in \mathbf{V} \wedge \mathbf{V}$ satisfying $L \cdot \infty^* = 0$ by $\mathbf{T}$. It follows that $\mathbf{T}$ has dimension 5. From the coordinate expressions for $\overline{L}$ and L^* it follows that $L \in \mathbf{T}$ if and only if $\overline{L} + L^* = 0$.

Let $A \in \mathbf{L}$, $a, b \in \mathbf{V}$. Then

$$(a \wedge b)^A \cdot \infty^* = aA\,b\,\infty^* + a\,bA\,\infty^* = aA\overline{b} - bA\overline{a},$$

and this is 0 whenever $A \in \mathbf{J}$. Hence $\mathbf{T}$ is a 5-dimensional $\mathbf{J}$-module (but not an $\mathbf{L}$-module).

The space $\mathbf{T}$ can also be interpreted as a subspace of $\mathbf{L}$. For $L \in \mathbf{V} \wedge \mathbf{V}$ define $\hat{L}$ to be the following linear transformation

$$\hat{L} : a \mapsto a\hat{L} \stackrel{\text{def}}{=} L\overline{a}, \quad \text{for } a \in \mathbf{V},$$

Then

$$\begin{aligned} \operatorname{Tr} \hat{L} &= e_1\hat{L}\eta_1 + \cdots + e_4\hat{L}\eta_4 = L\overline{e_1}\eta_1 + \cdots + L\overline{e_4}\eta_4 \\ &= -L[2,1] + L[1,2] - L[4,3] + L[3,4] = 2L \cdot \infty^*. \end{aligned}$$

Hence $\hat{L} \in \mathbf{L}$ when $L \in \mathbf{T}$.

Consider $L = a \wedge b$ and let $A \in \mathbf{J}$. Let $c \in \mathbf{V}$, then $c\hat{L} = (a\overline{c})b - (b\overline{c})a$ and

$$\begin{aligned} c\widehat{L^A} &= (aA\overline{c})b - (b\overline{c})aA + (a\overline{c})bA - (bA\overline{c})a \\ &= (a\overline{c})bA - (b\overline{c})aA + (cA\overline{a})b - (cA\overline{b})a \\ &= c\hat{L}A - cA\hat{L} = c[\hat{L}, A] = c\hat{L}^A. \end{aligned}$$

This proves that $\mathbf{T}$ and $\widehat{\mathbf{T}}$ are equivalent as $\mathbf{J}$-modules.

We define a geometry $\mathcal{B}$ of points and lines as follows :

- The *points* of $\mathcal{B}$ are the points of $\mathcal{A}$.
- The *lines* of $\mathcal{B}$ are the lines KL of $\mathcal{A}$ for which L belongs to $\mathbf{T}$.

Incidence between points and lines of $\mathcal{B}$ is the same as in $\mathcal{A}$.

Proposition A.5 *Let KL denote a line of $\mathcal{A}$. Then KL is a line of $\mathcal{B}$ if and only if one of the following conditions holds :*

1. $\overline{L} \circ L = 0$,
2. $L = a \wedge b$ *with* $a, b \in \mathbf{V}$ *such that* $a\overline{b} = 0$, $a \in \mathbf{V}$.

Proposition A.6 *Let $a, b \in \mathbf{V} - \{0\}$, $\alpha, \beta \in \mathbf{V}^* - \{0\}$ and $L, M \in \mathbf{T} - \{0\}$ be such that $L^\# = M^\# = 0$. Then*

1. *The points Ka and Kb are collinear if and only if $a\overline{b} = 0$.*
2. *The point Ka is incident with the line KL if and only if $L\overline{a} = 0$.*
3. *If a point Ka is not incident with a line KL, then $K\,L\overline{a}$ is the unique point incident with KL and collinear to Ka and $K(L\overline{a} \wedge a)$ is the unique line incident with Ka that intersects KL.*

The reader will have recognized the geometry $\mathcal{B}$ as the symplectic (generalized) quadrangle $W(K)$.

A.3 The case of characteristic 2

When the characteristic of the base field K is equal to 2 the module $\mathbf{T}$ is no longer irreducible. Indeed, it may easily be derived that in this case $\infty^A = 0$

for all $A \in \mathbf{J}$. It therefore makes sense to consider the quotient module $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{T}/K\infty$ which is now a 4-dimensional module for $\mathbf{J}$.

Many of the operations on $\mathbf{T}$ also have meaning on $\mathbf{Q}$. For example, because $\infty \cdot L^* = 0$ and $L \cdot \infty^* = 0$ for all $L \in \mathbf{T}$, the dot product $(L + K\infty) \cdot (M + K\infty)^* \stackrel{\text{def}}{=} L \cdot M^*$ is well-defined for any two elements of $\mathbf{Q}$. Likewise, given $a \in \mathbf{V}$ we have $\infty\overline{a} \wedge a = \overline{\overline{a}} \wedge a = a \wedge a = 0$, and hence also $(L + K\infty)\overline{a} \wedge a \stackrel{\text{def}}{=} L\overline{a} \wedge a$ is independent of the choice of the representative.

We introduce the following linear transformation $\mu : \mathbf{V} \to \mathbf{Q}$:

$$\begin{array}{rclrcl} \mu(e_1) & = & e_1 \wedge e_4 + K\infty, \qquad & \mu(e_2) & = & e_2 \wedge e_3 + K\infty, \\ \mu(e_3) & = & e_2 \wedge e_4 + K\infty, \qquad & \mu(e_4) & = & e_1 \wedge e_3 + K\infty. \end{array}$$

The following proposition indicates that μ acts as a kind of 'orthogonal' transformation between the modules $\mathbf{V}$ and $\mathbf{Q}$.

Proposition A.7 *Let $a, b \in \mathbf{V}$. Then $\mu(a) \cdot \mu(b)^* = a\overline{b}$.*

It is a well-known fact that the generalized quadrangle $\mathcal{B}$ is self-dual when $\operatorname{char} K = 2$, and it would therefore not be unexpected if this property would be reflected in the form of a *polarity* which interchanges the modules $\mathbf{V}$ and $\mathbf{Q}$. The map μ is a likely candidate, but it will turn out that we need to be more inventive. One reason is that the operation $L\overline{a}$ which is used to represent incidence in $\mathcal{B}$ cannot be easily generalized to elements $L + K\infty$ of $\mathbf{Q}$.

We will instead introduce yet another operator $Q(\cdot)$. Let $a \in \mathbf{V}$. Define $Q(a)$ to be the unique bilinear form on $\mathbf{V}^*$ satisfying

$$Q(a)\overline{b}\overline{c} = (\mu(b)\overline{a} \wedge a) \cdot \mu(c)^*, \quad \text{for all } b, c \in \mathbf{V}.$$

It can be proved that $Q(a)\overline{b}\overline{b} = 0$ for all $a, b \in \mathbf{V}$, and hence $Q(a)$ is antisymmetric. We may compute the coordinates of $Q(a)$ from the definition. For example, we have

$$\begin{aligned}
Q(a)[1,2] = Q(a)\eta_1\eta_2 = Q(a)\overline{e_2 e_1} &= (\mu(e_2)\overline{a} \wedge a) \cdot \mu(e_1)^* \\
&= [(e_2 \wedge e_3)\overline{a} \wedge a] \cdot (e_1 \wedge e_4)^* \\
&= [(e_2\overline{a})e_3 + (e_3\overline{a})e_2] \cdot (\eta_2 \wedge \eta_3) \\
&= (e_2\overline{a})\eta_2 + (e_3\overline{a})\eta_3 = a[1]a[2] + a[4]a[3].
\end{aligned}$$

In a similar way we obtain

$$\begin{aligned}
Q(a)[1,2] &= a[1]a[2] + a[3]a[4], & Q(a)[3,4] &= a[1]a[2] + a[3]a[4], \\
Q(a)[1,3] &= a[4]^2, & Q(a)[2,3] &= a[2]^2, \\
Q(a)[1,4] &= a[1]^2, & Q(a)[2,4] &= a[3]^2.
\end{aligned}$$

It follows that $Q(a)[1,2] + Q(a)[3,4] = 0$ and hence $Q(a) \in \mathbf{T}$. Also

$$Q(a)^\# = (a[1]a[2] + a[3]a[4])^2 + a[1]^2a[2]^2 + a[3]^2a[4]^2 = 0,$$

so $Q(a)$ is isotropic.

From the coordinate expression we also derive the following identity :

$$\mu^{-1}(Q(a) + K\infty) = a^{\text{frob}},$$

where a^{frob} denotes the vector $(a[1]^2 \ \cdots \ a[4]^2)$, i.e., the vector obtained by applying the Frobenius morphism to the coordinates of a.

Proposition A.8 *Let $a, b \in \mathbf{V}$, then*

1. $Q(a) = 0$ *if and only if* $a = 0$.
2. $Q(a) \cdot Q(b)^* = (a\overline{b})^2$.
3. $Q(Q(a)\overline{b}) = Q(b)\overline{a}^{\text{frob}} \wedge a^{\text{frob}}$.

Comparing the last result with Proposition A.6, we obtain the following geometrical interpretation.

Proposition A.9 *The map defined by* $Ka \mapsto KQ(a)$, $KQ(a) \mapsto Ka^{\mathrm{frob}}$, *for* $a \in \mathbf{V}$, *is a* correlation *of* $\mathcal{B}$, *i.e., it maps points of* $\mathcal{B}$ *onto lines of* $\mathcal{B}$, *lines onto points and preserves incidence.*

Note that this map is quadratic (and not linear), and is not an involution.

A.4 The Suzuki-Tits ovoid

In what follows, we will restrict ourselves to fields K of characteristic 2 which have a field automorphism σ with the property

$$(k^\sigma)^\sigma = k^2, \qquad \text{for every } k \in K.$$

(Such fields are perfect, i.e., $K^2 = K$.) An automorphism with these properties is called a *Tits* automorphism of K.

We may now use σ to make the correlation of the previous section into a true polarity of $\mathcal{B}$, by interchanging a^σ with $Q(a)$. We leave it to the reader to verify that indeed $Q(a)\overline{b}^\sigma = 0$ if and only if $Q(b)\overline{a}^\sigma = 0$, for all $a, b \in \mathbf{V}$.

Consider the set $\mathcal{O}$ of all *absolute points* of this polarity, i.e., the points Ka of $\mathcal{B}$ such that a^σ is incident with $Q(a)$, or equivalently, such that $Q(a)\overline{a}^\sigma = 0$.

Expressed in coordinates, Ka belongs to $\mathcal{O}$ if and only if the following system of equations is satisfied :

$$\begin{array}{lllllllll}
a[1]^\sigma a[1]a[2] & + & a[1]^\sigma a[3]a[4] & + & a[3]^\sigma a[1]^2 & + & a[4]^\sigma a[4]^2 & = 0, & (i) \\
a[2]^\sigma a[1]a[2] & + & a[2]^\sigma a[3]a[4] & + & a[3]^\sigma a[3]^2 & + & a[4]^\sigma a[2]^2 & = 0, & (ii) \\
a[1]^\sigma a[2]^2 & + & a[2]^\sigma a[4]^2 & + & a[3]^\sigma a[1]a[2] & + & a[3]^\sigma a[3]a[4] & = 0, & (iii) \\
a[1]^\sigma a[3]^2 & + & a[2]^\sigma a[1]^2 & + & a[4]^\sigma a[1]a[2] & + & a[4]^\sigma a[3]a[4] & = 0. & (iv)
\end{array}$$

These equations are not independent. For instance, we find

$$a[1](iv)^\sigma = a[4]^\sigma(i) + (i)^\sigma, \qquad a[1]^\sigma(iii) = a[3]^\sigma(i) + (iv)^\sigma,$$

and therefore equations (i) and (ii) are sufficient to define $\mathcal{O}$ when $a[1] \neq 0$. In that case, setting $t = a[3]$, $u = a[4]$, the first two equations reduce to

$$\begin{array}{rcl} a[2] + tu + t^\sigma + u^{2+\sigma} & = & 0, \\ a[2]^{1+\sigma} + a[2]^\sigma tu + t^{2+\sigma} + u^\sigma a[2]^2 & = & 0. \end{array}$$

The first equation yields a value for $a[2]$ which may be substituted into the second equation and it turns out that this does not induce any further restrictions on the parameters t and u.

When $a[1] = 0$ we have $a[4] = 0$ by (i) and then $a[3] = 0$ by (ii).

This proves that the set $\mathcal{O}$ consists of all points of the form

$$K(0,1,0,0), \text{ or } K(1, tu + t^\sigma + u^{2+\sigma}, t, u), \text{ with } t, u \in K.$$

This is the standard parameter representation [26, Section 7.6.12] for the points of the *Suzuki-Tits ovoid*, which proves that $\mathcal{O}$ is indeed what we intended.

B Een algebraïsche aanpak van de perfecte veralgemeende achthoeken van Ree-Tits en verwante meetkundes

1 Inleiding en overzicht

Van alle veralgemeende veelhoeken die verband houden met algebraïsche groepen zijn de veralgemeende achthoeken wellicht het minst grondig bestudeerd. Misschien is dit omdat er slechts één familie van voorbeelden van dergelijke objecten is gekend of omdat deze zogenaamde veralgemeende achthoeken van *Ree-Tits* en hun inbeddingen in een projectieve ruimte niet gemakkelijk kunnen worden geconstrueerd.

De gebruikelijke manier waarop een Ree-Tits-achthoek wordt gedefinieerd, is als een cosetmeetkunde van een Ree-groep. Deze groep construeert men door de Chevalley-groep van type F_4 over een veld K van karakteristiek 2 te *vlechten* (Engels: *to twist*). De Ree-Tits-achthoek kan ook worden geconstrueerd met behulp van een speciale coördinatisatie van de hand van H. van Maldeghem [26] die dan zelf weer is gebaseerd op eigenschappen van de Ree-groepen, namelijk de commutatierelaties uit [24].

Wat er voor de Ree-Tits-achthoek ontbreekt en wel bestaat voor alle andere klassieke veralgemeende veelhoeken, is een expliciete inbedding in één of andere projectieve ruimte. Het hoofddoel van deze tekst is om precies een dergelijke inbedding op te stellen (in een 25-dimensionale projectieve ruimte) waarbij de punten (en rechten) van een Ree-Tits-achthoek een deelverzameling vormen van de punten (en rechten) uit de projectieve ruimte. We leiden expliciete 'formules' af die je op de projectieve coördinaten van een punt kunt toepassen om te bepalen of het al dan niet tot de achthoek behoort. Bovendien beschrijven we hoe je aan de hand van de coördinaten van twee punten van de achthoek kan bepalen of ze al dan niet collinear zijn, en meer algemeen, wat hun onderlinge afstand is.

Deze 'formules' zijn niet zo elementair als bij de andere veralgemeende veelhoeken. We bewijzen bijvoorbeeld dat een punt Ke tot de Ree-Tits-achthoek behoort als en slechts als zowel $e^2 = 0$ als $q([e, \mathbf{V}], e) = 0$. De eerste van deze uitdrukking staat voor een stelsel van 26 kwadratische vergelijkingen in 26 variabelen (de 26 coördinaten van e). De tweede uitdrukking correspondeert met 676 veeltermenvergelijkingen van de derde graad in 26 variabelen, waaraan bovendien nog een veldautomorfisme σ van K te pas komt. (Vergelijk dit bijvoorbeeld met de formules die je nodig hebt om de punten van een split-Cayley zeshoek te definiëren: één enkele vergelijking in zeven onbekenden volstaat hier.)

In deze tekst wordt de Ree-Tits-achthoek, en de metasymplectische ruimte waartoe zijn punten en rechten behoren, vanuit drie verschillende perspectieven bekeken: groepentheoretisch, meetkundig en algebraïsch, waarbij we ons vooral concentreren op het laatste aspect. We doen dit in vier opeenvolgende stappen: eerst behandelen we de Lie-algebra (en groep) van type E_6 voor een algemeen veld (Hoofdstuk 2), dan beschouwen we de subalgebra van type F_4 in algemene karakteristiek (Hoofdstuk 3) en daarna meer specifiek voor karakteristiek 2 (Hoofdstuk 4). Tot slot bekomen we een beschrijving van de veralgemeende Ree-Tits-achthoek voor een perfect veld (Hoofdstuk 5).

Appendix A illustreert de algemene technieken uit de tekst op een veel eenvoudiger voorbeeld: we reduceren de Lie-algebra van type A_3 naar één van type B_2 en bekomen uiteindelijk de structuur van een *Suzuki-Tits-ovoïde*. Deze appendix staat los van de rest van de tekst.

• De nummering en betiteling van de paragrafen in deze samenvatting loopt parallel met die van de hoofdstukken en paragrafen in de tekst.

2 De Lie-algebra van type E_6 en verwante structuren

2.1 Een wortelsysteem van type E_6

We hebben ervoor gekozen het bekende wortelsysteem van type E_6 te introduceren op een minder conventionele manier. We baseren ons hierbij op het feit dat de Weyl-groep van type E_6 kan worden geïnterpreteerd als automorfismegroep van de veralgemeende vierhoek $\mathcal{Q} = Q^-(5,2)$ met puntenverzameling $\mathcal{P}$ en rechtenverzameling $\mathcal{L}$.

De 27 punten van $\mathcal{Q}$ kunnen we representeren als vectoren in een 6-dimensionale reële vectorrruimte $\mathbf{P}$. Het Euclidisch inproduct van twee punten p, q in die ruimte geeft aan of ze adjacent zijn of niet, op de volgende manier:

$$p \cdot q = \begin{cases} 4/3, & \text{als en slechts als } p = q, \\ 1/3, & \text{als en slechts als } p \not\perp q, \\ -2/3, & \text{als en slechts als } p \sim q. \end{cases} \tag{2.2}$$

De 72 wortels van het wortelsysteem Φ van type E_6 zijn alle verschillen $p - q$ van punten p, q waarvoor $p \cdot q = 1/3$. Uit de waarden van de inproducten van wortels en punten, of van wortels onderling, kunnen opnieuw heel wat combinatorische eigenschappen worden afgeleid:

$r \cdot s =$	als en slechts als
2	$r = s$
1	$r - s \in \Phi$
0	$r - s, r + s \notin \Phi$
-1	$r + s \in \Phi$
-2	$r = -s$

$p \cdot r =$	als en slechts als
1	$p - r \in \mathcal{P}$
0	$p + r, p - r \notin \mathcal{P}$
-1	$p + r \in \mathcal{P}$

voor alle $r, s \in \Phi, p \in \mathcal{P}$. (2.5)

Met elke wortel $r \in \Phi$ komen zes zogenaamde basispunten $p_0, \ldots, p_5 \in \mathcal{P}$ overeen die we kunnen gebruiken om $\mathbf{P}$ te coördinatiseren. Punten krijgen dan coördinaten van de vorm 300000, 211111, 111122, of permutaties hiervan (een streepje boven getal moet gelezen worden als een minteken). Wortels hebben coördinaten van de vorm 111111, 111222, 330000, 111222, 111111, of permutaties hiervan.

Elke wortel r bepaalt een orthogonale spiegeling w_r in $\mathbf{P}$ die r afbeeldt op $-r$ en elk element invariant laat van het hypervlak dat loodrecht staat op r. De groep voortgebracht door deze spiegelingen is de Weyl-groep $W(\mathsf{E}_6)$ van het wortelsysteem. Deze groep bezit belangrijke transitiviteitseigenschappen. Zo is hij bijvoorbeeld transitief op alle paren uit $\mathcal{P} \times \mathcal{P}$, $\Phi \times \mathcal{P}$ of $\Phi \times \Phi$ met een gegeven inproduct en op alle (geördende) cocliques (verzamelingen van onderling niet-adjacente punten) van grootte 2, 3, 4 en 6 (Propositie 2.5). Voor cocliques van grootte 5 zijn er twee banen, afhankelijk van of ze tot een coclique van grootte 6 kunnen worden uitgebreid, of niet.

Neem een vast automorfisme ω van $\mathcal{Q}$, van orde 3 en met de bijkomende eigenschap dat het de punten van $\mathcal{P}$ in 9 banen verdeelt die elk samenvallen met een rechte. Deze negen rechten vormen een spread Σ. Door gebruik te maken van deze spread, kan je de 6 basispunten van elke wortel r partitioneren in 3 positieve en 3 negatieve basispunten:

$$\begin{aligned} &p \cdot r = -1,\ \ p \cdot r\omega = 1,\ \ p \cdot r\omega^2 = 0,\ \ \text{wanneer } p \text{ een positief basispunt is,} \\ &p \cdot r = -1,\ \ p \cdot r\omega = 0,\ \ p \cdot r\omega^2 = 1,\ \ \text{wanneer } p \text{ een negatief basispunt is.} \end{aligned} \tag{2.13}$$

Het belang van het onderscheid tussen beide soorten basispunten zal tot uiting komen in de volgende paragrafen.

2.2 De Lie-algebra van type E_6 over K

Beschouw een 27-dimensionale vectorruimte $\mathbf{V}$ over een veld K met basisvectoren e_p die elk met een verschillend punt $p \in \mathcal{P}$ overeenkomen. De corresponderende duale ruimte $\mathbf{V}^*$ heeft canonische basisvectoren η_q (met $q \in \mathcal{P}$) die voldoen aan $e_p\eta_q = \delta_{pq}$. Element van $\mathrm{Hom}(\mathbf{V}, \mathbf{V})$ zullen we identificeren met matrices die rechts inwerken op $\mathbf{V}$ en links of $\mathbf{V}^*$.

Op $\mathbf{V}$ introduceren we de volgende trilineaire vorm:

$$\langle e_i, e_j, e_k \rangle \stackrel{\text{def}}{=} \begin{cases} \ \ 1, & \text{als } \{i,j,k\} \in \Sigma, \\ -1, & \text{als } \{i,j,k\} \in \mathcal{L} - \Sigma, \\ \ \ 0, & \text{in alle andere gevallen.} \end{cases} \tag{2.15}$$

Met behulp van deze vorm kunnen we ook het bilineair product $\times$ definiëren dat een paar $a, b \in \mathbf{V}$ afbeeldt op een element $a \times b$ uit $\mathbf{V}^*$ en voldoet aan de eigenschap $c(a \times b) \stackrel{\text{def}}{=} \langle c, a, b\rangle$, voor alle $c \in \mathbf{V}$. Daarnaast introduceren we ook een kwadratische operator $\cdot^{\#}$ en een cubische vorm D: $a^{\#} = \frac{1}{2} a \times a$, $D(a) = \frac{1}{3} a a^{\#} = \frac{1}{6}\langle a, a, a\rangle$. (Deze definities kunnen ook een betekenis krijgen wanneer char $K = 2$ of 3.) Voor elk van deze bewerkingen bestaat er ook een 'duaal' equivalent voor $\mathbf{V}^*$.

De belangrijkste eigenschap van deze operatoren is de volgende (Propositie 2.10):

$$\begin{aligned}(a \times b) \times (c \times d) \;&+\; (a \times c) \times (b \times d) \;+\; (a \times d) \times (b \times c) \\ &= \; \langle a, b, c\rangle d \;+\; \langle b, c, d\rangle a \;+\; \langle c, d, a\rangle b \;+\; \langle d, a, b\rangle c,\end{aligned} \tag{2.24}$$

voor alle $a, b, c, d \in \mathbf{V}$. De tekst bevat vele andere formules die hiervan onmiddellijk kunnen worden afgeleid.

We kunnen nu de Lie-algebra $\widehat{\mathbf{L}}$ op de volgende manier definiëren: zij bestaat uit alle lineaire transformaties A die voldoen aan de eigenschap

$$a \times aA + Aa^{\#} + \tau(A)a^{\#} = 0, \qquad \text{voor alle } a \in \mathbf{V}, \tag{2.41}$$

waarbij $\tau(A)$ een element is van K dat enkel afhangt van A. We bewijzen gemakkelijk dat τ een lineaire functionaal is en dat $\widehat{\mathbf{L}}$ een Lie-algebra is over K met de gebruikelijke definitie van de Lie-haak ($[A, B] \stackrel{\text{def}}{=} AB - BA$). Er geldt $\tau([A, B]) = 0$, voor alle $A, B \in \widehat{\mathbf{L}}$ en dus is de kern $\mathbf{L}$ van τ opnieuw een Lie algebra.

De algebra $\widehat{\mathbf{L}}$ kan ook gedefinieerd worden als de algebra voortgebracht door alle lineaire transformaties $a * \alpha$ met $a \in \mathbf{V}$ en $\alpha \in \mathbf{V}^*$, waarbij

$$b(a * \alpha)\beta \stackrel{\text{def}}{=} (b\alpha)(a\beta) - (\alpha \times \beta)(a \times b), \qquad \text{voor alle } b \in \mathbf{V}, \beta \in \mathbf{V}^*. \tag{2.34}$$

Dat beide definities van $\widehat{\mathbf{L}}$ equivalent zijn, wordt bewezen in Stelling 2.19. (We hebben $\tau(a * \alpha) = a\alpha$.)

Propositie 2.22 toont aan dat $\mathbf{L}$ de gekende Lie-algebra is van type E_6 over K en dat we een Chevalley-basis voor deze algebra kunnen construeren aan de hand van elementen van de vorm

$$E_r \stackrel{\text{def}}{=} e_q * \eta_p, \qquad H_r \stackrel{\text{def}}{=} e_q * \eta_q - e_p * \eta_p, \tag{2.44, 2.48}$$

waarbij $p, q \in \mathcal{P}$, $r \in \Phi$ en $r = q - p$.

Met behulp van deze basiselementen definiëren we ook nog een inproduct op $\mathbf{L}$, als volgt (zij $r, s \in \Phi$):

$$\begin{aligned} E_r \cdot E_s &\stackrel{\text{def}}{=} \begin{cases} -1 & \text{wanneer } r = -s, \\ 0 & \text{wanneer } r \neq -s. \end{cases} \\ E_r \cdot H_s = H_s \cdot E_r &\stackrel{\text{def}}{=} 0 \\ H_r \cdot H_s &\stackrel{\text{def}}{=} r \cdot s. \end{aligned} \tag{2.68}$$

(Deze definitie kan gemakkelijk uitgebreid worden naar $\widehat{\mathbf{L}}$.) Er volgt andere dat $(a * \alpha) \cdot A = aA\alpha$ en dat $A \cdot \mathbf{1} = \tau(A)$, voor alle $a \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ en $A \in \widehat{\mathbf{L}}$.

2.3 Isotrope elementen

We noemen een $e \in \mathbf{V}$ *isotroop* wanneer $e^\# = 0$. Op analoge wijze noemen we $\eta \in \mathbf{V}^*$ isotroop wanneer $\eta^\# = 0$. Een element E van de Lie-algebra $\mathbf{L}$ heet isotroop wanneer we het kunnen schrijven als $E = e * \eta$ waarbij zowel e als η isotroop zijn, en bovendien $e\eta = 0$.

Typische voorbeelden van isotrope elementen zijn de basiselementen e_p (resp. η_q) van $\mathbf{V}$ (resp. $\mathbf{V}^*$) en de elementen E_r uit de Chevalley-basis van $\mathbf{L}$.

Isotrope elemenen hebben vele interessante eigenschappen (Proposities 2.27 en 2.28), waarvan we hier slechts enkele vermelden: zij $e \in \mathbf{V}$ en $E \in \mathbf{L}$ isotroop, zij $a, b \in \mathbf{V}$, $A \in \mathbf{L}$, dan geldt $(e \times a) \times (e \times b) = \langle e, a, b \rangle e$, $(aE)^\# = 0$, $aE \times bE = 0$ en $EAE = (E \cdot A)E$.

Een verwant begrip is dat van *isotrope deelruimte* van $\mathbf{V}$: een deelruimte heet isotroop als en slechts als al zijn elementen isotroop zijn. Uit bovenstaande volgt bijvoorbeeld dat $\mathbf{V}E$ steeds een isotrope deelruimte is van $\mathbf{V}$ wanneer $E \in \mathbf{L}$ isotroop is.

2.4 Automorfismen

An automorfisme g is een lineaire afbeelding van $\mathbf{V}$ (en bij extensie, van $\mathbf{V}^*$ en $\mathrm{Hom}(\mathbf{V},\mathbf{V})$) die de algebraïsche basisbewerkingen op $\mathbf{V}$ 'bewaart': voor elke $a,b,c \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ en $A \in \mathrm{Hom}(\mathbf{V},\mathbf{V})$ geldt $a^g\alpha^g = a\alpha$, $a^gA^g\alpha^g = aA\alpha$, $(a^g)^\# = (a^\#)^g$, $a^g \times b^g = (a\times b)^g$, $\langle a^g, b^g, c^g\rangle = \langle a,b,c\rangle$, $a^g * \alpha^g = (a*\alpha)^g$, ... Automorfismen beelden isotrope elementen af op isotrope elementen en laten $\mathbf{L}$ invariant.

In Propositie 2.30 introduceren we een basisvoorbeeld van een dergelijk automorfisme, namelijk de lineaire afbeelding $x(E)$, met E een isotroop element van $\mathbf{L}$, gedefinieerd op de volgende manier:

$$\begin{aligned} a^{x(E)} &= a - aE, && \text{voor alle } a \in \mathbf{V}, \\ \alpha^{x(E)} &= \alpha + E\alpha, && \text{voor alle } \alpha \in \mathbf{V}^*, \\ A^{x(E)} &= A + [E,A] - (E\cdot A)E, && \text{voor alle } A \in \widehat{\mathbf{L}}. \end{aligned} \tag{2.84}$$

De groep die door alle dergelijke $x(E)$ wordt voortgebracht, noteren we als $\widehat{\mathsf{E}}_6(K)$. Dit is een zogenaamde *Chevalley-groep* van type E_6.

Stellingen 2.34–2.38 beschrijven de banen van $\widehat{\mathsf{E}}_6(K)$ op de isotrope elementen van $\mathbf{V}$, $\mathbf{V}^*$ en $\mathbf{L}$ (er is telkens één niet-triviale baan) en op *paren* van dergelijke isotrope elementen. Deze banen blijken sterk verwant aan de banen van de Weyl-groep $W(\mathsf{E}_6)$ op paren van punten en wortels.

Zo zijn er bijvoorbeeld drie soorten banen op paren van niet-triviale isotrope elementen van $\mathbf{V}$, namelijk banen met representanten (e_p, e_q) met $p \not\sim q$, (e_p, e_q) met $p \sim q$ en $(e_p, \ell e_p)$, waarbij $p,q \in \mathcal{P}$ en $\ell \in K - \{0\}$ (Stelling 2.35). Deze corresponderen met de drie banen van $W(\mathsf{E}_6)$ op $\mathcal{P}\times\mathcal{P}$. Dezelfde banen kunnen ook worden vastgelegd met behulp van algebraïsche bewerkingen op $\mathbf{V}$: in het eerste geval gaat het om paren (e,f) waarvoor $e \times f \neq 0$, in het tweede geval is $e\times f = 0$ maar $Ke \neq Kf$, en in het derde geval hebben we $f = \ell e$. De andere stellingen bieden gelijkaardige informatie in de andere gevallen.

In zekere zin zijn deze stellingen de meest belangrijke uit het hoofdstuk, omdat ze toelaten algemene eigenschappen van (paren) isotrope elementen af te leiden door ze enkel te bewijzen voor isotrope elementen van de vorm e_p, η_q

of E_r. Zo rekenen we bijvoorbeeld gemakkelijk uit dat $\dim \mathbf{V}E_r = 6$ en hieruit volgt dan onmiddellijk dat $\dim \mathbf{V}E = 6$ voor elk isotroop element E van $\mathbf{L}$.

2.5 Nog meer E_6-modules

Behalve $\mathbf{V}$, $\mathbf{V}^*$ en $\mathbf{L}$ bestuderen we ook nog enkele andere modules voor de Lie-algebra $\mathbf{L}$, namelijk $\mathbf{V} \wedge \mathbf{V}$ (en de duale $\mathbf{V}^* \wedge \mathbf{V}^*$), $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ (die in zekere zin zelf-duaal blijkt te zijn) en $\mathrm{Hom}(\mathbf{V}, \mathbf{V})$. Bij elk van deze modules horen weer enkele nieuwe bewerkingen en kan ook het concept 'isotroop' worden ingevoerd.

Zo zijn isotrope elementen van $\mathbf{V} \wedge \mathbf{V}$ van de vorm $e \wedge f$ met $e, f \in \mathbf{V}$ isotroop en $e \times f = 0$, of in andere woorden, met $Ke + Kf$ een isotrope deelruimte van $\mathbf{V}$. (De bewerking '$\wedge$' is het welbekende antisymmetrische 'wedge'-product.) De niet-triviale isotrope elementen van $\mathbf{V} \wedge \mathbf{V}$ vormen opnieuw één enkele baan voor $\widehat{\mathsf{E}}_6(K)$. Op analoge wijze zijn de isotrope elementen van $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ van de vorm $d \wedge e \wedge f$, waarbij $Kd + Ke + Kf$ een isotrope deelruimte is van $\mathbf{V}$. Ook de niet-triviale elementen van de vorm $\eta \otimes e \in \mathrm{Hom}(\mathbf{V}, \mathbf{V})$ vormen een baan van $\widehat{\mathsf{E}}_6(K)$, met $e \in \mathbf{V}$, $\eta \in \mathbf{V}^*$ isotrope elementen die voldoen aan $e\eta = 0$ en $e * \eta = 0$. (De bewerking '$\otimes$' is het welbekende 'uitproduct'.)

Deze isotrope elementen kunnen ook op andere manieren worden bekomen: zo kan je bovenstaande $\eta \otimes e$ ook steeds schrijven als EF met E, F isotrope elementen van $\mathbf{L}$ waarvoor $EF = FE$. Ook voor isotrope elementen van $\mathbf{V} \wedge \mathbf{V}$ en $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$ bestaan er alternatieve schrijfwijzen die gebruik maken van de nieuwe operatoren die hier werden ingevoerd.

2.6 De meetkunde $\mathcal{E}$

Het einddoel van Hoofdstuk 2 is de introductie van de meetkunde $\mathcal{E}$ en aan te tonen dat deze meetkunde inderdaad van het type E_6 is (Proposities 2.56–2.57). Deze meetkunde bezit de volgende 6 verschillende *types*:

1. De *punten* van $\mathcal{E}$ zijn de isotrope 1-ruimten van $\mathbf{V}$, t.t.z., de verzamelingen Ke waarbij e een isotroop element is van $\mathbf{V} - \{0\}$.

2. De *rechten* van $\mathcal{E}$ zijn de isotrope 1-ruimten van $\mathbf{V} \wedge \mathbf{V}$, t.t.z., de verzamelingen $K(e \wedge f)$ waarbij e, f een isotrope 2-ruimte van $\mathbf{V}$ voortbrengen.

3. De *vlakken* van $\mathcal{E}$ zijn de isotrope 1-ruimten van $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$, t.t.z., de verzamelingen $K(d \wedge e \wedge f)$ waarbij d, e, f een isotrope 3-ruimte van $\mathbf{V}$ voortbrengen.

4. De *duale rechten* van $\mathcal{E}$ zijn de isotrope 1-ruimten van $\mathbf{V}^* \wedge \mathbf{V}^*$, t.t.z., de verzamelingen $K(\eta \wedge \varphi)$ waarbij η, φ een isotrope 2-ruimte van $\mathbf{V}^*$ voortbrengen.

5. De *duale punten* van $\mathcal{E}$ zijn de isotrope 1-ruimten van $\mathbf{V}^*$, t.t.z., de verzamelingen $K\eta$ waarbij η een isotroop element is van $\mathbf{V}^* - \{0\}$.

0. De *simplexen* van $\mathcal{E}$ zijn de isotrope 1-ruimten van $\mathbf{L}$, t.t.z., de verzamelingen KE waarbij E een isotroop element is van $\mathbf{L} - \{0\}$.

Incidentie tussen types kan uitgedrukt worden met behulp van de verschillende algebraïsche bewerkingen, zoals aangegeven in de volgende tabel:

	$e \in \mathbf{V}$	$L \in \mathbf{V} \wedge \mathbf{V}$	$P \in \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$	$\Lambda \in \mathbf{V}^* \wedge \mathbf{V}^*$	$\eta \in \mathbf{V}^*$
$E \in \mathbf{L}$	$e \in \mathbf{V}E$ $E \times e = 0$	$L\mathbf{V}^* \leq \mathbf{V}E$ $E \times L\mathbf{V}^* = 0$	$P\mathbf{V}^*\mathbf{V}^* \leq \mathbf{V}E$ $E \times P\mathbf{V}^*\mathbf{V}^* = 0$	$\mathbf{V}\Lambda \leq E\mathbf{V}^*$ $\mathbf{V}\Lambda \times E = 0$	$\eta \in E\mathbf{V}^*$ $\eta \times E = 0$
e		$e \in L\mathbf{V}^*$	$e \in P\mathbf{V}^*\mathbf{V}^*$	$\mathbf{V}\Lambda \leq e \times \mathbf{V}$ $e * \mathbf{V}\Lambda = 0$	$e \in \eta \times \mathbf{V}^*$ $\eta \in e \times \mathbf{V}$ $e * \eta = 0$
L			$L\mathbf{V}^* \leq P\mathbf{V}^*$	$L\mathbf{V}^* * \mathbf{V}\Lambda = 0$	$L\mathbf{V}^* \leq \eta \times \mathbf{V}^*$ $L\mathbf{V}^* * \eta = 0$
P				$P\mathbf{V}^*\mathbf{V}^* * \mathbf{V}\Lambda = 0$	$P\mathbf{V}^*\mathbf{V}^* \leq \eta \times \mathbf{V}^*$ $P\mathbf{V}^*\mathbf{V}^* * \eta = 0$
Λ					$\eta \in \mathbf{V}\Lambda$

Alle elementen e, L, P, Λ, η en E in deze tabel zijn isotroop. Incidentie is symmetrisch en anti-reflexief. Wanneer een cel meerdere relaties bevat dan zijn ze equivalent.

3 De Lie-algebra van type F_4 en verwante structuren

3.1 Een wortelsysteem van type F_4

Kies een vaste rechte L_∞ in de veralgemeende vierhoek $\mathcal{Q}$ en noem ze *de rechte op oneindig*. Schrijf $\mathcal{P}^* \stackrel{\text{def}}{=} \mathcal{P} - L_\infty$. Met behulp van deze rechte construeren we een lineaire transformatie $\bar{\cdot}$ op de volgende manier: elk punt van z van L_∞ wordt afgebeeld op $-z$ en elk punt p van $\mathcal{P}^*$ wordt afgebeeld op $\overline{p} = p + p_\infty$ waarbij p_∞ het unieke punt is van L_∞ dat adjacent is met p (de punten $\{p, p_\infty, -\overline{p}\}$ vormen de unieke rechte door p die L_∞ snijdt). De afbeelding $p \mapsto -\overline{p}$ is een automorfisme van $\mathcal{Q}$ die Σ invariant laat, punten afbeeldt op punten en wortels op wortels.

De verzameling $\Phi_F \stackrel{\text{def}}{=} \{\frac{1}{2}(r + \overline{r}) \mid r \in \Phi\}$ vormt een wortelsysteem van type F_4 (Propositie 3.6) in een 4-dimensionale reële deelruimte $\mathbf{P}_F$ van $\mathbf{P}$. Φ_F kan men ook beschouwen als een projectie van Φ op deze ruimte, door de afbeelding $x \mapsto (x + \overline{x})/2$.

Het wortelsysteem Φ_F bevat 24 zogenaamd *lange* wortels (de elementen van $\Phi_F \cap \Phi$) en 24 *korte* wortels (die niet tot Φ behoren). De verzamelingen van lange en korte wortels noteren we respectievelijk als Φ_L en Φ_S. Met een geschikte basis voor $\mathbf{P}_F$ krijgen de lange wortels coördinaten van de vorm $(\pm 1, \pm 1, 0, 0)$ (of permutaties hiervan) en corresponderen de korte wortels met viertallen van de vorm $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ of $(\pm 1, 0, 0, 0)$ (of permutaties hiervan). Elke $s \in \Phi_S$ is een projectie $s = (p + \overline{p})/2$ van een uniek punt $p \in \mathcal{P}^*$. Daarenboven kunnen we s ook schrijven als projectie $s = (r + \overline{r})/2$ van een paar $\{r, \overline{r}\} \subseteq \Phi - \Phi_L$. Er is precies één dergelijk paar voor elke korte wortel.

Omdat niet alle wortels van Φ_F even lang zijn, gebruiken we in deze context bij voorkeur het product $\langle r, s \rangle \stackrel{\text{def}}{=} 2r \cdot s / r \cdot r$ in plaats van het gewone inproduct $r \cdot s$. Toegepast op twee wortels van Φ_F heeft dit product steeds een gehele waarde uit het gebied $-2, \ldots, +2$.

De Weyl-groep $W(\mathsf{F}_4)$ van Φ_F bezit opnieuw enkele belangrijke transitiviteitseigenschappen: hij is transitief op de korte wortels, op de lange wortels, op paren (r,s) van korte wortels met een gegeven waarde van $\langle r,s\rangle$, op paren van lange wortels (r,s) met een gegeven $\langle r,s\rangle$, en analoog voor gemengde paren. De groep kan ook op een zodanige manier gedefinieerd worden dat hij inwerkt op $\mathbf{P}$ (in plaats van op het kleinere $\mathbf{P}_F$) en dan vormen ook $\mathcal{P}^*$ en L_∞ banen van $W(\mathsf{F}_4)$ (Propositie 3.8).

3.2 De Lie-algebra van type F_4 over K

Kies een vast element $\infty \in \mathbf{V}$ met de eigenschap $D(\infty) = 1$ en definieer de lineaire operator $\bar{\cdot}$ als volgt:

$$\begin{aligned} \overline{a} &\stackrel{\text{def}}{=} (a\infty^\#)\infty^\# - \infty \times a, && \text{voor } a \in \mathbf{V}, \\ \overline{\alpha} &\stackrel{\text{def}}{=} (\infty\alpha)\infty - \infty^\# \times \alpha, && \text{voor } \alpha \in \mathbf{V}^*. \end{aligned} \tag{3.18}$$

Deze lineaire transformatie kan op een eenvoudige manier worden uitgebreid naar andere $\mathbf{L}$-modules en gedraagt zich als een soort *polariteit* voor de algebraische bewerkingen op deze modules. Voor $a,b \in \mathbf{V}$, $\alpha \in \mathbf{V}^*$ en $A,B \in \mathbf{L}$ gelden bijvoorbeeld $\overline{\overline{a}} = a$, $\overline{\overline{\alpha}} = \alpha$, $\overline{a \times b} = \overline{a} \times \overline{b}$, $a\overline{A}\alpha = \alpha A\overline{a}$, $\overline{a * \alpha} = \overline{\alpha} * \overline{a}$, $\overline{A} \cdot \overline{B} = A \cdot B$, $\overline{a \wedge b} = \overline{b} \wedge \overline{a}$, … Isotrope elementen worden steeds afgebeeld op isotrope elementen.

We definiëren nu de Lie-algebra $\mathbf{J}$ als de deelalgebra van $\mathbf{L}$ waarvan de elementen A voldoen aan één van de volgende (equivalente) eigenschappen:

$$\infty A = 0, A\overline{\infty} = 0 \text{ of } aA\overline{a} = 0 \text{ voor elke } a \in \mathbf{V}.$$

Elk element van $\mathbf{J}$ voldoet dan automatisch aan de eigenschap $A + \overline{A} = 0$.

Propositie 3.19 toont aan dat $\mathbf{J}$ de gekende Lie-algebra is van type F_4 over K en dat we een Chevalley-basis voor deze algebra kunnen construeren aan de hand van elementen E_r, H_r met $r \in \Phi_L$ samen met nieuwe basiselementen van de vorm

$$E_s \stackrel{\text{def}}{=} E_u + E_{\overline{u}} = E_u - \overline{E_u}, \qquad H_s \stackrel{\text{def}}{=} H_u + H_{\overline{u}} = H_u - \overline{H_u}, \tag{3.38, 3.41}$$

met $s \in \Phi_S$ en $u \in \Phi$ zodanig dat $s = (u + \overline{u})/2$.

De deelruimte **W** van **V** waarvan de elementen a voldoen aan $a\overline{\infty} = 0$ is een 26-dimensionale **J**-module. De elementen e_p behoren tot **W** wanneer $p \in \mathcal{P}^*$ maar dit is niet het geval voor $p \in L_\infty$.

3.3 Automorfismen van W

Bij de studie van de Lie-algebra van type F_4 blijkt de eerdere definitie van automorfisme niet langer sterk genoeg. In Hoofdstuk 3 beperken we ons daarom tot zogenaamde *automorfismen van* **W**. Dit zijn automorfismen die voldoen aan de bijkomende eigenschap $\infty^g = \infty$ en daardoor meteen ook aan

$$\overline{a}^g = \overline{a^g}, \quad \overline{A}^g = \overline{A^g}, \quad \alpha^g = \overline{\alpha^g}. \tag{3.48}$$

Voor een automorfisme g van **W** geldt ook dat $\mathbf{W}^g = \mathbf{W}$ en $\mathbf{J}^g = \mathbf{J}$.

Is E een isotroop element van **L** dat ook tot **J** behoort, dan is $x(E)$ een automorfisme van **W**. Behoort E niet tot **J** dan is

$$x'(E) \stackrel{\text{def}}{=} x(E)x(-\overline{E}) = x(-\overline{E})x(E) \tag{3.49}$$

een automorfisme van **W**. De groep voortgebracht door deze beide types van automorfisme van **W** noteren we als $\widehat{\mathsf{F}}_4(K)$ en is een Chevalley-groep van type F_4.

Stellingen 3.25–3.28 beschrijven de banen van $\widehat{\mathsf{F}}_4(K)$ op de isotrope elementen van **W** en **J** (er is telkens één niet-triviale baan) en op paren van dergelijke isotrope elementen. In twee gevallen zijn de banen van $\widehat{\mathsf{F}}_4(K)$ eenvoudigweg de doorsneden van banen van $\widehat{\mathsf{E}}_6(K)$ met $\mathbf{W} \times \mathbf{J}$ en $\mathbf{J} \times \mathbf{J}$.

Bij de banen op isotrope paren $(e, f) \in \mathbf{W} \times \mathbf{W}$ daarentegen zijn er 5 soorten banen, wat 2 meer is dan bij de gelijkaardige situatie voor de algebra van type E_6. (Dit houdt verband met het feit dat de Weyl-groep $W(\mathsf{F}_4)$ vijf banen heeft op $\mathcal{P}^* \times \mathcal{P}^*$.)

De corresponderende paren (e, f) voldoen nu aan

1. $e\overline{f} = \ell \in K - \{0\}$,
2. $e\overline{f} = 0$ maar $e \times f \neq 0$,
3. $e\overline{f} = 0$, $e \times f = 0$ maar $e * \overline{f} \neq 0$,
4. $e\overline{f} = 0$, $e \times f = 0$, $e * \overline{f} = 0$ maar $Ke \neq Kf$,
5. $e = \ell f$ met $\ell \in K - \{0\}$.

Opnieuw zijn deze stellingen heel belangrijk omdat ze toelaten algemene eigenschappen van isotrope elementen af te leiden door deze enkel te bewijzen voor eenvoudige voorbeelden zoals e_p en E_r met $p \in \mathcal{P}^*$, $r \in \Phi_L$.

3.4 F-isotrope deelruimten

Ook het begrip 'isotrope deelruimte' dient te worden ingeperkt wanneer we werken met F_4-modules. We noemen een deelruimte van $\mathbf{W}$ *F-isotroop* wanneer ze isotroop is en bovendien geldt dat $e * \overline{f} = 0$ voor elk paar elementen e, f uit die deelruimte. Een element $e \wedge f$ (resp. $d \wedge e \wedge f$) uit $\mathbf{W} \wedge \mathbf{W}$ (resp. $\mathbf{W} \wedge \mathbf{W} \wedge \mathbf{W}$) noemen we F-isotroop als de corresponderende deelruimte F-isotroop is. Niet alle isotrope elementen uit $\mathbf{W} \wedge \mathbf{W}$ en $\mathbf{W} \wedge \mathbf{W} \wedge \mathbf{W}$ zijn ook F-isotroop.

3.5 De meetkunde $\mathcal{F}$

Het einddoel van Hoofdstuk 3 is de introductie van de meetkunde $\mathcal{F}$ en aan te tonen dat deze meetkunde inderdaad van het type F_4 is (Proposities 3.37–3.39). Deze meetkunde bezit de volgende 4 verschillende *types*:

1. De *hyperrechten* van $\mathcal{F}$ zijn de isotrope 1-ruimten van $\mathbf{J}$, t.t.z., de verzamelingen KE waarbij E een isotroop element is van $\mathbf{J} - \{0\}$.

2. De *vlakken* van $\mathcal{F}$ zijn de 1-ruimten KP van $\mathbf{W} \wedge \mathbf{W} \wedge \mathbf{W}$ waarbij P F-isotroop is.

3. De *rechten* van $\mathcal{F}$ zijn de 1-ruimten KL van $\mathbf{W} \wedge \mathbf{W}$ waarbij L F-isotroop is.

4. De *punten* van $\mathcal{F}$ zijn de isotrope 1-ruimten van $\mathbf{V}$, t.t.z., de verzamelingen Ke waarbij e een isotroop element is van $\mathbf{W} - \{0\}$.

Punten, rechten, vlakken en hyperrechten van $\mathcal{F}$ kunnen ook worden geïnterpreteerd als punten, rechten, vlakken en simplexen van $\mathcal{E}$, met dezelfde definitie van incidentie. Niet alle rechten of vlakken van $\mathcal{E}$ zijn echter rechten of vlakken van $\mathcal{F}$, zelfs wanneer al hun punten ook punten zijn van $\mathcal{F}$.

Tot slot van Hoofdstuk 3 bewijzen we nog (Stelling 3.41) dat er vijf relaties mogelijk zijn tussen punten Ke en Kf van $\mathcal{F}$, en dat deze relaties kunnen worden uitgedrukt met behulp van de gekende algebraische bewerkingen. Twee punten zijn ofwel *overstaand*, *schieroverstaand*, *cohyperlineair*, *collineair* of *gelijk* aan elkaar. Er bestaat een 1–1-correspondentie tussen deze begrippen en de vijf banen van $\widehat{\mathsf{F}}_4(K)$ die we hierboven hebben opgesomd in paragraaf 3.3.

4 De Lie-algebra van type F_4 wanneer $\operatorname{char} K = 2$

In Hoofdstukken 4 en 5 beperken we ons tot een veld K met karakteristiek 2.

4.1 Twee Lie-algebras isomorf met W

Zij $a, b \in \mathbf{V}$. Definieer

$$\begin{aligned}
S(a) &\stackrel{\text{def}}{=} a * \overline{\infty} + (a\overline{\infty})\mathbf{1}, \\
[a,b] = [b,a] &\stackrel{\text{def}}{=} aS(b) = bS(a) = (a\overline{\infty})b + (b\overline{\infty})a + (a \times b) \times \overline{\infty}, \\
a^2 &\stackrel{\text{def}}{=} (a\overline{\infty})a + a^{\#} \times \overline{\infty} = (a \times \infty)^{\#} + (\infty a^{\#})\infty.
\end{aligned} \tag{4.1}$$

De bewerking $\cdot^2$ voldoet aan $(a+b)^2 = a^2 + [a,b] + b^2$ voor alle $a, b \in \mathbf{V}$. M.a.w., dit is de kwadratische operator die op een natuurlijke manier is verbonden met de bilineaire Lie-haak (die nu symmetrisch is omdat char $K = 2$). Een element e van $\mathbf{W}$ is isotroop als en slechts als $e^2 = 0$.

De Lie-haak maakt van $\mathbf{W}$ een Lie-algebra die isomorf is met de Lie-algebra $S(\mathbf{W})$, een ideaal van $\mathbf{J}$ (Proposities 4.2–4.3). De algebra $\mathbf{J}$ is dus niet langer irreduciebel. Zij nu $s \in \Phi_S$, $p \in \mathcal{P}^*$, zodat $s = (p + \overline{p})/2$, dan schrijven we $e_s \stackrel{\text{def}}{=} e_p$ en $h_s \stackrel{\text{def}}{=} [e_s, e_{-s}] = e_{p_\infty} + \infty$. Met deze notatie geldt $S(e_s) = E_s$ en $S(h_s) = H_s$. De elementen van de vorm E_s en H_s kunnen dienen als basiselementen van $S(\mathbf{W})$.

Het ligt voor de hand om ook de quotient-algebra $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{J}/S(\mathbf{W})$ van dichterbij te bekijken. Propositie 4.6 toont aan dat veel van de basisbewerkingen (zoals Lie-haak, kwadratische operator, inproduct) ook in $\mathbf{Q}$ een zinnige definitie kunnen krijgen. Het blijkt zelfs dat $\mathbf{Q}$ als Lie-algebra isomorf is met $S(\mathbf{W})$, maar vooraleer we dit kunnen bewijzen dienen we eerst nog enkele nieuwe begrippen te introduceren.

Beschouw de lineaire transformatie † van $\mathbf{P}_4$ die inwerkt op de basiselementen op de volgende manier

$$\mathtt{1000}^\dagger \stackrel{\text{def}}{=} \mathtt{\bar{1}001}, \quad \mathtt{0100}^\dagger \stackrel{\text{def}}{=} \mathtt{0110}, \quad \mathtt{0010}^\dagger \stackrel{\text{def}}{=} \mathtt{01\bar{1}0}, \quad \mathtt{0001}^\dagger \stackrel{\text{def}}{=} \mathtt{1001}.$$

(We gebruiken een verkorte notatie voor coördinaten van wortels: $\mathtt{\bar{1}001}$ staat bijvoorbeeld voor $(-1, 0, 0, 1)$.)

Er geldt $r^\dagger \cdot s^\dagger = 2r \cdot s$, $r^{\dagger\dagger} = 2r$ en $\langle r, s \rangle = \langle r^\dagger, s^\dagger \rangle$. Elke korte wortel wordt door † afgebeeld op een lange wortel, en elke lange wortel op het dubbele van een korte wortel.

Definieer nu de $\mu : \mathbf{V} \to \mathbf{Q}$ als volgt:

$$\begin{aligned} \mu(e_s) &\stackrel{\text{def}}{=} E_{s^\dagger} + S(\mathbf{W}), \\ \mu(h_s) &\stackrel{\text{def}}{=} H_{s^\dagger} + S(\mathbf{W}). \end{aligned} \tag{4.12}$$

Stelling 4.10 bewijst dat μ voldoet aan

$$\begin{array}{llllll} \mu(\infty) & = & 0, & \mu(a^2) & = & \mu(a)^2, \\ \mu([a,b]) & = & [\mu(a),\mu(b)], & \mu(a)\cdot\mu(b) & = & a\overline{b}, \end{array} \tag{4.13}$$

voor elke $a, b \in \mathbf{V}$. Er volgt ook dat μ een isomorfisme is tussen $\mathbf{W}$ en $\mathbf{Q}$.

4.2 Dualiteit

Zij e een isotroop element van $\mathbf{W}$. Zij $Q(e)$ het unieke element van $\mathrm{Hom}(\mathbf{V},\mathbf{V})$ dat voldoet aan

$$aQ(e)\overline{b} \stackrel{\text{def}}{=} eAB\overline{e} \quad \text{voor } a,b \in \mathbf{V},\, A \in \mu(a)+S(\mathbf{W}),\, B \in \mu(b)+S(\mathbf{W}). \tag{4.14}$$

(De waarde van de uitdrukking $sAB\overline{e}$ is onafhankelijk van de specifieke keuze voor A en B wanneer e isotroop is.)

De kwadratische operator $Q(\cdot)$ beeldt isotrope elementen e van $\mathbf{W}$ af op isotrope elementen $Q(e)$ van $\mathbf{J}$ en behoudt in zekere zin ook vele van de eigenschappen van paren (e, f) van isotrope elementen van $\mathbf{W}$ (Stelling 4.12). We hebben bijvoorbeeld:

- $Q(e)\cdot Q(f) = (e\overline{f})^2$,
- als $e\overline{f} = 0$ dan $Q([e,f]) = [Q(e), Q(f)]$,
- als $[e,f] = 0$ dan geldt $Q(e)Q(f) = 0$ als en slechts als $e * \overline{f} = 0$,
- als $e * \overline{f} = 0$ dan $Q(e+f) = Q(e) + Q(f)$.

Noteren we e^{frob} voor het element dat je bekomt door het Frobenius-morfisme $k \mapsto k^2$ toe te passen op de coördinaten van e, dan bewijzen we bovendien dat

$$Q(e) = \mu(e^{\text{frob}}) \bmod S(\mathbf{W}), \tag{4.17}$$

en

$$Q(eQ(f)) = f^{\text{frob}}Q(e) * \overline{f}^{\text{frob}}, \tag{4.18}$$

voor alle isotrope elementen e en f uit $\mathbf{W}$.

4.3 Het graafendomorfisme van $\widehat{\mathsf{F}}_4(K)$

Zij e een isotroop element van $\mathbf{W}$. Definieer

$$x(e) : \mathbf{V} \to \mathbf{V} : a \mapsto a^{x(e)} \stackrel{\text{def}}{=} a + [e, a] + (a\bar{e})e. \tag{4.20}$$

Het element $x(e)$ behoort tot $\widehat{\mathsf{F}}_4(K)$ (steeds onder de voorwaarde char $K = 2$).

Propositie 4.17 legt een verband tussen de elementen $x(e)$ van deze vorm, en de elementen $x(E)$ met E een isotroop element van $\mathbf{J}$: zij $g \in \widehat{\mathsf{F}}_4(K)$ en definieer

$$g^\dagger \stackrel{\text{def}}{=} \mu g \mu^{-1}, \tag{4.28}$$

dan geldt $g^\dagger \in \widehat{\mathsf{F}}_4(K)$, en

$$x(e)^\dagger = x(Q(e)), \qquad x(E)^\dagger = x(e'), \tag{4.29}$$

voor elk isotroop element $e \in \mathbf{W}$ en $E \in \mathbf{J}$, waarbij e' het unieke element voorstelt van $\mathbf{W}$ waarvoor $E = \mu(e')$.

De afbeelding $\dagger$ is het zogenaamde *graafendomorfisme* van $\widehat{\mathsf{F}}_4(K)$. Propositie 4.18 toont het verband aan tussen dit endomorfisme en de operator $Q(\cdot)$:

$$Q(e^g) = Q(e)^{g^\dagger}, \quad \text{voor elke isotrope } e \in \mathbf{W} \text{ en elke } g \in \widehat{\mathsf{F}}_4(K). \tag{4.34}$$

5 De perfecte veralgemeende achthoeken van Ree-Tits

In Hoofdstuk 5 veronderstellen we dat het basisveld K karakteristiek 2 heeft, *perfect* is (m.a.w., dat $K^2 = K$) en een zogenaamd *Tits-automorfisme* σ bezit, m.a.w., een automorfisme met de eigenschap $(k^\sigma)^\sigma = k^2$, voor alle $k \in K$. Is K een eindig veld, dan bestaat er een Tits-automorfisme als en slechts als de orde van K een oneven macht is van 2.

We schrijven e^σ voor het element dat je bekomt wanneer je σ toepast op alle coördinaten van $e \in \mathbf{W}$. Is e isotroop dan is ook e^σ isotroop en geldt $Q(e^\sigma) = Q(e)^\sigma$.

5.1 Een wortelsysteem van type $^2\mathsf{F}_4$

Net zoals we in Hoofdstuk 3 het wortelsysteem Φ_F hebben geconstrueerd als een projectie van Φ, definiëren we nu het wortelsysteem Φ_O (van type $^2\mathsf{F}_4$) als projectie van Φ_F:

$$\Phi_O \stackrel{\text{def}}{=} \{\sqrt{2}r + r^\dagger | r \in \Phi_S\}.$$

(Dit wortelsysteem wordt afgebeeld in Figuur 5.1 op bladzijde 194.)

Afhankelijk van de waarde van $r \cdot r^\dagger$ (1, 0 of -1) hebben de wortels $\sqrt{2}r + r^\dagger$ drie mogelijke lengtes (lang, intermediair of kort). De hoeken tussen twee verschillende wortels zijn steeds veelvouden van $\pi/8$.

Zij $s = \sqrt{2}r + r^\dagger$ een wortel van Φ_O dan definiëren we $w''_s \stackrel{\text{def}}{=} w'_r w_{r^\dagger}$ wanneer $r \cdot r^\dagger = 0$ en $w''_s \stackrel{\text{def}}{=} w'_r w_{r^\dagger} w'_r w_{r^\dagger}$ in andere gevallen. De groep voortgebracht door alle elementen w''_s van deze vorm noteren we als $W(^2\mathsf{F}_4)$ en fungeert als een soort Weyl-groep van Φ_O, isomorf met de dihedrale groep van orde 16.

$W(^2\mathsf{F}_4)$ verdeelt de wortels van Φ_O in drie banen (afhankelijk van hun lengte). Twee wortelparen zijn equivalent onder $W(^2\mathsf{F}_4)$ als en slechts als ze corresponderende lengtes hebben en hetzelfde inproduct.

5.2 Octagonaliteit en de Ree-groep $^2\widehat{\mathsf{F}}_4(K)$

Beschouw $a, e \in \mathbf{W}$ met e isotroop. Definieer de operator $q(\cdot,\cdot)$ als volgt:

$$q(a,e) \stackrel{\text{def}}{=} aQ(e)^{\sigma/2} = aQ(e^{\sigma/2}) \tag{5.6}$$

en schrijf $q(\mathbf{V}, e)$ voor de verzameling van elementen $q(a,e)$ met $a \in \mathbf{V}$.

We noemen $e \in \mathbf{W}$ *semi-octagonaal* wanneer e isotroop is en $q(e,e) = 0$. We noemen e *octagonaal* wanneer e isotroop is en $e \in q(\mathbf{V}, e)$ (en dan is e meteen ook semi-octagonaal). We noemen e *para-octagonaal* wanneer e isotroop is maar niet semi-octagonaal. Zij $r \in \Phi_S$, dan is e_r para-octagonaal wanneer $r \cdot r^\dagger = -1$, semi-octagonaal wanneer $r \cdot r^\dagger >= 0$ en octagonaal wanneer $r \cdot r^\dagger = +1$.

Vooraleer we verdergaan dienen we nog één nieuwe bewerking in te voeren: zijn $e, f \in \mathbf{W}$ isotrope elementen waarvoor $[e, f] = 0$, dan definiëren we $c(e, f)$ als het unieke element uit $\mathbf{W}$ waarvoor $\mu(c(e, f)) = (e * \overline{f})^{\sigma/2}$.

In dit hoofdstuk zijn we voornamelijk geïnteresseerd in automorfismen g van $\mathbf{W}$ die ook nog aan de volgende eigenschap voldoen:

$$(a^g)^\sigma = (a^\sigma)^{g^\dagger}, \qquad \text{voor alle } a \in \mathbf{W}. \tag{5.8}$$

Propositie 5.4 stelt dat een dergelijk automorfisme voldoet aan $q(a^g, e^g) = q(a,e)^g$, $q(\mathbf{V}, e^g) = q(\mathbf{V}, e)^g$ en $c(e^g, f^g) = c(e, f)^g$, voor alle $a, e, f \in \mathbf{W}$ waarvoor deze uitdrukkingen gedefinieerd zijn. Een voorbeeld wordt gegeven door het automorfisme

$$y(e) \stackrel{\text{def}}{=} x(Q(e)^{\sigma/2})x(e)x(q(e,e)), \tag{5.9}$$

met $e \in \mathbf{W}$ isotroop (Propositie 5.5). De deelgroep van $\widehat{\mathsf{F}}_4(K)$ voortgebracht door alle groepselementen van deze vorm noteren we als ${}^2\widehat{\mathsf{F}}_4(K)$ en is een zogenaamde *Ree-Tits-groep* of *vervlochten* (Engels: twisted) *Chevalley-groep* van type ${}^2\mathsf{F}_4$. Deze groep bewaart de begrippen octagonaliteit, semi- en para-octagonaliteit.

Vooraleer we een studie maken van de Ree-Tits-groep zelf, bekijken we eerst enkele van haar kleinere deelgroepen. Hiertoe is het handig om paren (k_0, k_1), (ℓ_0, ℓ_1) van elementen van K te behandelen als elementen $\boldsymbol{k}$, $\boldsymbol{\ell}$ van een algebraïsche structuur $K_\sigma^{(2)}$ waarop we een optelling, een spoor en een norm definiëren op de volgende manier:

$$\begin{aligned} \boldsymbol{k} \oplus \boldsymbol{\ell} &\stackrel{\text{def}}{=} (k_0 + \ell_0, k_1 + \ell_1 + k_0^\sigma \ell_0) \\ T(\boldsymbol{k}) &\stackrel{\text{def}}{=} k_0^{1+\sigma} + k_1, \\ N(\boldsymbol{k}) &\stackrel{\text{def}}{=} k_0^{2+\sigma} + k_0 k_1 + k_1^\sigma = k_0 T(k) + k_1^\sigma = k_0 k_1 + T(k)^\sigma. \end{aligned} \tag{5.18}$$

Is $e \in \mathbf{W}$ isotroop, dan schrijven we

$$y(\boldsymbol{k}e) \stackrel{\text{def}}{=} y(k_0e)y(k_1q(e,e)), \tag{5.20}$$

en dan bewijst Propositie 5.6 dat

$$y(\boldsymbol{k}e)y(\boldsymbol{\ell}e) = y((\boldsymbol{k} \oplus \boldsymbol{\ell})e), \qquad \text{voor alle } \boldsymbol{k}, \boldsymbol{\ell} \in K_\sigma^{(2)}, \tag{5.21}$$

en bijgevolg dat de verzameling $y(K_\sigma^{(2)}e)$ een deelgroep vormt van ${}^2\widehat{\mathsf{F}}_4(K)$.

5.3 De Suzuki-Tits-ovoïde en de Suzuki-groep

Beschouw twee octagonale elementen $d, f \in \mathbf{W}$ met $d\overline{f} = 1$. Schrijf $e \stackrel{\text{def}}{=} q(f,d)$, $g \stackrel{\text{def}}{=} q(d,f)$ en $h \stackrel{\text{def}}{=} [d,f]$. Zij $\mathbf{B}(d,f)$ de deelruimte van $\mathbf{W}$ voortgebracht door d, e, f, g, h. Dan is $\mathbf{B}(d,f)$ een *deelalgebra* van $\mathbf{W}$.

De deelgroep van $\widehat{\mathsf{F}}_4(K)$ die wordt voortgebracht door $y(K_\sigma^{(2)}e)$ en $y(K_\sigma^{(2)}g)$ noteren we als $\mathrm{Suz}(d,f)$ en is een zogenaamde *Suzuki-groep* of *vervlochten Chevalley-groep* van type ${}^2\mathsf{B}_2$. Deze groep werkt transitief op octagonale elementen van $\mathbf{B}(d,f) - \{0\}$ en op de paren (b_1, b_2) van dergelijke elementen waarvoor $b_1\overline{b}_2 = 1$ (Stelling 5.7). De stabilisator van een octagonaal element is een deelgroep van de vorm $y(K_\sigma^{(2)}b)$.

De verzameling 1-ruimten Kb van $\mathbf{B}(d,f)$ waarbij b octagonaal is, noemen we een *Suzuki-Tits-ovoïde* van de 4-dimensionale projectieve ruimte geassocieerd met $\mathbf{B}(d,f)$. De term 'ovoïde' duidt aan dat drie dergelijke 1-ruimten nooit tot een gemeenschappelijke 2-ruimte zullen behoren.

5.4 Transitiviteitseigenschappen van ${}^2\widehat{\mathsf{F}}_4(K)$

Beschouw $\mathbf{B}_1 = \mathbf{B}(e_{\mathtt{0100}}, e_{\mathtt{0\bar{1}00}})$ and $\mathbf{B}_2 = \mathbf{B}(e_{\mathtt{0001}}, e_{\mathtt{000\bar{1}}})$ met corresponderende Suzuki-groepen $\mathrm{Suz}_1 = \mathrm{Suz}(\mathbf{B}_1)$ en $\mathrm{Suz}_2 = \mathrm{Suz}(\mathbf{B}_2)$. $\mathbf{B}_1$ en $\mathbf{B}_2$ snijden in de 1-ruimte $Kh_{\mathtt{1000}}$ en genereren dus samen een 9-dimensionale deelruimte $\mathbf{B}_1 + \mathbf{B}_2$. (Deze ruimte is een Lie-algebra van type B_4). De deelgroep Suz_1 laat elk element van $\mathbf{B}_2$ invariant, en de deelgroep Suz_2 laat elk element van $\mathbf{B}_1$ invariant. We combineren deze beide groepen samen met een element $N \in \widehat{\mathsf{F}}_4(K)$ dat $\mathbf{B}_1$ en $\mathbf{B}_2$ omwisselt, tot een groep $2\mathrm{Suz}^2(\mathbf{B}_1, \mathbf{B}_2)$.

Stelling 5.10 vertelt ons dat de banen van deze groep op de semi-octagonale elementen van $\mathbf{B}_1 + \mathbf{B}_2$ de volgende zijn:

1. De triviale baan $\{0\}$,

2. Een baan van niet-triviale octagonale elementen. Deze elementen behoren ofwel tot $\mathbf{B}_1$, ofwel tot $\mathbf{B}_2$.

3. Een baan van semi-octagonale elementen die niet octagonaal zijn. Dat zijn precies die elementen die kunnen geschreven worden als de som van een niet-triviaal octagonaal element uit $\mathbf{B}_1$ en een niet-triviaal octagonaal element uit $\mathbf{B}_2$.

Deze eigenschappen vormen een eerste stap in het bewijs van Stelling 5.11 die de banen bepaalt van de grotere groep ${}^2\widehat{\mathsf{F}}_4(K)$ op alle semi-octagonale elementen van $\mathbf{W}$. We vinden opnieuw drie banen:

1. De triviale baan $\{0\}$,

2. Een baan van niet-triviale octagonale elementen.

3. Een baan van semi-octagonale elementen die niet octagonaal zijn.

5.5 Paren octagonale elementen

En zo komen we stilaan bij de voornaamste resultaten uit Hoofdstuk 5.

Stelling 5.16 beschrijft de banen van ${}^2\widehat{\mathsf{F}}_4(K)$ op paren niet-triviale octagonale elementen uit $\mathbf{W}$. Net zoals in Hoofdstuk 3 zijn er 5 soorten banen: elke baan van ${}^2\widehat{\mathsf{F}}_4(K)$ kan je bekomen door uit één van de banen van $\widehat{\mathsf{F}}_4(K)$ op paren van isotrope elementen enkel de paren van octagonale elementen te selecteren.

Dit betekent dat je dezelfde algebraische bewerkingen kan gebruiken als voorheen om te bepalen in welke baan een specifiek paar octagonale elementen (e, f) zich bevindt. Je kan echter ook gebruik maken van de nieuwe operatoren uit dit hoofdstuk. Zo blijkt bijvoorbeeld dat $q(e, f) = 0$ als en slechts als $[e, f] = 0$ en $f \in q(\mathbf{V}, e)$ als en slechts als $[e, f] = 0$ en $e * \overline{f} = 0$.

We noemen een deelruimte van **W** *octagonaal* als en slechts als al zijn elementen octagonaal zijn. Het blijkt dat octagonale deelruimten steeds F-isotroop te zijn en hoogstens dimensie 2 kunnen hebben.

5.6 De Ree-Tits-achthoek

Het hoofddoel van Hoofdstuk 5 (en van het volledige werk) is de introductie van de meetkunde $\mathcal{O}$:

1. De *punten* van $\mathcal{O}$ zijn de octagonale 1-ruimtes van **W**.
2. De *rechten* van $\mathcal{O}$ zijn de octagonale 2-ruimtes van **W**.

Alle punten en rechten van $\mathcal{O}$ zijn dus ook punten en rechten van $\mathcal{F}$ en collineariteit in $\mathcal{O}$ is dit keer wel dezelfde als in $\mathcal{F}$. Stelling 5.19 vertelt ons dat $\mathcal{O}$ een veralgemeende achthoek is en dat puntenparen met een onderlinge *afstand* die gelijk is aan 8, 6, 4, 2 of 0 respectievelijk overstaand, schieroverstaand, cohyperlinear, collinear dan wel gelijk zijn.

Tot slot tonen Proposities 5.21 en 5.22 aan dat $\mathcal{O}$ een zogenaamde *Moufang-veelhoek* is en bijgevolg niets anders dan de perfecte veralgemeende achthoek $\mathsf{O}(K, \sigma)$ van Ree-Tits.

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List of symbols

Below we give a list of symbols from Chapters 1–5 together with a reference to the page where they are introduced. (Note that Appendix A assigns a different meaning to some of these symbols.)

ϵ_{ijk}	Equal to $\langle e_i, e_j, e_k \rangle$ $(-1, 0$ or $1)$	pg. 46
η_q	Canonical base vector of $\mathbf{V}^*$ associated with $-q \in -\mathcal{P}$	45
μ	Isomorphism between $\mathbf{W}$ and $\mathbf{Q}$	175
Φ	Root system of type E_6	31
Φ_F	Root system of type F_4	120
Φ_S, Φ_L	Short roots, long roots of Φ_F	121
Φ_O	'Root system' of type ${}^2\mathsf{F}_4$	193
π_i	Fundamental root for Φ	36
π_i'	Fundamental weight for Φ	37
ψ_i	Fundamental root for Φ_F	120
ψ_i'	Fundamental weight for Φ_F	125
σ	Tits endomorphism of K	191
Σ	Regular spread of $\mathcal{Q}$	41
ω	Defining automorphism for Σ	41
$\mathbf{1}$	Identity transformation on $\mathbf{V}$	50
∞	Element of $\mathbf{V}$ that defines $\mathbf{W}$ and $\mathbf{J}$	127
$\bar{\mathtt{1}}\ \bar{\mathtt{2}}\ \bar{\mathtt{3}}$	Root coordinate shorthand for -1,-2 or -3	33
$\mathtt{+}\ \ \mathtt{-}$	Root coordinate shorthand for $1/2$ and $-1/2$	123
$p \sim q$	p is collinear with q (in $\mathcal{Q}$)	27
$p \perp q$	$p \sim q$ or $p = q$	27
$p \cdot q$	Inner product on $\mathbf{P}$	30
p_∞	Unique point of L_∞ adjacent to $p \in \mathcal{P}^*$	116
r^*	Co-root corresponding to $r \in \Phi$	6
Φ^*	Dual root system of Φ	6
$\langle r, s \rangle$	Cartan integer for roots r, s	36
$a[p]$	'pth' coordinate of $a \in \mathbf{V}$, i.e., $a\eta_p$	45
$\alpha[p]$	'pth' coordinate of $\alpha \in \mathbf{V}^*$, i.e., $e_p\alpha$	45
$A[p,q]$	'p,qth' entry of matrix $A \in \widehat{\mathbf{L}}$, i.e., $e_p A \eta_q$	45
$\bar{\cdot}$	Involution used to create Φ_F from Φ	115
$\bar{\cdot}$	Polarity which 'folds' $\mathbf{L}$ into $\mathbf{J}$	127
$\cdot^\dagger$	Linear map from Φ_S to Φ_L	174
$\cdot^\dagger$	Graph endomorphism of $\widehat{\mathsf{F}}_4(K)$	189
$\cdot^{\mathrm{frob}}$	Frobenius map	182
$\boldsymbol{k} \oplus \boldsymbol{\ell}$	Group operation in $K_\sigma^{(2)}$	201

The following are 'generic' symbols, i.e., symbols that are generally used to represent a certain kind of element, but do not indicate one specific object. Because there are only so many letters and fonts, several of these symbols are sometimes used to mean something different from what is stated here. We hope their meaning shall always be clear from context.

a, b, c	Elements of $\mathbf{V}$
A, B, C	Elements of $\mathbf{L}$
e, f	Isotropic elements of $\mathbf{V}$
E, F	Isotropic elements of $\mathbf{L}$
g	Element of a Chevalley group
k, ℓ	Elements of K
$\boldsymbol{k}, \boldsymbol{\ell}$	Elements of $K_\sigma^{(2)}$
L	Element of $\mathbf{V} \wedge \mathbf{V}$
p, q	Points of $\mathcal{P}$
P	Element of $\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$
r, s, t	Roots of Φ
z, z', z'', z_i	Points of L_∞
α, β, γ	Elements of $\mathbf{V}^*$
η, φ	Isotropic elements of $\mathbf{V}^*$
Λ	Element of $\mathbf{V}^* \wedge \mathbf{V}^*$
Π	Element of $\mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$

We end with a list of all operations on Lie algebra modules which were introduced in the main text.

$a\alpha$	$\mathbf{V}, \mathbf{V}^* \to K$	bilinear	pg. 45
$\langle a, b, c\rangle$	$\mathbf{V}, \mathbf{V}, \mathbf{V} \to K$	trilinear, symmetric	45
$\langle \alpha, \beta, \gamma\rangle$	$\mathbf{V}^*, \mathbf{V}^*, \mathbf{V}^* \to K$	trilinear, symmetric	46
$a \times b$	$\mathbf{V}, \mathbf{V} \to \mathbf{V}^*$	bilinear, symmetric	46
$\alpha \times \beta$	$\mathbf{V}^*, \mathbf{V}^* \to \mathbf{V}$	bilinear, symmetric	46
$a^\#$	$\mathbf{V} \to \mathbf{V}^*$	quadratic	46
$\alpha^\#$	$\mathbf{V}^* \to \mathbf{V}$	quadratic	46
$D(a)$	$\mathbf{V} \to K$	cubic	46
$D(\alpha)$	$\mathbf{V}^* \to K$	cubic	46
aA	$\mathbf{V}, \widehat{\mathbf{L}} \to \mathbf{V}$	bilinear	45
$A\alpha$	$\widehat{\mathbf{L}}, \mathbf{V}^* \to \mathbf{V}^*$	bilinear	45
$[A, B]$	$\widehat{\mathbf{L}}, \widehat{\mathbf{L}} \to \widehat{\mathbf{L}}$	bilinear, antisymmetric	51

$a * \alpha$	$\mathbf{V}, \mathbf{V}^* \to \widehat{\mathbf{L}}$	bilinear	pg. 50
$\tau(A)$	$\widehat{\mathbf{L}} \to K$	linear	51
$A \cdot B$	$\widehat{\mathbf{L}}, \widehat{\mathbf{L}} \to K$	bilinear, symmetric	62
$L\alpha$	$\mathbf{V} \wedge \mathbf{V}, \mathbf{V}^* \to \mathbf{V}$	bilinear	99
$a\Lambda$	$\mathbf{V}, \mathbf{V}^* \wedge \mathbf{V}^* \to \mathbf{V}^*$	bilinear	99
$a \wedge b$	$\mathbf{V}, \mathbf{V} \to \mathbf{V} \wedge \mathbf{V}$	bilinear, antisymmetric	99
$\alpha \wedge \beta$	$\mathbf{V}^*, \mathbf{V}^* \to \mathbf{V}^* \wedge \mathbf{V}^*$	bilinear, antisymmetric	99
$\alpha \times A$	$\mathbf{V}^*, \mathbf{L} \to \mathbf{V} \wedge \mathbf{V}$	bilinear, needs $A\alpha = 0$	100
$A \times a$	$\mathbf{L}, \mathbf{V} \to \mathbf{V}^* \wedge \mathbf{V}^*$	bilinear, needs $aA = 0$	100
$P\alpha$	$\mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}, \mathbf{V}^* \to \mathbf{V} \wedge \mathbf{V}$	bilinear	102
$a\Pi$	$\mathbf{V}, \mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^* \to \mathbf{V}^* \wedge \mathbf{V}^*$	bilinear	103
$a \wedge b \wedge c$	$\mathbf{V}, \mathbf{V}, \mathbf{V} \to \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$	trilinear, antisymmetric	103
$\alpha \wedge \beta \wedge \gamma$	$\mathbf{V}^*, \mathbf{V}^*, \mathbf{V}^* \to \mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$	trilinear, antisymmetric	103
$A \times B$	$\mathbf{L}, \mathbf{L} \to \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$	bilinear, antisymmetric, needs $AB = 0$	99
$(A \times B)^*$	$\mathbf{L}, \mathbf{L} \to \mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$	bilinear, antisymmetric, needs $AB = 0$	103
$X(a, b, c)$	$\mathbf{V}, \mathbf{V}, \mathbf{V} \to \mathbf{V}^* \wedge \mathbf{V}^* \wedge \mathbf{V}^*$	trilinear, antisymmetric, needs $Ka + Kb + Kc$ to be isotropic	104
$X(\alpha, \beta, \gamma)$	$\mathbf{V}^*, \mathbf{V}^*, \mathbf{V}^* \to \mathbf{V} \wedge \mathbf{V} \wedge \mathbf{V}$	trilinear, antisymmetric, needs $K\alpha + K\beta + K\gamma$ to be isotropic	104
$\alpha \otimes a$	$\mathbf{V}^*, \mathbf{V} \to K$	bilinear	108
$\overline{a}$	$\mathbf{W} \to \mathbf{W}^*$	linear	127
$a\overline{b}$	$\mathbf{W}, \mathbf{W} \to K$	bilinear, symmetric	
$a * \overline{b}$	$\mathbf{W}, \mathbf{W} \to \mathbf{J}$	bilinear	
$S(a)$	$\mathbf{W} \to \mathbf{J}$	linear	166
$[a, b]$	$\mathbf{W}, \mathbf{W} \to \mathbf{W}$	bilinear, (anti)symmetric	166
a^2	$\mathbf{W} \to \mathbf{W}$	quadratic	166
$Q(e)$	$\mathbf{W} \to \mathbf{J}$	quadratic, needs e to be isotropic	177
$q(a, e)$	$\mathbf{W}, \mathbf{W} \to \mathbf{W}$	$q(ka, \ell e) = k\ell^\sigma q(a, e)$, needs e to be isotropic	195
$c(e, f)$	$\mathbf{W}, \mathbf{W} \to \mathbf{W}$	$c(ke, \ell f) = k^{\sigma/2}\ell^{\sigma/2}c(e, f)$, needs $Ke + Kf$ to be isotropic	196

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