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Classifications of blocking set related structures in Galois geometries

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Preface

For a little more than three years, blocking sets and related structures have occupied my mind. Many of the ideas and thoughts I had during these years are collected in this thesis. Its aim is to add to the knowledge of several geometrical structures by exploring their connection with blocking sets.

In the first chapter, the field of research is situated. Basic definitions, notations and some important theorems concerning finite projective and polar spaces are recalled. Some objects living in these spaces are described. Amongst those, blocking sets are handled in most detail, since they are crucial in the study in subsequent chapters.

In Chapters 2 and 3, classification results for a certain type of blocking sets are obtained. The blocking sets considered are equivalent to certain optimal linear codes: linear codes meeting the Griesmer bound. Hence, the classification results on blocking sets, which in this context are called minihypers, immediately translate into classification results on linear codes meeting the Griesmer bound. The results in Chapter 3 are refinements of those in Chapter 2 in the case that the blocking sets satisfy some further conditions.

The link with linear codes is not the only reason for the study of minihypers in Chapters 2 and 3. In the following chapters, the classification results are used to study other geometrical structures.

Partial t-spreads and t-covers of finite projective and polar spaces are related to blocking sets. That is why the results from Chapters 2 and 3 can be used in Chapter 4 to study partial t-spreads and minimal t-covers. They yield extendibility results for partial t-spreads and allow to describe the structure of the set of multiple points of t-covers. Also divisibility conditions for the existence of t-spreads in finite classical polar spaces are obtained and constructions for small minimal t-covers are presented.

In Chapter 5, the attention is shifted towards partial ovoids and blocking sets in finite classical polar spaces and to partial ovoids of the split Cayley hexagon. Theorems are presented that allow to lift bounds on the size of partial ovoids and blocking sets in a given dimension to bounds on the size of partial ovoids and blocking sets in higher dimensional spaces. Also small blocking sets of $W_{2n+1}(q)$ are studied. As in Chapter 4, the results from Chapters 2 and 3 turn out to be useful: they allow to prove a new upper bound for the size of a partial ovoid of the Hermitian variety $H(4, q^2)$ and to restrict the possibilities for the size of a maximal partial ovoid of the split Cayley hexagon.

The relation between blocking sets and Cameron-Liebler line classes is explored in Chapter 6, and it is this relation that allows to prove new nonexistence results for Cameron-Liebler line classes.

In Appendix A, two theorems on blocking sets are proved that were used in previous chapters. The first one shows that a small blocking set in a higher dimensional projective space contains a planar blocking set. The second one classifies the smallest double blocking sets in $PG(2, 4)$.

The second appendix presents two ways of constructing maximal sets of mutually orthogonal Latin squares. One way is to start with maximal partial spreads of finite projective spaces, the other one uses (non)existence results on spreads and ovoids of hyperbolic quadrics.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

The objects studied in this thesis live in finite projective spaces and in substructures of these spaces. This first chapter consists of two parts. The first part briefly describes different objects and related terminology that will be essential. The second part discusses blocking sets in some more detail.

Many of the objects and theorems from Section 1.1 exist in the general case, but here only the finite case is considered.

1.1 Finite affine, projective and polar spaces

For more information on projective spaces, see Hirschfeld [68]. More information on finite polar spaces can be found in Hirschfeld and Thas [70], while generalised quadrangles are studied in detail in Payne and Thas [91].

1.1.1 Finite projective spaces and substructures

Let $\mathrm{GF}(q)$ denote the finite field of order q, q a prime power, and let $V(n+1, q)$ denote the $(n+1)$-dimensional vector space over $\mathrm{GF}(q)$.

The *n-dimensional projective space over* $\mathrm{GF}(q)$, denoted by $\mathrm{PG}(n, q)$, is defined as follows. It is the pair (D, I), where D denotes the set of all subspaces of $V(n+1, q)$ and where I, the incidence relation, is containment. A subspace U of dimension $i+1$, $i \geq -1$, of $V(n+1, q)$ is said to have *dimension i* considered as element of $\mathrm{PG}(n, q)$. It is called an *i-dimensional subspace*, or simply an *i-space*, of $\mathrm{PG}(n, q)$. A 0-space, respectively 1-space, 2-space, 3-space, $(n-1)$-space, of $\mathrm{PG}(n, q)$ is called a *point*, respectively *line*, *plane*, *solid*, *hyperplane*, of $\mathrm{PG}(n, q)$. The (-1)-space is called the *empty* *space*. Often, subspaces of $\mathrm{PG}(n,q)$ are identified with the set of points contained in them. In what follows, this will be done without further notice.

Since a point of $\mathrm{PG}(n,q)$ corresponds to a vector line in $V(n+1,q)$, a point P in $\mathrm{PG}(n,q)$ can be represented by a nonzero vector $\bar{x}$ in $V(n+1,q)$; this point is denoted by $P(\bar{x})$. Two nonzero vectors represent the same point if and only if they are a scalar multiple of each other. Similarly, a hyperplane H is defined by a linear equation: it is a set of points whose representing vectors $\bar{x}=(x_0,x_1,\ldots,x_n)$ satisfy an equation $u_0x_0+u_1x_1+\ldots+u_nx_n=0$ for some $\bar{u}=(u_0,u_1,\ldots,u_n)$ in $V(n+1,q)\setminus\{\bar{0}\}$; it is denoted by $H(\bar{u})$. An m-space is a set of points whose representing vectors $\bar{x}=(x_0,x_1,\ldots,x_n)$ satisfy an equation $\bar{x}A=0$, where A is an $(n+1)\times(n-m)$-matrix over $\mathrm{GF}(q)$ of rank $n-m$.

Let π_r and π_s be two subspaces of $\mathrm{PG}(n,q)$. Then $\langle\pi_r,\pi_s\rangle$ denotes the subspace generated by π_r and π_s. If P and Q are two distinct points of $\mathrm{PG}(n,q)$, then the line $\langle P,Q\rangle$ joining P and Q is sometimes denoted by PQ. In general, if A and B are sets of points in $\mathrm{PG}(n,q)$, then AB denotes the *cone* with *vertex* A and *base* B, i.e., the union of the set of points of all lines joining an element of A with an element of B. In particular, if A and B are subspaces of $\mathrm{PG}(n,q)$, then $AB=\langle A,B\rangle$. The *Grassmann identity* gives the relation between the dimensions of two subspaces π_r and π_s and the dimensions of their intersection and the subspace generated by them:

$$\dim(\pi_r)+\dim(\pi_s)=\dim(\langle\pi_r,\pi_s\rangle)+\dim(\pi_r\cap\pi_s).$$

In this thesis, many counting arguments will be used. The following identities will prove to be useful.

Theorem 1.1.1 *Let* $\mathrm{PG}^{(r)}(n,q)$ *denote the set of all* r*-spaces in* $\mathrm{PG}(n,q)$*. For* $0\le r\le n$*,*

$$|\mathrm{PG}^{(r)}(n,q)|=\frac{\prod_{i=n-r+1}^{n+1}(q^i-1)}{\prod_{i=1}^{r+1}(q^i-1)}.$$

Let $\chi(s,r;n,q)$ *denote the number of* r*-spaces containing a given* s*-space of* $\mathrm{PG}(n,q)$*. For* $0\le s\le r\le n$*,*

$$\chi(s,r;n,q)=\frac{\prod_{i=r-s+1}^{n-s}(q^i-1)}{\prod_{i=1}^{n-r}(q^i-1)}.$$

Proof See e.g. [68, p. 85]. □

As $\mathrm{PG}(n,q)$ is often identified with its set of points, $|\mathrm{PG}(n,q)|$ denotes $|\mathrm{PG}^{(0)}(n,q)|$. By the above, $|\mathrm{PG}(n,q)|=(q^{n+1}-1)/(q-1)$; this number is often denoted by θ_n or $\theta(n)$, and in the theory of minihypers, see Chapters 2 and 3, by v_{n+1}.

Tangents and secants

Let B be a set of points in the projective plane $\mathrm{PG}(2,q)$. A *tangent* to B is a line of $\mathrm{PG}(2,q)$ intersecting B in exactly one point. A *secant* to B is a line of $\mathrm{PG}(2,q)$ containing at least two points of B. An i-secant to B is a line intersecting B in exactly i points; sometimes i is allowed to equal 0 or 1, in which case the i-secant is not really a secant.

Polarities

Let S and S' be two spaces $\mathrm{PG}(n,q)$, $n \geq 2$. A *collineation* $\varphi : S \to S'$ is an incidence-preserving bijection, i.e., for any two subspaces π_r and π_s of S, $\pi_r \subset \pi_s$ if and only if $\pi_r^\varphi \subset \pi_s^\varphi$. Now suppose that S' is the dual space of S, superimposed on S, i.e., the points of S' are the hyperplanes of S, the hyperplanes of S' are the points of S, the r-spaces of S' are the $(n-r-1)$-spaces of S, and incidence is reverse containment. A *polarity of* $\mathrm{PG}(n,q)$ is a collineation from S to S' with period two. Hence, it is an involutory incidence-reversing permutation of $\mathrm{PG}(n,q)$. If φ is a polarity, then the image of a point, respectively a hyperplane, is called its *polar* (hyperplane), respectively its *pole*. Two points, respectively hyperplanes, that lie in one another's polar, respectively contain one another's pole, are called *conjugate*. A point or hyperplane that is conjugate to itself is called *self-conjugate*.

A polarity is uniquely defined if for each point $P(\bar{x})$ its image $H(\bar{u})$ is defined. It is known, see e.g. [68, p.34], that a polarity φ of $\mathrm{PG}(n,q)$ can always be represented as

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix} = A \begin{pmatrix} x_0^\theta \\ x_1^\theta \\ \vdots \\ x_n^\theta \end{pmatrix}$$

or shorter $\bar{u}^T = A\bar{x}^{T^\theta}$, for some involutory field automorphism θ of $\mathrm{GF}(q)$ and some nonsingular $(n+1)\times(n+1)$-matrix A over $\mathrm{GF}(q)$ satisfying $A^T = \pm A$ if $\theta = 1$ and $A^{T^\theta} = A$ if $\theta \neq 1$.

Note that $\mathrm{GF}(q)$ has a nontrivial involutory automorphism if and only if q is a square. If q is a square, then $\mathrm{GF}(q)$ has a unique nontrivial involutory automorphism $\theta : \mathrm{GF}(q) \to \mathrm{GF}(q) : x \mapsto x^\theta = x^{\sqrt{q}}$.

The following terminology is used.

1. $\mathrm{char}(\mathrm{GF}(q)) > 2$
   - If $A^T = A$ and $\theta = 1$, then φ is called an *ordinary polarity* or an *orthogonal polarity*. The set of self-conjugate points forms a quadric.

• If n is odd, $A^T = -A$, and $\theta = 1$, then φ is called a *null polarity* or a *symplectic polarity.* All points of $\mathrm{PG}(n,q)$ are self-conjugate.
• If q is a square, $A^{T^\theta} = A$ and $\theta : x \mapsto x^{\sqrt{q}}$, then φ is called a *Hermitian polarity* or a *unitary polarity.* The set of self-conjugate points forms a Hermitian variety.

2. $\mathrm{char}(\mathrm{GF}(q)) = 2$
• If $A^T = A$, $\theta = 1$, and not all diagonal elements of A equal zero, then φ is called a *pseudo-polarity.* The set of self-conjugate points forms a hyperplane of $\mathrm{PG}(n,q)$.
• If $A^T = A$, $\theta = 1$, and all diagonal elements of A equal zero, then φ is called a *null polarity* or a *symplectic polarity.* All points of $\mathrm{PG}(n,q)$ are self-conjugate.
• If q is a square, $A^{T^\theta} = A$ and $\theta : x \mapsto x^{\sqrt{q}}$, then φ is called a *Hermitian polarity* or a *unitary polarity.* The set of self-conjugate points forms a Hermitian variety.

Varieties

If F is a nonzero homogeneous polynomial in $\mathrm{GF}(q)[x_0, x_1, \ldots, x_n]$, then the *variety* $V(F)$ is the set of all projective points $P(\bar{x})$ satisfying $F(\bar{x}) = 0$. If F is a non-degenerate quadratic form in $\mathrm{GF}(q)[x_0, x_1, \ldots, x_n]$, then $V(F)$ is a *nonsingular quadric* in $\mathrm{PG}(n,q)$. Over $\mathrm{GF}(q^2)$, a *Hermitian form* F is an element of $\mathrm{GF}(q^2)[x_0, x_1, \ldots, x_n]$ such that $F = \bar{x}H(\bar{x}^q)^T$ where $H = (h_{ij}) \neq 0$ is an $(n+1) \times (n+1)$-matrix over $\mathrm{GF}(q^2)$ satisfying $h_{ij}^q = h_{ji}$. If F is a non-degenerate Hermitian form in $\mathrm{GF}(q^2)[x_0, x_1, \ldots, x_n]$, then $V(F)$ is a *nonsingular Hermitian variety* in $\mathrm{PG}(n,q^2)$. The standard forms for nonsingular quadrics and Hermitian varieties are given in Table 1.1. In $\mathrm{PG}(2n,q)$, there is—up to collineations—only one nonsingular quadric, the *parabolic quadric* $\mathrm{Q}(2n,q)$. In $\mathrm{PG}(2n+1,q)$, there are—up to collineations—exactly two nonsingular quadrics, the *hyperbolic quadric* $\mathrm{Q}^+(2n+1,q)$ and the *elliptic quadric* $\mathrm{Q}^-(2n+1,q)$. In $\mathrm{PG}(n,q^2)$, there is—up to collineations—only one nonsingular *Hermitian variety* $\mathrm{H}(n,q^2)$.

Unitals

A *unital* U in $\mathrm{PG}(2,q)$, q square, is a set of $q\sqrt{q}+1$ points such that every line intersects U in either 1 or $\sqrt{q}+1$ points. The *classical unital* is the

space	variety	standard form
$\mathrm{PG}(2n,q)$	$\mathrm{Q}(2n,q)$	$V(x_0^2+x_1x_2+\ldots+x_{2n-1}x_{2n})$
$\mathrm{PG}(2n+1,q)$	$\mathrm{Q}^+(2n+1,q)$	$V(x_0x_1+x_2x_3+\ldots+x_{2n}x_{2n+1})$
$\mathrm{PG}(2n+1,q)$	$\mathrm{Q}^-(2n+1,q)$	$V(f(x_0,x_1)+x_2x_3+\ldots+x_{2n}x_{2n+1})$
		f is an irreducible quadratic polynomial over $\mathrm{GF}(q)$
$\mathrm{PG}(n,q^2)$	$\mathrm{H}(n,q^2)$	$V(x_0^{q+1}+x_1^{q+1}+\ldots+x_n^{q+1})$

Table 1.1: Standard forms of nonsingular quadrics and Hermitian varieties

set of points of a *Hermitian curve*, i.e., a nonsingular Hermitian variety in $\mathrm{PG}(2,q)$, see above.

The trace function

If $\mathrm{GF}(q_0)$ is a subfield of $\mathrm{GF}(q)$, $q=q_0^d$, then the *trace function* $\mathrm{Tr}_{q\to q_0}$ from $\mathrm{GF}(q)$ to $\mathrm{GF}(q_0)$ is defined as follows:

$$\mathrm{Tr}_{q\to q_0}:\mathrm{GF}(q)\to\mathrm{GF}(q_0):x\mapsto x+x^{q_0}+x^{q_0^2}+\ldots+x^{q_0^{d-1}}.$$

Grassmann coordinates

Although Grassmann coordinates can be introduced for subspaces of arbitrary dimension in $\mathrm{PG}(n,q)$, see e.g. [70], here they will only be considered for lines in $\mathrm{PG}(n,q)$.

So, let l be a line in $\mathrm{PG}(n,q)$, $n\geq 2$, and let $P(x_0,x_1,\ldots,x_n)$ and $Q(y_0,y_1,\ldots,y_n)$ be distinct points on l. For $0\leq i<j\leq n$, let p_{ij} denote the element $x_iy_j-x_jy_i$ of $\mathrm{GF}(q)$. Then the $\binom{n+1}{2}$-tuple

$$(p_{ij})_{0\leq i<j\leq n}:=(p_{01},\ldots,p_{0n},p_{12},\ldots,p_{1n},\ldots,p_{n-1,n})$$

defines a point P_l in $\mathrm{PG}(\binom{n+1}{2}-1,q)$. One easily checks that the $\binom{n+1}{2}$-tuple (p_{ij}) is, up to a scalar multiple, independent of the choice of the points P and Q on l. Hence, the point P_l is uniquely defined by the line l. The coordinates of P_l are called the *Grassmann coordinates of* l. In the special case $n=3$, the Grassmann coordinates are also called *Plücker coordinates*. In this case, the set of points of $\mathrm{PG}(5,q)$ whose coordinates are Plücker coordinates for lines of $\mathrm{PG}(3,q)$ is the set of points of the hyperbolic quadric $\mathrm{Q}^+(5,q)$, which is sometimes called the *Klein quadric*.

1.1.2 Finite affine spaces

The $(n+1)$-*dimensional affine space over* GF(q), AG$(n+1,q)$, is the pair (D,I), where D is the set of all cosets of subspaces of $V(n+1,q)$ and the incidence relation I is containment. A coset of a subspace U of dimension i, $i \geq -1$, of $V(n+1,q)$ is said to have *dimension* i considered as element of AG(n,q). It is called an i-*dimensional subspace*, or simply an i-*space*, of AG$(n+1,q)$. A 0-space, respectively 1-space, 2-space, 3-space, n-space, of AG$(n+1,q)$ is called a *point*, respectively *line*, *plane*, *solid*, *hyperplane*, of AG$(n+1,q)$. The (-1)-space is called the *empty space*. As in the projective case, subspaces of AG$(n+1,q)$ are often identified with the set of points contained in them.

AG(n,q) can be constructed from PG(n,q) in the following way. Let H_∞ be a hyperplane of PG(n,q) and define the set D as follows: $D = \{U \setminus H_\infty : U \text{ is a subspace of PG}(n,q)\}$. Let the incidence relation I be containment. Then $(D,I) \cong$ AG(n,q). So, in a way, PG(n,q) can be represented as PG$(n,q) =$ AG$(n,q) \cup H_\infty$. If this representation is used, AG(n,q) is called the *affine part*, while H_∞ is called the *hyperplane at infinity*.

1.1.3 Finite polar spaces

A *finite polar space* $\mathcal{P}$ *of rank* k, $k \geq 3$, consists of a finite set P whose elements are called *points* and a set of subsets of P, called *subspaces*, satisfying the following properties.

1. A subspace with the subspaces contained in it is isomorphic with a PG(d,q) for some $-1 \leq d \leq k-1$; such a subspace is said to have dimension d.

2. The intersection of any two subspaces is a subspace.

3. Given a subspace π of dimension $k-1$ and a point $P \in \mathsf{P} \setminus \pi$, there exists a unique subspace π' containing P that intersects π in a subspace of dimension $k-2$. The subspace $\pi' \cap \pi$ consists of all points of π that lie in subspaces of dimension 1 containing P.

4. There exist disjoint subspaces of dimension $k-1$.

The *projective index* of a finite polar space of rank k is by definition the integer $k-1$.

A *finite generalised quadrangle* or *finite polar space* $\mathcal{Q}$ *of rank* 2 *with order* (s,t), $s,t \geq 1$, is an incidence structure $\mathcal{S} = (\mathsf{P}, \mathsf{B}, \mathtt{I})$ in which P and B are finite nonempty disjoint sets of objects, respectively called *points* and *lines*, and where $\mathtt{I}$ is a symmetric incidence relation, $\mathtt{I} \subseteq (\mathsf{P} \times \mathsf{B}) \cup (\mathsf{B} \times \mathsf{P})$, satisfying the following properties.

1. Each point is incident with $t+1$ lines.

2. Each line is incident with $s+1$ points.

3. Given two points P and Q, $P \neq Q$, there is at most one line l satisfying $P \mathtt{I} l \mathtt{I} Q$.

4. Given a point P and a line l, $P \not\mathtt{I} l$, there exists a unique pair $(Q, m) \in \mathsf{P} \times \mathsf{B}$ satisfying $P \mathtt{I} m \mathtt{I} Q \mathtt{I} l$.

A generalised quadrangle with order (s, t) is often denoted by $\mathrm{GQ}(s, t)$. The following terminology and notations are used. Let $\mathcal{Q} = (\mathsf{P}, \mathsf{B}, \mathtt{I})$ be a $\mathrm{GQ}(s, t)$. Two points P and Q of $\mathcal{Q}$ are called *collinear*, denoted by $P \sim Q$, if there exists a line l incident with both. Dually, two lines l and m are called *concurrent*, denoted by $l \sim m$, if there exists a point P incident with both. If A is a set of points of $\mathcal{Q}$, then $A^\perp$ denotes the set $\{P \in \mathsf{P} : P \sim Q \text{ for all } Q \in A\}$. Similarly, if B is a set of lines of $\mathcal{Q}$, then $B^\perp$ denotes the set $\{l \in \mathsf{B} : l \sim m \text{ for all } m \in B\}$. If A is a set of points of $\mathcal{Q}$ or a set of lines of $\mathcal{Q}$, then the set $(A^\perp)^\perp$ is sometimes denoted by $A^{\perp\!\!\perp}$.

Finite classical polar spaces

The *finite classical polar spaces* of rank at least two are:

1. $\mathrm{W}_{2n+1}(q)$, the polar space arising from a symplectic polarity φ of $\mathrm{PG}(2n+1, q)$, $n \geq 1$. It consists of the subspaces π of $\mathrm{PG}(2n+1, q)$ satisfying $\pi \subseteq \pi^\varphi$.

2. $\mathrm{Q}^-(2n+1, q)$, the polar space arising from a nonsingular elliptic quadric of $\mathrm{PG}(2n+1, q)$, $n \geq 2$. It consists of the subspaces of $\mathrm{PG}(2n+1, q)$ whose point set is contained in the quadric.

3. $\mathrm{Q}(2n, q)$, the polar space arising from a nonsingular (parabolic) quadric of $\mathrm{PG}(2n, q)$, $n \geq 2$. It consists of the subspaces of $\mathrm{PG}(2n, q)$ whose point set is contained in the quadric.

4. $\mathrm{Q}^+(2n+1, q)$, the polar space arising from a nonsingular hyperbolic quadric of $\mathrm{PG}(2n+1, q)$, $n \geq 1$. It consists of the subspaces of $\mathrm{PG}(2n+1, q)$ whose point set is contained in the quadric.

$\mathcal{P}$	$\mathrm{rk}(\mathcal{P})$	$\lvert\mathcal{P}\rvert$	$\lvert\mathcal{G}(\mathcal{P})\rvert$
$\mathrm{W}_{2n+1}(q)$	$n+1$	$\frac{q^{2n+2}-1}{q-1}$	$(q+1)(q^2+1)\dots(q^{n+1}+1)$
$\mathrm{Q}^-(2n+1,q)$	n	$\frac{(q^n-1)(q^{n+1}+1)}{q-1}$	$(q^2+1)(q^3+1)\dots(q^{n+1}+1)$
$\mathrm{Q}(2n,q)$	n	$\frac{q^{2n}-1}{q-1}$	$(q+1)(q^2+1)\dots(q^n+1)$
$\mathrm{Q}^+(2n+1,q)$	$n+1$	$\frac{(q^n+1)(q^{n+1}-1)}{q-1}$	$2(q+1)(q^2+1)\dots(q^n+1)$
$\mathrm{H}(2n,q^2)$	n	$\frac{(q^{2n}-1)(q^{2n+1}+1)}{q^2-1}$	$(q^3+1)(q^5+1)\dots(q^{2n+1}+1)$
$\mathrm{H}(2n+1,q^2)$	$n+1$	$\frac{(q^{2n+2}-1)(q^{2n+1}+1)}{q^2-1}$	$(q+1)(q^3+1)\dots(q^{2n+1}+1)$

Table 1.2: Finite classical polar spaces: rank, number of points and number of generators

5. $\mathrm{H}(n,q^2)$, the polar space arising from a nonsingular Hermitian variety in $\mathrm{PG}(n,q^2)$, $n \geq 3$. It consists of the subspaces of $\mathrm{PG}(n,q^2)$ whose point set is contained in the variety.

Except for the quadrics in even characteristic, for each finite classical polar space $\mathcal{P}$ in $\mathrm{PG}(n,q)$, there exists a polarity φ of $\mathrm{PG}(n,q)$ such that $\mathcal{P}$ consists of the subspaces π of $\mathrm{PG}(n,q)$ that satisfy $\pi \subseteq \pi^\varphi$. If $\mathcal{P}$ is a nonsingular quadric in $\mathrm{PG}(n,q)$, n odd, q even, then there exists a polarity φ such that all subspaces π of $\mathcal{P}$ satisfy $\pi \subseteq \pi^\varphi$; however, they are not the only subspaces of $\mathrm{PG}(n,q)$ that satisfy this property. The polarity corresponding to a finite classical polar space will often be denoted by $\perp$.

A *generator* of a finite classical polar space $\mathcal{P}$ is a maximal totally isotropic or maximal singular subspace of $\mathcal{P}$, i.e., it is a subspace of $\mathcal{P}$ of dimension $k-1$, where $k-1$ is the projective index of $\mathcal{P}$. The set of all generators of $\mathcal{P}$ is denoted by $\mathcal{G}(\mathcal{P})$. In Table 1.2, the finite classical polar spaces are listed with their rank, number of points and number of generators.

Let $\mathcal{P}$ be a non-singular quadric or Hermitian variety in $\mathrm{PG}(n,q)$. For each point P of $\mathcal{P}$, there exists a *tangent hyperplane to* $\mathcal{P}$ *at* P, which is denoted by $T_P(\mathcal{P})$ and which is the hyperplane of $\mathrm{PG}(n,q)$ that consists of all the lines of $\mathrm{PG}(n,q)$ through P that either are contained in $\mathcal{P}$ or that intersect $\mathcal{P}$ only in P. If π_t is a t-space of $\mathcal{P}$, then $T_{\pi_t}(\mathcal{P})$, the *tangent space to* $\mathcal{P}$ *at* π_t, is by definition the $(n-t-1)$-space of $\mathrm{PG}(n,q)$ that is the intersection of all the tangent hyperplanes at points of π_t. The tangent space $T_{\pi_t}(\mathcal{P})$ intersects $\mathcal{P}$ in a cone $\pi_t\mathcal{P}'$ with vertex π_t and base $\mathcal{P}'$ a non-singular quadric or Hermitian variety of the same type as $\mathcal{P}$ in an $(n-2t-2)$-space skew to π_t. If $\mathcal{P}$ is not the parabolic quadric in even characteristic, then

$T_{\pi_t}(\mathcal{P})$ is the image of π_t under $\perp$, the polarity associated with $\mathcal{P}$.

Other finite polar spaces?

There are several generalised quadrangles known that are not classical—for an overview, see K. Thas [118]—but there exist no other finite polar spaces of rank $k > 2$ than the classical ones.

Theorem 1.1.2 (Veldkamp [125], Tits [121]) *All finite polar spaces of rank at least three are classical.*

Isomorphism results

To finish this section, some isomorphism results are mentioned. If $\mathcal{Q} = (\mathsf{P}, \mathsf{B}, \mathtt{I})$ is a GQ(s,t), then $\mathcal{Q}^D$ denotes the point-line dual of $\mathcal{Q}$, i.e., $\mathcal{Q}^D = (\mathsf{B}, \mathsf{P}, \mathtt{I})$ is a GQ(t,s).

Theorem 1.1.3 (Payne and Thas [91, §3.2])

1. *If q is even, then* $\mathrm{Q}(2n,q)$ *is isomorphic to* $\mathrm{W}_{2n-1}(q)$.
2. $\mathrm{W}_3(q)$ *is isomorphic to* $\mathrm{Q}(4,q)^D$.
3. $\mathrm{W}_3(q)$ *is isomorphic to* $\mathrm{W}_3(q)^D$ *if and only if q is even.*
4. $\mathrm{Q}^-(5,q)$ *is isomorphic to* $\mathrm{H}(3,q^2)^D$.

1.2 Blocking sets

Since blocking sets will play a crucial role in the thesis, this introductory section is quite extensive. It is however by no means complete. More information can be found in the survey papers [15, 16] by Blokhuis.

A *blocking set* in $\mathrm{PG}(2,q)$ is a set of points in $\mathrm{PG}(2,q)$ that intersects every line. Blocking sets that have no proper subset that is a blocking set are called *minimal*. It is not hard to see that a blocking set contains at least $q+1$ points and that a blocking set of size $q+1$ is necessarily a line. Blocking sets that contain a line are called *trivial*. The projective plane $\mathrm{PG}(2,2)$ has no nontrivial blocking set, but all other projective planes do. In $\mathrm{PG}(2,q)$, q odd, respectively $q > 2$ even, there exist so-called *projective triangles*, respectively *projective triads*. These are minimal blocking sets of size $3(q+1)/2$, respectively $(3q+2)/2$, all whose points lie on the sides of a triangle, respectively three concurrent lines, see [40, 25].

The size of a nontrivial minimal blocking set in $\mathrm{PG}(2,q)$ must lie somewhere in the interval $[q+2, q^2+q+1]$, but not all values are possible. The following theorem gives an upper and a lower bound for the size of a nontrivial minimal blocking set in $\mathrm{PG}(2,q)$.

Theorem 1.2.1 *Let B be a minimal nontrivial blocking set in* $\mathrm{PG}(2,q)$. *Then*

1. **(Bruen [24])** $|B| \geq q + \sqrt{q} + 1$, *with equality if and only if B is a Baer subplane.*

2. **(Bruen and Thas [30])** $|B| \leq q\sqrt{q} + 1$, *with equality if and only if B is a unital.*

Clearly, these lower and upper bounds can only be reached when q is a square. Substantial improvements to the lower bound when q is not a square are presented in the next theorem. Not much is known on blocking sets close to the upper bound.

Notation 1.2.2 Let p be a prime. Then c_p equals $2^{-1/3}$ when $p \in \{2,3\}$ and 1 when $p \geq 5$.

Theorem 1.2.3 *Let B be a nontrivial blocking set of* $\mathrm{PG}(2,q)$, $q > 2$.

1. **(Blokhuis [14])** *If q is a prime, then* $|B| \geq 3(q+1)/2$.

2. **(Blokhuis [15], Blokhuis et al. [22])** *If $q = p^{2e+1}$, p prime, $e \geq 1$, then* $|B| \geq \max(q+1+p^{e+1}, q+1+c_p q^{2/3})$.

The bound in the first case is sharp, since the projective triangle has size $3(q+1)/2$. In the second case, the bound is sharp for certain values of q; examples attaining it will be presented in Subsections 1.2.1 and 1.2.2.

What about blocking sets not too close to these upper and lower bounds? The following paragraph is copied from [15].

> One would expect the situation to be roughly the following. For a large interval, say roughly from $2q$ to $q\sqrt{q} - c \cdot q$ for some (possibly large) constant c, there exist minimal blocking sets of every possible size. So, the excluded cardinalities are some small ones, all at most $2q$, and on the other hand some large ones, all close to $q\sqrt{q}$.

1.2.1 Small blocking sets

A blocking set is called *small* if it contains less than $3(q+1)/2$ points. The following theorems show that not for all values of $b \in [q+\epsilon_q, 3(q+1)/2[$, small minimal blocking sets of size b exist. Here $q+\epsilon_q$ denotes the size of the smallest nontrivial blocking sets of $\mathrm{PG}(2,q)$.

Theorems 1.2.4 and 1.2.5 consider the specific cases where q is a square and a cube. Theorem 1.2.6 restricts the possibilities for b when $q=p^h$, $p \geq 7$ prime.

Theorem 1.2.4 *Let B be a blocking set in $\mathrm{PG}(2,q)$, q square, containing neither a line nor a Baer subplane.*

1. **(Blokhuis et al. [22])** *If $q > 16$, $q = p^h$, p prime, then $|B| \geq q+1+c_p q^{2/3}$.*

2. **(Szőnyi [107])** *If $q=p^2$, p prime, then $|B| \geq 3(q+1)/2$.*

Theorem 1.2.5 *Let $q=q_0^3$, $q_0=p^{h_0}$, $p \geq 7$ prime, $h_0 \geq 1$.*

1. **(Polverino [95, 97], Polverino and Storme [98])** *In $\mathrm{PG}(2,q)$, the smallest minimal nontrivial blocking sets that are not Baer subplanes are:*
(a) a minimal blocking set of size $q+q_0^2+1$, projectively equivalent to the set $K=\{(x,T(x),1) : x \in \mathrm{GF}(q)\} \cup \{(x,T(x),0) : x \in \mathrm{GF}(q) \setminus \{0\}\}$, with $T=\mathrm{Tr}_{q\to q_0}$ the trace function from $\mathrm{GF}(q)$ to $\mathrm{GF}(q_0)$;
(b) a minimal blocking set of size $q+q_0^2+q_0+1$, projectively equivalent to the set $K=\{(x,x^{q_0},1) : x \in \mathrm{GF}(q)\} \cup \{(x,x^{q_0},0) : x \in \mathrm{GF}(q)\setminus\{0\}\}$.

2. **(Polverino [97])** *If $h_0=1$, then these are the only minimal nontrivial blocking sets of size smaller than $3(q+1)/2$.*

These blocking sets have the following structure. The first one has one point that lies on q_0+1 (q_0^2+1)-secants. It is called the *vertex* of the blocking set. All other points of the blocking set lie on one (q_0^2+1)-secant and q_0^2 (q_0+1)-secants. The second blocking set has one $(q_0^2+q_0+1)$-secant and for the remainder only (q_0+1)-secants and tangents.

Let B be a small minimal blocking set in $\mathrm{PG}(2,q)$, $q=p^h$, p prime. The *exponent* e of B is the largest integer for which every line of $\mathrm{PG}(2,q)$ intersects B in $1 \pmod{p^e}$ points. Szőnyi [107] shows that the possible sizes of small minimal nontrivial blocking sets are restricted to certain intervals. If B is such a blocking set, then the interval $|B|$ lies in depends on the exponent

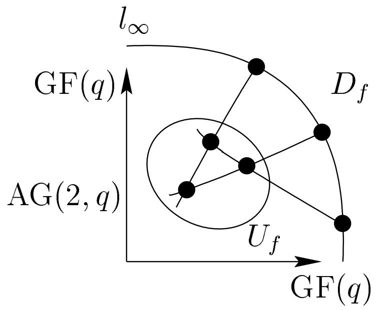

Figure 1.1: A Rédei type blocking set

of the blocking set. The lower bounds for these intervals were improved upon by Blokhuis [15], while the upper bounds were refined by Polverino [96].

Theorem 1.2.6 (Szőnyi [107], Blokhuis [15], Polverino [96]) *Let B be a small minimal nontrivial blocking set in* $\mathrm{PG}(2,q)$, *$q=p^h$, $p \geq 7$ prime, with exponent e. Then $1 \leq e \leq h/2$ and*

$$q+1+\frac{q+p^e}{p^e+1} \leq |B| \leq \frac{1+(q+1)(p^e+1)-\sqrt{\Delta}}{2},$$

where $\Delta = [1+(q+1)(p^e+1)]^2 - 4(p^e+1)(q^2+q+1)$.

For fixed q, these intervals are skew and nonempty. Moreover, each interval contains an integer, and in between any two intervals an integer can be found.

1.2.2 Rédei type blocking sets

One easily sees that if B is a nontrivial blocking set, and l is a line containing m points of B, then $|B| \geq q+m$. It suffices to consider the lines through a point of $l \setminus B$. A blocking set of size $q+m$ for which there exists a line intersecting it in m points, is called a *Rédei type* blocking set.

All Rédei type blocking sets can be constructed in the following way, see Bruen and Thas [30]. It is depicted in Figure 1.1. Let $\mathrm{PG}(2,q) = \mathrm{AG}(2,q) \cup l_\infty$. Let U be a set of q points in $\mathrm{AG}(2,q)$. Then $U \cup \{uv \cap l_\infty : u \neq v \in U\}$ is a Rédei type blocking set of size $q+m$, with m points on the line at infinity, where m is the number of directions determined by U. In practice, U is chosen in such a way that it does not determine the vertical direction. This means that U has exactly one point on every vertical line. Hence U can be considered as the graph of a function $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$

and $U = U_f = \{(x, f(x)) : x \in \mathrm{GF}(q)\}$. The set D_f of directions determined by f is given by

$$D_f = \{\frac{f(x) - f(y)}{x - y} : x \neq y \in \mathrm{GF}(q)\}.$$

The size of the blocking set determined by a function $f : \mathrm{GF}(q) \to \mathrm{GF}(q)$ is given by $|U_f| + |D_f| = q + |D_f|$. Below, some examples are given.

Examples 1.2.7 (see e.g. [16])

1. Let $q = q_0^d$ and $f : x \mapsto \mathrm{Tr}_{q \to q_0}(x)$. Then $|D_f| = q/q_0 + 1$ and the corresponding blocking set has size $q + q_0^{d-1} + 1$. Note that the blocking set in Theorem 1.2.5.1(a) is a special case of this one.
2. Let $q = q_0^d$ and $f : x \mapsto x^{q_0}$. Then $|D_f| = (q-1)/(q_0-1)$ and the corresponding blocking set has size $q + q_0^{d-1} + q_0^{d-2} + \ldots + 1$. Note that the blocking set in Theorem 1.2.5.1(b) is a special case of this one.
3. Let q be odd and $f : x \mapsto x^{(q+1)/2}$. Then $|D_f| = (q+3)/2$ and the corresponding blocking set has size $3(q+1)/2$. It is the projective triangle.

Successive articles [99, 18, 17, 2] have pinned down the possible sizes of D_f in increasingly smaller intervals, resulting in the following theorem.

Theorem 1.2.8 (Ball [2]) *Let f be a function from $\mathrm{GF}(q)$ to $\mathrm{GF}(q)$, $q = p^h$ for some prime p. Let e be maximal such that any line with a direction determined by D_f is incident with a multiple of p^e points of the graph of f. Then $|D_f| \equiv 1 \pmod{p^e}$ and one of the following holds:*

1. *$p^e = 1$ and $(q+3)/2 \leq |D_f| \leq q$;*
2. *$\mathrm{GF}(p^e) = \mathrm{GF}(q_0)$ is a subfield of $\mathrm{GF}(q)$ and $q/q_0 + 1 \leq |D_f| \leq (q-1)/(q_0-1)$;*
3. *$p^e = q$ and $|D_f| = 1$.*

Moreover if $q^e > 2$, then the graph of f is $\mathrm{GF}(p^e)$-linear.

Note that the bounds in cases 2 and 3 are sharp. Examples 1.2.7.1 and 1.2.7.2 attain the ones in case 2, while a linear function attains the one from case 3. For q odd, the lower bound in case 1 is sharp, see Example 1.2.7.3, while it is easy to see that the upper bound is sharp for all $q \neq 2$.

Remark 1.2.9 From the examples above, one might get the impression that all small minimal blocking sets are of Rédei type. This is not true: Lunardon [80], and Polito and Polverino [94] construct small minimal blocking sets that are not of Rédei type.

1.2.3 Multiple blocking sets

An *s-fold blocking set* in $\mathrm{PG}(2,q)$ is a set of points that intersects every line in at least s points. It is called *minimal* if no proper subset is an s-fold blocking set. A 1-fold blocking set is simply called a *blocking set*, while a 2-fold, respectively 3-fold, blocking set is sometimes called a *double*, respectively *triple*, blocking set. The following theorem indicates that, to obtain an s-fold blocking set of small cardinality with $s > 1$, it is no longer interesting to include a line in the set. In this way, there exists no such thing as a trivial multiple blocking set.

Theorem 1.2.10 *Let B be an s-fold blocking set of* $\mathrm{PG}(2,q)$*, $s > 1$.*

1. **(Bruen [27])** *If B contains a line, then $|B| \geq sq + q - s + 2$.*
2. **(Ball [3])** *If B does not contain a line, then $|B| \geq sq + \sqrt{sq} + 1$.*

If s is not too large, substantial improvements to this theorem have been obtained for general q. Also, for q a square and s not too large, the smallest minimal s-fold blocking sets are classified.

Theorem 1.2.11 (Blokhuis et al. [22]) *Let B be an s-fold blocking set in* $\mathrm{PG}(2,q)$ *of size $s(q+1)+c$ for some $s > 1$. For a prime p, let $c_p = 2^{-1/3}$ for $p \in \{2,3\}$ and $c_p = 1$ for $p > 3$.*

1. *If $q = p^{2d+1}$ and $s < q/2 - c_p q^{2/3}/2$, then $c > c_p q^{2/3}$.*
2. *If q is a square, $s < q^{1/4}/2$ and $c < c_p q^{2/3}$, then $c \geq s\sqrt{q}$ and B contains the union of s disjoint Baer subplanes.*
3. *If $q = p^2$ and $s < q^{1/4}/2$ and $c < p\lceil \frac{1}{4} + \sqrt{\frac{p+1}{2}} \rceil$, then $c \geq s\sqrt{q}$ and B contains the union of s disjoint Baer subplanes.*

Note that if B is nontrivial, then this theorem also holds when $s = 1$, see Theorems 1.2.3 and 1.2.4.

Remark 1.2.12 In [3], a table with the sizes of the smallest s-fold blocking sets in $\mathrm{PG}(2,q)$, $s > 1$, q small, can be found. Many examples of such blocking sets are described in [6, 4, 3].

An s-fold blocking set is called *small* if its size is smaller than $sq+(q+3)/2$. As was the case for 1-fold blocking sets, see Theorem 1.2.6, the sizes of small s-fold blocking sets are restricted to certain intervals.

Theorem 1.2.13 (Lovász and Szőnyi [79]) *Let B be a small minimal s-fold blocking set in* $\mathrm{PG}(2,q)$, $q = p^h$, *p prime, $s < p$ with $q > q_0(s)$.*

1. *Then*

$$sq + s + p^e \left\lceil \frac{q/p^e + 1}{p^e + 1} \right\rceil \leq |B| \leq \frac{1 + (q+1)(2s-1+p^e) - \sqrt{\Delta}}{2},$$

   where $\Delta = [1 + (q+1)(2s-1+p^e)]^2 - 4(s^2 + sp^e)(q^2 + q + 1)$, *for some integer* $1 \leq e \leq h/2$.

2. *If $|B|$ lies in the interval belonging to e, then each line intersects B in $s \pmod{p^e}$ points.*

1.2.4 Blocking sets in higher dimensional spaces

In this subsection, only a few results are mentioned. More can be found in the survey article [69] by Hirschfeld and Storme.

A *blocking set with respect to t-spaces* in $\mathrm{PG}(n,q)$ is a set of points that intersects every t-space. In the literature, such a blocking set is sometimes called an *$(n-t)$-blocking set* in $\mathrm{PG}(n,q)$.

Theorem 1.2.14 (Bose and Burton [23]) *If B is a blocking set with respect to t-spaces in* $\mathrm{PG}(n,q)$, *then* $|B| \geq |\mathrm{PG}(n-t,q)|$. *Equality holds if and only if B is an $(n-t)$-space.*

A blocking set with respect to t-spaces that contains an $(n-t)$-space is called *trivial*. The smallest nontrivial blocking sets with respect to t-spaces are characterised in the following theorem.

Theorem 1.2.15 (Beutelspacher [13], Heim [65]) *In* $\mathrm{PG}(n,q)$, *the smallest nontrivial blocking sets with respect to t-spaces are cones with vertex an $(n-t-2)$-space π_{n-t-2} and base a nontrivial blocking set of minimal cardinality in a plane skew to π_{n-t-2}.*

In $\mathrm{PG}(n,q)$, a blocking set with respect to hyperlanes is simply called a *blocking set*. For this case, Theorem 1.2.15 was already proved by Bruen in [26]. This result was improved upon.

Theorem 1.2.16 **(Storme and Weiner [105])** *Let B be a blocking set in* $\mathrm{PG}(n, q)$*,* $n \geq 3$*,* $q = p^h$ *square,* $p > 3$ *prime, of cardinality smaller than or equal to the cardinality of the second smallest nontrivial blocking set in* $\mathrm{PG}(2, q)$*. Then B contains a line or a planar blocking set of* $\mathrm{PG}(n, q)$*.*

In Chapter 3 a result of this kind will be useful. It is however a bit unfortunate that the cases $p = 2, 3$ are excluded, since using this theorem would also limit the ensuing results to $p > 3$. Strengthening the condition on the size of B, it is possible to prove the theorem for general q square, excluding only the cases $q = 4$ and $q = 9$. The proof is based on the proof of Theorem 1.2.16 in [105].

Theorem 1.2.17 *Let B be a blocking set in* $\mathrm{PG}(n, q)$*,* $n \geq 3$*,* $q = p^h$ *square,* p *prime,* $q \geq 16$*, of cardinality smaller than* $q + c_p q^{2/3}$*. Then B contains a line or a Baer subplane of* $\mathrm{PG}(n, q)$*.*

Proof See Section A.1. $\square$

Chapter 2

Minihypers

Minihypers were introduced by Hamada and Tamari in [64]. Usually, specific classes of minihypers are studied because of their connection with linear codes meeting the Griesmer bound. In this chapter, classification results on a particular class of minihypers are presented. They were published in *Designs, Codes and Cryptography* in *P. Govaerts and L. Storme, On a particular class of minihypers and its applications. I. The result for general* q [56]. These results translate immediately into classification results on linear codes meeting the Griesmer bound, but that is not the main reason for their study here. As will be shown in Chapters 4 and 5, they will be useful in proving various new results on partial spreads and covers of finite projective spaces, and on partial spreads, covers and partial ovoids of finite polar spaces.

2.1 Preliminaries

Definition 2.1.1 An $\{f, m; n, q\}$*-minihyper* is a pair (F, w), where F is a subset of the point set of $\mathrm{PG}(n, q)$ and w is a weight function $w : \mathrm{PG}(n, q) \to \mathbb{N} : P \mapsto w(P)$, satisfying

1. $w(P) > 0 \Leftrightarrow P \in F$,
2. $\sum_{P \in F} w(P) = f$, and
3. $\min\{\sum_{P \in H} w(P) : H \in \mathrm{PG}^{(n-1)}(n, q)\} = m$.

Remark 2.1.2 It is clear that a minihyper (F, w) is uniquely defined by its weight function w. We hold on to the notation (F, w), since often minihypers without weights, i.e., minihypers where w is a mapping onto $\{0, 1\}$, are studied, in which case the minihyper can be identified with the set F and is simply denoted by F.

In what follows, expressions such as "*Consider l points of the minihyper* (F, w)" will be encountered. What is meant is the following "*Consider a weight function* $w' : \mathrm{PG}(n, q) \to \mathbb{N}$ *satisfying* $\sum_{P \in \mathrm{PG}(n,q)} w'(P) = l$ *and* $w'(P) \leq w(P)$ *for each point* P". This terminology originates from the case where the minihyper is one without weights, and is not unnatural when the minihyper (F, w) is considered as a multiset. Following this terminology, a point of a minihyper (F, w) is a point of $\mathrm{PG}(n, q)$ with positive weight, and a set of points of $\mathrm{PG}(n, q)$ is said to be contained in (F, w) if it is contained in F. For a subspace π of $\mathrm{PG}(n, q)$ and a minihyper (F, w), the minihyper obtained by restricting w to π is denoted by $(F, w) \cap \pi$. The parameter f of the minihyper (F, w) is denoted by $|(F, w)|$.

2.1.1 Minihypers and blocking sets

Minihypers without weights are simply multiple blocking sets in $\mathrm{PG}(n, q)$. Indeed, from the definition, an $\{f, m; n, q\}$-minihyper F is an m-fold blocking set in $\mathrm{PG}(n, q)$. Conversely, every m-fold blocking set B in $\mathrm{PG}(n, q)$ that is not an $(m + 1)$-fold blocking set, is a $\{|B|, m; n, q\}$-minihyper.

2.1.2 Linear codes meeting the Griesmer bound

In this subsection, the relation between a certain class of minihypers and linear codes meeting the Griesmer bound is explained. Since the coding theoretical aspect of the minihypers will not be used in this thesis, the information provided here is very concise. For an introduction to coding theory, see Hill [66].

A *linear* $[\tilde{n}, k, d; q]$*-code* C is a k-dimensional subspace of the $\tilde{n}$-dimensional vector space $V(\tilde{n}, q)$ over $GF(q)$ having minimum Hamming distance d.

From an economical point of view, it is interesting to use linear codes having a minimal length $\tilde{n}$ for given k, d and q. Every linear $[\tilde{n}, k, d; q]$-code satisfies $\tilde{n} \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil$, see [58, 104]. This inequality is known as the *Griesmer bound.*

Suppose that q is a prime power and $d \geq 1$, $k \geq 2$. Then d can be written in an unique way as $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i$, with $\theta \geq 1$ and $0 \leq \zeta_i \leq q - 1$, $i = 0, 1, \ldots, k - 2$. Using such an expression for d, the Griesmer bound for an $[\tilde{n}, k, d; q]$-code can be expressed as $\tilde{n} \geq \theta v_k - \sum_{i=0}^{k-2} \zeta_i v_{i+1}$.

Theorem 2.1.3 (Hamada [60])[1] *Let q be a prime power and let k, θ and ζ_i, $i = 0, 1, \ldots, k - 2$, be integers satisfying $k \geq 3$, $\theta \geq 1$, $0 \leq \zeta_i \leq q - 1$ and*

[1] See also Remark 2.1.4

$(\zeta_0, \zeta_1, \ldots, \zeta_{k-2}) \neq \bar{0}$. *Let* $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i$. *Then there is a one-to-one correspondence between the set of all nonequivalent* $[\tilde{n}, k, d; q]$*-codes meeting the Griesmer bound and the set of all*

$$\left\{\sum_{i=0}^{k-2} \zeta_i v_{i+1}, \sum_{i=0}^{k-2} \zeta_i v_i; k-1, q\right\}\text{-}minihypers\ (F, w)$$

satisfying $w(P) \leq \theta$ *for each point* P *in* $\mathrm{PG}(k-1, q)$.

Let C be an $[\tilde{n}, k, d; q]$-code meeting the Griesmer bound and let $G = (\bar{g_1}^T \cdots \bar{g_{\tilde{n}}}^T)$ be a $(k \times \tilde{n})$-generator matrix of C. Then, for each $i \in \{1, 2, \ldots, \tilde{n}\}$, $\bar{g_i}$ is a nonzero vector in $V(k, q)$; hence it defines a point $P(\bar{g_i})$ in $\mathrm{PG}(k-1, q)$. Now define a weight function $w' : \mathrm{PG}(k-1, q) \to \mathbb{N}$:

$$w'(P) = |\{i \in \{1, 2, \ldots, \tilde{n}\} : P = P(\bar{g_i})\}| .$$

If $d = \theta q^{k-1} - \sum_{i=0}^{k-2} \zeta_i q^i$, with θ and ζ_i, $i = 1, 2, \ldots, k-2$, as above, then $\max\{w'(P) : P \in \mathrm{PG}(k-1, q)\} = \theta$. Let $w : \mathrm{PG}(k-1, q) \to \mathbb{N} : P \mapsto w(P) = \theta - w'(P)$ be a weight function and let $F = \{P \in \mathrm{PG}(k-1, q) : w(P) > 0\}$. Then (F, w) is a $\{\sum_{i=0}^{k-2} \zeta_i v_{i+1}, \sum_{i=0}^{k-2} \zeta_i v_i; k-1, q\}$-minihyper.

Starting with a minihyper (F, w) with parameters of this type and an integer θ satisfying $\theta \geq \max\{w(P) : P \in \mathrm{PG}(k-1, q)\}$, this construction can be reversed in the obvious way.

Remark 2.1.4 In the literature, there are various references for Theorem 2.1.3, most of them to papers by Hamada (with or without co-authors). However, the idea of the correspondence between $[\tilde{n}, k, d; q]$-codes and multisets in $\mathrm{PG}(k-1, q)$ already appears in Belov [8], where many examples of linear codes meeting the Griesmer bound are constructed. These examples are codes, called *of Belov type*, arising from sets of disjoint subspaces in $\mathrm{PG}(k-1, q)$, at most $q-1$ of any given dimension.

2.1.3 Notations and preliminary results

Theorem 2.1.3 shows that, in order to characterise all $[\tilde{n}, k, d; q]$-codes meeting the Griesmer bound for given values of k, d and q, it suffices to solve the following problem.

Problem 2.1.5 Characterise all

$$\left\{\sum_{i=0}^{n-1} \zeta_i v_{i+1}, \sum_{i=0}^{n-1} \zeta_i v_i; n, q\right\}\text{-minihypers } (F, w)$$

satisfying $w(P) \leq \theta$ for each point P, for given values of n, q and ζ_i, $i = 0, 1, \ldots, n-1$.

It explains why minihypers with these parameters are of particular interest to coding theory, and why many papers studying them have appeared. In this section, some of the results from those papers are mentioned.

The following theorem shows some classification results on minihypers, so on linear codes meeting the Griesmer bound, for general values of n, q and ζ_i, with the restriction that $\sum \zeta_i$ is not too big. Note that the minihypers under consideration are minihypers without weights.

Theorem 2.1.6 *Let F be a $\{\sum_{i=0}^{n} \zeta_i v_{i+1}, \sum_{i=0}^{n} \zeta_i v_i; n, q\}$-minihyper.*

1. **(Hamada and Helleseth [62], Hamada and Maekawa [63])** *If $\sum_{i=0}^{n} \zeta_i \leq \sqrt{q}$, then F is the disjoint union of ζ_n n-spaces, ζ_{n-1} $(n-1)$-spaces, ..., ζ_0 points in* $\mathrm{PG}(n, q)$.

2. **(Ferret and Storme [51])** *If $\sum_{i=0}^{n} \zeta_i < 2\sqrt{q}$, $q > q_0$, then F consists of the disjoint union of either*
(a) *ζ_n n-spaces, ζ_{n-1} $(n-1)$-spaces, ..., and ζ_0 points, or*
(b) *one subgeometry* $\mathrm{PG}(2l+1, \sqrt{q})$, *for some integer l with $1 \leq l \leq n$, ζ_n n-spaces, ..., ζ_{l+1} $(l+1)$-spaces, $\zeta_l - \sqrt{q} - 1$ l-spaces, ζ_{l-1} $(l-1)$-spaces, ..., and ζ_0 points, or*
(c) *one subgeometry* $\mathrm{PG}(2l, \sqrt{q})$, *for some integer l with $1 \leq l \leq n$, ζ_n n-spaces, ..., ζ_{l+1} $(l+1)$-spaces, $\zeta_l - 1$ l-spaces, $\zeta_{l-1} - \sqrt{q}$ $(l-1)$-spaces, ..., and ζ_0 points.*

Notation 2.1.7 For the remainder of this chapter,

- $\mathrm{E}(n, q)$ denotes the set of all n-tuples $\bar{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_{n-1})$ of integers ζ_i such that $\bar{\zeta} \neq \bar{0}$ and $0 \leq \zeta_i \leq q-1$ for $i = 0, 1, \ldots, n-1$;

- $\bar{\mathrm{E}}(n, q)$ denotes the set of all n-tuples $\bar{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_{n-1})$ of integers ζ_i such that $\bar{\zeta} \neq \bar{0}$ and either
(a) $\bar{\zeta} \in \mathrm{E}(n, q)$, or
(b) $\zeta_0 = q$ and $(\zeta_1, \zeta_2, \ldots, \zeta_{n-1}) \in \mathrm{E}(n-1, q)$, or
(c) $\zeta_0 = \zeta_1 = \ldots = \zeta_{\lambda-1} = 0$, $\zeta_\lambda = q$, and $(\zeta_{\lambda+1}, \zeta_{\lambda+2}, \ldots, \zeta_{n-1}) \in \mathrm{E}(n-1-\lambda, q)$ for some integer λ in $\{1, 2, \ldots, n-1\}$;

- $\mathrm{E}_{\mathrm{ext}}(n, q)$ denotes the set of all $(n+1)$-tuples $\bar{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_{n-1}, \zeta_n)$ of integers ζ_i such that either
(a) $\bar{\zeta} = \bar{0}$, or
(b) $\zeta_n = 0$ and $(\zeta_0, \zeta_1, \ldots, \zeta_{n-1}) \in \bar{\mathrm{E}}(n, q)$, or
(c) $\bar{\zeta} = (0, 0, \ldots, 0, 1)$.

The notations $\mathrm{E}(n,q)$ and $\bar{\mathrm{E}}(n,q)$ were introduced by Hamada, while $\mathrm{E}_{\mathrm{ext}}(n,q)$ is added in order to state Theorem 2.1.8 and Lemma 2.1.9, straightforward generalisations of the corresponding theorem in [61] and lemma in [62]. Note that each integer α, $0 \leq \alpha \leq v_{n+1}$, can be written in a unique way as $\alpha = \sum_{i=0}^{n} \zeta_i v_{i+1}$, with $(\zeta_0, \zeta_1, \ldots, \zeta_n) \in \mathrm{E}_{\mathrm{ext}}(n,q)$.

Theorem 2.1.8 (cf. Hamada [61]) *Let* $(\zeta_0, \zeta_1, \ldots, \zeta_n)$ *belong to* $\mathrm{E}_{\mathrm{ext}}(n,q)$, $n \geq 1$.

1. *If* $m \geq \sum_{i=1}^{n} \zeta_i v_i$*, then* $f \geq \sum_{i=1}^{n} \zeta_i v_{i+1}$ *for any* $\{f,m;n,q\}$*-minihyper* (F,w).

2. *If* (F,w) *is a minihyper in* $\mathrm{PG}(n,q)$*satisfying* $|(F,w)| = \sum_{i=0}^{n} \zeta_i v_{i+1}$ *and* $|(F,w) \cap H| \geq \sum_{i=0}^{n} \zeta_i v_i$ *for every hyperplane* H *in* $\mathrm{PG}(n,q)$*, then* (F,w) *is a* $\{\sum_{i=0}^{n} \zeta_i v_{i+1}, \sum_{i=0}^{n} \zeta_i v_i; n, q\}$*-minihyper.*

Lemma 2.1.9 (cf. Hamada and Helleseth [62]) *Suppose that* (F,w) *is a* $\{\sum_{i=0}^{n} \zeta_i v_{i+1}, \sum_{i=0}^{n} \zeta_i v_i; n, q\}$*-minihyper for some integers* n*,* q *and* ζ_i*,* $i = 0, 1, \ldots, n$*, such that* $n \geq 1$*,* $\sum_{i=0}^{n} \zeta_i = h$*,* $h \leq q$*, and* $(\zeta_0, \zeta_1, \ldots, \zeta_n) \in \mathrm{E}_{\mathrm{ext}}(n,q)$.

(i) *If there exists a hyperplane* H *in* $\mathrm{PG}(n,q)$ *such that* $|(F,w) \cap H| = \sum_{i=0}^{n-1} m_i v_{i+1}$ *for some ordered set* $(m_0, m_1, \ldots, m_{n-1})$ *in* $\mathrm{E}_{\mathrm{ext}}(n-1,q)$*, then* $(F,w) \cap H$ *is a* $\{\sum_{i=0}^{n-1} m_i v_{i+1}, \sum_{i=0}^{n-1} m_i v_i; n-1, q\}$*-minihyper in* H.

(ii) *There exists no hyperplane* H *in* $\mathrm{PG}(n,q)$ *satisfying* $|(F,w) \cap H| = \sum_{i=0}^{n-1} m_i v_{i+1}$ *for any ordered set* $(m_0, m_1, \ldots, m_{n-1})$ *in* $\mathrm{E}_{\mathrm{ext}}(n-1,q)$ *with* $\sum_{i=0}^{n-1} m_i > h$.

(iii) *In the case where* $\zeta_0 = 0$ *and* $q \geq 2h-1$*, there is no hyperplane* H *in* $\mathrm{PG}(n,q)$ *satisfying* $|(F,w) \cap H| = \sum_{i=0}^{n-1} m_i v_{i+1}$ *for any ordered set* $(m_0, m_1, \ldots, m_{n-1})$ *in* $\mathrm{E}_{\mathrm{ext}}(n-1,q)$ *with* $\sum_{i=0}^{n-1} m_i < h$.

Remark 2.1.10 In the original papers of Hamada, and Hamada and Helleseth, Theorem 2.1.8 and Lemma 2.1.9 were stated for minihypers without weights and $\bar{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_{n-1}, 0)$ with $(\zeta_0, \zeta_1, \ldots, \zeta_{n-1}) \in \bar{\mathrm{E}}(n,q)$. However, the proofs provided there are also valid when weights are introduced, and it is not hard to check that the results also hold for $\bar{\zeta} \in \{\bar{0}, (0,0,\ldots,0,1)\}$.

The reason for the generalisation of the original theorem and lemma is that it allows us to state the following corollary.

Corollary 2.1.11 *Let (F, w) be a $\{\sum_{i=0}^{n} \zeta_i v_{i+1}, \sum_{i=0}^{n} \zeta_i v_i; n, q\}$-minihyper satisfying $n \geq 1$, $\sum_{i=0}^{n} \zeta_i = h \leq q$ and $(\zeta_0, \zeta_1, \ldots, \zeta_n) \in \mathrm{E}_{\mathrm{ext}}(n, q)$. Suppose π_r is an r-dimensional subspace of $\mathrm{PG}(n, q)$, $1 \leq r \leq n$, such that $|\pi_r \cap (F, w)| = \sum_{i=0}^{r} m_i v_{i+1}$ for some $(m_0, m_1, \ldots, m_r) \in \mathrm{E}_{\mathrm{ext}}(r, q)$. Then $\pi_r \cap (F, w)$ is a $\{\sum_{i=0}^{r} m_i v_{i+1}, \sum_{i=0}^{r} m_i v_i; r, q\}$-minihyper satisfying $\sum_{i=0}^{r} m_i \leq h$.*

Proof The result clearly holds for $r = n$, and, by Lemma 2.1.9, for $r = n-1$. So assume it holds for all $(r+1)$-spaces and let π_r be an r-space, $1 \leq r \leq n-2$, such that $|\pi_r \cap (F, w)| = \sum_{i=0}^{r} m_i v_{i+1}$ for some $(m_0, m_1, \ldots, m_r) \in \mathrm{E}_{\mathrm{ext}}(r, q)$.

There are v_{n-r} $(r+1)$-spaces through π_r. Together they cover all points of (F, w). Hence, the average number of points of (F, w) in an $(r+1)$-space through π_r equals

$$\frac{\sum_{i=0}^{n} \zeta_i v_{i+1} - \sum_{i=0}^{r} m_i v_{i+1}}{v_{n-r}} + \sum_{i=0}^{r} m_i v_{i+1}. \tag{2.1}$$

Using the fact that $\sum_{i=0}^{n} \zeta_i v_{i+1} \leq v_{n+1}$ and $\sum_{i=0}^{r} m_i v_{i+1} \leq v_{r+1}$, straightforward calculations show that (2.1) is smaller than $v_{r+2} + 1$. Hence, there exists an $(r+1)$-space π_{r+1} through π_r containing at most v_{r+2} points of (F, w). So, there exists an $(r+2)$-tuple $(l_0, l_1, \ldots, l_{r+1}) \in \mathrm{E}_{\mathrm{ext}}(r+1, q)$ such that $|(F, w) \cap \pi_{r+1}| = \sum_{i=0}^{r+1} l_i v_{i+1}$. By assumption, $\pi_{r+1} \cap (F, w)$ is a $\{\sum_{i=0}^{r+1} l_i v_{i+1}, \sum_{i=0}^{r} l_i v_i; r+1, q\}$-minihyper satisfying $\sum_{i=0}^{r+1} l_i \leq h$. Applying Lemma 2.1.9 on π_{r+1} and its hyperplane π_r yields the desired result: $\pi_r \cap (F, w)$ is a $\{\sum_{i=0}^{r} m_i v_{i+1}, \sum_{i=0}^{r} m_i v_i; r, q\}$-minihyper satisfying $\sum_{i=0}^{r} m_i \leq h$. □

Corollary 2.1.12 *Let F be a $\{\sum_{i=0}^{n} \zeta_i v_{i+1}, \sum_{i=0}^{n} \zeta_i v_i; n, q\}$-minihyper satisfying $n \geq 1$, $\sum_{i=0}^{n} \zeta_i = h \leq q$ and $(\zeta_0, \zeta_1, \ldots, \zeta_n) \in \mathrm{E}_{\mathrm{ext}}(n, q)$. Then every r-space π_r, $1 \leq r \leq n$, intersects F in a $\{\sum_{i=0}^{r} m_i v_{i+1}, \sum_{i=0}^{r} m_i v_i; r, q\}$-minihyper $F \cap \pi_r$ satisfying $\sum_{i=0}^{r} m_i \leq h$.*

Proof In this case, the minihyper F is a set of points. Let π_r be an arbitrary r-space in $\mathrm{PG}(n, q)$. Then π_r intersects F in $\alpha \leq v_{r+1}$ points. But every number $\alpha \leq v_{r+1}$ can be written in a unique way as $\alpha = \sum_{i=0}^{r+1} m_i v_{i+1}$ with $(m_0, m_1, \ldots, m_{r+1}) \in \mathrm{E}_{\mathrm{ext}}(r+1, q)$. Applying Corollary 2.1.11 yields the result. □

Theorem 2.1.13, respectively Theorem 2.1.14, gives some more intersection properties of these minihypers, respectively some intersection properties of subspaces of $\mathrm{PG}(n, q)$.

Theorem 2.1.13 **(Hamada [61])** *Let m be any integer such that $1 \leq m <$*

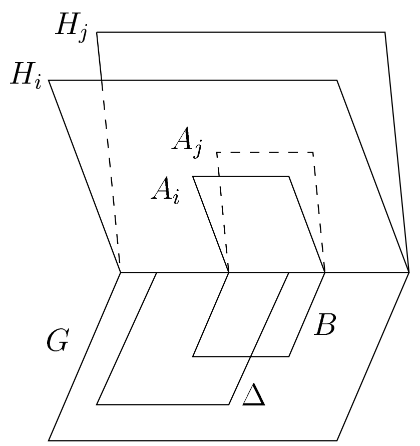

Figure 2.1: The configuration of Theorem 2.1.14

n. If there exists a

$$\left\{\sum_{i=0}^{n-1}\zeta_i v_{i+1}, \sum_{i=0}^{n-1}\zeta_i v_i; n, q\right\}\text{-minihyper}$$

(F, w) for some ordered set $(\zeta_0, \zeta_1, \ldots, \zeta_{n-1})$ in $\mathrm{E}(n, q)$, then:

1. *$|(F, w) \cap \Omega| \geq \sum_{i=m-1}^{n-1} \zeta_i v_{i+1-m}$ for any $(n-m)$-space Ω in $\mathrm{PG}(n, q)$ and the equality holds for some $(n-m)$-space Ω in $\mathrm{PG}(n, q)$.*

2. *In the special case $m = 2$, $|(F, w) \cap \Delta| \geq \sum_{i=1}^{n-1} \zeta_i v_{i-1}$ for any $(n-2)$-space Δ in $\mathrm{PG}(n, q)$ and $|(F, w) \cap G| = \sum_{i=1}^{n-1} \zeta_i v_{i-1}$ for some $(n-2)$-space G in $\mathrm{PG}(n, q)$. Let H_j, $j = 0, 1, \ldots, q$, be the $q+1$ hyperplanes in $\mathrm{PG}(n, q)$ that contain G. Then $(F, w) \cap H_j$ is a*

$$\left\{\delta_j + \sum_{i=1}^{n-1}\zeta_i v_i, \sum_{i=1}^{n-1}\zeta_i v_{i-1}; n, q\right\}\text{-minihyper}$$

in H_j for $j = 0, 1, \ldots, q$, where the δ_j are some non-negative integers such that $\sum_{j=0}^{q} \delta_j = \zeta_0$.

Theorem 2.1.14 (Hamada and Maekawa [63]) *Let G be any $(n-2)$-space in $\mathrm{PG}(n, q)$ and let $H_0, H_1, \ldots, H_q$ be the $q+1$ hyperplanes in $\mathrm{PG}(n, q)$ that contain G. Let $2 \leq \lambda < n$, let B be a $(\lambda-2)$-space in G and let A_i be a $(\lambda-1)$-space in H_i, $i = 0, 1, \ldots, q$, such that $G \cap A_i = B$, see Figure 2.1. For*

$j = 2, 3, \ldots, q$, denote the $(\lambda-1)$-space $H_j \cap \langle A_0, A_1\rangle$ by E_j. Suppose that α, $2 \le \alpha \le q$, is an integer for which $E_\alpha \cap A_\alpha = B$. Let Δ be an $(n-3)$-space in G and let $D_\alpha = G \cap \langle E_\alpha, A_\alpha\rangle$.

1. *In the case $B \subset \Delta$ and $D_\alpha \not\subset \Delta$, let $\Pi_1, \Pi_2, \ldots, \Pi_q$ be the q hyperplanes in $\mathrm{PG}(n, q)$ different from H_α that contain the $(n-2)$-space $\langle \Delta, E_\alpha\rangle$. Then there exists a hyperplane Π in $\{\Pi_1, \Pi_2, \ldots, \Pi_q\}$ such that $|\Pi \cap (\bigcup_{i=0}^{q} A_i)| = v_{\lambda-1}$.*

2. *In the case $B \not\subset \Delta$, $|\langle \Delta, E_\alpha\rangle \cap (\bigcup_{i=0}^{q} A_i)| = v_\lambda$.*

Remark 2.1.15 1. In the original statement of Theorem 2.1.14 in [63], there is a small error in the notations of the second case.

2. For sake of completeness, in Theorem 2.1.14 a third case could be added. If $D_\alpha \subset \Delta$ (this implies that $B \subset \Delta$) and if $\Pi_1, \Pi_2, \ldots, \Pi_q$ are the q hyperplanes in $\mathrm{PG}(n, q)$ different from H_α that contain the $(n-2)$-space $\langle \Delta, E_\alpha\rangle$, then there exists a hyperplane Π in $\{\Pi_1, \Pi_2, \ldots, \Pi_q\}$ such that $|\Pi \cap (\bigcup_{i=0}^{q} A_i)| = v_\lambda$.

2.1.4 Statement of the problem

In this chapter and the next one, a subcase of Problem 2.1.5 will be considered.

Problem 2.1.16 Characterise all

$$\{\delta v_{\mu+1}, \delta v_\mu; n, q\}\text{-minihypers } (F, w)$$

satisfying $w(P) \le \theta$ for each point P, for given values of n, q, μ and δ.

A *sum* of t-dimensional subspaces is a weight function $w : \mathrm{PG}^{(t)}(n, q) \to \mathbb{N} : \pi_t \mapsto w(\pi_t)$. Such a sum induces a weight function on subspaces of smaller dimension. Let π_r be a subspace of dimension $r < t$, then $w(\pi_r) = \sum_{\pi \in \mathrm{PG}^{(t)}(n,q), \pi \supset \pi_r} w(\pi)$. In particular, the weight of a point is the sum of the weights of the t-spaces passing through it. In the case that w is a mapping onto $\{0, 1\}$, the sum w can be identified with the set A of t-spaces with weight 1.

Example 2.1.17 Let $W_1, W_2, \ldots, W_\delta$ be δ μ-spaces in $\mathrm{PG}(n, q)$, where $1 \le \delta \le q - 1$ and $1 \le \mu \le n - 1$. For each point P in $\mathrm{PG}(n, q)$, let $w(P)$ denote the number of μ-flats W in $\{W_1, W_2, \ldots, W_\delta\}$ such that $P \in W$, and let F be the set of points P in $\mathrm{PG}(n, q)$ for which $w(P) \ge 1$. Then (F, w) is a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper.

It will be shown in Theorems 2.2.5 and 2.2.7 that, in a way, the converse of the result in the previous example holds—under certain restrictions on δ and μ. The concept of a sum of μ-spaces was introduced because the μ-spaces need not be distinct.

2.2 The classification result

Lemma 2.2.1 *Let (F, w) be a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper satisfying $\delta \geq 0$, $0 \leq \mu \leq n-1$. If $\pi_{n-\mu-1}$ is an $(n-\mu-1)$-space containing no points of (F, w), then all hyperplanes through $\pi_{n-\mu-1}$ contain exactly δv_μ points of (F, w).*

Proof Clearly, the lemma holds for $\delta = 0$. So assume $\delta > 0$. Hence $\mu > 0$ and $n \geq 2$. There are $v_{\mu+1}$ hyperplanes through $\pi_{n-\mu-1}$, and each point of (F, w) occurs in v_μ of these. Thus the average number of points of (F, w) in the hyperplanes through $\pi_{n-\mu-1}$ is $|(F, w)| v_\mu / v_{\mu+1} = \delta v_\mu$, which is the minimum number of points of (F, w) in a hyperplane. Therefore all hyperplanes containing $\pi_{n-\mu-1}$ contain exactly δv_μ points of (F, w). $\square$

Corollary 2.2.2 *Suppose (F, w) is a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper satisfying $\delta \geq 0$ and $0 \leq \mu \leq n-1$. If H is a hyperplane containing more than δv_μ points of (F, w), then every $(n-\mu-1)$-space in H contains at least one point of (F, w).*

Proof Suppose that in H there exists an $(n-\mu-1)$-space disjoint from (F, w). By Lemma 2.2.1, H contains exactly δv_μ points of (F, w), a contradiction. $\square$

Lemma 2.2.3 *Let (F, w) be a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper satisfying $1 \leq \delta \leq (q+1)/2$, $0 \leq \mu \leq n-1$, and containing a μ-space π_μ. Then the minihyper (F', w') defined by the weight function w', where*

- *$w'(P) = w(P) - 1$, for $P \in \pi_\mu$, and*
- *$w'(P) = w(P)$, for $P \in \mathrm{PG}(n, q) \setminus \pi_\mu$,*

is a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_\mu; n, q\}$-minihyper.

Proof The lemma holds obviously for $\delta = 1$, $\mu = 0$ or $n < 2$, so let $\delta > 1$, $\mu > 0$ and $n \geq 2$.

It is clear that (F', w') is a $\{(\delta-1)v_{\mu+1}, \geq 0; n, q\}$-minihyper. We will show that it is a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_\mu; n, q\}$-minihyper. Suppose that this is not the case. By Theorem 2.1.8, this is equivalent to supposing that there exists a hyperplane H containing less than $(\delta-1)v_\mu$ points of (F',w'). If $|H \cap \pi_\mu| = v_\mu$, then $|(F',w') \cap H| = |(F,w) \cap H| - |H \cap \pi_\mu| \geq (\delta-1)v_\mu$, a contradiction. Hence $|H \cap \pi_\mu| = v_{\mu+1}$, so H contains π_μ. Since $v_{\mu+1} \leq |(F,w) \cap H| < v_{\mu+1} + (\delta-1)v_\mu < 2v_{\mu+1}$, there exists a point P in π_μ with weight $w(P) = 1$.

Consider this point P in the μ-space π_μ. Less than $(\delta-1)v_\mu$ points of (F,w) are in H but not in π_μ. There exists an $(n-\mu-1)$-space through P in H containing no other points of (F,w). This can be seen in the following way.

Clearly, if $\mu = n-1$, then this $(n-\mu-1)$-space is the point P itself. So, assume that $\mu < n-1$. Since the number of lines through P that lie in H but not in π_μ equals

$$\frac{q^{n-1}-1}{q-1} - \frac{q^\mu - 1}{q-1} \geq q^\mu > (\delta-1)v_\mu$$

there exists a line π_1 through P in H containing no other points of (F,w). Now assume that through P there exists an r-space π_r, $1 \leq r < n-\mu-1$, contained in H, that contains no other points of (F,w). In H, there are $v_{n-1-r} - v_\mu \geq q^\mu > (\delta-1)v_\mu$ $(r+1)$-spaces through π_r that intersect π_μ in P. Hence, at least one of them, say π_{r+1}, contains no point of (F,w) different from P. Repeating this argument for increasing values of r, a chain $\pi_1 \subset \pi_2 \subset \ldots \subset \pi_{n-\mu-1}$ of subspaces π_i in H is obtained that contain P but no other point of (F,w).

Let $\pi_{n-\mu-1}$ be such an $(n-\mu-1)$-space and denote the $v_{\mu+1}$ hyperplanes in $\mathrm{PG}(n,q)$ through $\pi_{n-\mu-1}$ by $H =: H_0, H_1, H_2, \ldots, H_m$, where $m = v_{\mu+1} - 1$. Without loss of generality, we can assume that $|(F,w) \cap H_1| \geq |(F,w) \cap H_2| \geq \ldots \geq |(F,w) \cap H_m| \geq \delta v_\mu$. Since (i) $(F,w) \cap \pi_{n-\mu-1}$ consists of only one point P, which has weight one, and which is contained in every hyperplane H_i, and (ii) $|(F,w) \cap (\mathrm{PG}(n,q) \setminus \pi_{n-\mu-1})| = \delta v_{\mu+1} - 1$ and every point in $\mathrm{PG}(n,q) \setminus \pi_{n-\mu-1}$ is contained in v_μ hyperplanes H_i, (2.2) holds. It is obtained by counting the size of the set $\{(P, H_i) : P \in H_i, P \in \mathrm{PG}(n,q), 0 \leq i \leq m\}$, where each pair (P, H_i) is counted $w(P)$ times.

$$|(F,w) \cap H| + \sum_{i=1}^{m} |(F,w) \cap H_i| = v_{\mu+1} + v_\mu(\delta v_{\mu+1} - 1). \tag{2.2}$$

If $|(F,w) \cap H_1| = \delta v_\mu$, it follows that $|(F,w) \cap H| = v_{\mu+1} + v_\mu(\delta v_{\mu+1} - 1) - m\delta v_\mu = v_{\mu+1} + (\delta-1)v_\mu$, a contradiction. Hence $|(F,w) \cap H_1| > \delta v_\mu$. By Corollary 2.2.2, the points of (F,w) must block every $(n-\mu-1)$-space in H_1. The smallest set blocking every $(n-\mu-1)$-space in $\mathrm{PG}(n-1,q)$ is a μ-space in $\mathrm{PG}(n-1,q)$, having $v_{\mu+1}$ points, see Theorem 1.2.14. Thus $|H_1 \cap (F,w)| \geq v_{\mu+1}$.

Let x be the greatest integer in $\{1,2,\ldots,m\}$ for which $|(F,w)\cap H_x| > \delta v_\mu$. Then it follows from the above results that $x \geq 1$ and $|(F,w)\cap H_i| \geq v_{\mu+1}$ for $i = 1,2,\ldots,x$. Hence, by (2.2), $v_{\mu+1} + v_\mu(\delta v_{\mu+1} - 1) \geq (x+1)v_{\mu+1} + (v_{\mu+1} - x - 1)\delta v_\mu$. Since $v_{\mu+1} > \delta v_\mu$, it follows that $x \leq (\delta-1)v_\mu/((q-\delta)v_\mu + 1) < (\delta-1)/(q-\delta)$. But $1 \leq \delta \leq (q+1)/2$, which implies that $x < 1$, a contradiction.

It can be concluded that the hyperplane H does not exist and that (F',w') is indeed a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_\mu; n, q\}$-minihyper. $\square$

Notation 2.2.4 For $q = 2$, let ϵ_q equal 2. For $q > 2$, let $q + \epsilon_q$ denote the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2,q)$.

As mentioned in Section 1.2, in $\mathrm{PG}(2,q)$, q odd, respectively $q \neq 2$ even, there exist nontrivial blocking sets of size $3(q+1)/2$, respectively $(3q+2)/2$. Hence, for given q, if δ is an integer with $\delta < \epsilon_q$, then $\delta \leq (q+1)/2$ and every blocking set of size $q+\delta$ in $\mathrm{PG}(2,q)$ contains a line.

Theorem 2.2.5 *If (F,w) is a $\{\delta(q+1), \delta; n, q\}$-minihyper satisfying $0 \leq \delta < \epsilon_q$, then w is the weight function induced on the points of $\mathrm{PG}(n,q)$ by a sum of δ lines. Moreover, this sum is unique.*

Proof It is clear that the theorem holds for $\delta = 0$ or $n < 2$, so let $\delta > 0$ and $n \geq 2$. We proceed in two steps.

1. Assume all points of (F,w) have weight greater than or equal to $k \geq 1$ and there exists a point P with weight k. Consider an $(n-2)$-dimensional subspace π_{n-2} through P containing exactly k points of (F,w), i.e., k times the point P; the existence of such a subspace π_{n-2} is proved in the same way as the existence of $\pi_{n-\mu-1}$ was proved in Lemma 2.2.3. Let $H_0, H_1, \ldots, H_q$ be the $q+1$ hyperplanes through π_{n-2}. Then the following equation holds:

$$\sum_{i=0}^{q} |(F,w)\cap H_i| = |(F,w)| + q|(F,w)\cap \pi_{n-2}| = \delta(q+1) + qk.$$

   Therefore there exists a hyperplane H_i, say H_0, that contains more than δ points of (F,w). By Corollary 2.2.2, the points of $(F,w)\cap H_0$ form a blocking set in H_0. Since each hyperplane H_i contains at least $\delta - k$ points of (F,w) outside π_{n-2}, H_0 contains at most $\delta(q+1) - q(\delta-k)$ points of (F,w). All these points have weight at least k, hence, considered as points of $\mathrm{PG}(n,q)$, there are at most $\frac{\delta(q+1)-q(\delta-k)}{k} \leq q+\delta$

of them. Using Theorem 1.2.15, it can be concluded that the blocking set $(F,w) \cap H_0$ in H_0 contains a line, and since P is the only point of (F,w) in π_{n-2}, this line must pass through P.

2. Let P be a point of (F,w) with minimal weight. In step one, it was shown that there exists a line l through P completely contained in (F,w). Now construct a new minihyper (F',w') in $\mathrm{PG}(n,q)$ in the following way:

    - for P' in $\mathrm{PG}(n,q) \setminus l$: $w'(P') = w(P')$, and
    - for P' on l: $w'(P') = w(P') - 1$.

    By Lemma 2.2.3, (F',w') is a $\{(\delta-1)(q+1), \delta-1; n, q\}$-minihyper.

Repeating steps 1 and 2, the minihyper can be downsized until all points have weight zero. This implies that (F,w) is induced by a sum of δ lines.

Since $\delta \leq (q+1)/2$, there is only one sum of δ lines that induces (F,w). Indeed, if w_1 and w_2 were two such sums, and if $w_1(l) > w_2(l)$ for some line l in $\mathrm{PG}(n,q)$, then through each of the points of l there would pass a line $m \neq l$ for which $w_1(m) < w_2(m)$. Hence δ would be at least $q+1$, a contradiction. □

Remark 2.2.6 For minihypers without weights satisfying $n \geq 3$, $\delta < \sqrt{q}+1$, this result was already known, see Theorem 2.1.6.1.

Theorem 2.2.7 *If (F,w) is a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper satisfying $0 \leq \delta < \epsilon_q$ and $\mu \leq n-1$, then w is the weight function induced on the points of $\mathrm{PG}(n,q)$ by a sum of δ μ-spaces. Moreover, this sum is unique.*

Proof Note that the theorem clearly holds for $\mu = 0$ and that Theorem 2.2.5 states the result for $\mu = 1$. To obtain the general result, induction on μ will be used. So, let (F,w) be a $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper satisfying $1 < \mu \leq n-1$ and suppose the theorem holds for all positive integers μ' smaller than μ.

By Theorem 2.1.13, there exists an $(n-2)$-space G in $\mathrm{PG}(n,q)$ such that $|(F,w) \cap G| = \delta v_{\mu-1}$, and any $(n-2)$-space G in $\mathrm{PG}(n,q)$ for which $|(F,w) \cap G| = \delta v_{\mu-1}$ satisfies the following. Let $H_0, \ldots, H_q$ be the $q+1$ hyperplanes through G. Then $(F,w) \cap H_i$ is a $\{\delta v_\mu, \delta v_{\mu-1}; n, q\}$-minihyper with weights in H_i. Using the induction hypothesis, $(F,w) \cap H_i$ is a sum of δ $(\mu-1)$-spaces.

Let G' be an $(n-2)$-space such that $|(F,w) \cap G'| = \delta v_{\mu-1}$. Let P be a point of (F,w) with minimal weight. If $P \notin G'$, then let G be an $(n-2)$-space containing P and satisfying $|(F,w) \cap G| = \delta v_{\mu-1}$. To see that such a space G exists, suppose that $P \in H \setminus G'$, where H is one of the hyperplanes through G'. From the above, $H \cap (F, w)$ is a sum of δ $(\mu-1)$-spaces. Since there are v_{n-1}, respectively $v_{n-\mu}$, $v_{n-\mu-1}$, $(n-2)$-spaces in H through P, respectively through P and a $(\mu-1)$-space in H containing P, through P and a $(\mu-1)$-space in H not containing P, and since $\mu > 1$, there exists an $(n-2)$-space G through P in H that intersects each one of the δ $(\mu-1)$-spaces in a $(\mu-2)$-space. Thus G contains $\delta v_{\mu-1}$ points of (F, w).

Denote the $q+1$ hyperplanes through G by $H_0, \ldots, H_q$. From the above, $(F, w) \cap H_i$ is a sum of δ $(\mu-1)$-spaces $A_{i1}, A_{i2}, \ldots, A_{i\delta}$, and this sum is unique. Moreover, each one of these $(\mu-1)$-spaces intersects G in a $(\mu-2)$-space, such that $G \cap (F, w)$ is a sum of δ $(\mu-2)$-spaces $B_1, B_2, \ldots, B_\delta$, which is also unique. Number the spaces B_j and A_{ij} in such a way that $B_j \subset A_{ij}$ for all $j \in \{1, \ldots, \delta\}$ and $i \in \{0, 1, \ldots, q\}$.

Case 1. The point P has weight one. There is exactly one $(\mu-2)$-space in $G \cap (F, w)$ through P, e.g. B_1, and exactly one $(\mu-1)$-space in $H_i \cap (F, w)$ through P. By the convention above, this $(\mu-1)$-space is A_{i1}. Now suppose the $(\mu-1)$-spaces $A_{01}, A_{11}, \ldots, A_{q1}$ do not form a μ-space through P, i.e., suppose $\bigcup_{i=0}^{q} A_{i1} \neq \langle A_{01}, A_{11} \rangle$. Let $E_i = H_i \cap \langle A_{01}, A_{11} \rangle$. Then there exists an integer $\alpha \in \{2, \ldots, q\}$ such that $E_\alpha \neq A_{\alpha 1}$. Let $D_\alpha = G \cap \langle E_\alpha, A_{\alpha 1} \rangle$. This is a $(\mu-1)$-space in G containing B_1. As B_j, $j = 2, \ldots, \delta$, intersects B_1 in a subspace in G with dimension at most $\mu-3$, it follows that $\langle B_j, B_1 \rangle$ has dimension at least $\mu-1$.

Since the number of $(n-3)$-spaces in G through B_1, respectively through a $(\mu-1)$-space in G, through a subspace of dimension greater than $\mu-1$, equals $v_{n-\mu}$, respectively equals $v_{n-\mu-1}$, is smaller than $v_{n-\mu-1}$, and since $\delta v_{n-\mu-1} < v_{n-\mu}$, there exists an $(n-3)$-space Δ in G satisfying $B_1 \subset \Delta, B_2 \not\subset \Delta, \ldots, B_\delta \not\subset \Delta$ and $D_\alpha \not\subset \Delta$. This configuration is depicted in Figure 1.

By Theorem 2.1.14, there exists a hyperplane Π through $\langle \Delta, E_\alpha \rangle$ such that $|\Pi \cap (\bigcup_{i=0}^{q} A_{i1})| = v_{\mu-1}$. Now consider a $(\mu-1)$-space A_{ij}, where $j > 1$. It intersects G in the $(\mu-2)$-space B_j. Since $\Pi \cap G = \Delta$, it can be concluded that $\Pi \cap A_{ij}$, $j > 1$, is a $(\mu-2)$-space. Now count the points of (F, w) in Π: there are at most $v_{\mu-1} + (\delta-1)(q+1)v_{\mu-1} < \delta v_\mu$ of them, a contradiction. We conclude that the spaces $A_{01}, \ldots, A_{q1}$ form a μ-space through P.

Case 2. The point P has weight k > 1. In H_i, $i = 0, \ldots, q$, all $(\mu-1)$-spaces have weight zero or at least k. Indeed, suppose this is not the case, suppose there is a $(\mu-1)$-space π in H_0 with positive weight at most $k-1$. Since each point in this $(\mu-1)$-space has weight at least k, we need more than $q > \delta$ other $(\mu-1)$-spaces in H_0 with weight greater than zero to cover the points of π. This is impossible. In H_i, there are exactly k

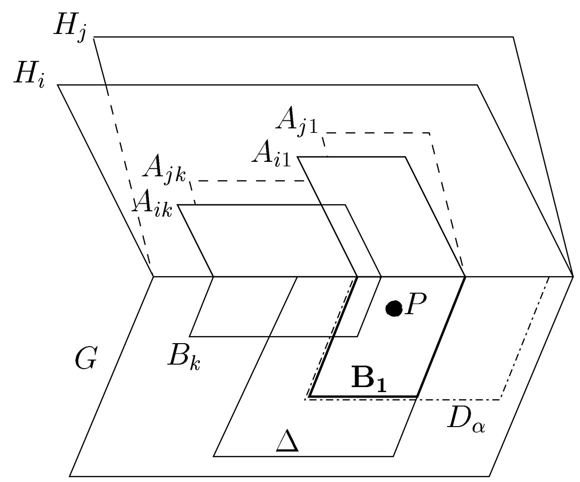

Figure 2.2: Reconstructing the μ-spaces
For given k, the $(\mu-1)$-spaces A_{lk} in H_l, $0 \leq l \leq q$, through B_k form a μ-space.

$(\mu-1)$-spaces through P. Since the weight of a point of (F, w) is at least k, these k $(\mu-1)$-spaces must in fact be one $(\mu-1)$-space A_{i1} counted k times. The other $(\mu-1)$-spaces in H_i intersect G in $(\mu-2)$-spaces not containing P. Repeating the arguments from Case 1, it follows that the k-fold $(\mu-1)$-spaces A_{i1} form a k-fold μ-space through P.

In both cases, there exists a μ-space through P. By Lemma 2.2.3, this μ-space can be deleted, resulting in a $\{(\delta-1)v_{\mu+1}, (\delta-1)v_\mu; n, q\}$-minihyper (F', w').

Repeating this technique for downsizing the minihyper, the desired result is obtained: (F, w) is a sum of δ μ-spaces. Uniqueness of this sum can be proved in the same way as the uniqueness of the sum of lines in Theorem 2.2.5.

□

Chapter 3

More minihypers: Improvements for q square

The results in the previous chapter are weakest in the case that q is a square, since in that case the smallest nontrivial blocking sets have size $q + \sqrt{q} + 1$. However, these blocking sets in $\mathrm{PG}(2, q)$ have a very nice structure, and there are several results on the size of the second smallest nontrivial blocking sets in $\mathrm{PG}(2, q)$, q square, see Section 1.2. These two facts allow to improve Theorem 2.2.7 in the case that q is a square under the condition that all points of the minihyper have weight one. The results collected in this chapter were published in *Journal of Combinatorial Theory. Series A* in *P. Govaerts and L. Storme, On a particular class of minihypers and its applications. II. Improvements for q square* [55].

3.1 Introduction

In this chapter, improvements to Theorem 2.2.7 are obtained in the case that q is a square. However, one extra assumption is made: the minihypers under consideration are minihypers without weights. Remember that such minihypers are denoted by F, the set of points with weight one. So, it is the aim to classify

$$\{\delta v_{\mu+1}, \delta v_{\mu}; n, q\}\text{-minihypers } F$$

in $\mathrm{PG}(n, q)$, q square, for all $\delta \leq a$ for some integer a, where the intention is (of course) to obtain a classification for a as large as possible. If $\delta \leq \sqrt{q}$, then, by Theorem 2.1.6.1, such a minihyper consists of a disjoint union of μ-spaces. For larger δ, other examples exist.

Example 3.1.1 Let q be a square, $1 \leq \delta \leq q-1$, $1 \leq \mu \leq (n-1)/2$ and $k \leq \delta/(\sqrt{q}+1)$. Let $D_1, D_2, \ldots, D_k$ be k mutually disjoint subspaces $\mathrm{PG}(2\mu+1, \sqrt{q})$ in $\mathrm{PG}(n,q)$ and let $W_1, W_2, \ldots, W_l$, $l = \delta - k(\sqrt{q}+1)$, be l mutually disjoint μ-spaces in $\mathrm{PG}(n,q)$ that are skew to $\cup_{i=1}^{k} D_i$. Then $F = (\cup_{i=1}^{k} D_i) \cup (\cup_{i=1}^{l} W_i)$ is a $\{\delta v_{\mu+1}, \delta v_{\mu}; n, q\}$-minihyper, since $|\mathrm{PG}(2\mu+1, \sqrt{q})| = (\sqrt{q}+1)v_{\mu+1}$, and since a hyperplane intersects D_i in a subspace $\mathrm{PG}(s, \sqrt{q})$ for some $s \in \{2\mu-1, 2\mu, 2\mu+1\}$.

It will be shown in Theorem 3.4.1 that, for δ not too large, the converse holds: a $\{\delta v_{\mu+1}, \delta v_{\mu}; n, q\}$-minihyper F in $\mathrm{PG}(n,q)$, q square, consists of the disjoint union of μ-spaces and subspaces $\mathrm{PG}(2\mu+1, \sqrt{q})$.

Although the minihypers that will be studied are minihypers without weights, due to the use of projection arguments, weighted minihypers will appear. For those, the following terminology will be used. A *simple point* of a minihyper (F, w) is a point with weight one, while a *multiple point* of (F, w) is a point with weight at least two.

3.2 The case $\mu = 1$

The case $\mu = 1$—the smallest nontrivial possibility for μ—is handled first. The results obtained in this section will be used to study the situation for larger μ.

To obtain the classification of $\{\delta(q+1), \delta; n, q\}$-minihypers F, a projection argument will be used. For $n > 3$, the minihyper F will be projected from a point onto a hyperplane, yielding a new minihyper (F', w) in $\mathrm{PG}(n-1, q)$. By then, the structure of (F', w) will be known, and (F', w) will be "lifted" to the original minihyper F. The knowledge of the structure of (F', w) will prove to be sufficient to determine the structure of F.

For $n > 4$, the new minihyper (F', w) will be, like F, one without weights, but for $n = 4$, it will not be possible to exclude the existence of multiple points in (F', w), see Lemma 3.2.1. That is why, in the study of the minihyper in three dimensions, (a small number of) multiple points will be allowed.

Because of the strategy chosen, the cases $n = 3$ and $n = 4$ are handled separately in Subsections 3.2.1 and 3.2.2, after which the general case is handled by induction in Subsection 3.2.3.

Lemma 3.2.1 *Consider a $\{\delta(q+1), \delta; n, q\}$-minihyper F without weights, where $q > 16$ and $\delta < q^{5/8}/\sqrt{2}+1$. If $n > 4$, then F can be projected from a point into a hyperplane resulting in a $\{\delta(q+1), \delta; n-1, q\}$-minihyper F'. If $n = 4$, then F can be projected from a point into a solid resulting in a*

$\{\delta(q+1), \delta; 3, q\}$-minihyper (F', w) with less than $q^{1/4}/2 + 1$ multiple points (counted according to their weight).

Proof The number of secants to F, counting a secant containing m points of F precisely $\binom{m}{2}$ times, equals $\delta(q+1)(\delta(q+1)-1)/2$. The sum of these secants, counting a point that lies on m secants m times, contains less than $\delta^2(q+1)^3/2$ points of $\mathrm{PG}(n, q)$. This number is smaller than $(q^{1/4}/4+1/2)v_5$.

In the case $n > 4$, there exists a point R not lying on any secant. Projecting F from R onto a hyperplane not containing R, a $\{\delta(q+1), \delta; n-1, q\}$-minihyper F' is obtained.

In the case $n = 4$, there exists a point R lying on less than $q^{1/4}/4 + 1/2$ secants, where a secant is counted $\binom{m}{2}$ times if it contains m points of F. But since $m \geq 2$ implies that $\binom{m}{2} \geq m/2$, the union of these secants through R contains less than $q^{1/4}/2+1$ points of F. Thus, if F is projected from R onto a solid not containing R, then the resulting structure is a $\{\delta(q+1), \delta; 3, q\}$-minihyper (F', w) with less than $q^{1/4}/2+1$ multiple points (counted according to their weight). $\square$

Remark 3.2.2 If (F, w) is a $\{\delta(q+1), \delta; n, q\}$-minihyper, $\delta \leq q$, then by Corollary 2.1.11, every plane π intersects it in an $\{m_1(q+1)+m_0, m_1; 2, q\}$-minihyper (F', w') for some $(m_0, m_1) \in \bar{\mathrm{E}}(2, q) \cup \{(0, 0)\}$ with $m_0 + m_1 \leq \delta$. Denote by $m_1(\pi)$ the integer m_1 corresponding to the plane π. If $m_1(\pi)$ is zero, then π is said to be *poor*; if π is not poor, then it is called *rich*.

Since these planar intersections of the minihyper will show up in the proofs in this section, they are studied in the following lemma. The parameter c_p from Section 1.2 is used. It equals $2^{-1/3}$ when $p \leq 3$ and 1 otherwise.

Lemma 3.2.3 *If F is an $\{m_1(q+1)+m_0, m_1; 2, q\}$-minihyper (without weights), $q > 16$ square, $q = p^h$, p prime, and $m_0 + m_1 = \alpha < q^{5/8}/\sqrt{2}+1$, then $m_1 < c_p q^{1/6}$ and either F contains a disjoint union of m_1 Baer subplanes or $m_1 = 1$ and F contains a line.*

Proof Note that $q \geq 16$ implies that $q^{5/8}/\sqrt{2}+1 \leq c_p q^{2/3}$. Four cases can be distinguished.

Case 1. Assume $\mathbf{m_1 = 1}$. The conditions of Theorem 1.2.4 are fulfilled. Hence F contains a line or a Baer subplane. Note that $m_1 < c_p q^{1/6}$.

Case 2. Assume $\mathbf{2 \leq m_1 < q^{1/4}/2}$. In this case, Theorem 1.2.11 can be applied. It states that F contains the disjoint union of m_1 Baer subplanes. Hence $|F| \geq m_1(q+\sqrt{q}+1)$. But $|F| = m_1(q+1)+m_0$, implying that $m_0 \geq \sqrt{q}\, m_1$. Therefore $\alpha = m_0 + m_1 \geq (\sqrt{q}+1)m_1$. Since $\alpha < c_p q^{2/3}$, it follows that $m_1 < c_p q^{1/6}$.

Case 3. Assume $\mathbf{m_1 \geq q^{1/4}/2}$, and F contains no line. Theorem 1.2.10 says that $|F| \geq m_1q + \sqrt{m_1q} + 1$. Substituting the second appearance of m_1 in this expression by the lower bound $m_1 \geq q^{1/4}/2$, and using the fact that $|F| = m_1q + m_1 + m_0$, yields $m_0 + m_1 \geq q^{5/8}/\sqrt{2} + 1$, a contradiction. Hence this case cannot occur.

Case 4. Assume $\mathbf{m_1 \geq q^{1/4}/2}$, and F contains a line. Then Theorem 1.2.10 states that $|F| \geq m_1q + q - m_1 + 2$. But $q > 16$, implying $q > 2q^{2/3} \geq 2\alpha \geq 2m_1 + m_0$, such that $|F| > m_1(q+1) + m_0$, a contradiction.

This concludes the proof of the lemma. □

Remark 3.2.4 One easily sees that if, under the conditions of Lemma 3.2.3, F contains the disjoint union of m_1 Baer subplanes, then F cannot contain a line.

3.2.1 The smallest dimension

In this subsection $\{\delta(q+1), \delta; 3, q\}$-minihypers (F, w) with $\delta \leq (q+1)/2$ are studied. Here (F, w) will always denote such a minihyper, but, progressing in the subsection, further assumptions will be made. By Lemma 2.1.9, if π is a plane, then π intersects (F, w) in a $\{m_1(q+1)+m_0, m_1; 2, q\}$-minihyper satisfying $m_0 + m_1 = \delta$. Hence, for such a minihyper, every plane π satisfies

$$|(F, w) \cap \pi| = qm_1(\pi) + \delta. \tag{3.1}$$

Lemma 3.2.5 *If l is a line containing α points of (F, w), and a plane π is counted $m_1(\pi)$ times, then there are exactly α planes through l.*

Proof Let $\pi_i, i = 0, \ldots, q$, be the $q + 1$ planes in $\mathrm{PG}(3, q)$ containing l. Then

$$\sum_{i=0}^{q} |(F, w) \cap \pi_i| = \alpha(q+1) + \delta(q+1) - \alpha.$$

By (3.1),

$$\sum_{i=0}^{q} (q\, m_1(\pi_i) + \delta) = (\alpha + \delta)q + \delta.$$

Thus $\sum_{i=0}^{q} m_1(\pi_i) = \alpha$. □

Lemma 3.2.6 *Through a point P of weight α, if a plane π is counted $m_1(\pi)$ times, there are exactly $\alpha q + \delta$ planes.*

Proof Consider the set $\{(P', \pi) : \pi \text{ a plane}, \pi \ni P, P' \in \mathrm{PG}(3, q), P' \in \pi\}$. Count the elements of this set in the following way. A pair (P', π) is counted $w(P')$ times. Then

$$\alpha(q^2 + q + 1) + (\delta(q + 1) - \alpha)(q + 1) = \delta(q^2 + q + 1) + q \sum_{\pi \ni P} m_1(\pi).$$

Therefore $\sum_{\pi \ni P} m_1(\pi) = \alpha q + \delta$. □

Lemma 3.2.7 *A line l containing a point P not in (F, w), contains at most δ points of (F, w).*

Proof Suppose l contains α points of (F, w). By Lemma 3.2.5, counting a plane $m_1(\pi)$ times, there pass exactly α planes through l. Each one of them contains P. By Lemma 3.2.6, counting a plane $m_1(\pi)$ times, there are exactly δ planes passing through P, implying that $\alpha \leq \delta$. □

If P is a point of $\mathrm{PG}(3, q)$, q square, then a *Baer cone* $\mathcal{B}$ with vertex P is a set of points that is the union of $q + \sqrt{q} + 1$ lines on P that form a Baer subplane in the quotient space on P. These $q + \sqrt{q} + 1$ lines are called the *lines of the cone*, while the *planes of the cone* are the $q + \sqrt{q} + 1$ planes on P that contain $\sqrt{q} + 1$ of these lines.

Lemma 3.2.8 *Suppose q is a square and (F, w) is a $\{\delta(q + 1), \delta; 3, q\}$-minihyper with $\delta < \epsilon'_q$, where $q + \epsilon'_q$ denotes the size of the second smallest nontrivial minimal blocking sets in* $\mathrm{PG}(2, q)$*. Suppose furthermore that (F, w) contains no line. If P is a point of (F, w) with minimal weight, then the set of rich planes through P contains the set of planes of a Baer cone $\mathcal{B}$ with vertex P.*

Proof Note that, by the condition imposed on δ, $\delta < (q + 1)/2$. Hence the previous lemmas can be applied.

Case 1. The point P has weight one. By Lemma 3.2.6, there exist at most $q + \delta$ planes π through P that satisfy $m_1(\pi) \geq 1$.

By Lemma 3.2.5, every line l through P lies in at least one of these planes. Thus, in the quotient geometry of P—describe it by using a plane π not through P—the lines corresponding to the planes through P sharing at least a 1-fold blocking set with (F, w), i.e., the planes with $m_1 \geq 1$, form a dual blocking set B. Since $|B| \leq q + \delta$, B contains a dual line or a dual Baer subplane. Suppose B contains a dual line, i.e., B contains all lines through some point R of π. Then there are $q + 1$ planes with $m_1 \geq 1$

through the line PR, a contradiction since, by Lemma 3.2.7, this implies that all points of PR are points of (F, w). So B contains a Baer subplane, and the set of rich planes through P contains the set planes of a Baer cone $\mathcal{B}$ with vertex P.

Case 2. The point P has weight $\alpha > 1$. A plane intersects (F, w) in a $\{m_1(q+1) + m_0, m_1; 2, q\}$-minihyper, but since the minimal weight of a point in the minihyper is α, if a plane π satisfies $m_1(\pi) > 0$, then $m_1(\pi) \geq \alpha$. By Lemma 3.2.6, $\sum_{\pi \ni P} m_1(\pi) = \alpha q + \delta$, such that there exist at most $q + \delta$ such planes.

The rest of the arguments can be copied from Case 1.

This concludes the proof. □

Lemma 3.2.9 *Suppose (F, w) is a minihyper satisfying the conditions from Lemma* 3.2.8. *Assume furthermore that (F, w) has a simple point P and that $\delta < q^{5/8}/\sqrt{2} + 1$. Let $\mathcal{B}$ be the Baer cone with vertex P from Lemma* 3.2.8. *Then every plane E_i of $\mathcal{B}$ with $m_1(E_i) = 1$ contains a unique Baer subplane $B(E_i)$ consisting of points of (F, w), and this Baer subplane is contained in $\mathcal{B}$.*

Proof The proof of this lemma is based on the proof of Lemma 2.2 of [87]. Denote by $E_0, E_1, \ldots, E_{q+\sqrt{q}}$ the planes of $\mathcal{B}$ and by a_i, $i = 0, 1, \ldots, q + \sqrt{q}$, the number of points of (F, w) in E_i that are not on $\mathcal{B}$. Keeping in mind that a point outside $\mathcal{B}$ lies on exactly one plane of $\mathcal{B}$, while a point of $\mathcal{B} \setminus \{P\}$ lies on exactly $\sqrt{q} + 1$ planes of $\mathcal{B}$, the points of (F, w) can be counted, resulting in

$$\delta(q+1) = 1 + \sum_{i=0}^{q+\sqrt{q}} a_i + \frac{1}{\sqrt{q}+1} \sum_{i=0}^{q+\sqrt{q}} (q m_1(E_i) + \delta - 1 - a_i),$$

or

$$\sum_{i=0}^{q+\sqrt{q}} a_i = \delta q + \sqrt{q} - \sqrt{q} \sum_{i=0}^{q+\sqrt{q}} m_1(E_i). \tag{3.2}$$

Since through P, there are exactly $q + \delta$ (not necessarily distinct) rich planes, there are at most $\delta - \sqrt{q} - 1$ different planes π of $\mathcal{B}$ with $m_1(\pi) > 1$.

Let E_{i_0} be a plane of $\mathcal{B}$ with $m_1(E_{i_0}) = 1$, and denote the lines of $\mathcal{B}$ in E_{i_0} by $l_0, l_1, \ldots, l_{\sqrt{q}}$. Since $m_1(E_{i_0}) = 1$, by Lemma 3.2.3, the intersection $E_{i_0} \cap (F, w)$ contains a unique Baer subplane $B(E_{i_0}) = \pi_B$. Therefore $|l_j \cap \pi_B| \in \{1, \sqrt{q} + 1\}$, for $i \in \{0, 1, \ldots, \sqrt{q}\}$.

If $a_{i_0} < q - \sqrt{q}$, then $|\pi_B \cap \mathcal{B}| > 2\sqrt{q} + 1$. Thus, in this case at least two lines l_j contain more than one point of π_B. So, they are lines of π_B, such that $P \in \pi_B$, implying that there are at least three lines l_j that contain more than one point of π_B. It follows that $\pi_B \subset \mathcal{B}$.

Since (3.2) implies that $\sum_{i=0}^{q+\sqrt{q}} a_i < \delta(q-\sqrt{q})$, it holds that $B(E_i) \not\subset \mathcal{B}$ for at most $A := (\sum_{i=0}^{q+\sqrt{q}} a_i)/(q-\sqrt{q}) < \delta$ planes E_i of $\mathcal{B}$ with $m_1(E_i) = 1$.

Now suppose that E_{i_1} is a plane of $\mathcal{B}$ with $m_1(E_{i_1}) = 1$ and that $B(E_{i_1})$ is not contained in $\mathcal{B}$. Then $a_{i_1} \geq q - \sqrt{q}$ such that E_{i_1} contains at most $\delta + \sqrt{q}$ points of (F,w) in $\mathcal{B}$. Denote the lines of $\mathcal{B}$ in E_{i_1} by $l'_0, l'_1, \ldots, l'_{\sqrt{q}}$. Then at most $1 + \delta/\sqrt{q}$ of these lines can contain at least $\sqrt{q}+1$ points of (F,w). Thus at least $\sqrt{q} - \delta/\sqrt{q}$ of these lines contain less than $1+\sqrt{q}$ points of (F,w). For such a line l'_j, no plane E_i through l'_j can contain a Baer subplane of points of (F,w) contained in $\mathcal{B}$. Counting the number of planes of $\mathcal{B}$ that do not contain a Baer subplane contained in $\mathcal{B}$ consisting entirely of points of (F,w) yields

$$A + \delta - \sqrt{q} - 1 \geq 1 + (\sqrt{q} - \frac{\delta}{\sqrt{q}})\sqrt{q},$$

implying that $3\delta \geq q + \sqrt{q} + 2$, a contradiction. $\square$

Lemma 3.2.10 *Suppose (F,w) is a minihyper satisfying the conditions from Lemma 3.2.9. Through every simple point P of (F,w), there exists a Baer subgeometry $D := \mathrm{PG}(3,\sqrt{q})$ consisting entirely of points of (F,w). Furthermore, this Baer subgeometry is unique.*

Proof Let $\mathcal{B}$ denote the Baer cone with vertex P from Lemma 3.2.9. Let $E_0, E_1, \ldots, E_{r-1}$ be the planes of $\mathcal{B}$ satisfying $m_1(E_i) = 1$. Note that $r \geq q - \delta + 2\sqrt{q} + 2$.

Let $\pi \in \{E_0, E_1, \ldots, E_{r-1}\}$ and denote the lines of $\mathcal{B}$ in π by $l_0, l_1, \ldots, l_{\sqrt{q}}$. Suppose α of these lines contain more than one Baer subline consisting of points of (F,w). Then $|\pi \cap (F,w)| = q+\delta \geq q+\sqrt{q}+1+\alpha(\sqrt{q}-1)$, such that $\alpha < q^{1/8}/\sqrt{2}$. Call the lines of $\mathcal{B}$ in π containing exactly one Baer subline consisting of points of (F,w) *good* lines.

Let π and π' be two distinct elements of $\{E_0, E_1, \ldots, E_{r-1}\}$ intersecting in a good line. Denote by B, respectively B', the Baer subplane of π, respectively π', consisting of points of (F,w). Define D as the subspace $\mathrm{PG}(3,\sqrt{q})$ spanned by B and B'. The good lines of π and π' define more than $(\sqrt{q} - q^{1/8}/\sqrt{2})^2$ planes of $\mathcal{B}$ intersecting π as well as π' in a good line. Thus there are at least $q - 3q^{5/8}/\sqrt{2} + \sqrt{q} + q^{1/4}/2$ planes E_i of $\mathcal{B}$ with $m_1(E_i) = 1$ that intersect π as well as π' in a good line. Since the Baer subplanes of (F,w) in these planes have two Baer sublines in common with D, they are contained in D.

Let π^* be one of these planes. Then there exists some line of π^* on P that is contained in at least $(q - 3q^{5/8}/\sqrt{2} + \sqrt{q} + q^{1/4}/2 - 1)/(\sqrt{q} + 1) > \sqrt{q} - 3q^{1/8}/\sqrt{2}$ planes of $\mathcal{B}$ whose Baer subplane is contained in D. Therefore, more than $q - 3q^{5/8}/\sqrt{2} + \sqrt{q} + 1$ lines of $\mathcal{B}$ have a Baer subline consisting of points of (F, w) that is contained in D. Denote these lines by $m_0, m_1, \ldots, m_\alpha$.

Suppose that there exists a point P' of D that does not belong to (F, w). Then it lies on $\delta < q^{5/8}/\sqrt{2} + 1$ rich planes. The q planes of D through P' but not through P, intersect each of the lines $m_0, m_1, \ldots, m_\alpha$ in a point of (F, w). Therefore they contain more than $q - 3q^{5/8}/\sqrt{2} + \sqrt{q} + 1$ points of (F, w). Since this number is greater than δ, these planes are rich. So, there are more than δ rich planes through P', implying that $P' \in (F, w)$, a contradiction.

Hence all points of D belong to (F, w). It remains to be shown that this subgeometry D is unique. Suppose that this is not the case, i.e., suppose that D_1 and D_2 are two distinct subspaces $\mathrm{PG}(3, \sqrt{q})$ on the simple point P consisting entirely of points of (F, w). Then D_1 and D_2 share at most a Baer subplane and a point. Two distinct Baer planes, one of D_1 and one of D_2, intersect in at most a Baer line and a point, and thus contain at least $2q + 2\sqrt{q} + 2 - \sqrt{q} - 2$, which is greater than $q + \delta$, points of (F, w).

Now take a look at all the planes of D_1 and D_2 on P. A plane π of D_1 that is not a plane of D_2 (and vice versa) contains more than δ points of (F, w) and thus satisfies $m_1(\pi) \geq 1$. A plane π of D_1 that is also a plane of D_2 has more than $q + \delta$ points of (F, w) and thus $m_1(\pi) \geq 2$, unless for maybe one plane whose Baer subplane belongs to D_1 as well as to D_2. This means that there are at least $2q + 2\sqrt{q} + 1$ rich planes (counted according to their weight) on P, a number greater than $q + \delta$, a contradiction. □

Theorem 3.2.11 *If (F, w) is a $\{\delta(q+1), \delta; 3, q\}$-minihyper, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2} + 1$, containing less than $q^{1/4}/2 + 1$ multiple points (counted according to their weight), then (F, w) is a sum of lines and Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$, and this sum is unique.*

Proof By Lemma 2.2.3, if (F, w) contains a line, then it can be deleted in order to obtain a new minihyper with parameters $\{(\delta - 1)(q + 1), \delta - 1; 3, q\}$. This process can be repeated until a minihyper not containing any lines is obtained.

So, suppose that (F, w) does not contain a line. Let P be a simple point of (F, w). By Lemma 3.2.10, there exists a unique Baer subspace $D = \mathrm{PG}(3, \sqrt{q})$ consisting entirely of points of (F, w) on P.

Construct in the following way a new minihyper (F', w') defined by the weight function w',

- $w'(P) = w(P) - 1$, for $P \in D$, and

- $w'(P) = w(P)$, for $P \in \mathrm{PG}(3,q) \setminus D$.

Then (F', w') is a $\{(\delta - \sqrt{q} - 1)(q+1), \delta - \sqrt{q} - 1; 3, q\}$-minihyper.

For suppose that this is not the case. Then, by Theorem 2.1.8, there exists a plane π that contains less than $\delta - \sqrt{q} - 1$ points of (F', w'). Let $|\pi \cap (F, w)| = m_1(q+1) + m_0$, with $m_0 + m_1 = \delta$.

Case 1. $\pi \cap \mathbf{D} = \mathrm{PG}(\mathbf{2}, \sqrt{\mathbf{q}})$.

Deleting D, π contains $(m_1 - 1)q + m_0 + m_1 - \sqrt{q} - 1$ points of (F', w'). If m_1 were zero, then π could not have contained $\mathrm{PG}(2, \sqrt{q})$. Therefore, this number is at least $\delta - \sqrt{q} - 1$. Thus, the next case has to occur.

Case 2. $\pi \cap \mathbf{D} = \mathrm{PG}(\mathbf{1}, \sqrt{\mathbf{q}})$.

Deleting D, π contains $m_1(q+1) + m_0 - \sqrt{q} - 1 \geq \delta - \sqrt{q} - 1$ points of (F', w'), a contradiction.

So, a subspace $\mathrm{PG}(3, \sqrt{q})$ contained in (F, w) can be deleted, resulting in a similar minihyper. Deleting subspaces $\mathrm{PG}(3, \sqrt{q})$ until there are no more simple points, what remains is a $\{\delta^*(q+1), \delta^*; 3, q\}$-minihyper, $\delta^* \geq 0$, with nothing but multiple points. But the number of multiple points is smaller than $q^{1/4}/2 + 1$, implying $\delta^* = 0$.

It follows that (F, w) is a sum of lines and subspaces $\mathrm{PG}(3, \sqrt{q})$. Since $\delta \leq q^{5/8}/\sqrt{2} + 1$, there are at most $q^{5/8}/\sqrt{2} + 1$ lines and $q^{1/8}/\sqrt{2}$ subspaces $\mathrm{PG}(3, \sqrt{q})$ contained in this sum. A line not contained in this sum cannot be covered by the lines and subspaces $\mathrm{PG}(3, \sqrt{q})$ of this sum; also, a subspace $\mathrm{PG}(3, \sqrt{q})$ not contained in this sum cannot be covered by the lines and subspaces $\mathrm{PG}(3, \sqrt{q})$ of this sum. Hence (F, w) can be written in a unique way as a sum of lines and subspaces $\mathrm{PG}(3, \sqrt{q})$. □

3.2.2 One dimension up

We continue with the study of $\{\delta(q+1), \delta; 4, q\}$-minihypers F, minihypers without weights, where $q > 16$ is a square and $\delta < q^{5/8}/\sqrt{2} + 1$.

Lemma 3.2.1 states that F can be projected from a point $R \notin F$ into a solid not containing R, resulting in a $\{\delta(q+1), \delta; 3, q\}$-minihyper (F', w) with less than $q^{1/4}/2 + 1$ multiple points. Such a minihyper (F', w) is—as shown in the preceding subsection—a sum of lines and subspaces $\mathrm{PG}(3, \sqrt{q})$. In this subsection the minihyper (F', w) shall be "lifted" from $\mathrm{PG}(3, q)$ to the minihyper F in $\mathrm{PG}(4, q)$.

Lemma 3.2.12 *There is a one-to-one correspondence between lines of F and lines of (F', w).*

Proof Clearly, if l is a line of F in $\mathrm{PG}(4, q)$, then l is projected onto a line of (F', w) in $\mathrm{PG}(3, q)$.

Now suppose l is a line of (F', w) in $\mathrm{PG}(3, q)$. Since there are less than $q^{1/4}/2 + 1$ multiple points in (F', w), in the minihyper F in $\mathrm{PG}(4, q)$, the plane Rl contains less than $q + 1 + q^{1/4}/2 + 1 < q + \sqrt{q} + 1$ points of F. By Corollary 2.1.12, this plane contains a line consisting of points of F. Clearly, this line is projected onto l. □

So every line of (F', w) is the projection of a line of F. Removing one by one all lines of F, by Lemma 2.2.3, a $\{\delta^*(q + 1), \delta^*; 4, q\}$-minihyper F^* is obtained. Denote this new minihyper again by F and the corresponding minihyper in $\mathrm{PG}(3, q)$ by (F', w). Both then do not contain any line.

Lemma 3.2.13 *Every point of F is contained in at least two Baer subplanes that are completely contained in F.*

Proof Let P be a point of F. Since there are v_4 lines through P but only $\delta(q + 1) - 1$ points of F different from P, there exists a line l through P containing no other point of F. On l, there are v_3 planes. So, through l and therefore through P, there exists a plane π containing no point of F other than P. If all solids on π would contain exactly δ points of F, then $|F|$ would equal $1 + (q + 1)(\delta - 1)$, a contradiction. Thus, on π, there exists a solid π_3 containing more than δ points of F. By Lemma 2.1.9, it contains at least $q + 1$ points, and, by Corollary 2.2.2, the points of $\pi_3 \cap F$ block all planes in π_3. But π_3 contains at most $q + \delta$ points of F (since all other solids on π must also contain at least δ points) and no line of F. Therefore it must contain a Baer subplane π_B consisting of points of F, see Theorem 1.2.17. Since $F \cap \pi = \{P\}$, the point P is contained in π_B.

Denote by π' the plane of $\mathrm{PG}(4, q)$ that contains π_B. Let m be a line containing P but no other point of F, and $m \not\subset \pi'$. On m, there are q^2+q+1 planes, q^2 of which intersect π' only in the point P. Since there are less than $\delta(q+1)$ points of F outside π', at least one of them, say π^*, contains no point of F other than P. As in the previous argument, there exists a solid on π^* that contains a Baer subplane π'_B consisting of points of F. Also as above, P is contained in π'_B. Clearly, π'_B is distinct from π_B. □

Lemma 3.2.14 *Through every point of F that is projected onto a simple point of (F', w), there exists a subspace $\mathrm{PG}(3, \sqrt{q})$ consisting entirely of points of F.*

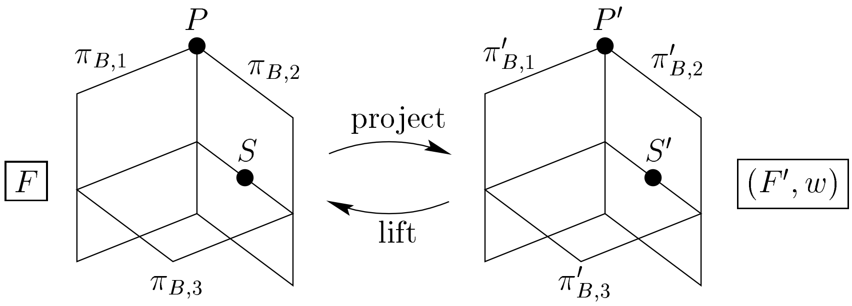

Figure 3.1: Projecting and "lifting" the minihyper
The Baer 3-spaces from the projected minihyper (F', w) can be lifted to reconstruct the original minihyper F.

Proof Let P' be a simple point of (F', w) and let D' be the unique subspace $\mathrm{PG}(3, \sqrt{q})$ consisting of points of (F', w) that contains P'. Let P be the point of F that is projected onto P'. Let $\pi_{B,1}$ and $\pi_{B,2}$ be two Baer subplanes through P that consist of points of F, see Lemma 3.2.13. Their projections, $\pi'_{B,1}$ and $\pi'_{B,2}$, are Baer subplanes of (F', w). Since a Baer subplane consisting of points of (F', w) is contained in exactly one subspace $\mathrm{PG}(3, \sqrt{q})$ consisting of points of (F', w), both $\pi'_{B,1}$ and $\pi'_{B,2}$ must be contained in D', which implies that $\pi'_{B,1}$ and $\pi'_{B,2}$ intersect in a Baer subline. As there are less than $q^{1/4}/2+1$ multiple points, also $\pi_{B,1}$ and $\pi_{B,2}$ intersect in a Baer subline. Denote the subspace $\mathrm{PG}(3, \sqrt{q})$ that is spanned by $\pi_{B,1}$ and $\pi_{B,2}$ by D.

Let S be a point of $\pi_{B,2} \setminus \pi_{B,1}$ that is projected onto a simple point S', and let $\pi_{B,3}$ be a second Baer subplane consisting of points of F through S, see Lemma 3.2.13. As above, $\pi_{B,3}$ intersects $\pi_{B,1}$ and $\pi_{B,2}$ in a Baer subline, implying that $\pi_{B,3}$ is contained in D. This situation is depicted in Figure 3.1.

Now let V' be any simple point of $D' \setminus (\pi'_{B,1} \cup \pi'_{B,2} \cup \pi'_{B,3})$. Let V be the point of F that is projected onto V. There exists a Baer subplane π_B consisting of points of F through V. Let π'_B be the projection of π_B. Then π'_B is a plane of D' and intersects either

(i) $\pi'_{B,1}$ and $\pi'_{B,2}$ in two different Baer sublines, in which case π_B intersects $\pi_{B,1}$ and $\pi_{B,2}$ in two Baer sublines and thus $\pi_B \subset D$; or

(ii) $\pi'_{B,1}$ and $\pi'_{B,2}$ in $\pi'_{B,1} \cap \pi'_{B,2}$, in which case it has to intersect $\pi'_{B,3}$ in some other Baer subline, implying that π_B intersects π_1 and π_3 in two different Baer sublines and $\pi_B \subset D$.

In both cases, V is contained in D. Therefore, D contains more than $q\sqrt{q}+$

$q + \sqrt{q} - q^{1/4}/2$ points of F.

Suppose D is not completely contained in F, i.e., suppose there exists a point $W \in D \setminus F$. There exist $q + \sqrt{q} + 1$ planes through W that intersect D in a Baer subplane. In each such plane, there are $q - \sqrt{q}$ lines that intersect D only in W, and two such planes intersect in a line that contains a Baer subline of D. Thus there exist $(q + \sqrt{q} + 1)(q - \sqrt{q}) = q^2 - \sqrt{q}$ lines in planes of D that intersect D only in W. But since this number is greater than $\delta(q + 1)$, at least one of these lines, say l, is skew to F. Let π be the plane through l that intersects D in a Baer subplane. Denote this Baer subplane by π_B. On l, there exists a plane π' skew to F. All hyperplanes through π' contain exactly δ points of F. But one of these solids has to contain π, which contains π_B. Since every Baer subplane of D contains more than $q + \sqrt{q} - q^{1/4}/2$ points of F, a contradiction is obtained. □

Theorem 3.2.15 *If F is a $\{\delta(q + 1), \delta; 4, q\}$-minihyper, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2} + 1$, then F is a unique disjoint union of lines and subspaces $\mathrm{PG}(3, \sqrt{q})$.*

Proof For the moment, assume that the minihyper F does not contain any line. Let P be a point of F that is projected onto a multiple point of (F', w) (if it exists). There exists a Baer subplane $\pi_{B,1}$ consisting of points of F and containing P. Since (F', w) contains less than $q^{1/4}/4 + 1/2$ distinct multiple points, there exists a point P_2 of $\pi_{B,1}$ that is projected on a simple point of (F', w). There exists a second Baer subplane $\pi_{B,2}$ consisting of points of F on P_2, and $\mathrm{PG}(3, \sqrt{q}) := \langle \pi_{B,1}, \pi_{B,2} \rangle$ is contained in F. Therefore, through every point of F, there exists a Baer subspace $\mathrm{PG}(3, \sqrt{q})$ consisting of points of F.

Now suppose that there exists a point P of F that is contained in two distinct subspaces $\mathrm{PG}(3, \sqrt{q})$ of F. Clearly, a subspace $\mathrm{PG}(3, \sqrt{q})$ of F is projected onto a subspace $\mathrm{PG}(3, \sqrt{q})$ of (F', w). Hence P', the projection of P, is a multiple point. Therefore, there has to exist another point P_2 of F that is projected onto P. Through this new point there exists a subspace $\mathrm{PG}(3, \sqrt{q})$ of F, which is projected onto a third subspace $\mathrm{PG}(3, \sqrt{q})$ through P'. (Note that P_2 cannot be contained in one of the first two Baer subspaces, since otherwise PP_2 would contain at least $\sqrt{q} + 1$ points of a subspace $\mathrm{PG}(3, \sqrt{q})$ of F and (F', w) would have a point with weight at least $\sqrt{q} + 1$.) It follows that P' has weight at least three, implying that there exists another point P_3 of F mapped onto P'. Continuing along this line of thought, a contradiction is obtained the moment that the weight of P exceeds $q^{1/4}/2 + 1$. Hence P is contained in a unique subspace $\mathrm{PG}(3, \sqrt{q})$ consisting of points of F.

Recall that in the proof of Theorem 3.2.11, it was demonstrated how a subspace $\mathrm{PG}(3,\sqrt{q})$ can be deleted from a minihyper. Resuming the study of the original minihyper F, it is now clear that F can be written as a disjoint union of lines and subspaces $\mathrm{PG}(3,\sqrt{q})$. An argument similar to the one in the previous subsection shows that this union is unique. □

3.2.3 Arbitrary dimensions

Induction on the dimension is used to settle the case $\mu = 1$.

Theorem 3.2.16 *If F is a $\{\delta(q+1),\delta;n,q\}$-minihyper, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2}+1$, $n \geq 3$, then F is a unique disjoint union of lines and subspaces $\mathrm{PG}(3,\sqrt{q})$.*

Proof The theorem holds for $n = 3$ by Theorem 3.2.11 and for $n = 4$ by Theorem 3.2.15. So suppose $n \geq 5$, and suppose it holds for all $n' < n$.

By Lemma 3.2.1, F can be projected from a point $R \notin F$ onto a minihyper F' in $\mathrm{PG}(n-1,q)$, $R \notin \mathrm{PG}(n-1,q)$. By induction, F' is a unique disjoint union of lines and subspaces $\mathrm{PG}(3,\sqrt{q})$.

If l is a line of F', then Rl is a plane in $\mathrm{PG}(n,q)$ containing exactly $q+1$ points of F. By Corollary 2.1.12, it contains a line, which can be deleted with the standard procedure, Lemma 2.2.3. Conversely, if l is a line of F, then it is projected onto a line of F'. From now on, assume that F nor F' contains a line.

So F' is a unique disjoint union of subspaces $\mathrm{PG}(3,\sqrt{q})$. Let P be a point of F. Through P, there exists an $(n-2)$-space containing no other points of F and there exists a hyperplane H containing this $(n-2)$-space and satisfying $\delta < |F \cap H| \leq q+\delta$. By Corollary 2.2.2 and Theorem 1.2.17, this hyperplane has to contain a Baer subplane π_B consisting entirely of points of F and containing P. Let π be the plane containing π_B. Since the number of lines through P not contained in π equals $v_n - v_2 > \delta(q+1)$, there exists a line π_1 on P not in π containing no other points of F. A similar argument shows that there exists a plane π_2 on P containing no other points of F and intersecting π in P. Continuing along this line of thought, the existence of an $(n-2)$-space π_{n-2} containing P but no other points of F and intersecting π in P is demonstrated. As above, through π_{n-2} there exists a hyperplane H^* containing a Baer subplane containing P and consisting entirely of points of F. Since H^* intersects π in a line, this Baer subplane is different from π_B.

Taking into account that F' has no multiple points, and using the techniques from the proof of Lemma 3.2.14, it is seen that through every point of F there exists a subspace $\mathrm{PG}(3,\sqrt{q})$ consisting entirely of points of F.

Now the study of the original minihyper that may contain lines can be resumed. As in the proof of Theorem 3.2.15, F is a unique disjoint union of lines and subspaces $\mathrm{PG}(3, \sqrt{q})$. $\square$

3.3 The case $\mu = 2$

The second case is handled. In this section, F denotes a $\{\delta(q^2+q+1), \delta(q+1); n, q\}$-minihyper, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2}+1$, and $n \geq 5$.

Lemma 3.3.1 *If P is a point of F that is contained in two lines of F, then the plane spanned by these two lines consists entirely of points of F.*

Proof Denote by π the plane generated by these two lines. Then $|\pi \cap F| = m_1(q+1) + m_0$ for some pair $(m_1, m_0) \in \bar{\mathrm{E}}(2, q)$ or $\pi \subset F$.

Suppose the first possibility occurs. Then $m_1 + m_0 \leq \delta$, and $\pi \cap F$ is a $\{m_1(q+1) + m_0, m_1; 2, q\}$-minihyper. By Lemma 3.2.3 and Remark 3.2.4, $\pi \cap F$ contains a union of m_1 Baer subplanes, but no lines, a contradiction.

Hence, the second possibility must occur, and π is contained in F. $\square$

By the standard argument, Lemma 2.2.3, a plane π contained in F can be deleted from F, resulting in a new $\{(\delta-1)(q^2+q+1), (\delta-1)(q+1); n, q\}$-minihyper. So, from now on, assume that no point of F is contained in two lines of F. The following lemma shows that this implies that F contains no lines.

Remark 3.3.2 In $\mathrm{PG}(3, q)$, a plane intersects a subspace $\mathrm{PG}(3, \sqrt{q})$ in at least a Baer subline. Therefore, if Δ is an $(n-2)$-space containing exactly δ points of F, if H is a hyperplane containing Δ, and if D is a subspace $\mathrm{PG}(3, \sqrt{q})$ of $F \cap H$, then $D \cap \Delta$ is a Baer subline in Δ.

Lemma 3.3.3 *If F contains no planes, then it contains no lines.*

Proof Suppose F contains a line l but no plane, and consider a point P on l. Through P, there exists an $(n-2)$-space Δ that intersects F in exactly δ points. Let $H_0, H_1, \ldots, H_q$ be the $q+1$ hyperplanes containing Δ. By Theorem 2.1.13, they intersect F in $\{\delta(q+1), \delta; n-1, q\}$-minihypers. As seen in the previous section, such a minihyper is a unique disjoint union of lines and subspaces $\mathrm{PG}(3, \sqrt{q})$.

Suppose, without loss of generality, that l is contained in H_0. Then, since F contains no plane, in H_i, $i = 1, 2, \ldots, q$, there exists a subspace $\mathrm{PG}(3, \sqrt{q})$ of F that contains P. Denote this subspace $\mathrm{PG}(3, \sqrt{q})$ by E_i.

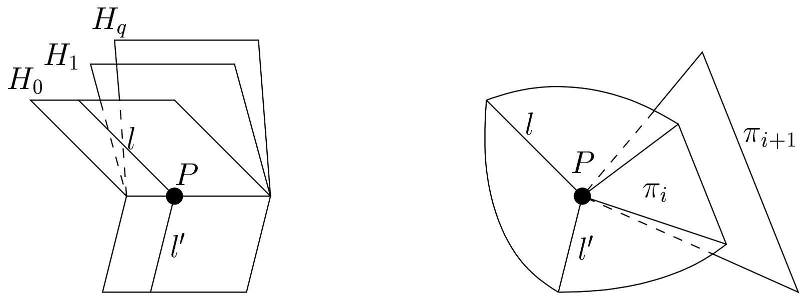

Figure 3.2: The minihyper without planes: two steps from the proof of Lemma 3.3.3

Every point Q of $F \cap \Delta$ lies on a unique Baer subline consisting of points of $F \cap \Delta$. Indeed, every point of $F \cap \Delta$ lies in at most one line completely contained in F and $|F \cap \Delta| = \delta < q^{5/8}/\sqrt{2}+1$. Therefore there exists a hyperplane H through Δ intersecting F in a disjoint union of Baer subgeometries $\mathrm{PG}(3, \sqrt{q})$. Hence, $F \cap \Delta$ is the disjoint union of $\delta/(\sqrt{q}+1) < q^{1/8}/\sqrt{2}+1$ Baer sublines, see Remark 3.3.2. Note that this implies that $\delta \geq \sqrt{q}+1$. Since two distinct Baer sublines intersect in at most two points, it is impossible that Q lies on a second Baer subline contained in $F \cap \Delta$.

Denote the Baer subline of F in Δ on P by l'_B and the line (over $\mathrm{GF}(q^2)$) containing it by l'. The present configuration is depicted in Figure 3.2. There exists an $(n-2)$-space Δ' through P containing exactly δ points of F satisfying $l' \not\subset \langle \Delta', l \rangle$. This can be seen as follows.

Since there are v_n lines through P, $\delta(q^2+q+1)-1$ points in $F \setminus \{P\}$ and $q+1$ lines through P in $\langle l', l \rangle$, there exists a line π_1 on P satisfying $\pi_1 \cap F = \{P\}$ and $\pi_1 \not\subset \langle l', l \rangle$. Hence, $l' \not\subset \langle \pi_1, l \rangle$. Similarly, there exists a plane π_2 containing π_1 satisfying $\pi_2 \cap F = \{P\}$ and $\pi_2 \not\subset \langle l', \pi_1, l \rangle$. Hence $l' \not\subset \langle \pi_2, l \rangle$. Continuing this argument inductively, see Figure 3.2, one sees that there exists an $(n-3)$-space π_{n-3} through π_{n-4} skew to $F \setminus \{P\}$ such that $\pi_{n-3} \not\subset \langle l', \pi_{n-4}, l \rangle$. Hence $l' \not\subset \langle \pi_{n-3}, l \rangle$.

There are q^2+q+1 $(n-2)$-spaces containing π_{n-3}.

- Since each one of them contains at least δ points of F, the number of points of F that needs to be distributed among these $(n-2)$-spaces, after they have all received their minimum number of δ points, is q^2+q.
- By Corollary 2.1.12, an $(n-2)$-space that contains more than δ points of F contains at least $q+1$ points of F.

Taking into account that $\delta \geq \sqrt{q}+1$, it follows that there are less than $q+2\delta$ $(n-2)$-spaces through π_{n-3} with more than δ points of F. There are $q+1$

$(n-2)$-spaces through π_{n-3} in $\langle l', \pi_{n-3}, l\rangle$. Since $q+2\delta+q+1 < q^2+q+1$, there exists an $(n-2)$-space Δ' through P containing exactly δ points of F and satisfying $\Delta' \not\subset \langle l', \pi_{n-3}, l\rangle$. This subspace Δ' satisfies $l' \not\subset \langle \Delta', l\rangle$.

All hyperplanes through Δ' intersect F in a $\{\delta(q+1), \delta; n-1, q\}$-minihyper. Let H be the hyperplane through Δ' containing l. Then H intersects H_i in an $(n-2)$-space Δ_i containing P, $i = 1, 2, \ldots, q$. But $\Delta_i \cap E_i$ is at least a Baer subline. Let $l_{B,i}$ be a Baer subline through P in $\Delta_i \cap E_i$. Now suppose that $l_{B,i} = l_{B,j}$ for some $i \neq j$. Then this Baer line lies in $H_i \cap H_j = \Delta$. Hence it equals l'_B, a contradiction, since $l' \not\subseteq \langle \Delta', l\rangle = H$. Therefore $l_{B,i} \neq l_{B,j}$ for $i \neq j$, and the following configuration exists.

In H, which intersects F in a unique disjoint union of lines and subspaces $\mathrm{PG}(3, \sqrt{q})$, there exists a point P of F contained in a line l of F and q Baer sublines $l_{B,i}$, $i = 1, 2, \ldots, q$, of F.

- *Baer 3-spaces.* The hyperplane H contains at most $\delta/(\sqrt{q}+1) < q^{1/8}/\sqrt{2}+1$ subgeometries $\mathrm{PG}(3, \sqrt{q})$ of F. Such a subspace $\mathrm{PG}(3, \sqrt{q})$ contains at most $q+1$ points of F on the q Baer sublines; this occurs when it contains two points on one of these Baer sublines and one point on all the others. For, if it would intersect two of these q Baer sublines in two points, then it would also contain their intersection point, which is the point P; a contradiction, since P is already contained in the line l. The subspaces $\mathrm{PG}(3, \sqrt{q})$ therefore contain less than $(q^{1/8}/\sqrt{2}+1)(q+1)$ points on $l_{B,1}, l_{B,2}, \ldots, l_{B,q}$. In total there are at least $q\sqrt{q}$ points of F, different from P, on these lines. Hence, there are more than $qq^{1/4}$ points left that should be covered by lines of $H \cap F$.

- *Lines.* The hyperplane H contains at most δ lines of F. Hence, there exists a line m of F containing at least $qq^{1/4}/\delta > q^{5/8}$ points of F on the Baer sublines $l_{B,1}, l_{B,2}, \ldots, l_{B,q}$. This line cannot contain two points of the same Baer subline $l_{B,i}$. Thus, the following configuration is obtained in a plane $\pi = \langle m, P\rangle$, see Figure 3.3.

  1. A point $P \in F$;
  2. a line m not containing P and consisting entirely of points of F;
  3. more than $q^{5/8}(\sqrt{q}-1)+1+q+1$ points of F in π.

  This implies that π intersects F in more than $q+\delta$ points. By Lemma 3.2.3, π intersects F in at least a 2-fold blocking set, and therefore contains a disjoint union of Baer subplanes. This is impossible by Remark 3.2.4.

So, assuming that F contains no plane is the same as assuming that F contains no line. $\square$

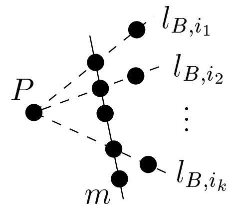

Figure 3.3: The minihyper without planes: an impossible configuration
Note that $k > q^{5/8}$.

Lemma 3.3.4 *Suppose F' is a $\{\delta(q+1), \delta; n-1, q\}$-minihyper, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2}+1$ containing no lines. If l_B is a Baer subline contained in F', then l_B is contained in a unique subspace $\mathrm{PG}(3, \sqrt{q})$ of F'.*

Proof The number of Baer 3-subspaces in F' not containing l_B is not big enough to cover l_B. □

Lemma 3.3.5 *If F contains no planes and P is a point of F, then P is contained in a unique subspace $\mathrm{PG}(5, \sqrt{q})$ of F.*

Proof Let Δ be an $(n-2)$-space on P containing exactly δ points of F and denote the $q+1$ hyperplanes on Δ by $H_0, H_1, \ldots, H_q$. By Lemma 3.3.3, they intersect F in $\{\delta(q+1), \delta; n-1, q\}$-minihypers that are disjoint unions of subspaces $\mathrm{PG}(3, \sqrt{q})$. Looking in a hyperplane H_i, it can be seen that P lies on a unique Baer subline of $F \cap \Delta$, and that $F \cap \Delta$ is a uniquely determined union of disjoint Baer sublines. Let l_B be the Baer subline of $F \cap \Delta$ containing P, and denote by $\mathrm{PG}(3, \sqrt{q})_1$, respectively $\mathrm{PG}(3, \sqrt{q})_2$, the subspace $\mathrm{PG}(3, \sqrt{q})$ of $H_1 \cap F$, respectively $H_2 \cap F$, containing P.

Let l' be a line in H_1 containing P and intersecting $\mathrm{PG}(3, \sqrt{q})_1$ in a Baer subline l'_B different from l_B. Note that $|F \cap l'| \leq \delta$. Through l', there exists an $(n-2)$-space Δ' containing exactly δ points of F. Each hyperplane through Δ' intersects F in a $\{\delta(q+1), \delta; n-1, q\}$-minihyper, i.e., in a disjoint union of subspaces $\mathrm{PG}(3, \sqrt{q})$, and none of these hyperplanes can contain $\mathrm{PG}(3, \sqrt{q})_2$. Otherwise, P would lie on two distinct Baer sublines (one on l' and one on $\mathrm{PG}(3, \sqrt{q})_2 \cap \Delta'$), which is false. So each one of these hyperplanes intersects $\mathrm{PG}(3, \sqrt{q})_2$ in a Baer subline or a Baer subplane. Therefore in each hyperplane through Δ', there exists a subspace $\mathrm{PG}(3, \sqrt{q})$ of F containing the Baer subline l'_B and this Baer subline or Baer subplane of $\mathrm{PG}(3, \sqrt{q})_2$. Therefore $\mathrm{PG}(4, \sqrt{q}) := \langle l'_B, \mathrm{PG}(3, \sqrt{q})_2 \rangle$ is contained in F.

Now, letting the line l' vary, one sees that $\langle \mathrm{PG}(3, \sqrt{q})_1, \mathrm{PG}(3, \sqrt{q})_2 \rangle =: \mathrm{PG}(5, \sqrt{q})$ is contained in F. Hence, P is contained in a subspace $\mathrm{PG}(5, \sqrt{q})$

of F. It remains to be shown that this subspace $\mathrm{PG}(5, \sqrt{q})$ is unique.

Suppose P is contained in two subspaces $\mathrm{PG}(5, \sqrt{q})$: $\mathrm{PG}(5, \sqrt{q})_1$ and $\mathrm{PG}(5, \sqrt{q})_2$. Let Δ be an $(n-2)$-space containing P and satisfying $|\Delta \cap F| = \delta$. Then Δ intersects $\mathrm{PG}(5, \sqrt{q})_i$ in a Baer subline $l_{B,i}$, $i = 1, 2$. Since $l_{B,1} \cap l_{B,2} \neq \emptyset$, it follows that $l_{B,1} = l_{B,2}$. Now let $H_0, H_1, \ldots, H_q$ be the hyperplanes on Δ. They all contain exactly $\delta(q+1)$ points of F. Denote by $\mathrm{PG}(3, \sqrt{q})_{i,j}$ the intersection of $\mathrm{PG}(5, \sqrt{q})_i$ and H_j, $i = 1, 2$ and $j = 0, 1, \ldots, q$. Then $\mathrm{PG}(3, \sqrt{q})_{1,j} \cap \mathrm{PG}(3, \sqrt{q})_{2,j} \supset l_{B,1}$ $(= l_{B,2})$, and H_j intersects F in a unique disjoint union of subspaces $\mathrm{PG}(3, \sqrt{q})$, $j = 0, 1, \ldots, q$. Hence, $\mathrm{PG}(3, \sqrt{q})_{1,j} = \mathrm{PG}(3, \sqrt{q})_{2,j}$ for $j = 0, 1, \ldots, q$, such that $\mathrm{PG}(5, \sqrt{q})_1 = \mathrm{PG}(5, \sqrt{q})_2$.

Therefore, this subspace $\mathrm{PG}(5, \sqrt{q})$ is unique. □

Now return to the general case where the minihyper F is allowed to contain planes.

Theorem 3.3.6 *If F is a $\{\delta(q^2 + q + 1), \delta(q + 1); n, q\}$-minihyper, $q > 16$, $\delta < q^{5/8}/\sqrt{2} + 1$, then F is a unique disjoint union of planes and subspaces $\mathrm{PG}(5, \sqrt{q})$.*

Proof If F contains planes, these can be deleted, see Lemma 2.2.3, and the remaining points form a disjoint union of subspaces $\mathrm{PG}(5, \sqrt{q})$. Hence, F is a disjoint union $\mathcal{U}$ of planes and subspaces $\mathrm{PG}(5, \sqrt{q})$. It remains to be shown that this union is unique.

Suppose that in F a subspace $\mathrm{PG}(5, \sqrt{q}) =: D$ exists that does not belong to $\mathcal{U}$. Clearly, a plane can contain at most $q + \sqrt{q} + 1$ points of D. Therefore, by removing planes, at most $\delta(q + \sqrt{q} + 1)$ points of D can be removed. What is left of F is a disjoint union of less than $q^{1/8}/\sqrt{2} + 1$ subspaces $\mathrm{PG}(5, \sqrt{q})$. Such a subspace $\mathrm{PG}(5, \sqrt{q})$ can contain at most $q^2 + q\sqrt{q} + q + \sqrt{q} + 2$ points of D. Therefore not all points of D can be points of F, a contradiction.

Similarly, there cannot exist a plane contained in F that does not belong to $\mathcal{U}$. □

3.4 The general case

In this section, $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihypers are studied, where $q > 16$ is a square and $\delta < q^{5/8}/\sqrt{2} + 1$.

Theorem 3.4.1 *A $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper F, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2} + 1$, $\mu \geq 1$, $2\mu + 1 \leq n$, is a unique disjoint union of μ-spaces and subgeometries $\mathrm{PG}(2\mu + 1, \sqrt{q})$.*

Proof In the previous sections, the theorem was proved for $\mu \in \{1, 2\}$. So, assume that $\mu \geq 3$ and that it holds for all $\mu' < \mu$. Let Δ be an $(n-2)$-space containing exactly $\delta v_{\mu-1}$ points of F. Then, by Corollary 2.1.11 or 2.1.12, $\Delta \cap F$ is a $\{\delta v_{\mu-1}, \delta v_{\mu-2}; n-2, q\}$-minihyper. By induction, it is a disjoint union of $(\mu-2)$-spaces and subspaces $\mathrm{PG}(2\mu-3, \sqrt{q})$. Denote by $H_0, H_1, \ldots, H_q$ the $q+1$ hyperplanes containing Δ. Then $F \cap H_i$ is a $\{\delta v_\mu, \delta v_{\mu-1}; n-1, q\}$-minihyper for $i = 0, 1, \ldots, q$, i.e., a disjoint union of $(\mu-1)$-spaces and subspaces $\mathrm{PG}(2\mu-1, \sqrt{q})$.

Since these unions are unique, if A is a $(\mu-2)$-space, respectively a subspace $\mathrm{PG}(2\mu-3, \sqrt{q})$, of F in Δ, then each hyperplane H_i contains a $(\mu-1)$-space, respectively a subspace $\mathrm{PG}(2\mu-1, \sqrt{q})$, of F that contains A.

Suppose A is a $(\mu-2)$-space in $\Delta \cap F$. Let B_1, respectively B_2, be the $(\mu-1)$-space in $H_1 \cap F$, respectively $H_2 \cap F$, that contains A. If l_1 and l_2 are two lines of B_1 and B_2 intersecting in a point of Δ, then, by the same argument as in Lemma 3.3.1, the plane $\langle l_1, l_2 \rangle$ is contained in F. It follows that the μ-space $\langle B_1, B_2 \rangle$ is contained in F. It can be removed with the standard argument of Lemma 2.2.3.

So, assume that $\Delta \cap F$ is a union of subspaces $\mathrm{PG}(2\mu-3, \sqrt{q})$. Let G be a subspace $\mathrm{PG}(2\mu-3, \sqrt{q})$ in $\Delta \cap F$ and let E_1, respectively E_2, be the subspace $\mathrm{PG}(2\mu-1, \sqrt{q})$ of $H_1 \cap F$, respectively $H_2 \cap F$, containing it. Now let l be a line in H_1, $l \not\subset \Delta$, intersecting E_1 in a Baer subline l_B containing a point P of G. Note that $|l \cap F| \leq \delta$. Through l, there exists an $(n-\mu-1)$-space $\pi_{n-\mu-1}$ such that $\pi_{n-\mu-1} \cap F = l \cap F$. If an $(n-\mu)$-space contains more than δ points of F, then it contains at least $q+1$ points of F. Therefore there exists an $(n-\mu)$-space $\pi_{n-\mu}$ through $\pi_{n-\mu-1}$ intersecting F in exactly δ points. All $(n-2)$-spaces through $\pi_{n-\mu}$ intersect F in $\delta v_{\mu-1}$ points. Let Δ' be such an $(n-2)$-space. All hyperplanes through Δ', denote them by $H'_0, H'_1, \ldots, H'_q$, intersect F in a $\{\delta v_\mu, \delta v_{\mu-1}; n, q\}$-minihyper, a disjoint union of subspaces $\mathrm{PG}(2\mu-1, \sqrt{q})$.

Hence in H'_i, there exists a subspace $\mathrm{PG}(2\mu-1, \sqrt{q}) =: E'_i$ of F, intersecting l in the Baer subline l_B. Now H'_i intersects H_2 in an $(n-2)$-space, which in its turn intersects E'_i in a subspace $\mathrm{PG}(2\mu-2, \sqrt{q})$ or $\mathrm{PG}(2\mu-3, \sqrt{q})$. Since this subspace $\mathrm{PG}(2\mu-2, \sqrt{q})$ or $\mathrm{PG}(2\mu-3, \sqrt{q})$ contains P, it must be contained in E_2. In the first case, $\langle \mathrm{PG}(2\mu-2, \sqrt{q}), l_B \rangle$ is contained in F; in the second case, $\langle \mathrm{PG}(2\mu-3, \sqrt{q}), l_B \rangle$ is contained in F. But repeating this argument for H'_i, $i = 0, 1, \ldots, q$, proves that $\langle l_B, E_2 \rangle =: \mathrm{PG}(2\mu, \sqrt{q})$ is contained in F. Now the line l can be varied, so $\langle E_1, E_2 \rangle =: \mathrm{PG}(2\mu+1, \sqrt{q})$ is contained in F.

Therefore every point of F is contained in a subspace $\mathrm{PG}(2\mu+1, \sqrt{q})$ of F.

Now suppose that some point P of F is contained in two subspaces $\mathrm{PG}(2\mu+1,\sqrt{q})$, say D_1 and D_2, of F. Let π_{n-2} be an $(n-2)$-space containing P and intersecting F in $\delta v_{\mu-1}$ points. The $q+1$ hyperplanes H_i, $i=0,1,\ldots,q$, on π_{n-2} intersect F in $\{\delta v_\mu, \delta v_{\mu-1}; n-1, q\}$-minihypers, which are unique disjoint unions of $(\mu-1)$-spaces and subspaces $\mathrm{PG}(2\mu-1,\sqrt{q})$. Now D_1 and D_2 intersect H_i in subspaces $\mathrm{PG}(2\mu-1,\sqrt{q})$, denote them by $E_{i,1}$ and $E_{i,2}$, but since they both contain P, they must be equal. This holds for $i=0,1,\ldots,q$. Hence, the subspaces $\mathrm{PG}(2\mu+1,\sqrt{q})$ are disjoint.

It can be concluded that F is a disjoint union of μ-spaces and subspaces $\mathrm{PG}(2\mu+1,\sqrt{q})$. By counting arguments, similar to those used in the previous sections, this union is unique. $\square$

Now the main result of this chapter easily follows.

Theorem 3.4.2 *A $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper F, $q > 16$ square, $\delta < q^{5/8}/\sqrt{2}+1$, $\mu \geq 1$, is a unique disjoint union of μ-spaces and subgeometries $\mathrm{PG}(2\mu+1,\sqrt{q})$.*

Proof The condition $2\mu+1 \leq n$ from Theorem 3.4.1 is not necessary. Indeed, suppose that F is a minihyper with the correct parameters in $\mathrm{PG}(n,q)$, $n < 2\mu+1$. Embed $\mathrm{PG}(n,q)$ in $\mathrm{PG}(n',q)$ for some $n' \geq 2\mu+1$. In $\mathrm{PG}(n',q)$, F is still a minihyper, and it has the same parameters as in $\mathrm{PG}(n,q)$. Therefore F is as described in Theorem 3.4.1. $\square$

Chapter 4

Partial spreads and covers

In this chapter, (non)existence results on partial t-spreads and t-covers in finite projective and polar spaces are presented. Some of these are applications of the classification results on minihypers from Chapters 2 and 3. The results from Section 4.2 and Subsection 4.3.1 were published in [56] and [55], the papers containing the results on minihypers from Chapters 2 and 3. The results from Sections 4.5 and 4.6 and those from Subsection 4.7.1 were published in *European Journal of Combinatorics* in *P. Govaerts, L. Storme, and H. Van Maldeghem, On a particular class of minihypers and its applications. III. Applications* [57].

There will be two introductory sections in this chapter. Section 4.1 will consider the terminology and some known results on partial t-spreads and t-covers in finite projective spaces; Section 4.4 will do the same for partial t-spreads and t-covers in finite classical polar spaces.

4.1 Introduction for projective spaces

A *partial t-spread* of $\mathrm{PG}(n,q)$ is a set of mutually disjoint t-spaces in $\mathrm{PG}(n,q)$. A *t-cover* of $\mathrm{PG}(n,q)$ is a set of t-spaces of $\mathrm{PG}(n,q)$ that covers the point set of $\mathrm{PG}(n,q)$. A *t-spread* of $\mathrm{PG}(n,q)$ is a set of t-spaces in $\mathrm{PG}(n,q)$ that is a partial t-spread as well as a t-cover of $\mathrm{PG}(n,q)$. In other words, a t-spread of $\mathrm{PG}(n,q)$ is a set of t-spaces in $\mathrm{PG}(n,q)$ that partitions the point set of $\mathrm{PG}(n,q)$. If n is odd and $t=(n+1)/2$, then a (partial)t-spread, respectively t-cover, of $\mathrm{PG}(n,q)$ is simply called a *(partial) spread*, respectively *cover*, of $\mathrm{PG}(n,q)$.

When studying these structures, the following questions turn up in a natural way.

Question 4.1.1 When do t-spreads in $\mathrm{PG}(n,q)$ exist?

Question 4.1.2 If t-spreads do not exist, what is the size of the sets "closest" to it? That is, what is the size of the largest partial t-spreads and the size of the smallest t-covers in $\mathrm{PG}(n, q)$?

Question 4.1.1 is answered in Subsection 4.1.1.

For t-covers, Question 4.1.2 has been answered satisfactorily: the size of the smallest t-covers in $\mathrm{PG}(n, q)$ is known, examples are known, and the structure of the set of points that are covered more than once is also known. These results are discussed in Subsection 4.1.3. For partial t-spreads less is known: for some cases the size of the largest partial t-spreads of $\mathrm{PG}(n, q)$ is known, but for others there is a substantial gap between the best known upper bounds and the largest known examples, see Subsection 4.1.3.

Given a nonempty partial t-spread $\mathcal{S}'$ of $\mathrm{PG}(n, q)$, new partial t-spreads can be constructed by removing elements from $\mathcal{S}'$. A partial t-spread that cannot be constructed in this way from a larger partial t-spread is called *maximal*. So, a maximal partial t-spread is a partial t-spread that cannot be extended to a larger one. Similarly, a *minimal* t-cover $\mathcal{C}$ is a t-cover that cannot be constructed by adding a t-space to a smaller t-cover, i.e., removing any element of $\mathcal{C}$ yields a set that is no longer a t-cover.

Once the questions on the size of the largest partial t-spreads and smallest t-covers are solved, the following questions can be asked.

Question 4.1.3 What are the possible sizes of maximal partial t-spreads and minimal t-covers in $\mathrm{PG}(n, q)$?

Although already several results towards the solution of this question are known, in this chapter, only the situation "close to" t-spreads will be considered, that is, only large partial t-spreads and small t-covers will be studied.

4.1.1 t-Spreads

Since a t-spread of $\mathrm{PG}(n, q)$ partitions the point set of $\mathrm{PG}(n, q)$, it can only exist if $|\mathrm{PG}(t, q)|$ divides $|\mathrm{PG}(n, q)|$. This condition is equivalent to the condition that $t+1$ divides $n+1$, see e.g. Lemma 4.5.1. But also the converse is well-known to be true, see e.g. [39, p. 29]: if $t+1$ divides $n+1$, then there exist t-spreads in $\mathrm{PG}(n, q)$. Below, a proof is included since it is constructive and not long.

Theorem 4.1.4 $\mathrm{PG}(n, q)$ *has a t-spread if and only if* $(t+1)$ *divides* $(n+1)$.

Proof (from [68, p. 93]) It suffices to show that if $t+1$ divides $n+1$, then $\mathrm{PG}(n,q)$ has a t-spread. So, assume $n+1=k(t+1)$. If F is an irreducible polynomial of degree $t+1$ over $\mathrm{GF}(q)$ and α is a root of F in $\mathrm{GF}(q^{t+1})$, then every element χ of $\mathrm{GF}(q^{t+1})$ can be written $\chi = x_0 + x_1\alpha + \ldots + x_t\alpha^t$, where $x_i \in \mathrm{GF}(q)$, for all $i \in \{0,\ldots,t\}$. Hence, k elements $\chi_0, \chi_1, \ldots, \chi_{k-1}$ of $\mathrm{GF}(q^{t+1})$ can be written $\chi_i = x_{i0} + x_{i1}\alpha + \ldots + x_{it}\alpha^t$, where $x_{ij} \in \mathrm{GF}(q)$, $i \in \{0,\ldots,k-1\}$, $j \in \{0,\ldots,t\}$. The $n+1$ elements x_{ij} will be interpreted as coordinates of a point in $\mathrm{PG}(n,q)$. Thus each point of $\mathrm{PG}(n,q)$ is given by a k-tuple $(\chi_0,\ldots,\chi_{k-1})$ of elements of $\mathrm{GF}(q^{t+1})$.

Let $\tau_0,\ldots,\tau_{k-1}$ be any elements, not all zero, of $\mathrm{GF}(q^{t+1})$. Then the equations $\chi_0/\tau_0 = \chi_1/\tau_1 = \ldots = \chi_{k-1}/\tau_{k-1}$ define a t-space π_t in $\mathrm{PG}(n,q)$. Each k-tuple $\tau = (\tau_0,\ldots,\tau_{k-1})$ corresponds to a point $P(\tau)$ in $\mathrm{PG}(k-1,q^{t+1})$. As $P(\tau)$ varies in $\mathrm{PG}(k-1,q^{t+1})$, so the corresponding t-space π_t in $\mathrm{PG}(n,q)$ varies through a partition of $\mathrm{PG}(n,q)$.

Indeed, every point of $\mathrm{PG}(n,q)$ lies in one of the t-spaces π_t in $\mathrm{PG}(n,q)$ thus defined. As the number of points of $\mathrm{PG}(n,q)$ matches the number of points in $\mathrm{PG}(k-1,q^{t+1})$ times the number of points in $\mathrm{PG}(t,q)$, the t-spaces π_t in $\mathrm{PG}(n,q)$ are disjoint and form a t-spread in $\mathrm{PG}(n,q)$. $\square$

Let $\mathcal{S}$ be a t-spread in $\mathrm{PG}(n,q)$ and let U be a subspace of $\mathrm{PG}(n,q)$. Then $\mathcal{S}$ is said to *induce a spread* in U if $U \cap V \in \{\emptyset, V\}$ for every element V of $\mathcal{S}$. A t-spread $\mathcal{S}$ is called *geometric* if $\mathcal{S}$ induces a spread in $\langle V, V'\rangle$ for any two elements V, V' of $\mathcal{S}$. Segre [100] shows that a geometric t-spread of $\mathrm{PG}(k(t+1)-1,q)$ gives rise to a projective space $\mathcal{J}(\mathcal{S})$ of dimension $k-1$ and order q^{t+1} in the following way: the points of $\mathcal{J}(\mathcal{S})$ are the elements of $\mathcal{S}$, the blocks of $\mathcal{J}(\mathcal{S})$ are the subspaces $\langle V, V'\rangle$ for any two distinct elements V, V' of $\mathcal{S}$ and incidence is inherited from $\mathrm{PG}(k(t+1)-1,q)$.

A *t-regulus* in $\mathrm{PG}(2t+1,q)$ is a set $\mathcal{R}$ of $q+1$ mutually skew t-spaces with the property that every line intersecting three elements of $\mathcal{R}$ intersects all elements of $\mathcal{R}$. If $t=1$, then it is simply called a *regulus*. A line intersecting all elements of $\mathcal{R}$ is called a *transversal* of $\mathcal{R}$ and the set of transversals of $\mathcal{R}$ is denoted by $\mathcal{R}^T$. If $t=1$, then also $\mathcal{R}^T$ is a regulus which is called the *opposite regulus* of $\mathcal{R}$ and which is denoted by $\mathcal{R}^{\mathrm{opp}}$. It is known, see e.g. [39, p. 221], that for any three mutually disjoint t-spaces V_1, V_2 and V_3 in $\mathrm{PG}(2t+1,q)$, there exists a unique t-regulus $\mathcal{R}(V_1,V_2,V_3)$ containing V_1, V_2 and V_3. A t-spread $\mathcal{S}$ in $\mathrm{PG}(2t+1,q)$ is called *regular* if for every triple (V_1,V_2,V_3) of elements of $\mathcal{S}$, the whole regulus $\mathcal{R}(V_1,V_2,V_3)$ is contained in $\mathcal{S}$.

Concerning the existence of regular and geometric t-spreads, the following is known, see e.g. [100].

Theorem 4.1.5 1. $\mathrm{PG}(n,q)$ *contains a geometric* t*-spread if and only if* $t+1$ *divides* $n+1$.

2. *If* $\mathcal{S}$ *is a geometric* t*-spread in* $\mathrm{PG}(n,q)$ *with* $n > 2t+1$, *then* $\mathcal{S}$ *induces a regular* t*-spread in* $\langle V, V' \rangle$ *for any two distinct elements* V, V' *of* $\mathcal{S}$.

3. *Every finite projective space of dimension* $2t+1$ *contains a regular* t*-spread.*

Remark 4.1.6 Part 3 of this theorem is an immediate corollary of parts 1 and 2.

4.1.2 Partial t-spreads

Remember the notation v_{k+1} for $|\mathrm{PG}(k,q)|$.

Clearly, a partial t-spread in $\mathrm{PG}(n,q)$ contains at most $\lfloor v_{n+1}/v_{t+1} \rfloor$ t-spaces. The *deficiency* δ of a partial t-spread $\mathcal{S}'$ is the number of elements $\mathcal{S}'$ has less than this upper bound, so $\delta = \lfloor v_{n+1}/v_{t+1} \rfloor - |\mathcal{S}'|$. In the particular case that $t+1$ divides $n+1$, it is the number of elements that $\mathcal{S}'$ has less than a t-spread of $\mathrm{PG}(n,q)$. The *holes* of a partial t-spread $\mathcal{S}'$ are those points of $\mathrm{PG}(n,q)$ that are not covered by $\mathcal{S}'$, i.e., those points that lie in no element of $\mathcal{S}'$. In the particular case that $t+1$ divides $n+1$, the number of holes equals δv_{t+1}, where δ is the deficiency of the partial t-spread.

Theorem 4.1.7 *Let* $n+1 = k(t+1)+r$, $1 \leq r \leq t$. *Suppose* $\mathcal{S}'$ *is a partial* t*-spread of* $\mathrm{PG}(n,q)$ *of size* $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} - s$.

1. **(Beutelspacher [11])** *If* $r = 1$, *then* $s \geq q-1$.

2. **(Drake and Freeman [41])** *If* $r > 1$, *then* $s \geq \lfloor \theta \rfloor + 1$, *where* $2\theta = \sqrt{1+4q^{t+1}(q^{t+1}-q^r)} - (2q^{t+1} - 2q^r + 1)$.

To get a clearer view on the upper bound of Drake and Freeman, the value of θ can be approximated.

Corollary 4.1.8 *Let* $n+1 = k(t+1)+r$, $1 \leq r \leq t$. *Suppose* $\mathcal{S}'$ *is a partial* t*-spread of* $\mathrm{PG}(n,q)$ *of size* $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} - s$.

1. *If* $r = 1$, *then* $s \geq q^r - 1$.

2. *If* $r > 1$ *and* $t+1 \geq 2r$, *then* $s \geq \frac{q^r}{2} - 1$.

3. *If* $r > 1$ *and* $t+1 < 2r$, *then* $s \geq \frac{q^r}{2} - \frac{q^{2r-t-1}}{2} + 1$.

In [11], Beutelspacher gives a construction for partial t-spreads in $\mathrm{PG}(n,q)$, $n+1 = k(t+1)+r$, of size $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} - q^r + 1$. Let

$$U_{t+r} \subseteq U_{2(t+1)+r-1} \subseteq \ldots \subseteq U_{k(t+1)+r-1} = \mathrm{PG}(n,q)$$

be a chain of subspaces in $\mathrm{PG}(n,q)$, $\dim(U_{i(t+1)+r-1}) = i(t+1)+r-1$, $i = 1, 2, \ldots, k$. Take a partition $\mathcal{S}_j$ by t-spaces of $U_{(j+1)(t+1)+r-1} \setminus U_{j(t+1)+r-1}$ for each $j \in \{1, \ldots, k-1\}$. Let π_t be a t-space in U_{t+r}. Then

$$\mathcal{S}' = \cup_{1 \leq j \leq k-1} \mathcal{S}_j \cup \{\pi_t\}$$

is a maximal partial t-spread of size $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} - q^r + 1$ in $\mathrm{PG}(n,q)$. This is the largest known example.

Hence, for $r = 1$, Beutelspacher's bound is sharp, and for $r > 1$, Drake and Freeman's bound is approximately halfway in between the trivial upper bound and the largest known example.

Partial spreads in $\mathrm{PG}(2t+1,q)$

When $\mathrm{PG}(n,q)$ has a t-spread, then there is a gap between the size of a t-spread and the size of the largest maximal partial t-spreads different from a t-spread. Various results concerning the size of this gap have been obtained, especially in the case where $\mathrm{PG}(n,q)$ has spreads, i.e., in the case where $n = 2t+1$.

Theorem 4.1.9 (Mesner [83]) *If $\mathcal{S}'$ is a maximal partial spread with deficiency $\delta > 0$ in $\mathrm{PG}(3,q)$, then $\delta \geq \sqrt{q}+1$. In the case of equality, the set of holes forms a subspace $\mathrm{PG}(3,\sqrt{q})$.*

In the same article [83], an example of such a maximal partial spread is presented: a maximal partial spread of size 14 in $\mathrm{PG}(3,4)$; its lines cover the points of $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$. The maximal partial spreads with deficiency 3 in $\mathrm{PG}(3,4)$ were classified by van Dam [122]. Blokhuis and Metsch [19] prove that for $q > 4$ no maximal partial spreads with deficiency $\sqrt{q}+1$ exist. An outline of their proof is given in the proof of Theorem 4.2.3, since the same technique works for the case considered in Theorem 4.2.3.2.

Several improvements to Theorem 4.1.9 are now known. The following one works for any non-square prime power q and is based on an extendibility result for nets [84].

Theorem 4.1.10 (Metsch [84]) *If $\mathcal{S}'$ is a maximal partial spread of $\mathrm{PG}(3,q)$, q not a square, with deficiency $\delta > 0$, then δ satisfies $8\delta^3 - 18\delta^2 + 8\delta + 4 \geq 3q^2$.*

Also for q a square, improvements to Theorem 4.1.9 were obtained. Remember that, for q square, $q+\epsilon'_q$ denotes the size of the second smallest nontrivial blocking sets in $\mathrm{PG}(2,q)$.

Theorem 4.1.11 (Metsch and Storme [87]) *Suppose $\mathcal{S}'$ is a maximal partial spread of $\mathrm{PG}(3,q)$, $q>4$ square, with deficiency $\delta>0$. If $\delta<\epsilon'_q$, then $\delta=k(\sqrt{q}+1)$ for some integer $k\geq 2$, and the set of holes of $\mathcal{S}'$ is the disjoint union of k Baer subgeometries $\mathrm{PG}(3,\sqrt{q})$.*

Substituting the bounds from Theorem 1.2.4, the following corollary is obtained.

Corollary 4.1.12 *Suppose $\mathcal{S}'$ is a maximal partial spread of $\mathrm{PG}(3,q)$, $q>4$ square, with deficiency $\delta>0$. If either*

1. *$q>16$ and $\delta<c_pq^{2/3}+1$, or*
2. *$q=p^2$, p prime, and $\delta\leq(q+1)/2$,*

then $\delta=k(\sqrt{q}+1)$ for some integer $k\geq 2$, and the set of holes of $\mathcal{S}'$ is the disjoint union of k Baer subgeometries $\mathrm{PG}(3,\sqrt{q})$.

Also for q a cube, Theorem 4.1.9 was improved upon.

Theorem 4.1.13 (Metsch and Storme [87]) *Suppose $\mathcal{S}'$ is a maximal partial spread of $\mathrm{PG}(3,q)$, with deficiency $\delta>0$.*

1. *If $q=q_0^3$, $q_0=p^{h_0}$, $h_0\geq 1$ odd, $p\geq 7$ prime, and $\delta\leq q_0^2+q_0+1$, then $\delta=q_0^2+q_0+1$, and the set of holes forms a projected subgeometry $\mathrm{PG}(5,q_0)$ in $\mathrm{PG}(3,q_0^3)$.*
2. *If $q=q_0^3$, $q_0=p^{h_0}$, $h_0\geq 2$ even, $p\geq 7$ prime, and $\delta\leq q_0^2+q_0+1$, then either $\delta=k(\sqrt{q}+1)$ for some integer $k\geq 2$, and the set of holes of $\mathcal{S}'$ is the disjoint union of k Baer subgeometries $\mathrm{PG}(3,\sqrt{q})$, or $\delta=q_0^2+q_0+1$ and the set of holes forms a projected subgeometry $\mathrm{PG}(5,q_0)$ in $\mathrm{PG}(3,q_0^3)$.*

In fact, Theorem 4.1.10 also holds for partial t-spreads in $\mathrm{PG}(2t+1,q)$, but in the cases where $t\geq 2$, Metsch and Storme improved upon it. See also Remark 4.2.5.

Theorem 4.1.14 (Metsch and Storme [87]) *Suppose $\mathcal{S}'$ is a maximal partial t-spread of $\mathrm{PG}(2t+1,q)$, $t\geq 2$, with deficiency δ satisfying $3q^{t+1}>8\delta^3-16\delta^2+8\delta+4$.*

1. *If $q = p^h$, p prime, $h \geq 3$ prime, and $\delta \leq q-2$, then $\delta \equiv 0 \pmod{v_h}$, and the set of holes is the disjoint union of projected subgeometries* $\mathrm{PG}(h(t+1)-1, p)$.

2. *If $q = p^{rs}$, p prime, r the largest nontrivial divisor of rs, $s > 1$, and $\delta \leq (q-1)/(q^{r'}-1)-1$, where r' is the largest divisor of rs different from r, then $\delta \equiv 0 \pmod{v_{rs}/v_r}$, and the set of holes is the disjoint union of projected subgeometries* $\mathrm{PG}(s(t+1)-1, p^r)$.

4.1.3 t-Covers

The *excess* ε of a t-cover $\mathcal{C}$ in $\mathrm{PG}(n,q)$ is defined as follows: $\varepsilon = |\mathcal{C}| - \lceil v_{n+1}/v_{t+1} \rceil$, where v_{k+1}, as defined in Chapter 2, equals $|\mathrm{PG}(k,q)|$. It is the number of elements $\mathcal{C}$ has more than the lower bound $\lceil v_{n+1}/v_{t+1} \rceil$ on $|\mathcal{C}|$. In the particular case that $t+1$ divides $n+1$, it is the number of elements that $\mathcal{C}$ has more than a t-spread of $\mathrm{PG}(n,q)$. The *multiple points* of a t-cover are the points of $\mathrm{PG}(n,q)$ that are covered more than once, i.e., that lie in more than one element of the t-cover. The *surplus* of a point of $\mathrm{PG}(n,q)$ is the number of elements of $\mathcal{C}$ that contain it minus one. Sometimes, surplus is considered as a weight function mapping a point P of $\mathrm{PG}(n,q)$ onto a nonnegative integer surplus(P). In the particular case that $t+1$ divides $n+1$, the number of multiple points—counted according their surplus—equals εv_{t+1}, where ε is the excess of the t-cover. Considering surplus as a function, it can easily be modified to act on any set A of points in $\mathrm{PG}(n,q)$: $\mathrm{surplus}(A) = \sum_{P \in A} \mathrm{surplus}(P)$. In particular, if $t+1$ divides $n+1$, then $\mathrm{surplus}(\mathrm{PG}(n,q)) = \varepsilon v_{t+1}$.

Theorem 4.1.15 (Beutelspacher [12]) *Let $n+1 = k(t+1)+r$, $1 \leq r \leq t$. If $\mathcal{C}$ is a t-cover of* $\mathrm{PG}(n,q)$, *then* $|\mathcal{C}| \geq q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} + 1$.

In the same article, Beutelspacher gives examples of t-covers reaching this lower bound for every q, n and t. Let

$$U_{t+r} \subseteq U_{2(t+1)+r-1} \subseteq \ldots \subseteq U_{k(t+1)+r-1} = \mathrm{PG}(n,q)$$

be a chain of subspaces in $\mathrm{PG}(n,q)$, $\dim(U_{i(t+1)+r-1}) = i(t+1)+r-1$, $i = 1, 2, \ldots, k$. Take a partition $\mathcal{S}_j$ by t-spaces of $U_{(j+1)(t+1)+r-1} \setminus U_{j(t+1)+r-1}$ for each $j \in \{1, \ldots, k-1\}$. Consider a $(t-r)$-space U_{t-r} in U_{t+r} and let $\mathcal{S} = \{\pi^0_{r-1}, \pi^1_{r-1}, \ldots, \pi^{q^r}_{r-1}\}$ be an $(r-1)$-spread in the quotient geometry of U_{t-r} in U_{t+r}. Then U_{t+r} can be covered by the q^r+1 t-spaces $\langle U_{t-r}, \pi^k_{r-1} \rangle$,

$k = 0, 1, \dots q^r$. Call this cover $\mathcal{C}'$. Then

$$\mathcal{C} = \mathcal{C}' \cup \left(\bigcup_{1 \leq j \leq k-1} \mathcal{S}_j \right)$$

is a t-cover of $\mathrm{PG}(n, q)$ of size $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} + 1$.

In a way, as the following theorem of Eisfeld shows, all t-covers meeting the lower bound "look like" the examples given by Beutelspacher.

Theorem 4.1.16 (Eisfeld [46]) *If $\mathcal{C}$ is a t-cover of $\mathrm{PG}(n, q)$, $n + 1 = k(t+1)+r$, $1 \leq r \leq t$, of size $q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} + 1$, then there exists a $(t-r)$-space U of $\mathrm{PG}(n, q)$ such that every point of $\mathrm{PG}(n, q) \setminus U$ is contained in exactly one element of $\mathcal{C}$ and every point of U is contained in exactly $q^r + 1$ elements of $\mathcal{C}$.*

4.2 Partial t-spreads in projective spaces

In Theorem 4.2.2, it is shown that the holes of a partial t-spread are distributed in a special way. They form a minihyper in $\mathrm{PG}(n, q)$ with the same parameters as the minihypers studied in Chapters 2 and 3. Hence, the results from those chapters allow to make some observations on the structure of the partial t-spreads.

Notation 4.2.1 If a and b are two integers, then $a \mid b$ denotes a *divides* b.

Theorem 4.2.2 *Let $\mathcal{S}'$ be a partial t-spread of $\mathrm{PG}(n, q)$, $t+1 \mid n+1$, with deficiency $\delta < q$, and let F be the set of holes of $\mathcal{S}'$. Then F is a $\{\delta v_{t+1}, \delta v_t; n, q\}$-minihyper.*

Proof Let $\mathcal{S}$ be a t-spread of $\mathrm{PG}(n, q)$ and let H be a hyperplane of $\mathrm{PG}(n, q)$. Suppose α elements of $\mathcal{S}$ are contained in H and β elements of $\mathcal{S}$ intersect H in a $(t-1)$-space. Then

$$\alpha + \beta = \frac{q^{n+1} - 1}{q^{t+1} - 1}, \tag{4.1}$$

$$\alpha \frac{q^{t+1} - 1}{q - 1} + \beta \frac{q^t - 1}{q - 1} = \frac{q^n - 1}{q - 1}. \tag{4.2}$$

Now suppose α' elements of $\mathcal{S}'$ are contained in H and β' elements of $\mathcal{S}'$ intersect H in a $(t-1)$-space. Then

$$\alpha' + \beta' = \frac{q^{n+1} - 1}{q^{t+1} - 1} - \delta, \tag{4.3}$$

$$\alpha'\frac{q^{t+1}-1}{q-1}+\beta'\frac{q^t-1}{q-1}\leq\frac{q^n-1}{q-1}. \tag{4.4}$$

Three cases can be distinguished.

Case 1. Assume $\alpha' > \alpha$. Let $\alpha' = \alpha + a$. Substituting this equation in (4.4) and taking (4.2) into account, $\frac{\beta-\beta'}{a} \geq \frac{q^{t+1}-1}{q^t-1}$ is obtained. The right hand side of this inequality is greater than q, such that $\beta \geq \beta' + aq + 1$. Adding α' to both sides of this inequality leads to $\delta \geq a(q-1)+1$, which implies $\delta \geq q$, a contradiction.

Case 2. Assume $\alpha' = \alpha$. In this case the number of holes in H equals $\delta\frac{q^t-1}{q-1} = \delta v_t$.

Case 3. Assume $\alpha' < \alpha$. Let $\alpha = \alpha' + a'$ and note that, by (4.2) and (4.3), $\beta' = \alpha + \beta - \delta - \alpha'$. The number of holes in H equals

$$\begin{aligned}
|F\cap H| &= v_n - \alpha' v_{t+1} - \beta' v_t\\
&= v_n - \alpha v_{t+1} + a' v_{t+1} - \alpha v_t - \beta v_t + \delta v_t + \alpha' v_t\\
&= v_n \underbrace{-\alpha v_{t+1} - \beta v_t}_{-v_n} + a' v_{t+1} - \alpha v_t + \delta v_t + \alpha v_t - a' v_t\\
&= \delta v_t + a'(v_{t+1} - v_t).
\end{aligned}$$

So, in this case, H contains more than δv_t holes.

Therefore, F consists of δv_{t+1} points and $|F\cap H| \geq \delta v_t$ for any hyperplane H. By Theorem 2.1.8, F is a $\{\delta v_{t+1}, \delta v_t; n, q\}$-minihyper. □

Recall the definition of ϵ_q from Notation 2.2.4.

Theorem 4.2.3 *Suppose $t+1$ divides $n+1$ and let $\mathcal{S}'$ be a maximal partial t-spread with deficiency δ in* PG(n,q)*.*

1. *If $\delta > 0$, then $\delta \geq \epsilon_q$.*
2. *If $q > 16$ is a square and $\delta < q^{5/8}/\sqrt{2}+1$, then $\delta \equiv 0 \pmod{\sqrt{q}+1}$ and the set of holes is the disjoint union of subgeometries* PG$(2t+1,\sqrt{q})$*. Moreover, if $\delta > 0$ and $n < \sqrt{q}+1$, then $\delta \geq 2(\sqrt{q}+1)$.*

Proof Let $\mathcal{S}'$ be such a maximal partial t-spread. By Theorem 4.2.2, the set of holes forms a $\{\delta v_{t+1}, \delta v_t; n, q\}$-minihyper. Now Theorems 2.2.7 and 3.4.2 can be applied.

1. If $\delta < \epsilon_q$, then the set of holes forms a disjoint union of δ t-spaces. Since $\mathcal{S}'$ is maximal, δ equals zero.

2. If $q > 16$ is a square and $\delta < q^{5/8}/\sqrt{2}+1$, then the set of holes is a disjoint union of t-spaces and subgeometries $\mathrm{PG}(2t+1, \sqrt{q})$. Since $\mathcal{S}'$ is maximal, it cannot contain a t-space. Therefore it consists of a disjoint union of subspaces $\mathrm{PG}(2t+1, \sqrt{q})$. Hence, $\delta \equiv 0 \pmod{\sqrt{q}+1}$.

   If $n < \sqrt{q}+1$, then the case $\delta = \sqrt{q}+1$ cannot occur. This follows from the weight argument of Blokhuis and Metsch [19]. They define a weight function $f : \mathrm{AG}(n, \tilde{q}^s) \to \mathrm{GF}(\tilde{q}^s)$ on the points $(x_1, \ldots, x_n)$ of $\mathrm{AG}(n, \tilde{q}^s)$ by $f(x_1, \ldots, x_n) = (\prod_{i=1}^n x_i)^{\tilde{q}-1}$. The weight of a set of points is the sum of the weights of the points in that set. In [19], it is proved that f has the following properties:
   (1) if $s > 1$, then the weight of $\mathrm{AG}(n, \tilde{q}^s)$ is zero;
   (2) if $n(\tilde{q}-1) < (\tilde{q}^s - 1)$, then the weight of every subspace of $\mathrm{AG}(n, \tilde{q}^s)$ is zero; and
   (3) if e is an integer, $1 \le e \le n$, then the weight of $\mathrm{AG}(e, \tilde{q})$ (with its natural embedding in $\mathrm{AG}(n, \tilde{q}^s)$) is non-zero.
   This weight function is then used to prove that for $\tilde{q} > 2$, it is not possible to partition the set of points outside a Baer subspace of $\mathrm{PG}(3, \tilde{q}^2)$ by lines. But this proof works just as well to prove that if $n < \sqrt{q}+1$, then it is impossible to partition the set of points of $\mathrm{PG}(n, q)$ outside a subspace $\mathrm{PG}(2t+1, \sqrt{q})$ by t-spaces. Indeed, suppose that such a partition exists and let H_∞ be a hyperplane intersecting the subspace $\mathrm{PG}(2t+1, \sqrt{q})$ in a subspace $\mathrm{PG}(2t, \sqrt{q})$ and write $\mathrm{PG}(n, q) = H_\infty \cup \mathrm{AG}(n, q)$. Then this partition induces a partition of $\mathrm{AG}(n, q) \setminus \mathrm{AG}(2t+1, \sqrt{q})$ by t-spaces. Now the weight of $\mathrm{AG}(n, q)$ can be calculated in two ways. By (1), it equals zero, but by (2) and (3) it is non-zero, a contradiction.

This concludes the proof. □

Substituting the bounds on blocking sets from Theorems 1.2.1 and 1.2.3, immediately yields the following corollary.

Corollary 4.2.4 *Suppose $t+1$ divides $n+1$ and let $\mathcal{S}'$ be a maximal partial t-spread with deficiency $\delta > 0$ in* $\mathrm{PG}(n, q)$.

1. *If q is a prime, then $\delta > (q+1)/2$.*
2. *If q is a square, then $\delta \ge \sqrt{q}+1$.*
3. *If $q = p^{2e+1}$, p prime, $e \ge 1$, then $\delta \ge \max(p^{e+1}+1, c_p q^{2/3}+1)$.*
4. *If $q > 16$ is a square and $\delta < q^{5/8}/\sqrt{2}+1$, then $\delta \equiv 0 \pmod{\sqrt{q}+1}$ and the set of holes is the disjoint union of subgeometries $\mathrm{PG}(2t+1, \sqrt{q})$. Moreover, if $\delta > 0$ and $n(\sqrt{q}-1) < q-1$, then $\delta \ge 2(\sqrt{q}+1)$.*

The preceding results are generalisations of the Metsch-Storme results on maximal partial 1-spreads in $\mathrm{PG}(3,q)$ and maximal partial t-spreads in $\mathrm{PG}(2t+1,q)$ that were mentioned before. They are generalisations in the sense that they hold for partial t-spreads in $\mathrm{PG}(n,q)$ for every t and n that satisfy $t+1 \mid n+1$. However, for $n = 2t+1$, the bounds on δ in the theorems of Metsch and Storme are better.

Remark 4.2.5 Note that the bounds on δ in Theorem 4.1.14 are quite a lot better than the ones from Theorem 4.2.3. Theorem 4.1.14 was obtained using the link between partial t-spreads in $\mathrm{PG}(2t+1,q)$ and translation nets, and using the extendibility result on nets from [84]. It is not clear how this strategy can be modified to apply it to partial t-spreads in $\mathrm{PG}(n,q)$, $t+1 \mid n+1$.

Improvements to Theorem 4.2.3 in the spirit of Theorem 4.1.13 were obtained by Ferret and Storme. They prove a new characterisation result on the corresponding minihypers and use Theorem 4.2.2 to translate it to a theorem on partial t-spreads.

Theorem 4.2.6 (Ferret and Storme [50]) *Suppose $\mathcal{S}'$ is a maximal partial t-spread with deficiency δ of $\mathrm{PG}(n,q)$, where $t+1 \mid n+1$. If $q = p^{3h_0}$, $h_0 \geq 1$, $p \geq 7$ prime, and $\delta \leq 2q^{2/3} - 4q^{1/3}$, then the set of holes of $\mathcal{S}'$ is the disjoint union of (projected) subspaces $\mathrm{PG}(3t+2, q^{1/3})$ and subspaces $\mathrm{PG}(2t+1, \sqrt{q})$.*

4.3 t-Covers in projective spaces

In the first part of this section, the result on minihypers from Chapter 2 is applied to t-covers of $\mathrm{PG}(n,q)$, giving a characterisation of the set of multiple points.

In the second part, small line covers of $\mathrm{PG}(4,q)$ are studied. The structure of the set of multiple points of the second-smallest line covers of $\mathrm{PG}(4,q)$ is determined and a construction for small minimal line covers of $\mathrm{PG}(4,q)$ is presented.

4.3.1 Application of the results on minihypers

Theorem 4.3.1 shows that the multiple points of a t-cover are distributed in a special way. They form a minihyper in $\mathrm{PG}(n,q)$ with the same parameters as the minihypers studied in Chapters 2. Hence, the results from that chapter allow to make some observations on the structure of the set of multiple points of a t-cover.

Theorem 4.3.1 *Let $\mathcal{C}$ be a t-cover of* $\mathrm{PG}(n,q)$*,* $t+1\,|\,n+1$*, with excess* $\varepsilon<q$*. Let F be the set of multiple points of $\mathcal{C}$ and let* $w(P)=\mathrm{surplus}(P)$ *for* $P\in\mathrm{PG}(n,q)$*. Then (F,w) is a* $\{\varepsilon v_{t+1},\varepsilon v_t;n,q\}$*-minihyper.*

Proof This proof closely resembles the proof of Theorem 4.2.2. Let $\mathcal{S}$ be a t-spread of $\mathrm{PG}(n,q)$ and let H be a hyperplane of $\mathrm{PG}(n,q)$. Suppose α elements of $\mathcal{S}$ are contained in H and β elements of $\mathcal{S}$ intersect H in a $(t-1)$-space. Then

$$\alpha+\beta=\frac{q^{n+1}-1}{q^{t+1}-1}, \tag{4.5}$$

$$\alpha\frac{q^{t+1}-1}{q-1}+\beta\frac{q^{t}-1}{q-1}=\frac{q^{n}-1}{q-1}. \tag{4.6}$$

Now suppose α' elements of $\mathcal{C}$ are contained in H and β' elements of $\mathcal{C}$ intersect H in a $(t-1)$-space. Then

$$\alpha'+\beta'=\frac{q^{n+1}-1}{q^{t+1}-1}+\varepsilon, \tag{4.7}$$

$$\alpha'\frac{q^{t+1}-1}{q-1}+\beta'\frac{q^{t}-1}{q-1}\geq\frac{q^{n}-1}{q-1}. \tag{4.8}$$

Again, three cases can be distinguished.

Case 1. Assume $\alpha'<\alpha$. Let $\alpha'+a=\alpha$. Substituting this equation in (4.8) and taking (4.6) into account, the inequality $\frac{\beta'-\beta}{a}\geq\frac{q^{t+1}-1}{q^t-1}$ is obtained, the right hand side of which is greater than q, such that $\beta'\geq\beta+aq+1$. Adding α' to both sides of this inequality leads to $\varepsilon\geq a(q-1)+1$, which implies $\varepsilon\geq q$, a contradiction.

Case 2. Assume $\alpha'=\alpha$. In this case $\sum_{P\in H}w(P)=\varepsilon\frac{q^t-1}{q-1}=\varepsilon v_t$.

Case 3. Assume $\alpha'>\alpha$. Let $\alpha'=\alpha+a'$. Since $a'\geq 1$, $\alpha'+\beta'=\alpha+\beta+\varepsilon$ and $\sum_{P\in H}w(P)=\alpha' v_{t+1}+\beta' v_t-v_n$, it follows from (4.6) that $\sum_{P\in H}w(P)=\varepsilon v_t+a'(v_{t+1}-v_t)>\varepsilon v_t$.

Hence, (F,w) consists of εv_{t+1} points—counted according their surplus—and $|(F,w)\cap H|\geq\varepsilon v_t$ for any hyperplane H. By Theorem 2.1.8, (F,w) is an $\{\varepsilon v_{t+1},\varepsilon v_t;n,q\}$-minihyper. □

Recall the the definition of ϵ_q from Notation 2.2.4.

Theorem 4.3.2 *Suppose $t+1 \mid n+1$ and $\varepsilon < \epsilon_q$. If $\mathcal{C}$ is a t-cover of $\mathrm{PG}(n,q)$ with excess ε, then its multiple points form a sum of ε t-spaces.*

Proof Immediate from Theorems 2.2.7 and 4.3.1. □

Remark 4.3.3 Theorem 4.3.2 was already proved for the case of a line cover in $\mathrm{PG}(3,q)$ in Blokhuis et al. [21].

Example 4.3.4 In [21], the authors give examples of minimal line covers in $\mathrm{PG}(3,q)$ of excess ε for each $\varepsilon \in \{0,1,\dots,q-1\} \cup \{0,2,4,\dots,2q\}$. All the examples given have the property that the multiple points form a sum of lines, also in the cases where $\varepsilon \geq \epsilon_q$. These examples can be used to construct minimal $(2n-1)$-covers of $\mathrm{PG}(4n-1,q)$.

Consider $\mathrm{PG}(3,q^n) \cong V(4,q^n) \pmod{\mathrm{GF}(q^n)}$. Identify $V(4,q^n)$ with $V(4n,q)$. Now in $V(4n,q)$, a point of $\mathrm{PG}(3,q^n)$, which is in fact a $V(1,q^n)(\equiv V(n,q))$, defines a $\mathrm{PG}(n-1,q)$. Similarly, a line of $\mathrm{PG}(3,q^n)$ defines a subspace $\mathrm{PG}(2n-1,q)$ of $\mathrm{PG}(4n-1,q)$.

The line covers of $\mathrm{PG}(3,q^n)$ therefore yield $(2n-1)$-covers of $\mathrm{PG}(4n-1,q)$. The lines of $\mathrm{PG}(3,q^n)$ that form the weighted sum of lines now become subspaces $\mathrm{PG}(2n-1,q)$ of $\mathrm{PG}(4n-1,q)$ that form a weighted sum with sum of the weights equal to δ.

4.3.2 Small line covers in $\mathrm{PG}(4,q)$

By Theorem 4.1.15, a line cover of $\mathrm{PG}(4,q)$ has size at least q^3+q+1, in which case, by Theorem 4.1.16, there exists a point P that is covered exactly $q+1$ times, while all other points are covered only once. In this subsection, line covers that are a little bit larger are studied.

Distribution of the multiple points

The distribution of the multiple points of a line cover of size q^3+q+2 in $\mathrm{PG}(4,q)$ is determined.

Suppose that $\mathcal{C}$ is a line cover of size q^3+q+2 in $\mathrm{PG}(4,q)$. Then $\mathrm{surplus}(\mathrm{PG}(4,q)) = 2q+1$. Let H be a hyperplane and suppose H contains x lines of $\mathcal{C}$. Counting points according their multiplicity, $xq+q^3+q+2$ points of H are covered by the elements of $\mathcal{C}$. Hence, $\mathrm{surplus}(H) \equiv 1 \pmod q$ such that $\mathrm{surplus}(H) \in \{1, q+1, 2q+1\}$.

Suppose π is a plane with surplus $q+a$, $a \geq 2$. Then all hyperplanes through π have surplus $2q+1$. Counting the surplus of $\mathrm{PG}(4,q)$ by counting the surplus in the hyperplanes through π yields $q+a+(q+1)(2q+1-q-a) = 2q+1$, implying $a = q+1$. Similarly, if π is a plane with surplus $a \geq 2$, then—since every hyperplane through π has surplus at least $q+1$—counting the surplus of $\mathrm{PG}(4,q)$ shows that $a \geq q$.

The same counting argument can be applied to lines and points of $\mathrm{PG}(4,q)$. In this way, a list with the possible surpluses of subspaces of $\mathrm{PG}(4,q)$ is obtained:

$$\begin{array}{lcll}
\mathrm{surplus}(P) & \in & \{0,1,q,q+1,2q+1\} & \text{for each point } P,\\
\mathrm{surplus}(l) & \in & \{0,1,q,q+1,2q+1\} & \text{for each line } l,\\
\mathrm{surplus}(\pi) & \in & \{0,1,q,q+1,2q+1\} & \text{for each plane } \pi,\\
\mathrm{surplus}(H) & \in & \{1,q+1,2q+1\} & \text{for each hyperplane } H,\\
\mathrm{surplus}(\Sigma) & = & 2q+1 & \text{for } \Sigma = \mathrm{PG}(4,q).
\end{array}$$

Case 1. There exists a point P with surplus $\mathbf{2q+1}$. Any hyperplane not containing P has surplus 0, a contradiction.

Case 2. There exists a point P with surplus $\mathbf{q+1}$. In this case, there exists no point P' with surplus q, since a hyperplane containing P' but not P would have surplus q. So, all points different from P with positive surplus have surplus 1. Call these points $P_1, P_2, \ldots, P_q$ and consider the line $PP_1 = l$. Suppose there exists a point P_i, $2 \leq i \leq q$, that does not lie on the line l. Then any hyperplane H containing l but not P_i satisfies $q+1 < \mathrm{surplus}(H) < 2q+1$, a contradiction. Hence, all points with positive surplus lie on the line l.

Case 3. There exists a point P with surplus q. Suppose there exists a second point P' with surplus q. Then there is one remaining point P'' with positive surplus. It has surplus 1. If P, P' and P'' are collinear, then a hyperplane intersecting PP' in P has surplus q, a contradiction. But if P, P' and P'' are not collinear, then a hyperplane containing PP' but not P'' has surplus $2q$, also a contradiction. Hence, there exists no second point with surplus q and all remaining points with positive surplus have surplus 1. Call these points $P_1, P_2, \ldots, P_{q+1}$ and denote the line P_1P_2 by l. Suppose that P_i does not lie on l for some $3 \leq i \leq q+1$ and consider a hyperplane H containing l but not P_i. If H contains P, then $q+1 < \mathrm{surplus}(H) < 2q+1$, a contradiction. But also if H does not contain P, a contradiction is obtained, since then $1 < \mathrm{surplus}(H) < q+1$. Hence the points $P_1, P_2, \ldots, P_{q+1}$ lie on a line l.

Case 4. All points with positive surplus have surplus 1.

Case 4.1. There exists a line l with surplus $q+1$. Consider a point P with surplus 1 that does not lie on l. The plane $\langle P, l\rangle$ has surplus greater than $q+1$, hence surplus $2q+1$, implying there are no points with positive surplus outside $\langle P, l\rangle$. Since $2q+1 < q^2+q+1$, there exists a line l' with surplus smaller than $q+1$ joining P to a point of l. Consider a hyperplane intersecting $\langle P, l\rangle$ in l'. This hyperplane has surplus at least $q+1$, implying that it contains a point with positive surplus outside $\langle P, l\rangle$, a contradiction.

Case 4.2. All lines with surplus greater than 1 *have surplus* q. Let l be a line with surplus q and let $P_1, P_2, \ldots, P_q$ be the points on l with surplus 1. Let P be a point outside l with surplus 1. Now consider a hyperplane H containing l but not P. It has surplus $q+1$. Hence, in H there is only one point P' outside l with positive surplus. But all lines joining P' to a point in $\{P_1, P_2, \ldots, P_q\}$ must have surplus at least q. Hence $q = 2$ and there are five points $P_1, P_2, \ldots, P_5$ with surplus 1. Note that no plane contains more than three of these points. For, such a plane would contain all points with positive surplus, and there would exist a hyperplane intersecting it in a line with surplus 2. Let $\pi = \langle P_1, P_2, P_3\rangle$. Since $H = \langle P_4, \pi\rangle$ contains more than $q+1$ points with positive surplus, it contains all points with positive surplus. Hence $P_1, P_2, \ldots, P_5$ lie in a hyperplane that any four of them generate. Such a set of points is the set of points of an elliptic quadric $\mathrm{Q}^-(3, q)$.

Theorem 4.3.5 *Suppose $\mathcal{C}$ is a line cover of* $\mathrm{PG}(4, q)$ *of size* q^3+q+2. *Then the multiple points are distributed in* $\mathrm{PG}(4, q)$ *in one of the following ways:*

1. *there exists one point P with surplus q and there is a line skew to P on which every point has surplus* 1, *or*

2. *there exists one point P with surplus $q+1$ and there is a line through P on which every point different from P has surplus* 1, *or*

3. $q = 2$ *and there are five points with surplus* 1 *that form an elliptic quadric* $\mathrm{Q}^-(3, q)$ *in a hyperplane.*

Remark 4.3.6 1. In [13], Beutelspacher notes that in $\mathrm{PG}(n, 2)$, $n \geq 3$, a set of five points in a 3-space such that any four of these points generate the 3-space, is a nontrivial blocking set with respect to hyperplanes.

2. Surely, there exist line covers of $\mathrm{PG}(4, q)$ whose multiple points are distributed as in cases 1 and 2 from Theorem 4.3.5. It suffices to take a line cover of minimal cardinality q^3+q+1 of $\mathrm{PG}(4, q)$ and to add a line skew to, respectively through, the unique multiple point. However, such a cover is not minimal. Below, minimal examples will be constructed.

3. Since in case 3 of Theorem 4.3.5 the set of multiple points contains no line, a line cover of size 12 in $PG(4, 2)$ with such a set of multiple points is necessarily minimal. A computer search, using the share package PG [36] for the computer algebra system GAP [52], shows that such covers in $PG(4, 2)$ do exist. The implementation of these searches can be found on the website `http://cage.rug.ac.be/~pg/thesis/`.

A construction

Let $PG(4, q)$ be embedded in $PG(5, q)$ and let π be a plane in $PG(4, q)$. Consider a regular plane spread $\mathcal{S}_2 = \{\pi, \pi_1, \ldots, \pi_{q^3}\}$ in $PG(5, q)$ containing π, see Theorem 4.1.5. The planes π_i of $\mathcal{S}_2$, $1 \leq i \leq q^3$ intersect $PG(4, q)$ in lines l_i. These lines l_i partition the point set of $PG(4, q) \setminus \pi$. Let $\mathcal{S}_1 = \{l_1, \ldots, l_{q^3}\}$. Consider two distinct lines l_{i_1} and l_{i_2}, $1 \leq i_1 < i_2 \leq q^3$. The corresponding planes π_{i_1} and π_{i_2} define, together with π, a regulus $\mathcal{R} = \{\pi, \pi_{i_1}, \ldots, \pi_{i_q}\}$ of planes in $\mathcal{S}_2$. The transversals to $\mathcal{R}$ define for each point on l_{i_1} a unique point on each element of $\mathcal{R}$. All these points lie in the 4-space $\langle \pi, l_{i_1} \rangle = PG(4, q)$, hence they lie on the lines $l_{i_2}, \ldots, l_{i_q}$ and one further line l in π. In this way, any two lines l_{i_1} and l_{i_2} of $\mathcal{S}_1$ define a regulus of lines, all of which but one are contained in $\mathcal{S}_1$; the remaining line is contained in π. Call such a line regulus *special*. Hence, $q^2(q^2 + q + 1)$ special reguli exist.

Suppose that there exist two special line reguli R_1 and R_2 that have two lines in common. If these lines both are lines of $\mathcal{S}_1$, then—since two such lines uniquely define a special regulus—the reguli are equal. If one of these two lines is a line l in π and the second one is a line l_i, for some $i \in \{1, \ldots, q^3\}$, then both R_1 and R_2 are contained in the 3-space $\langle l, l_i \rangle$. Consider the intersection of the planes of $\mathcal{S}_2$ with $\langle l, l_i \rangle$. A plane of $\mathcal{S}_2$ can neither be skew to it, nor be contained in it. Let α denote the number of planes of $\mathcal{S}_2$ intersecting $\langle l, l_i \rangle$ in a line and $\beta = q^3 + 1 - \alpha$ the number of planes of $\mathcal{S}_2$ intersecting it in a point. Since $\mathcal{S}_2$ is a spread of $PG(5, q)$, $\alpha(q+1) + (q^3 + 1 - \alpha) = q^3 + q^2 + q + 1$, implying $\alpha = q + 1$. So, also in this case $R_1 = R_2$.

Since there are q^3 lines of $\mathcal{S}_1$, and no two distinct special line reguli have more than one line in common, there are exactly q^2 special line reguli containing a given line $l \subseteq \pi$.

Using these special line reguli, minimal covers of $PG(4, q)$ can be constructed. Let P be a point in π and let $\mathcal{C}_1$ be the set of lines in π through P. Let l be a line in π. Let $R_1, R_2, \ldots, R_{q^2}$ be the special line reguli containing

l and consider the line cover

$$\mathcal{C}_1 \cup \left(\bigcup_{i=1}^{q^2} (R_i \setminus \{l\}) \right)$$

of size $q^3 + q + 1$. Now replace α of the sets $R_i \setminus \{l\}$ by R_i^{opp}, the opposite regulus of R_i. If P does not lie on l, then in this way a minimal cover of size $q^3 + q + 1 + \alpha$ can be obtained for every α in $\{0, 1, \dots, q^2\}$. If P lies on l, then in this way a minimal cover of size $q^3 + q + \alpha$ can be obtained for every α in $\{1, 2, \dots, q^2\}$, since in the cover obtained by the replacing procedure, the line l can be deleted.

Theorem 4.3.7 *There exist minimal line covers of size* $q^3 + q + 1 + \alpha$ *in* $\mathrm{PG}(4, q)$ *for all* α *in* $\{0, 1, \dots, q^2\}$.

Remark 4.3.8

1. If $P \in l$ and $\alpha = 2$ in the above construction, then a minimal line cover of size $q^3 + q + 2$ of $\mathrm{PG}(4, q)$ is obtained whose multiple points are distributed as in case 2 of Theorem 4.3.5. If $P \notin l$ and $\alpha = 1$ in the above construction, then a minimal line cover of size $q^3 + q + 2$ of $\mathrm{PG}(4, q)$ is obtained whose multiple points are distributed as in case 1 of Theorem 4.3.5.

2. In the construction above, the multiple points always form a sum of lines and q points. These q points are in fact q times the same point.

4.4 Introduction for polar spaces

Let $\mathcal{P}_n$ denote a finite classical polar space of rank k in $\mathrm{PG}(n, q)$ and let $1 \leq t \leq k - 1$. A *partial t-spread* of $\mathcal{P}_n$ is a set of mutually disjoint t-spaces on $\mathcal{P}_n$. A *t-cover* of $\mathcal{P}_n$ is a set of t-spaces on $\mathcal{P}_n$ that covers the point set of $\mathcal{P}_n$. A *t-spread* of $\mathcal{P}_n$ is a set of t-spaces on $\mathcal{P}_n$ that is a partial t-spread as well as a t-cover of $\mathcal{P}_n$. In other words, a t-spread of $\mathcal{P}_n$ is a set of t-spaces on $\mathcal{P}_n$ that partitions the point set of $\mathcal{P}_n$. If $t = k - 1$, i.e. when the elements of the partial t-spread or t-cover are generators of $\mathcal{P}_n$, then a (partial) t-spread, respectively t-cover, of $\mathcal{P}_n$ is simply called a *(partial) spread*, respectively *cover*, of $\mathcal{P}_n$.

Many articles have been published on the (non)existence of spreads of $\mathcal{P}_n$, but it is only recently—with a very limited number of exceptions—that papers have started to appear that study partial t-spreads or t-covers of $\mathcal{P}_n$ where $t \neq k - 1$ with k the rank of the polar space.

$\mathcal{P}$	$\mathrm{rk}(\mathcal{P})$	$s(\mathcal{P})$
$\mathrm{W}_{2n+1}(q)$	$n+1$	$q^{n+1}+1$
$\mathrm{Q}^-(2n+1,q)$	n	$q^{n+1}+1$
$\mathrm{Q}(2n,q)$	n	q^n+1
$\mathrm{Q}^+(2n+1,q)$	$n+1$	q^n+1
$\mathrm{H}(2n,q^2)$	n	$q^{2n+1}+1$
$\mathrm{H}(2n+1,q^2)$	$n+1$	$q^{2n+1}+1$

Table 4.1: Finite classical polar spaces: rank and size of a spread

Clearly, if $\mathcal{P}_n$ has a spread, then v_k divides $|\mathcal{P}_n|$. Using the values from Table 1.2, one easily checks that this is always the case and that the size of a hypothetical spread of $\mathcal{P}_n$, which will here be denoted by $s(\mathcal{P}_n)$, is as given in Table 4.1. However, unlike the projective case, divisibility of $|\mathcal{P}_n|$ by v_k does not imply the existence of a spread of $\mathcal{P}_n$. In Table 4.2, an overview is presented of the cases where it known whether $\mathcal{P}_n$ has a spread or not.

The *deficiency* δ of a partial t-spread $\mathcal{S}'$ of $\mathcal{P}_n$ equals by definition $\delta = \lfloor |\mathcal{P}_n|/v_{t+1} \rfloor - |\mathcal{S}'|$. The *holes* of a partial t-spread of $\mathcal{P}_n$ are those points of $\mathcal{P}_n$ that lie in no element of $\mathcal{S}'$. A partial t-spread of $\mathcal{P}_n$ is called *maximal* if it is not contained in a larger partial t-spread of $\mathcal{P}_n$.

The *excess* ε of a t-cover $\mathcal{C}$ of $\mathcal{P}_n$ equals by definition $\varepsilon = |\mathcal{C}| - \lceil |\mathcal{P}_n|/v_{t+1} \rceil$. The *multiple points* of a t-cover are the points of $\mathcal{P}_n$ that lie in more than one element of $\mathcal{C}$. The *surplus* of a point of $\mathcal{P}_n$ is the number of elements of $\mathcal{C}$ that contain it minus one. Sometimes, surplus is considered as a weight function mapping a point P of $\mathcal{P}_n$ onto a nonnegative integer $\mathrm{surplus}(P)$. A t-cover of $\mathcal{P}_n$ is called *minimal* if it has no proper subset that is a t-cover of $\mathcal{P}_n$.

Known results

The results on partial t-spreads and t-covers of finite classical polar spaces that are stronger than the ones obtained in Sections 4.6 and 4.7 are stated below.

Theorem 4.4.1 (J. A. Thas [114, 115])

1. *The polar spaces* $\mathrm{Q}^-(4n+1,q)$, $n \geq 1$, *and* $\mathrm{Q}^+(4n+3,q)$, $n \geq 0$, *both have linespreads.*

2. *If* $\mathcal{S}'$ *is a partial spread of* $\mathrm{Q}^+(4n+1,q)$, $n \geq 1$, *then* $|\mathcal{S}'| \leq 2$.

$\mathcal{P}$	Restrictions	Spreads	References
$\mathrm{W}_{2n+1}(q)$	—	yes	[1, 74, 81, 93], [110, 117]
$\mathrm{Q}^-(2n+1,q)$	q even	yes	[44, 111, 112]
	$n=2$	yes	[91, 113, 117]
$\mathrm{Q}(2n,q)$	q even	yes	[44, 111, 112, 117]
	$n=3$, q odd with q prime or $q \equiv 0$ or $2 \pmod 3$	yes	[34, 44, 74, 75, 76], [88, 101, 111, 112]
	$n=2m$, q odd	no	[109, 115]
$\mathrm{Q}^+(2n+1,q)$	$n=2m$	no	[70]
	$n=2m+1$, q even	yes	[44, 111, 112]
	$n=1$	yes	
	$n=3$, q odd with q prime or $q \equiv 0$ or $2 \pmod 3$	yes	[34, 44, 74, 75, 76], [88, 101, 111, 112]
$\mathrm{H}(2n,q^2)$	$n=2$, $q=2$	no	(*)
$\mathrm{H}(2n+1,q^2)$	—	no	[112, 115]

(*) Brouwer, unpublished

Table 4.2: Existence of spreads in finite classical polar spaces
The references are copied from [116].

3. *If* $\mathcal{S}'$ *is a partial spread of* $\mathrm{H}(2n+1,q^2)$, $n \geq 1$ *and* n *odd, then* $|\mathcal{S}'| \leq q^{2n+1} - q^{n+1} + q^n + 1$.

4. *If* $\mathcal{S}'$ *is a partial spread of* $\mathrm{H}(5,q^2)$, *then* $|\mathcal{S}'| \leq q^2(q^2+q-1)$.

Theorem 4.4.2 (Eisfeld et al. [47, 48], Eisfeld et al. [49])

1. *Let* $\mathcal{S}'$ *be a partial* $(n-1)$*-spread of* $\mathrm{Q}^+(2n+1,q)$. *Then* $|\mathcal{S}'| \leq q^3+q$ *for* $n=2$ *and* $|\mathcal{S}'| \leq q^{n+1}+q-1$ *for* $n>2$.

2. *Let* $\mathcal{C}$ *be an* $(n-1)$*-cover of* $\mathrm{Q}^+(2n+1,q)$. *Then* $|\mathcal{C}| \geq q^{n+1}+2q+1$. *For* q *even, this bound is sharp.*

3. *Let* $\mathcal{C}$ *be a plane cover of* $\mathrm{Q}^+(5,q)$. *Then* $|\mathcal{C}| \geq q^2+q$. *This bound is sharp.*

4. *Let* $\mathcal{C}$ *be a cover of* $\mathrm{Q}(4,q)$, q *odd. Then* $|\mathcal{C}| > q^2+1+(q-1)/3$.

5. *Let* $\mathcal{C}$ *be a cover of* $\mathrm{Q}(4,q)$, q *even,* $q \geq 32$, *of size* q^2+1+r, *where* $0 < r \leq \sqrt{q}$. *Then* $\mathcal{C}$ *contains a spread of* $\mathrm{Q}(4,q)$.

Theorem 4.4.3 (Ebert and Hirschfeld [45]) *The largest partial spreads in* $\mathrm{H}(3,9)$ *have size* 16.

4.5 t-Spreads in polar spaces

Clearly, if a polar space $\mathcal{P}$ admits a t-spread, then $|\mathrm{PG}(t,q)|$ divides $|\mathcal{P}|$. In this section, this condition is rewritten in a more convenient expression. In order to do this, two lemmas are stated, followed by the actual simplification of the divisibility condition. The greatest common divisor of two integers a and b is denoted by (a,b).

Lemma 4.5.1 *Let* a *and* b *be nonnegative integers,* $a+b \geq 1$. *Then* $(q^a-1, q^b-1) = q^{(a,b)}-1$.

Proof This lemma is well known. It can be proved as follows. The lemma clearly holds if $a=b$, $a=0$ or $b=0$, so suppose $a>b>0$. Let d and r be nonnegative integers that satisfy $a = d\cdot b + r$, $r<b$. Then $(a,b)=(b,r)$. Repeating this procedure, (a,b) is obtained the moment that r equals zero.

Executing the Euclidean division, one sees that $(q^a-1,q^b-1) = (q^b-1,q^r-1)$ where r satisfies $a = d\cdot b + r$, $r<b$. Therefore, the Euclidean algorithm can be applied directly to the exponents of q. This proves the lemma. □

Lemma 4.5.2 *Let a and b be nonnegative integers, $a+b \geq 1$. Then*

$$(q^a+1, q^b-1) = \begin{cases} q^{(a,b)}+1 & \text{if } a/(a,b) \text{ is odd and } b/(a,b) \text{ is even,} \\ d & \text{otherwise, where } \begin{cases} d=1 \text{ if } q \text{ is even,} \\ d=2 \text{ if } q \text{ is odd.} \end{cases} \end{cases}$$

Proof Induction on $a+b$ will be used. Therefore, the cases $a=0$ and $b=0$ are considered first. If $a=0$, then $(2, q^b-1)$ equals 1 if q is even and 2 if q is odd. This is in accordance with the lemma, since in this case $b/(a,b)=1$. If $b=0$, then $(q^a+1, 0) = q^a+1$. Clearly, in this case the conditions $a/(a,b)$ odd and $b/(a,b)$ even are satisfied.

For the induction process, the following equalities will be needed. All of them are obtained by using the algorithm of Euclides.

If $a \geq b$, then $(q^a+1, q^b-1) = (q^{a-b}+1, q^b-1)$. If $b > a$, then $(q^a+1, q^b-1) = (q^a+1, q^{b-a}+1)$. Since it is the intention to apply induction, the right hand side of this equation is not satisfactory. Therefore the following equation will be used: if $n \geq m$, then $(q^n+1, q^m+1) = (q^m+1, q^{n-m}-1)$. Thus, if $b > a$, then $b > 2a$ implies $(q^a+1, q^b-1) = (q^a+1, q^{b-2a}-1)$, and $b \leq 2a$ implies $(q^a+1, q^b-1) = (q^{b-a}+1, q^{2a-b}-1)$.

Now suppose that $a, b \geq 0$, $a \neq 0 \neq b$, and that the lemma holds for all a', b', where $a'+b' < a+b$. It remains to be shown that it also holds for a, b.

Suppose $a \geq b$. Then $(q^a+1, q^b-1) = (q^{a-b}+1, q^b-1)$, which equals by induction

$$\begin{cases} q^{(a-b,b)}+1 & \text{if } (a-b)/(a-b,b) \text{ is odd and } b/(a-b,b) \text{ is even,} \\ d & \text{otherwise, where } \begin{cases} d=1 \text{ if } q \text{ is even,} \\ d=2 \text{ if } q \text{ is odd.} \end{cases} \end{cases}$$

Note that $(a,b) = (a-b,b)$, implying $b/(a-b,b) = b/(a,b)$. Furthermore, under the assumption that $b/(a,b)$ is even, $a/(a,b)$ odd implies that $(a-b)/(a-b,b)$ is odd, and $(a-b)/(a-b,b)$ odd implies that $a/(a,b)$ is odd.

The cases $a < b \leq 2a$ and $b > 2a$ are handled in a similar way. □

Theorem 4.5.3 *Suppose $\mathcal{P}$ is a polar space that has a t-spread. If $\mathcal{P}$ is*

1. *a symplectic space* $\mathrm{W}_{2n+1}(q)$*, then* $t+1 \mid 2n+2$*;*
2. *a parabolic quadric* $\mathrm{Q}(2n,q)$*, then* $t+1 \mid 2n$*;*
3. *a hyperbolic quadric* $\mathrm{Q}^+(2n+1,q)$*, then* $t+1 \mid n+1$*;*
4. *an elliptic quadric* $\mathrm{Q}^-(2n+1,q)$*, then* $t+1 \mid n$*;*
5. *a Hermitian variety* $\mathrm{H}(2n,q^2)$*, then* $t+1 \mid n$*;*

6. *a Hermitian variety* $\mathrm{H}(2n+1,q^2)$*, then* $t+1 \,|\, n+1$*.*

Proof If the polar space $\mathcal{P}$ has a t-spread, then $(q^{t+1}-1)/(q-1)$ divides $|\mathcal{P}|$.

1. If $\mathcal{P} = \mathrm{W}_{2n+1}(q)$ has a t-spread, then it follows immediately from Table 1.2 and Lemma 4.5.1 that $t+1 \,|\, 2n+2$.

2. If $\mathcal{P} = \mathrm{Q}(2n,q)$ has a t-spread, then it follows immediately from Table 1.2 and Lemma 4.5.1 that $t+1 \,|\, 2n$.

3. Suppose $\mathcal{P} = \mathrm{Q}^+(2n+1,q)$ has a t-spread. Then $q^{t+1}-1 \,|\, (q^n+1)(q^{n+1}-1)$. If $t+1 \,|\, n+1$, then this condition is fulfilled.

   Now suppose that $t+1$ does not divide $n+1$. Denote $(t+1,n+1)$ by a and $(t+1,n)$ by b. By Lemma 4.5.2, $q^{t+1}-1 \,|\, (q^a-1)(q^b+1)$. Therefore $a+b \geq t+1$ and $ab \leq t+1$. We now consider possible solutions for $\{a,b\}$. If a, respectively b, equals one, then $b \in \{t,t+1\}$, respectively $a \in \{t,t+1\}$. If $\{a,b\} = \{1,t\}$, then $t \,|\, t+1$, implying that $t=1$, such that $(2,n+1) = (2,n) = 1$, a contradiction. If $\{a,b\} = \{1,t+1\}$, then, as $t+1$ does not divide $n+1$, a equals 1. Therefore $q^{t+1}-1 \,|\, (q-1)(q^{t+1}+1)$, a contradiction. If $a=b=2$, then 2 would divide n as well as $n+1$, a contradiction. Finally, if $\{a,b\} = \{x \geq 2, y > 2\}$, then $t+1 \leq a+b < ab \leq t+1$, a contradiction.

   Therefore we may conclude that $t+1 \,|\, n+1$.

4. Suppose $\mathcal{P} = \mathrm{Q}^-(2n+1,q)$ has a t-spread. Then $q^{t+1}-1 \,|\, (q^{n+1}+1)(q^n-1)$. An argument similar to the one in Case 3 shows that $t+1 \,|\, n$.

5. Suppose $\mathcal{P} = \mathrm{H}(2n,q^2)$ has a t-spread. Then $q^{2(t+1)}-1 \,|\, (q^{2n+1}+1)(q^{2n}-1)$. An argument similar to the one in Case 3 shows that $2t+2 \,|\, 2n$.

6. Suppose $\mathcal{P} = \mathrm{H}(2n+1,q^2)$ has a t-spread. Then $q^{2(t+1)}-1 \,|\, (q^{2n+2}-1)(q^{2n+1}+1)$. An argument similar to the one in Case 3 shows that $2t+2 \,|\, 2n+2$.

This concludes the proof of the theorem. □

In the following corollary, the case where $\mathcal{P} = \mathrm{H}(2n+1,q^2)$ is omitted, since J. A. Thas [112, 115] proved that $\mathrm{H}(2n+1,q^2)$ has no spread.

Corollary 4.5.4 (i) *Suppose that t is even and that $\mathcal{P} = \mathrm{W}_{2n+1}(q)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $t+1 \,|\, 2n+2$.*

(ii) *Suppose that t is even and that $\mathcal{P} = \mathrm{Q}(2n, q)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $t + 1 \mid 2n$.*

(iii) *Suppose $\mathcal{P} = \mathrm{Q}^+(2n + 1, q)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $t + 1 \mid n + 1$.*

(iv) *Suppose $\mathcal{P} = \mathrm{Q}^-(2n + 1, q)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $t + 1 \mid n$.*

(v) *Suppose $\mathcal{P} = \mathrm{H}(2n, q^2)$ has a spread. Then $\mathcal{P}$ has a t-spread if and only if $t + 1 \mid n$.*

Proof From the divisibility conditions, it follows that it is possible to construct a t-spread in each element of the spread. The union of such t-spreads forms a t-spread of $\mathcal{P}$. □

Remember the overview of the known results on (non)existence of spreads in polar spaces in Table 4.2. Using this overview and Corollary 4.5.4, the following results on the existence of t-spreads are obtained.

Corollary 4.5.5 1. *If $\mathcal{P} = \mathrm{W}_{2n+1}(q)$ and t is even, then $\mathcal{P}$ has a t-spread if and only if $t + 1 \mid 2n + 2$.*

2. *If t is even and $\mathcal{P} = \mathrm{Q}(2n, q)$ satisfies either $n \geq 2$ and q is even, or $n = 3$ and q is an odd prime, or $n = 3$, q is odd and $q \equiv 0$ or $2 \pmod 3$, then $\mathcal{P}$ has a t-spread if and only if $t + 1 \mid 2n$.*

3. *If $\mathcal{P} = \mathrm{Q}^+(2n + 1, q)$ satisfies either $n = 1$, or $n = 2n' + 1$, $n' \geq 1$ and q is even, or $n = 3$ and q is an odd prime, or $n = 3$, q is odd and $q \equiv 0$ or $2 \pmod 3$, then $\mathcal{P}$ has a t-spread if and only if $t + 1 \mid n + 1$.*

4. *If $\mathcal{P} = \mathrm{Q}^-(2n + 1, q)$ satisfies either $n = 2$, or $n \geq 2$ and q is even, then $\mathcal{P}$ has a t-spread if and only if $t + 1 \mid n$.*

4.6 Partial t-spreads in polar spaces

As was the case for projective spaces, see Section 4.2, the holes of a partial t-spread in a finite classical polar space are distributed in a special way. They form a minihyper in $\mathrm{PG}(n, q)$ with the same parameters as the minihypers studied in Chapters 2 and 3. Hence, also in this case, the results from those chapters will allow to make some observations on the structure of the partial t-spreads.

Theorem 4.6.1 *Let $\mathcal{P}$ be a classical polar space in* $\mathrm{PG}(n,q)$ *whose size admits a t-spread, i.e., that satisfies the necessary conditions of Theorem* 4.5.3. *If $\mathcal{S}'$ is a partial t-spread of $\mathcal{P}$ with deficiency $\delta < q$, then the set F of holes forms a $\{\delta v_{t+1}, \delta v_t; n, q\}$-minihyper.*

Proof The proof very closely resembles the proof of Theorem 4.2.2. Denote the number $|\mathcal{P}|(q-1)/(q^{t+1}-1)$ by σ, i.e., σ is the size of a hypothetical t-spread of $\mathcal{P}$. Let H be an arbitrary hyperplane of $\mathrm{PG}(n,q)$. Consider the system of equations

$$\alpha + \beta = \sigma, \tag{4.9}$$

$$\alpha v_{t+1} + \beta v_t = |H \cap \mathcal{P}|. \tag{4.10}$$

One verifies that for any classical polar space $\mathcal{P}$ and for any hyperplane H the solutions α, β to this system are integers. This means that the point set of $H \cap \mathcal{P}$ can be partitioned in α sets of size v_{t+1} and β sets of size v_t satisfying $\alpha + \beta = \sigma$. Below, such a partition will be denoted by A. Note that if $\mathcal{P}$ has a t-spread $\mathcal{S}$, then the set $\{\pi_t \cap H : \pi_t \in \mathcal{S}\}$ is such a partition.

Now suppose that H contains α' elements of $\mathcal{S}'$ and intersects β' elements of $\mathcal{S}'$ in a $(t-1)$-space. Then α' and β' satisfy

$$\alpha' + \beta' = \sigma - \delta, \tag{4.11}$$

$$\alpha' v_{t+1} + \beta' v_t \leq |H \cap \mathcal{P}|. \tag{4.12}$$

Let $\mathsf{A}' = \{\pi_t \cap H : \pi_t \in \mathcal{S}'\}$ and consider the following cases.

Case 1. Assume $\alpha' > \alpha$. Let $\alpha' = \alpha + a$ for some positive integer a. A partition A as above of $H \cap \mathcal{P}$ has α elements of size v_{t+1}. Using the inequality $aqv_t < av_{t+1}$, it follows that if A' has a elements of size v_{t+1} more than A, then the number β' of elements of A' that have size v_t is more than qa less than β, the number of elements of A that have size v_t. In other words, $\beta' < \beta - qa$. This means that $\alpha' + \beta'$, which equals $\alpha + \beta - \delta$, is smaller than $\alpha + \beta - (q-1)a$. Since a is greater than zero, this implies that the deficiency δ of $\mathcal{S}'$ is at least q, a contradiction.

Case 2. Assume $\alpha' = \alpha$. Then, by (4.11), $\beta' = \beta - \delta$, implying that the number of holes in H equals δv_t.

Case 3. Assume $\alpha' < \alpha$. If $\alpha' + \beta'$ would equal σ, then there would be at least $v_{t+1} - v_t$ holes in H. But since $\alpha' + \beta' < \sigma$, there are at least $v_{t+1} - v_t + v_t$ holes in H. This number is greater than qv_t, which in its turn is greater than δv_t.

So, every hyperplane of $\mathrm{PG}(n,q)$ contains at least δv_t holes. Clearly, the total number of holes in $\mathcal{P}$ is δv_{t+1}. Theorem 2.1.8 states that such a set is a $\{\delta v_{t+1}, \delta v_t; n, q\}$-minihyper. □

Remark 4.6.2 The proofs of the different cases are in fact the same proofs as in the respective cases of Theorem 4.2.2. They are only formulated in a slightly more geometrical way.

Remember the definition of ϵ_q in Notation 2.2.4.

Corollary 4.6.3 *Let $\mathcal{P}$ be a finite classical polar space in $\mathrm{PG}(n,q)$ whose size admits a t-spread. Suppose furthermore that if $\mathcal{P} = \mathrm{W}_n(q)$, then q is even.*

1. *Every partial t-spread $\mathcal{S}'$ of deficiency $\delta < \epsilon_q$ of $\mathcal{P}$ can be extended to a t-spread of $\mathcal{P}$.*
2. *Suppose $q > 16$ is a square, and $\delta < q^{5/8}/\sqrt{2} + 1$. If $\mathcal{S}'$ is a maximal partial t-spread of $\mathcal{P}$ of deficiency δ, then the set of holes forms a disjoint union of subgeometries $\mathrm{PG}(2t+1, \sqrt{q})$, implying $\delta \equiv 0 \pmod{\sqrt{q}+1}$.*

Proof Using Theorems 2.2.7 and 3.4.2, it is clear that these corollaries hold in the case that $\mathcal{P}$ is a quadric or a Hermitian variety. To see that they also hold for the remaining case, the symplectic space $\mathrm{W}_{2n+1}(q)$ with q even, it suffices to remember Theorem 1.1.3.1, which states that, for q even, $\mathrm{W}_{2n+1}(q)$ is isomorphic to $\mathrm{Q}(2n+2, q)$. □

Remark 4.6.4 Suppose that $n \leq \sqrt{q}$. If the point set of $\mathcal{P}$ or the point set of $\mathrm{PG}(n,q) \setminus \mathcal{P}$ can be partitioned by a set of subspaces of $\mathrm{PG}(n,q)$ that may have different dimensions (but greater than zero), then the weight argument of Blokhuis and Metsch [19] that was mentioned in the proof of Theorem 4.2.3 shows that also the case $\delta = \sqrt{q} + 1$ of Corollary 4.6.3.2 cannot occur. This holds in particular for $\mathcal{P} = \mathrm{W}_n(q)$, q even, since all points of $\mathrm{PG}(n,q)$ are absolute with respect to the polarity corresponding to $\mathcal{P}$.

Corollary 4.6.5 *If $q > 16$ is a square, $n \leq \sqrt{q}$, $t+1 \mid 2n+2$, and $\mathcal{S}'$ is a maximal partial t-spread of $\mathrm{W}_{2n+1}(q)$, q even, of deficiency $\delta < q^{5/8}/\sqrt{2} + 1$, then $\delta = k(\sqrt{q}+1)$ for some $k \geq 2$, and the set of holes forms a disjoint union of k subgeometries $\mathrm{PG}(2t+1, \sqrt{q})$.*

A nonsingular quadric Q_n in $\mathrm{PG}(n,q)$ cannot contain a Baer subspace of dimension d greater than the dimension of a generator of Q_n, since such a Baer subspace would generate a totally singular subspace of dimension d. Consequently, Corollary 4.6.3.2 can be refined in the case that $\mathcal{P}$ is a quadric.

Corollary 4.6.6 *Suppose* Q_n *is a nonsingular quadric in* $\mathrm{PG}(n, q)$ *whose size admits a* t*-spread, where* $q > 16$ *is a square, and* $2t+1$ *is greater than the dimension of a generator of* Q_n*. Then, every partial* t*-spread of deficiency* $\delta < q^{5/8}/\sqrt{2} + 1$ *can be extended to a* t*-spread of* Q_n*.*

Corollary 4.6.3 does not include the case $\mathcal{P} = \mathrm{W}_n(q)$, q odd. This case is considered separately, obtaining a result similar to the result on partial ovoids on the generalised hexagon $\mathrm{H}(q)$, see Subsection 5.4.2. Unfortunately, we have to restrict ourselves to partial n-spreads of $\mathrm{W}_{2n+1}(q)$.

Corollary 4.6.7 *Let* $\mathcal{S}'$ *be a maximal partial* n*-spread of* $\mathrm{W}_{2n+1}(q)$*,* q *odd, with deficiency* δ*. Suppose that either* $\delta < \epsilon_q$*, or* $q > 16$ *is a square and* $\delta < q^{5/8}/\sqrt{2} + 1$*. Then* δ *is even.*

Proof By the previous theorems, the set of holes is a unique disjoint union of n-spaces and—in the case that q is a square—subspaces $\mathrm{PG}(2n + 1, \sqrt{q})$. Note that each Baer subspace $\mathrm{PG}(2n + 1, \sqrt{q})$ yields an additional amount of $\sqrt{q} + 1$ to the deficiency. Since $\sqrt{q} + 1$ is even, these Baer subspaces can be omitted from the remainder of the discussion.

Note that if π_n is an n-space consisting entirely of holes, then π_n is one of the spaces of the unique disjoint union of n-spaces and subspaces $\mathrm{PG}(2n + 1, \sqrt{q})$. Otherwise δ would clearly have to be greater than q.

So, suppose π_n is an n-space consisting entirely of holes. Since $\mathcal{S}'$ is maximal, $\pi_n \neq \pi_n^\perp$. Let $P \in \pi_n^\perp$ and suppose P is covered by an element π'_n of $\mathcal{S}'$. Then $\pi'_n \subseteq P^\perp$ and $\pi_n \subseteq P^\perp$. But $P^\perp$ is $2n$-dimensional, implying that $\pi'_n \cap \pi_n \neq \emptyset$, a contradiction.

Therefore, also $\pi_n^\perp$ consists entirely of holes. As $\pi_n \neq \pi_n^\perp$, it follows that π_n and $\pi_n^\perp$ must be distinct n-spaces from the unique disjoint union of n-spaces and subspaces $\mathrm{PG}(2n + 1, \sqrt{q})$ that the set of holes consists of. Hence, each n-space π_n in the minihyper corresponding to $\mathcal{S}'$, is paired to a unique n-space $\pi_n^\perp$ in this minihyper. Therefore, the number of n-spaces in the minihyper is even. □

4.7 t-Covers in polar spaces

In the first part of this section, the result on minihypers from Chapter 2 is applied to t-covers of finite classical polar spaces giving a characterisation of the set of multiple points.

In the second part, the uniqueness of the smallest line cover of $\mathrm{Q}(4, 3)$ is proved. This cover will later on, in Section 5.5, be used to construct blocking sets of $\mathrm{W}_{2n+1}(q)$.

4.7.1 Application of the results on minihypers

As was the case for t-covers of projective spaces, the multiple points of a t-cover of a finite classical polar space are distributed in a special way. They form a minihyper in $\mathrm{PG}(n,q)$ with the same parameters as the minihypers studied in Chapter 2. Hence, the results from that chapter allow to make some observations on the structure of the set of multiple points of a t-cover.

Theorem 4.7.1 *Let $\mathcal{P}$ be a finite classical polar space in $\mathrm{PG}(n,q)$ whose size admits a t-spread, i.e., that satisfies the necessary conditions of Theorem 4.5.3. If $\mathcal{C}$ is a t-cover of $\mathcal{P}$ with excess $\varepsilon < q$, then the weight function $w(P) = surplus(P)$ for $P \in \mathcal{P}$ defines an $\{\varepsilon v_{t+1}, \varepsilon v_t; n, q\}$-minihyper (F, w), where F is the set of multiple points of $\mathcal{P}$, i.e., the set of points of $\mathcal{P}$ that are covered at least twice by elements of $\mathcal{C}$.*

Proof This proof is very similar to the proof of Theorem 4.3.1.

Denote the number $|\mathcal{P}|(q-1)/(q^{t+1}-1)$ by σ, i.e., σ is the size of a hypothetical t-spread of $\mathcal{P}$. Now let H be an arbitrary hyperplane of $\mathrm{PG}(n,q)$. Consider the system of equations

$$\alpha + \beta = \sigma,$$
$$\alpha v_{t+1} + \beta v_t = |H \cap \mathcal{P}|.$$

For any classical polar space $\mathcal{P}$ and for any hyperplane H, the solutions α, β to this system are integers.

Now suppose that H contains α' elements of $\mathcal{C}$ and intersects β' elements of $\mathcal{C}$ in a $(t-1)$-space. Then α' and β' satisfy

$$\alpha' + \beta' = \sigma + \varepsilon,$$
$$\alpha' v_{t+1} + \beta' v_t \geq |H \cap \mathcal{P}|.$$

As in the proof of Theorem 4.3.1, one verifies that $\alpha' \geq \alpha$. So, for each hyperplane H of $\mathrm{PG}(n,q)$, $\sum_{P \in H \cap \mathcal{P}} w(P) \geq \varepsilon v_t$. Clearly, $\sum_{P \in \mathcal{P}} w(P) = \varepsilon v_{t+1}$. Theorem 2.1.8 states that such a set is an $\{\varepsilon v_{t+1}, \varepsilon v_t; n, q\}$-minihyper. $\square$

Corollary 4.7.2 *Let $\mathcal{P}$ be a finite classical polar space in $\mathrm{PG}(n,q)$ whose size admits a t-spread. If $\mathcal{C}$ is a t-cover of $\mathcal{P}$ with excess $\varepsilon < \epsilon_q$, then the function surplus is the weight function induced on the point set of $\mathcal{P}$ by a sum of δ t-spaces.*

Remark 4.7.3 Corollary 4.7.2 was proved by Eisfeld et al. [49] in the special case that $\mathcal{P}$ is a finite classical generalised quadrangle, i.e., when $\mathcal{P}$ is either $\mathrm{Q}^+(3,q)$, $\mathrm{Q}(4,q)$, $\mathrm{Q}^-(5,q)$, $\mathrm{H}(3,q^2)$, $\mathrm{H}(4,q^2)$ or $\mathrm{W}_3(q)$ and $\mathcal{C}$ is a line cover of $\mathcal{P}$.

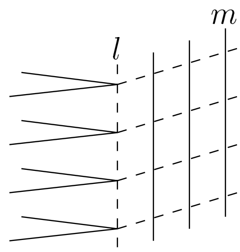

Figure 4.1: The smallest cover of Q$(4,3)$

4.7.2 The smallest cover of Q$(4,3)$

Remember from Section 4.4 the following theorem.

Theorem 4.7.4 (Eisfeld et al. [49]) *Let $\mathcal{C}$ be a cover on* Q$(4,q)$*, q odd. Then $|\mathcal{C}| > q^2+1+(q-1)/3$.*

In the same article [49], the following construction of a minimal cover of Q$(4,q)$ of size $q^2+1+(q-2)$ is given. Fix a line l on Q$(4,q)$ and consider the set $\mathcal{C}'$ of all q^2+q lines different from l that intersect l. Now let H be a 3-space containing l that intersects Q$(4,q)$ in a hyperbolic quadric Q$^+(3,q)$ and let $\mathcal{R}$ be the regulus of Q$^+(3,q)$ that does not contain l. Then the set $\mathcal{C} = (\mathcal{C}' \setminus \mathcal{R}) \cup (\mathcal{R}^{\mathrm{opp}} \setminus \{l\})$ is a minimal cover of Q$(4,q)$ of size $q^2+1+(q-2)$.

Theorem 4.7.5 (Blokhuis et al. [21]) *If $\mathcal{C}$ is a minimal cover with excess* 1 *of a generalised quadrangle, then there exists a line l in the generalised quadrangle that does not belong to the cover, all points of which are covered twice.*

Using this theorem, it is not hard to show that in the case $q=3$, the construction above gives the only cover of size 11 on Q$(4,3)$.

Theorem 4.7.6 *A cover $\mathcal{C}$ of* Q$(4,3)$ *has size at least* 11*. If $\mathcal{C}$ is a cover of* Q$(4,3)$ *of size* 11*, then there exist two disjoint lines l and m on* Q$(4,3)$ *such that $\mathcal{C}$ consists of, see also Figure* 4.1*,*

- *the lines different from l that intersect l but not m, and*
- *the lines different from l of the regulus on* Q$(4,3)$ *containing l and m.*

Proof By Theorem 4.7.5, there exists a line l on Q$(4,3)$, $l \notin \mathcal{C}$, such that through each point of l there pass exactly two lines of $\mathcal{C}$. Denote the remaining lines of Q$(4,3)$ intersecting l by m_i, $i=1,\ldots,4$. Every point of

$Q(4,3) \setminus l$ lies on exactly one line of $\mathcal{C}$. Let m be one of the three remaining lines of $\mathcal{C}$ not intersecting l. Then m contains a point on each of the lines m_i, $i = 1, \ldots, 4$. Looking only at m_1, m_2 and m_3, it follows that l and m belong to the opposite regulus of $\mathcal{R}(m_1, m_2, m_3)$. Clearly, also the two remaining lines of $\mathcal{C}$ belong to this regulus. □

This theorem can be dualised, see Theorem 5.5.1.

Remark 4.7.7 Very recently, the minimal covers of size 12 of $Q(4,3)$ were characterised, see [54]. They are always the known example. Using the notation of generalised quadrangles, they are a set $l^{\perp} \setminus \{l\}$ for some line l of $Q(4,3)$.

Chapter 5

Partial ovoids and blocking sets

Various results on partial ovoids and blocking sets are presented. Some of them are applications of the results on minihypers from Chapters 2 and 3. Those results are collected in Subsections 5.4.1 and 5.4.3, and were published in *European Journal of Combinatorics* in *P. Govaerts, L. Storme, and H. Van Maldeghem, On a particular class of minihypers and its applications. III. Applications* [57].

5.1 Introduction

Let $\mathcal{P}_m$ denote a finite classical polar space of rank k in $\mathrm{PG}(m, q^*)$, where $q^* = q^2$ if $\mathcal{P}_m$ is a Hermitian variety and $q^* = q$ otherwise. A *partial ovoid* O' of $\mathcal{P}_m$ is a set of points of $\mathcal{P}_m$ such that no generator of $\mathcal{P}_m$ contains more than one point of O'. A *blocking set* B of $\mathcal{P}_m$ is a set of points of $\mathcal{P}_m$ that has nonempty intersection with every generator of $\mathcal{P}_m$. A set O of points on $\mathcal{P}_m$ that is both a partial ovoid and a blocking set is called an *ovoid* of $\mathcal{P}_m$. Hence an ovoid of $\mathcal{P}_m$ is a set of points of $\mathcal{P}_m$ that intersects every generator of $\mathcal{P}_m$ in exactly one point.

Suppose that O is an ovoid of $\mathcal{P}_m$ and count the elements of the set $\{(P, G) : P \in O, G \in \mathcal{G}(\mathcal{P}_m), P \in G\}$, where $\mathcal{G}(\mathcal{P}_m)$ is the set of generators of $\mathcal{P}_m$, in two ways. The equality $|O| \cdot |\mathcal{G}(\mathcal{P}_{m-2})| = 1 \cdot |\mathcal{G}(\mathcal{P}_m)|$ is obtained. Hence, if $\mathcal{P}_m$ has an ovoid, then $|\mathcal{G}(\mathcal{P}_{m-2})|$ divides $|\mathcal{G}(\mathcal{P}_m)|$. Using the values from Table 1.2, one easily checks that this is always the case and that the size of a hypothetical ovoid of $\mathcal{P}_m$, which will here be denoted by $o(\mathcal{P}_m)$, is as given in Table 5.1. However, divisibility of $|\mathcal{G}(\mathcal{P}_m)|$ by $|\mathcal{G}(\mathcal{P}_{m-2})|$ does not imply the existence of an ovoid of $\mathcal{P}_m$. In Table 5.2 an overview is presented of the cases where it is known whether $\mathcal{P}_m$ has an ovoid or not.

$\mathcal{P}$	$\mathrm{rk}(\mathcal{P})$	$o(\mathcal{P})$
$\mathrm{W}_{2n+1}(q)$	$n+1$	$q^{n+1}+1$
$\mathrm{Q}^-(2n+1,q)$	n	$q^{n+1}+1$
$\mathrm{Q}(2n,q)$	n	q^n+1
$\mathrm{Q}^+(2n+1,q)$	$n+1$	q^n+1
$\mathrm{H}(2n,q^2)$	n	$q^{2n+1}+1$
$\mathrm{H}(2n+1,q^2)$	$n+1$	$q^{2n+1}+1$

Table 5.1: Finite classical polar spaces: rank and size of an ovoid

Remark 5.1.1 The size of a hypothetical ovoid of $\mathcal{P}$ equals the size of a hypothetical spread of $\mathcal{P}$, see Tables 4.2 and 5.2. This fact is explained by Shult and Thas [103], who introduce *m-systems*, certain sets of subspaces on finite classical polar spaces, the extremal cases of which are on one side ovoids and on the other side spreads of these polar spaces.

If O' is a partial ovoid of $\mathcal{P}_m$, then the *deficiency* of O' is by definition the number $o(\mathcal{P}_m) - |O'|$. A partial ovoid of $\mathcal{P}_m$ is called *maximal* if it is not contained in a larger partial ovoid of $\mathcal{P}_m$.

If B is blocking set of $\mathcal{P}_m$, then the *excess* of B is by definition the number $|B| - o(\mathcal{P}_m)$. A blocking set of $\mathcal{P}_m$ is called *minimal* if it has no proper subset that is a blocking set of $\mathcal{P}_m$.

5.2 Ovoids on $\mathrm{Q}(6,q)$

In the theory of minimal t-fold blocking sets, $t \bmod p$ results have proved to be very useful. Such results tell "how" a subspace intersects the minimal t-fold blocking sets: in $t \pmod p$ points. They make the blocking sets easier to handle and have made several classification theorems possible. Theorems 1.2.6 and 1.2.13 are examples of $t \bmod p$ results.

One can hope that similar results for other objects will prove to be equally fruitful. Ball [5] proves a $1 \bmod p$ result for ovoids on the quadric $\mathrm{Q}(4,q)$.

Theorem 5.2.1 (Ball [5]) *Let O be an ovoid on $\mathrm{Q}(4,q)$, $q = p^h$, p prime. Every elliptic quadric $\mathrm{Q}^-(3,q)$ on $\mathrm{Q}(4,q)$ intersects O in $1 \pmod p$ points.*

The following theorem uses the previous one to show that also for ovoids of $\mathrm{Q}(6,q)$, a $1 \bmod p$ property holds.

Theorem 5.2.2 *An ovoid O of $\mathrm{Q}(6,q)$, $q = p^h$, p prime, intersects every elliptic quadric $\mathrm{Q}^-(5,q)$ on $\mathrm{Q}(6,q)$ in $1 \pmod p$ points.*

$\mathcal{P}$	Restrictions	Ovoids	References
$W_{2n+1}(q)$	$n = 1$, q even	yes	[109]
	$n = 1$, q odd	no	[109]
	$n \geq 2$	no	[112]
$Q^-(2n + 1, q)$	—	no	[112]
$Q(2n, q)$	$n = 2$	yes	[74, 81, 93, 117]
	$n \geq 3$, q even	no	[112]
	$n = 3$, $q = 3^h$	yes	[74, 111, 112]
	$n = 3$, $q = 5, 7$	no	[90]
	$n \geq 4$, q odd	no	[59]
$Q^+(2n + 1, q)$	$n = 1, 2$	yes	[67]
	$n = 3$, q odd with q prime or $q \equiv 0$ or $2 \pmod 3$	yes	[34, 44, 74, 75, 76], [88, 101, 111, 112]
	$q = p^h$, p prime and (5.19) holds	no	[20]
$H(2n, q^2)$	—	no	[112]
$H(2n + 1, q^2)$	$n = 1$	yes	[91, 113, 117]
	$q = p^h$, p prime and (5.20) holds	no	[89]

Table 5.2: Existence of ovoids in finite classical polar spaces
The references are copied from [116].

Proof Let $P \in \mathrm{Q}(6,q) \setminus O$ and let π_4 be a 4-space in $T_P(\mathrm{Q}(6,q))$ that does not contain P. Then π_4 intersects $\mathrm{Q}(6,q)$ in a parabolic quadric Q_4 and $T_P(\mathrm{Q}(6,q)) \cap O$ is projected from P onto an ovoid O^* of Q_4. Every elliptic quadric $\mathrm{Q}^-(3,q)$ of Q_4 contains exactly $1 \pmod p$ points of O^*. Hence, for each point $P \in \mathrm{Q}(6,q) \setminus O$, every cone $P\mathrm{Q}^-(3,q)$ on $\mathrm{Q}(6,q)$ shares $1 \pmod p$ points with O.

Let Q_5^- be any elliptic quadric $\mathrm{Q}^-(5,q)$ on $\mathrm{Q}(6,q)$ and denote the hyperplane of $\mathrm{PG}(6,q)$ containing it by H. Consider a line l of $\mathrm{Q}^-(5,q)$ skew to O. Such a line exists since $\mathrm{Q}^-(5,q)$ has no ovoids, see Table 5.2. Denote the points on l by $P_0, P_1, \ldots, P_q$. The tangent hyperplanes $T_{P_i}(\mathrm{Q}(6,q))$ intersect H in 4-spaces $T_{P_i}(\mathsf{Q}_5^-)$ for each $i \in \{0,1,\ldots,q\}$. Let

$$\begin{aligned} \pi_3 &= T_{P_0}(\mathsf{Q}_5^-) \cap T_{P_1}(\mathsf{Q}_5^-) \\ &= T_{P_0}(\mathrm{Q}(6,q)) \cap T_{P_1}(\mathrm{Q}(6,q)) \cap H. \end{aligned}$$

Suppose $|\pi_3 \cap O| = x$. The 4-spaces $T_{P_i}(\mathsf{Q}_5^-) \subseteq T_{P_i}(\mathrm{Q}(6,q))$, $i = 0,1,\ldots,q$, define a pencil of 4-spaces through π_3 in H. Since $|T_{P_i}(\mathsf{Q}_5^-) \cap O| \equiv 1 \pmod p$, counting the number of points of O in this pencil of 4-spaces yields

$$\begin{aligned} |\mathsf{Q}_5^- \cap O| &\equiv (q+1)\cdot 1 - qx \pmod p \\ &\equiv 1 \pmod p. \end{aligned}$$

Since Q_5^- was chosen arbitrarily, the theorem is proved. □

Remark 5.2.3 It is not necessary to prove a similar result for higher dimensional parabolic quadrics, since it is known that such quadrics have no ovoids, see Table 5.2.

5.3 Inductive theorems

In this section, two theorems are proved. The first, respectively second, one "lifts" an upper, respectively a lower, bound on the size of partials ovoids, respectively blocking sets, of a finite classical polar space in a space with given dimension to an upper, respectively a lower, bound on the size of partial ovoids, respectively blocking sets, of finite classical polar spaces of the same type in higher dimensional spaces.

The same proof can be applied to lift bounds on the size of partial ovoids and to lift bounds on the size of covers. Also, the proof is independent of the type of the polar space considered. To fix ideas, suppose a lower bound on the size of minimal blocking sets of $\mathrm{H}(2n,q^2)$ exists, i.e., suppose every blocking set of $\mathrm{H}(2n,q^2)$ has size greater than $q^{2n+1}+1+\varepsilon$.

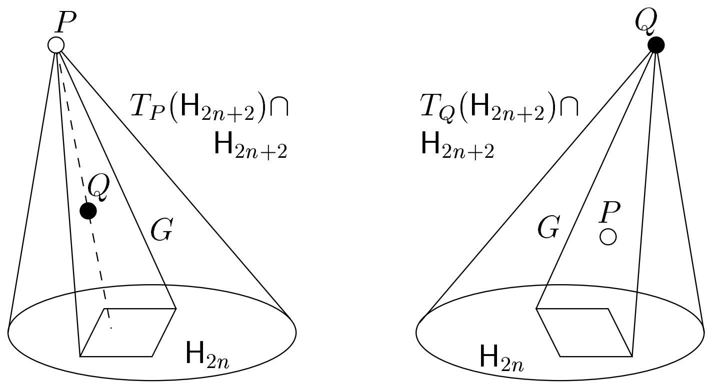

Figure 5.1: Two ways of counting the number of elements in the set X

Consider the variety $\mathrm{H}(2n+2, q^2)$ and denote it by H_{2n+2}. Let P be a point on H_{2n+2}. Remember that the tangent hyperplane at P to H_{2n+2} is denoted by $T_P(\mathsf{H}_{2n+2})$. If π_{2n} is a hyperplane in $T_P(\mathsf{H}_{2n+2})$ not containing P, then π_{2n} intersects H_{2n+2} in a Hermitian variety $\mathrm{H}(2n, q^2)$. And, if B is a blocking set of H_{2n+2} and $P \in \mathsf{H}_{2n+2} \setminus B$, then P projects the points of $B \cap T_P(\mathsf{H}_{2n+2})$ onto a blocking set of $\mathrm{H}(2n, q^2)$ in π_{2n}.

Now suppose B is a blocking set of H_{2n+2} of size $q^{2n+3}+1+\varepsilon'$ and consider the set

$$X = \{(P, Q, G) : P \in \mathsf{H}_{2n+2} \setminus B, Q \in B, G \in \mathcal{G}(\mathsf{H}_{2n+2}), PQ \subseteq G\}.$$

The size of this set can be counted in two ways. One way to count it is to start with the point P, then choose Q in $T_P(\mathsf{H}_{2n+2}) \cap B$ and finish with a generator G containing both P and Q. The second way to count the size of the set is to start with the point $Q \in B$, then take a generator G containing it, followed by a point $P \in G \setminus B$. These two approaches are depicted in Figure 5.1.

The size of the set X is given by

$$\begin{aligned} |X| \;&=\; \underbrace{\left(\frac{(q^{2n+3}+1)(q^{2n+2}-1)}{q^2-1} - q^{2n+3} - 1 - \varepsilon'\right)}_{\text{(i)}} \cdot \\ &\quad \underbrace{(q^{2n+1}+1+\mu)}_{\text{(ii)}}\underbrace{(q^3+1)(q^5+1)\dots(q^{2n-1}+1)}_{\text{(iii)}}, \end{aligned} \tag{5.1}$$

where (i) equals the number of points of $\mathsf{H}_{2n+2} \setminus B$, (ii) denotes the average number of points of B in a tangent hyperplane $T_P(\mathsf{H}_{2n+2})$ at a point $P \in$

$\mathsf{H}_{2n+2} \setminus B$, and (iii) equals the number of generators of H_{2n+2} through a line of H_{2n+2}.

An upper bound on the size of X is given by

$$|X| \leq \underbrace{(q^{2n+3}+1+\varepsilon')}_{\text{(i)}}\underbrace{(q^3+1)(q^5+1)\dots(q^{2n+1}+1)}_{\text{(ii)}}\underbrace{\left(\frac{q^{2(n+1)}-1}{q^2-1}-1\right)}_{\text{(iii)}}, \tag{5.2}$$

since (i) equals the number of points of B, (ii) is the number of generators of H_{2n+2} through a point of H_{2n+2}, and (iii) is the maximal number of points of $\mathsf{H}_{2n+2} \setminus B$ in a generator of H_{2n+2}.

By the assumption that every blocking set of $\mathrm{H}(2n, q^2)$ has size greater than $q^{2n+1}+1+\varepsilon$, it follows that $\mu > \varepsilon$. Substituting this inequality in (5.1) and comparing the thus obtained expression with (5.2) yields

$$\varepsilon' > \varepsilon \frac{q^{4n+3}+q^{4n+1}+\ldots+q^{2n+5}+q^{2n}+q^{2n-2}+\ldots+q^2}{q^{4n+1}+q^{4n-1}+\ldots+q^{2n+1}+q^{2n}+q^{2n-2}+\ldots+1+\varepsilon}. \tag{5.3}$$

The same counting argument can be applied to the other finite classical polar spaces. Suppose every blocking set of $\mathcal{P}_m$ has size greater than $o(\mathcal{P}_m)+\varepsilon$. Let B be a blocking set of $\mathcal{P}_{m+2}$ of size $o(\mathcal{P}_{m+2})+\varepsilon'$.

If $\mathcal{P}_m = \mathrm{H}(2n+1, q^2)$, then

$$\varepsilon' > \varepsilon \frac{q^{4n+5}+q^{4n+3}+\ldots+q^{2n+5}+q^{2n+2}+q^{2n}+\ldots+q^2}{q^{4n+3}+q^{4n+1}+\ldots+q^{2n+3}+q^{2n+2}+q^{2n+1}+q^{2n}+q^{2n-2}+\ldots+1+\varepsilon}; \tag{5.4}$$

if $\mathcal{P}_m = \mathrm{Q}^-(2n+1, q)$, then

$$\varepsilon' > \varepsilon \frac{q^{2n+2}+q^{2n+1}+\ldots+q^{n+3}+q^{n}+q^{n-1}+\ldots+q}{q^{2n+1}+q^{2n}+\ldots+1+\varepsilon}; \tag{5.5}$$

if $\mathcal{P}_m = \mathrm{Q}(2n, q)$, then

$$\varepsilon' > \varepsilon \frac{q^{2n+1}+q^{2n}+\ldots+q^{n+2}+q^{n}+q^{n-1}+\ldots+q}{q^{2n}+q^{2n-1}+\ldots+q^{n+1}+2q^{n}+q^{n-1}+q^{n-2}+\ldots+1+\varepsilon}; \tag{5.6}$$

if $\mathcal{P}_m = \mathrm{Q}^+(2n+1, q)$, then

$$\varepsilon' > \varepsilon \frac{q^{2n+2}+q^{2n+1}+\ldots+q}{q^{2n+1}+q^{2n}+\ldots+q^{n+2}+2q^{n+1}+2q^{n}+q^{n-1}+q^{n-2}+\ldots+1+\varepsilon}; \tag{5.7}$$

if $\mathcal{P}_m = \mathrm{W}_{2n+1}(q)$, then

$$\varepsilon' > \varepsilon \frac{q^{2n+3}+q^{2n+2}+\ldots+q^{n+3}+q^{n+1}+q^{n}+\ldots+q}{q^{2n+2}+q^{2n+1}+\ldots+q^{n+2}+2q^{n+1}+q^{n}+q^{n-1}+\ldots+1+\varepsilon}. \tag{5.8}$$

If ε is not too big, i.e., if some upper bound that is not too large exists for the size of a blocking set of $\mathcal{P}_m$, then the occurrence of ε in the denominator can be replaced by its upper bound and the new inequality can be approximated to obtain bounds on ε' that allow to immediately apply induction. Indeed, if in the respective cases—maintain the same ordering as above—$\varepsilon \le q^{4n-1}$, $\varepsilon \le q^{4n}$, $\varepsilon \le q^{2n}$, $\varepsilon \le q^{2n-1}$, $\varepsilon \le q^{2n}$ and $\varepsilon \le q^{2n+1}$, then $\varepsilon' > (q^* - 1)\varepsilon$, where $q^* = q^2$ if $\mathcal{P}_m$ is a Hermitian variety and $q^* = q$ in the other cases. Hence, Theorem 5.3.1 is proved. Since $\mathrm{Q}^+(5, q)$ has ovoids, for the polar spaces $\mathrm{Q}^+(2n + 1, q)$ the condition $n \ge 3$ is imposed.

Theorem 5.3.1 *Suppose that $\mathcal{P}_m$ is a finite classical polar space. Suppose that, if $\mathcal{P}_m = \mathrm{H}(2n, q^2)$, then $n \ge 2$ and $\varepsilon \le q^{4n-1}$; if $\mathcal{P}_m = \mathrm{H}(2n + 1, q^2)$, then $n \ge 1$ and $\varepsilon \le q^{4n}$; if $\mathcal{P}_m = \mathrm{Q}^-(2n + 1, q)$, then $n \ge 2$ and $\varepsilon \le q^{2n}$; if $\mathcal{P}_m = \mathrm{Q}(2n, q)$, then $n \ge 2$ and $\varepsilon \le q^{2n-1}$; if $\mathcal{P}_m = \mathrm{Q}^+(2n+1, q)$, then $n \ge 3$ and $\varepsilon \le q^{2n}$; if $\mathcal{P}_m = \mathrm{W}_{2n+1}(q)$, then $n \ge 1$ and $\varepsilon \le q^{2n+1}$. If every blocking set of $\mathcal{P}_m$ has size greater than $o(\mathcal{P}_m)+\varepsilon$, then, for each $i \ge 0$, every blocking set of $\mathcal{P}_{m+2i}$ has size greater than $o(\mathcal{P}_{m+2i}) + (q^* - 1)^i\varepsilon$, where $q^* = q^2$ if $\mathcal{P}_m$ is a Hermitian variety and $q^* = q$ in the other cases.*

The bounds on ε in Theorem 5.3.1 are no restriction, since for each of the polar spaces handled, there exist blocking sets whose size is smaller than ε_0, where ε_0 denotes the upper bound on ε from Theorem 5.3.1. Indeed, if $\mathcal{P}_m$ is a finite classical polar space with rank r in $\mathrm{PG}(m, q)$, then one easily checks that B, as defined below, is such a blocking set:

- if $\mathcal{P}_m$ is a Hermitian variety, then let $B = \mathcal{P}_m \cap \pi_{m-r+1}$, where π_{m-r+1} is an $(m - r + 1)$-space intersecting $\mathcal{P}_m$ in a nonsingular Hermitian variety;
- if $\mathcal{P}_m$ is a quadric, then let $B = \mathcal{P}_m \cap \pi_{m-r+1}$, where π_{m-r+1} is an $(m - r + 1)$-space intersecting $\mathcal{P}_m$ in a nonsingular quadric;
- if $\mathcal{P}_m$ is a symplectic space, then let $B = \pi_{m-r+1}$, where π_{m-r+1} is any $(m - r + 1)$-space in $\mathrm{PG}(m, q)$.

Hence the bounds on ε in Theorem 5.3.1 can be omitted.

Theorem 5.3.2 *Suppose that $\mathcal{P}_m$ is a finite classical polar space. Suppose that, if $\mathcal{P}_m = \mathrm{H}(2n, q^2)$, then $n \ge 2$; if $\mathcal{P}_m = \mathrm{H}(2n + 1, q^2)$, then $n \ge 1$; if $\mathcal{P}_m = \mathrm{Q}^-(2n + 1, q)$, then $n \ge 2$; if $\mathcal{P}_m = \mathrm{Q}(2n, q)$, then $n \ge 2$; if $\mathcal{P}_m = \mathrm{Q}^+(2n + 1, q)$, then $n \ge 3$; if $\mathcal{P}_m = \mathrm{W}_{2n+1}(q)$, then $n \ge 1$. If every blocking set of $\mathcal{P}_m$ has size greater than $o(\mathcal{P}_m) + \varepsilon$, then, for each $i \ge 0$, every blocking set of $\mathcal{P}_{m+2i}$ has size greater than $o(\mathcal{P}_{m+2i}) + (q^* - 1)^i\varepsilon$, where $q^* = q^2$ if $\mathcal{P}_m$ is a Hermitian variety and $q^* = q$ in the other cases.*

For partial ovoids of the finite classical polar spaces the same technique as above can be used. Suppose in these cases that every partial ovoid of $\mathcal{P}_m$ has size smaller than $o(\mathcal{P}_m) - \delta$ and that O is a partial ovoid of $\mathcal{P}_{m+2}$ of size $o(\mathcal{P}_{m+2}) - \delta'$. Then the inequalities obtained are very similar to the inequalities (5.3) throughout (5.8). It suffices to replace in the latter ε' by δ', the occurrence of ε in the numerator by δ and the occurrence of "$+\varepsilon$" in the denominator by "$-\delta$".

Also Theorem 5.3.1 can be copied. And, in the case of partial ovoids, it is immediately clear that the bound on δ disappears.

Theorem 5.3.3 *Suppose that $\mathcal{P}_m$ is a finite classical polar space. Suppose that, if $\mathcal{P}_m = \mathrm{H}(2n, q^2)$, then $n \geq 2$; if $\mathcal{P}_m = \mathrm{H}(2n+1, q^2)$, then $n \geq 1$; if $\mathcal{P}_m = \mathrm{Q}^-(2n+1, q)$, then $n \geq 2$; if $\mathcal{P}_m = \mathrm{Q}(2n, q)$, then $n \geq 2$; if $\mathcal{P}_m = \mathrm{Q}^+(2n+1, q)$, then $n \geq 3$; if $\mathcal{P}_m = \mathrm{W}_{2n+1}(q)$, then $n \geq 1$. If every partial ovoid of $\mathcal{P}_m$ has size smaller than $o(\mathcal{P}_m) - \delta$, then, for each $i \geq 0$, every partial ovoid of $\mathcal{P}_{m+2i}$ has size smaller than $o(\mathcal{P}_{m+2i}) - (q^* - 1)^i \delta$, where $q^* = q^2$ if $\mathcal{P}_m$ is a Hermitian variety and $q^* = q$ in the other cases.*

Remark 5.3.4 Note that the strict inequalities in this section can be replaced by non-strict inequalities, replacing "less than" by "at most" and "greater than" by "at least".

Other inductive bounds for the size of partial ovoids are known.

Theorem 5.3.5 (Klein [77]) *Let $O(\mathcal{P})$ denote the size of the largest partial ovoids of the finite classical polar space $\mathcal{P}$.*

(i) *For $n \geq 3$, $O(\mathrm{Q}^-(2n+1, q)) \leq \frac{q^n+1}{q^{n-1}+1}(O(\mathrm{Q}^-(2n-1, q)) - 2) + 2$.*

(ii) *For $n \geq 2$, $O(\mathrm{W}_{2n+1}(q)) \leq \frac{q^{2n}-1}{q^{2n-1}-1}(O(\mathrm{W}_{2n-1}(q)) - 2) + 2$.*

The upper bounds on the size of partial ovoids, respectively blocking sets, of finite classical polar spaces arising from application of these inductive theorems on known bounds are presented in Subsection 5.4.3, respectively Subsection 5.5.2.

5.4 Partial ovoids

This section consists of three parts. In the first one, an upper bound on the size of partial ovoids of $\mathrm{H}(4, q^2)$ is proved. In the second one, partial ovoids with small deficiency on the split Cayley hexagon $\mathrm{H}(q)$ are studied and a theorem reminiscent of Corollary 4.6.7 is obtained. The third part collects known upper bounds on the size of partial ovoids and applies Theorems 5.3.3 and 5.3.5 to lift them to higher dimensions. It also compares the new bounds with the Blokhuis-Moorhouse bounds.

5.4.1 Partial ovoids of $\mathrm{H}(4, q^2)$

A *k-cap* or *partial ovoid of size k* of a generalised quadrangle (GQ) is a set of k points on the GQ, no two of which are collinear. It is called *complete* or *maximal* if no points can be added to obtain a larger cap. A *blocking set* of a GQ is a set of points on the GQ that has nonempty intersection with every line. It is called *minimal* if the removal of a point from this set yields a set that no longer blocks every line of the GQ. A set of points of a $\mathrm{GQ}(s, t)$ that is a cap as well as a blocking set necessarily has size $st + 1$ and is called an *ovoid* of the GQ. If O' is a cap of a GQ $\mathcal{Q}$, then a *line external to* O', or simply an *external line*, is a line not containing any point of O'.

Since $\mathrm{H}(4, q^2)$ is a $\mathrm{GQ}(q^2, q^3)$, known results on k-caps of GQ's can be used in order to study partial ovoids on $\mathrm{H}(4, q^2)$.

Theorem 5.4.1 (Payne and Thas [91, §2.7]) *Suppose $\mathcal{Q}$ is a $\mathrm{GQ}(s, t)$.*

1. *Any $(st - \rho)$-cap of $\mathcal{Q}$ with $0 \leq \rho < t/s$ is contained in a uniquely defined ovoid of $\mathcal{Q}$. Hence if $\mathcal{Q}$ has no ovoid, then any k-cap of $\mathcal{Q}$ necessarily satisfies $k \leq st - t/s$.*

2. *Let O' be a complete $(st - t/s)$-cap of $\mathcal{Q} = (\mathsf{P}, \mathsf{B}, \mathtt{I})$. Let B' be the set of lines incident with no point of O'; let P' be the set of points on (at least one) line of B'; and let $\mathtt{I}'$ be the restriction of $\mathtt{I}$ to the points of P' and the lines of B'. Then $\mathcal{Q}' = (\mathsf{P}', \mathsf{B}', \mathtt{I}')$ is a subquadrangle of order $(s, t/s)$.*

From the theorem of Buekenhout and Lefèvre [31] and Theorem 5.4.1.2, it follows that if O' is a complete $(q^5 - q)$-cap of $\mathrm{H}(4, q^2)$, then the external lines to O' on $\mathrm{H}(4, q^2)$ form a Hermitian variety $\mathrm{H}(3, q^2)$. In the next corollary, it is shown that such caps do not exist.

Corollary 5.4.2 $\mathrm{H}(4, q^2)$ *has no complete cap of size $q^5 - q$.*

Proof Suppose that $\mathsf{H}_4 := \mathrm{H}(4, q^2)$ has a complete cap O' of size $q^5 - q$. Denote by H_3 the Hermitian variety $\mathrm{H}(3, q^2)$ that consists of the lines of H_4 external to O' and by π_3 the solid containing H_3. Let l be a line in π_3 that intersects H_3 in a Hermitian variety $\mathrm{H}(1, q^2)$; denote this variety by H_1. In π_3 there are $q^2 + 1$ planes through l, $q + 1$ of which intersect H_3 in a cone

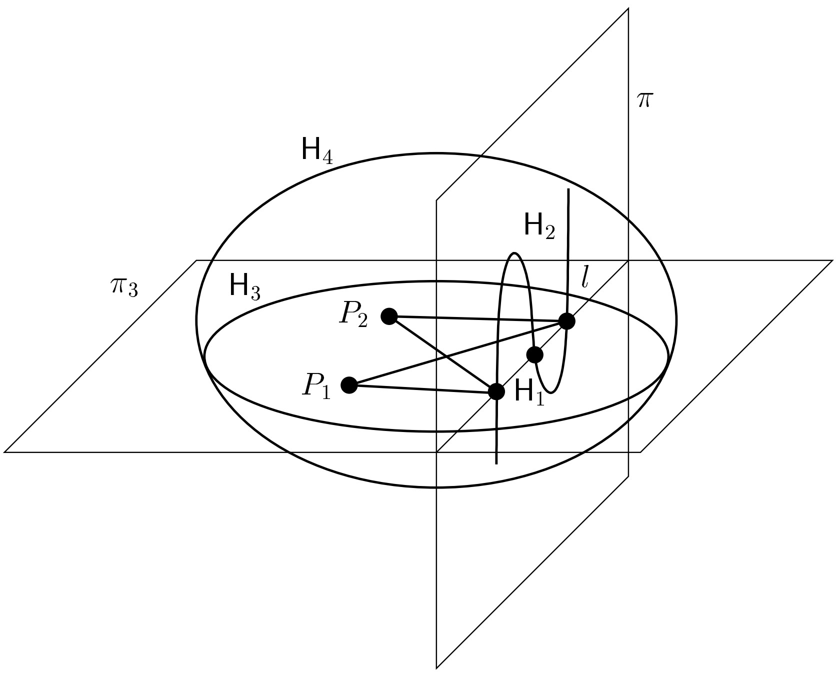

Figure 5.2: Intersections of $\mathrm{H}(4, q^2)$

$P\mathsf{H}_1$, for some $P \in \mathsf{H}_3$; the other $q^2 - q$ planes intersect H_3 in a Hermitian variety $\mathrm{H}(2, q^2)$.

Let P_1 and P_2 be two distinct points of H_3 such that the cones $P_1\mathsf{H}_1$ and $P_2\mathsf{H}_1$ lie on H_3. Now consider the solids $T_{P_1}(\mathsf{H}_4)$ and $T_{P_2}(\mathsf{H}_4)$. They intersect in a plane π containing neither P_1 nor P_2, but containing l. Clearly, π is not contained in π_3; so it intersects π_3 in l. The plane π intersects H_4 in a Hermitian variety $\mathrm{H}(2, q^2)$ containing H_1; denote this variety by H_2. The present configuration is depicted in Figure 5.2.

There are $q^2 + 1$ solids on π, $q + 1$ of which intersect H_4 in a cone $P_i'\mathsf{H}_2$, $i = 0, 1, \ldots, q$; the $q^2 - q$ other ones intersect H_4 in a Hermitian variety $\mathrm{H}(3, q^2)$. The vertices P_i', $i = 0, 1, \ldots, q$, are collinear and the line joining them is the polar line of π, a $(q+1)$-secant to H_4 that is skew to π. Since P_1 and P_2 lie on this line, all $q + 1$ of these points lie in H_3.

Now suppose π contains x points of O' and count the points of O' by counting the points of O' in the hyperplanes through π. This yields

$$q^5 - q = x + (q + 1)(q^3 - q - x) + (q^2 - q)(q^3 + 1 - x), \qquad (5.9)$$

since each one of the $q+1$ cones $P_i'\mathsf{H}_2$, $i = 0, 1, \ldots, q$, contains $q^3+1-(q+1)$ points of O' and each one of the q^2-q nonsingular Hermitian varieties $\mathrm{H}(3, q^2)$

through H_2 contains q^3+1 points of O' since they do not contain external lines to O'. From (5.9), x can be calculated, yielding $x = q - 1/q$, which is not an integer. A contradiction is obtained. □

Remark 5.4.3 This result was proved independently by K. Thas in [119].

As seen above, the external lines in $\mathrm{H}(4, q^2)$ to a hypothetical complete partial ovoid of size $q^5 - q$ would have formed a Hermitian variety $\mathrm{H}(3, q^2)$. One might suspect that, for a complete partial ovoid O' of size $q^5 - q - x$ of $\mathrm{H}(4, q^2)$, x small, and a given external line to O' in $\mathrm{H}(4, q^2)$, through this line there exists a Hermitian variety $\mathrm{H}(3, q^2)$ that contains many external lines. This actually happens, and this observation can be used to improve upon the bound on the size of partial ovoids of $\mathrm{H}(4, q^2)$ that is implied by Corollary 5.4.2. To obtain the new bound, once more results on caps of GQ's will be used.

Theorem 5.4.4 (Payne and Thas [91, §2.7]) *Suppose $\mathcal{Q}$ is a $\mathrm{GQ}(s, t)$ and let O' be an $(st - \rho)$-cap of $\mathcal{Q}$. Let B' be the set of lines of $\mathcal{Q}$ incident with no point of O'. Then every line of B' is concurrent with $t + \rho$ other lines of B'. If O' is complete, then any point on a line of B' is incident with at most ρ other lines of B'.*

Also a result on the extendibility of partial ovoids of $\mathrm{H}(3, q^2)$ will be applied. It is an immediate corollary to Corollary 4.6.3.1, since $\mathrm{H}(3, q^2)$ is the dual of $\mathrm{Q}^-(5, q)$, see Theorem 1.1.3.4. For the definition of ϵ_q, see Notation 2.2.4.

Corollary 5.4.5 *Every partial ovoid of $\mathrm{H}(3, q^2)$ of deficiency $\delta < \epsilon_q$ can be extended to an ovoid of $\mathrm{H}(3, q^2)$.*

Theorem 5.4.6 *If O' is a partial ovoid of $\mathrm{H}(4, q^2)$, then $|O'| < q^5 - (4q - 1)/3$.*

Proof In [112], J. A. Thas proves that $\mathrm{H}(4, q^2)$ has no ovoid. By Theorem 5.4.1.2 and Corollary 5.4.2, this implies that a partial ovoid of $\mathrm{H}(4, q^2)$ has size smaller than $q^5 - q$. This proves the theorem for $q \in \{2, 3\}$.

Now, suppose by way of contradiction that O' is a maximal partial ovoid of $\mathrm{H}(4, q^2)$ of size $q^5 - q - x$, $x \leq (q-1)/3$. By the arguments above, $x > 0$ and $q > 3$. The main part of this proof will consist of showing that through each external line to O', there exists a Hermitian variety $\mathrm{H}(3, q^2)$ containing more than $(q^4 + q^3 + xq^3 + x)/2 + q + 1$ external lines.

A counting argument shows that the number of external lines equals $(q + 1 + x)(q^3 + 1)$. Let l be an external line. Then through l, there exists a plane

π containing q points of O'. This plane intersects $\mathrm{H}(4,q^2)$ in a cone; denote its vertex by P. Denote the hyperplanes through π by $H_1, H_2, \ldots, H_{q^2+1}$, and define the deficiency δ_i of H_i in the following way: $\delta_i = q^3+1-|H_i \cap O'|$. Each hyperplane H_i has—since l contains no point of O'—a "deficiency of 1 in the plane π". Denoting the deficiency of H_i outside π by $\delta_i' = \delta_i - 1$, the points of O' can be counted, resulting in

$$q + \sum_{i=1}^{q^2+1} (q^3 - q - \delta_i') = q^5 - q - x,$$

or

$$\sum_{i=1}^{q^2+1} \delta_i' = q + x. \tag{5.10}$$

Now suppose q is an odd prime, respectively $q = p^{2e}$, $q = p^{2e+1}$; here p is a prime and e is a positive integer. By Corollary 5.4.5, any partial ovoid on $\mathrm{H}(3,q^2)$ of deficiency at most $(q+1)/2$, respectively p^e, p^{e+1}, can be extended to an ovoid of $\mathrm{H}(3,q^2)$. Suppose that ξ hyperplanes have a deficiency $\delta_i > (q+1)/2$, respectively $\delta_i > p^e$, $\delta_i > p^{e+1}$; then these satisfy $\delta_i' \geq (q+1)/2$, respectively $\delta_i' \geq p^e$, $\delta_i' \geq p^{e+1}$, such that, by (5.10)

$$\xi \leq 2\frac{q+x}{q+1}, \text{ respectively } \xi \leq p^e + \frac{x}{p^e}, \ \xi \leq p^e + \frac{x}{p^{e+1}}. \tag{5.11}$$

Substitution of $x < q$ yields

$$\xi < 4, \text{ respectively } \xi < 2p^e, \ \xi < 2p^e,$$

such that in all three cases ξ satisfies

$$\xi \leq q. \tag{5.12}$$

Three—not necessarily disjoint—types of hyperplanes H_i can be distinguished:

(i) a tangent hyperplane;

(ii) hyperplanes with deficiency greater than $(q+1)/2$, respectively p^e, p^{e+1};

(iii) non-tangent hyperplanes with deficiency at most $(q+1)/2$, respectively p^e, p^{e+1}.

The hyperplanes of type (iii) intersect $\mathrm{H}(4,q^2)$ in a Hermitian variety $\mathrm{H}(3,q^2)$ and O' in a cap of this variety that is extendible to an ovoid of this variety. Therefore, for each such hyperplane, there exists a point P' on l that lies on a pencil of $q+1$ external lines in this hyperplane. Taking into account that, as implied by Theorem 5.4.4, no point of l can lie on two of the aforementioned pencils, by (5.12), there are at least $q^2-q>3$ such points on l.

Let L, M, N be three points of l, each one lying on a pencil of $q+1$ external lines: $l,l_1,\dots,l_q$; $l,m_1,\dots,m_q$; $l,n_1,\dots,n_q$. Define an *E-line* as being an external line different from l that intersects a line l_i, a line m_j and a line n_k.

We now show that such an E-line exists. By Theorem 5.4.4, there are at least q^4 external lines not through L that intersect one of the lines l_i, $i=1,\dots,q$. Let γ denote the number of external lines skew to $\bigcup_{j\in\{1,\dots,q\}} m_j$. Amongst these are the external lines different from l that intersect l in a point different from M. There are at least q^3 such lines. Surely they are different from the q^4 external lines not containing L that intersect a line l_i. Let γ' denote the number of external lines skew to $\bigcup_{k=1}^{q} n_k$. Then, there are at least

$$q^4-(\gamma-q^3)-(\gamma'-q^3) \tag{5.13}$$

E-lines.

An upper bound on γ can be obtained as follows. Let ζ denote the number of external lines through M. Then $\zeta\in\{q+1,\dots,q+x+1\}$ and the number of external lines intersecting $\bigcup_{j=1}^{q} m_j$ equals, by Theorem 5.4.4, $\zeta(1-q)+q^4+q^2+qx+q$. The total number of external lines equals $(q^3+1)(q+x+1)$, implying $\gamma=(x+1)q^3-q^2+\zeta(q-1)-qx+x+1$. Therefore γ is maximal when ζ is maximal and

$$\gamma\le(x+1)q^3. \tag{5.14}$$

Clearly, this bound on γ is also a bound on γ'.

Substituting $x\le(q+1)/3$ in these bounds for γ and γ' and taking into account that $q>2$, it follows that (5.13) is greater than zero. Therefore E-lines exist.

Let l^* be an E-line. Then l^* intersects a line l_i, a line m_j, and a line n_k, say l_1, m_1, and n_1. Hence, l and l_1 lie in the solid $\langle l,l^*\rangle$, such that all lines l_i, $i=1,2,\dots,q$, lie in $\langle l,l^*\rangle$. Similarly, also the lines m_j and n_k, $j,k=1,2,\dots,q$, are contained in this solid.

This means that all lines l, l_i, m_j, n_k and all E-lines are contained in a common Hermitian variety $\mathrm{H}(3,q^2)$ and that the E-lines are exactly those external lines different from l that intersect both a line l_i and a line m_j. Denote the variety $\mathrm{H}(3,q^2)$ by H_3.

As seen before, there are at least q^4 external lines not through L that intersect one of the lines l_i, $i = 1, \ldots, q$. At least $q^4 - (\gamma - q^3)$ of those are E-lines. Therefore, in H_3, there are at least $1 + 3q + q^4 - (\gamma - q^3)$ external lines. By (5.14) and the fact that $x \leq (q-1)/3$, it follows that there are more than $(q^4 + q^3 + xq^3 + x)/2 + q + 1$ external lines in H_3.

Now consider an external line l' not in H_3. Note that since $x > 0$, such a line exists. Through l', there exists a Hermitian variety $\mathrm{H}(3, q^2)$, say H_3', containing more than $(q^4 + q^3 + xq^3 + x)/2 + q + 1$ external lines. But H_3 and H_3' have at most $q+1$ lines in common. This implies that there are more than

$$2\frac{q^4 + q^3 + xq^3 + x}{2} + q + 1 = (q + 1 + x)(q^3 + 1)$$

external lines. However, this number is the exact number of external lines, a contradiction. $\square$

In Subsection 5.4.3, this bound will be lifted to a bound on the size of partial ovoids of $\mathrm{H}(2n, q^2)$, $n \geq 3$.

5.4.2 Partial ovoids in the split Cayley hexagon

For information on generalised hexagons, see Van Maldeghem [123].

Let q be a prime power and let $\mathrm{H}(q)$ be the split Cayley hexagon, i.e., the generalised hexagon defined in the following way. The points of $\mathrm{H}(q)$ are the points of $\mathrm{PG}(6, q)$ on the quadric $\mathrm{Q}(6, q)$ with equation $x_0x_4 + x_1x_5 + x_2x_6 = x_3^2$; the lines are the lines of this quadric whose Grassmann coordinates satisfy the equations

$$\begin{array}{lll} p_{12} = p_{34}, & p_{23} = p_{45}, & p_{02} = -p_{35}, \\ p_{03} = p_{56}, & p_{01} = p_{36}, & p_{13} = -p_{46}, \end{array}$$

and incidence is the natural one. *Opposite* points of $\mathrm{H}(q)$ are points that are at distance 6 from each other in the incidence graph of $\mathrm{H}(q)$ (and that is also the maximal possible distance). The generalised hexagon $\mathrm{H}(q)$ has the property that the set of points collinear with a given point P in $\mathrm{H}(q)$ is the point set of a unique plane $P^\perp$ contained in $\mathrm{Q}(6, q)$. An *ovoid* of $\mathrm{H}(q)$ is a set of $q^3 + 1$ mutually opposite points. A simple counting argument yields that every point outside a given ovoid of $\mathrm{H}(q)$ is collinear with exactly one point of the ovoid, see also [123, Chapter 7]. Hence, if $\mathcal{O}$ is an ovoid of $\mathrm{H}(q)$, then the set of $q^3 + 1$ planes $P^\perp$, with $P \in \mathcal{O}$, is a plane spread of $\mathrm{Q}(6, q)$. A *partial ovoid* of $\mathrm{H}(q)$ is a set of mutually opposite points; it is called *maximal* if no point of $\mathrm{H}(q)$ is opposite every point of the partial ovoid. The *deficiency* of a partial ovoid containing k points is $\delta = q^3 + 1 - k$. Remember the definition of ϵ_q from Notation 2.2.4.

Theorem 5.4.7 *If the deficiency δ of a maximal partial ovoid of* H(q) *is smaller than ϵ_q, or if q is a square and δ is smaller than $q^{5/8}/\sqrt{2}+1$, then δ is even.*

Proof Let $\mathcal{O}'$ be a maximal partial ovoid of H(q) with deficiency δ satisfying the conditions above. The set of planes $P^\perp$, with $P \in \mathcal{O}'$, is a partial plane spread $\mathcal{S}'$ of Q$(6,q)$, and hence by Corollaries 4.6.3.1 and 4.6.6, there exists a set $\mathcal{S}^*$ of δ planes of Q$(6,q)$ such that $\mathcal{S}' \cup \mathcal{S}^*$ is a spread of Q$(6,q)$. Let π^* be any plane belonging to $\mathcal{S}^*$. If π^* were equal to a plane $P^\perp$, for some point P of H(q), then $\{P\} \cup \mathcal{O}'$ would be a partial ovoid, a contradiction. Hence, the point set of π^* defines a set of q^2+q+1 points of H(q) at mutual distance four in the incidence graph of H(q).

By the third paragraph of the proof of Theorem 6.3.1 of [123], these q^2+q+1 points are a subset of the point set of a subhexagon of order $(1,q)$ of H(q), the remaining points of which form a plane π of Q$(6,q)$. This plane is uniquely defined by the following property: the point set of π is the set of points of H(q) that are collinear with exactly $q+1$ points of π^*, and such a set of $q+1$ points of π^* is the point set of a line of π^*; all lines of π^* arise in this way. Note that, since there are $q+1$ lines of H(q) through every point of H(q), every line of H(q) containing a point of π contains a point of π^*, and vice versa.

Assume by way of contradiction that a point P of π belongs to an element π' of $\mathcal{S}'$. Let $\pi' = Q^\perp$, with $Q \in \mathcal{O}'$. The line PQ contains a unique point of π^*, a contradiction. Hence all points of π are contained in elements of $\mathcal{S}^*$, implying that $\pi \in \mathcal{S}^*$. Since π^* was arbitrary in $\mathcal{S}^*$, it can be concluded that δ is even. □

Corollary 5.4.8 *A partial ovoid of* H(q)*, q even, has size at most q^3-1.*

Proof The previous result says that every partial ovoid of H(q) of size q^3 can be extended to an ovoid. But for q even, H(q) has no ovoid, see Thas [112]. It can be concluded that a partial ovoid of H(q) has size at most q^3-1. □

For $q=2$, this bound is sharp. An example of a partial ovoid of H(2) consisting of seven points can be obtained using the description of H(2) from Van Maldeghem [124]. Let π be the projective plane of order two. The point set of H(2) is the set of points, lines and point-line pairs of π. There are two kinds of lines: (1) the triples $\{P, l, \{P, l\}\}$, with P a point of π incident with the line l of π; (2) the triples $\{\{P,l\},\{P_1,l_1\},\{P_2,l_2\}\}$, where P and

l are as above, and $\{P, P_1, P_2\}$ is the point set of l, while $\{l, l_1, l_2\}$ is the set of lines incident with P in π. Incidence is natural. By Proposition 3 of [124], two non-incident point-line pairs $\{P, l\}$ and $\{Q, m\}$ of π correspond to two opposite points in $\mathrm{H}(2)$ if and only if either P is incident with m or Q is incident with l, but not both. Now represent the point set of π as the integers modulo 7. The lines of π are the translates of the set $\{1, 2, 4\}$. It is easily checked that the seven translates of the point-line pair $\{0, \{1, 2, 4\}\}$ define a set of seven mutually opposite points in $\mathrm{H}(2)$, and hence a maximal partial ovoid.

5.4.3 More bounds on the sizes

The bounds on the size of partial ovoids of finite classical polar spaces are examined. The inductive arguments of Section 5.3 are applicable if the polar space under consideration has no ovoid. For an overview of (non)existence results for ovoids of finite classical polar spaces, see Table 5.2.

Applying triality, see e.g. [120], on Corollary 4.6.3.1 in the case that $\mathcal{P} = \mathrm{Q}^+(7, q)$, Theorem 5.4.9.1 is obtained.

Theorem 5.4.9 1. *If* $\mathrm{Q}^+(7, q)$ *has no ovoid, then a partial ovoid* O' *of* $\mathrm{Q}^+(7, q)$ *satisfies* $|O'| \leq q^3 + 1 - \epsilon_q$*. On the other hand, if* $\mathrm{Q}^+(7, q)$ *has an ovoid, then any partial ovoid of* $\mathrm{Q}^+(7, q)$ *of size* $q^3 + 1 - \delta$ *with* $\delta < \epsilon_q$ *can be extended to an ovoid of* $\mathrm{Q}^+(7, q)$*.*

In the case that q *is a square, these results can be improved by replacing in the previous statement "*ϵ_q*" by "*$q^{5/8}/\sqrt{2} + 1$*".*

2. **(Theorem 5.4.6)** *If* O' *is a partial ovoid of* $\mathrm{H}(4, q^2)$*, then* $|O'| < q^5 + 1 - 2/3(2q + 1)$*.*

3. **(Tallini [108])** *If* O' *is a partial ovoid of* $\mathrm{W}_3(q)$*,* q *odd, then* $|O'| \leq q^2 + 1 - q$*.*

4. **(Thas [115])** *If* O' *is a partial ovoid of* $\mathrm{W}_5(q)$*, then* $|O'| \leq q^3 + 1 - q + 1$*.*

5. **(Thas [115])** *If* O' *is a partial ovoid of* $\mathrm{Q}^-(5, q)$*, then* $|O'| \leq q^3 + 1 - q(q - 1)$*.*

Note that for all q for which it is at present known whether or not $\mathrm{Q}^+(7, q)$ has an ovoid, the answer is positive: for q even, $q = 3^h$, $q \equiv 2 \bmod 3$ or q an odd prime, $\mathrm{Q}^+(7, q)$ has an ovoid, see Table 5.2 for references.

Corollary 5.4.10 1. *If* $\mathrm{Q}^+(7,q)$ *has no ovoid, then a partial ovoid* O' *of* $\mathrm{Q}^+(2n+1,q)$, $n \geq 3$, *satisfies* $|O'| \leq q^n + 1 - (q-1)^{n-3}\epsilon_q$.

In the case that q *is a square, the result by replacing in the previous statement "*ϵ_q*" by "*$q^{5/8}/\sqrt{2}+1$*".*

2. *If* O' *is a partial ovoid of* $\mathrm{H}(2n,q^2)$, $n \geq 2$, *then* $|O'| < q^{2n+1} + 1 - 2/3(q^2-1)^{n-2}(2q+1)$.

3. *If* O' *is a partial ovoid of* $\mathrm{W}_{2n+1}(q)$, $n \geq 1$, q *odd, then* $|O'| \leq q^{n+1} + 1 - q^{n-1}(q+1) + 1$.

4. *If* O' *is a partial ovoid of* $\mathrm{W}_{2n+1}(q)$, $n \geq 2$, q *even, then* $|O'| \leq q^{n+1} + 1 - q^{n-1} + 1$.

5. *If* O' *is a partial ovoid of* $\mathrm{Q}^-(2n+1,q)$, $n \geq 2$, *then* $|O'| \leq q^{n+1} + 1 - q^{n-2}(q^2-q+1) + 1$.

Proof Apply Theorem 5.3.3 to the first two cases of Theorem 5.4.9 and apply Theorem 5.3.5 to the other cases of Theorem 5.4.9. □

Also the following upper bounds on the sizes k of partial ovoids—which are sometimes called *k-caps*—are known. Bounds (5.15), (5.16), and (5.17) were proved by Blokhuis and Moorhouse [20]; bound (5.18) by Moorhouse [89].

Theorem 5.4.11 (Blokhuis and Moorhouse [20], Moorhouse [89]) *If* K *is a* k*-cap of the finite classical polar space* $\mathcal{P}$*, naturally embedded in* $\mathrm{PG}(n,q)$ *with* $q = p^h$ *and* p *prime, then*

$$k \leq \binom{p+n-1}{n}^h + 1. \tag{5.15}$$

If $\mathcal{P}$ *arises from a quadric in* $\mathrm{PG}(n,q)$*, then* (5.15) *can be improved to*

$$k \leq \left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^h + 1. \tag{5.16}$$

If $\mathcal{P}$ *arises from a quadric in* $\mathrm{PG}(n,q)$ *and if* n *and* q *are both even, then* (5.16) *can be improved to*

$$k \leq n^h + 1. \tag{5.17}$$

If $\mathcal{P}$ *arises from a Hermitian variety in* $\mathrm{PG}(n,q^2)$, $q = p^h$, *then* (5.15) *is improved to*

$$k \leq \left[\binom{p+n-1}{n}^2 - \binom{p+n-2}{n}^2\right]^h + 1. \tag{5.18}$$

	$\mathrm{H}(2n,q^2)$	$\mathrm{W}_{2n+1}(q)$, q odd	$\mathrm{Q}^-(2n+1,q)$
n	p_0	p_0	p_0
1	–	3	–
2	5	5	11
3	11	11	13
4	17	17	19
5	23	23	29
6	29	29	37
7	37	41	47
8	47	53	59
9	59	59	71
10	71	73	83

Table 5.3: Comparing the bounds on the size of partial ovoids

Corollaries 5.4.12 (i) *If $q = p^h$, p prime, $n \geq 1$, and*

$$p^n > \binom{p+2n}{2n+1} - \binom{p+2n-2}{2n+1}, \tag{5.19}$$

then the polar space $\mathrm{Q}^+(2n+1,q)$ has no ovoid.

(ii) *If $q = p^h$, p prime, $n \geq 1$, and*

$$p^{2n+1} > \binom{p+2n}{2n+1}^2 - \binom{p+2n-1}{2n+1}^2, \tag{5.20}$$

then the polar space $\mathrm{H}(2n+1,q^2)$ has no ovoid.

Remark 5.4.13 It is not immediately clear when the bounds of Theorem 5.4.11 are better than those of Corollary 5.4.10, but the following holds. For $\mathrm{H}(2n,q^2)$, respectively $\mathrm{W}_{2n+1}(q)$, q odd, $\mathrm{Q}^-(2n+1,q)$, and a fixed n, there exists a prime p_0 such that for each $q = p^h$, $p \geq p_0$, the bound of Corollary 5.4.10 is better, but such that for each $q = p^h$, $p < p_0$, the bound of Theorem 5.4.11 is at least as good. A few of these values for p_0 are given in Table 5.3. For $\mathrm{W}_{2n+1}(q)$, q even, $n \geq 2$, the bound of Theorem 5.4.11 is always better.

5.5 Blocking sets

In the first part of this section, the smallest blocking set in $\mathrm{W}_3(3)$ is described and a (small) blocking set of $\mathrm{W}_{2n+1}(q)$ is constructed, while the second part

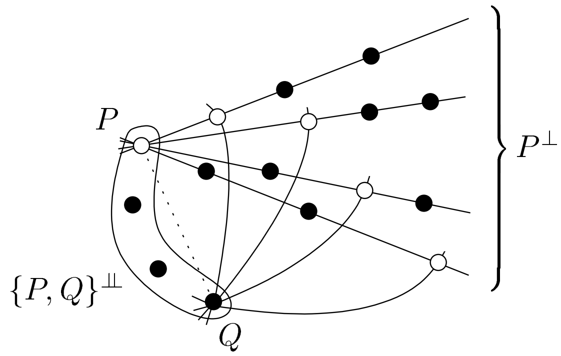

Figure 5.3: The smallest blocking set on $\mathrm{W}_3(3)$

discusses bounds on the sizes of blocking sets of finite classical polar spaces.

5.5.1 A small blocking set on $\mathrm{W}_{2n+1}(q)$

Remember Theorem 4.7.6. As $\mathrm{Q}(4,q)$ is the dual of $\mathrm{W}_3(q)$, see Theorem 1.1.3.2, dualising, the following result is obtained. It is stated in the language of generalised quadrangles.

Theorem 5.5.1 *A blocking set B of $\mathrm{W}_3(3)$ has size at least 11. If $|B| = 11$, then there exist two points P and Q on $\mathrm{W}_3(3)$, $P \not\sim Q$, such that B consists of, see also Figure 5.3,*

- *the points of $P^{\perp} \setminus (Q^{\perp} \cup \{P\})$, and*
- *the points of $\{P,Q\}^{\perp\!\!\perp} \setminus \{P\}$.*

This construction can be extended to a construction for general n and q.

Theorem 5.5.2 *The symplectic space $\mathrm{W}_{2n+1}(q)$ has a blocking set of size $q^{n+1} + q^n - q^{n-1}$.*

Proof Let π_{n-1} be a totally singular $(n-1)$-space and $\pi_{n+1} = \pi_{n-1}^{\perp}$. Take a point Q outside π_{n+1} and let $\pi'_n := \pi_{n+1} \cap Q^{\perp}$ and $\pi'_{n-2} = \pi_{n-1} \cap Q^{\perp}$. Let $\pi_n = \langle \pi_{n-1}, Q \rangle$. Then the set of points $B = (\pi_{n+1} \cup \pi_n) \setminus (\pi_{n-1} \cup \pi'_n)$, see Figure 5.4, is a blocking set of $\mathrm{W}_{2n+1}(q)$. Indeed, suppose by way of contradiction that there exists a totally singular n-space N skew to B. Now N intersects π_{n+1} in a subspace with dimension at least zero. Since N contains no point of B, $N \cap \pi_{n+1}$ is of the form $\langle \pi_a^*, \pi_b^* \rangle$, for some $\pi_a^* \subseteq \pi'_n \setminus \pi_{n-1}$, $\pi_b^* \subseteq \pi_{n-1}$, and $-1 \leq a \leq 1$, $-1 \leq b \leq n-1$.

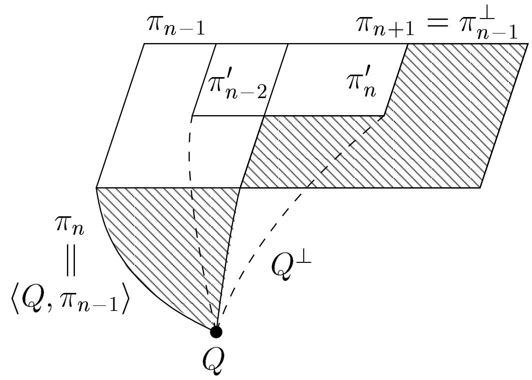

Figure 5.4: A small blocking set on $\mathrm{W}_{2n+1}(q)$

Case 1. Assume $\mathbf{a = -1}$. In this case, $N \cap \pi_{n+1}$ is contained in π_{n-1} and $N \cap \pi_{n+1} = \pi_b^*$. Since N is totally singular, $N \subseteq (\pi_b^*)^{\perp}$. Also $\pi_{n+1} \subseteq (\pi_b^*)^{\perp}$. But $\dim(N) = n$, $\dim(\pi_{n+1}) = n+1$ and $\dim((\pi_b^*)^{\perp}) = 2n - b$. Hence, $\dim(N \cap \pi_{n+1}) \geq b+1$, a contradiction.

Case 2. Assume $\mathbf{a = 0}$. In this case, N contains a point of $\pi'_n \setminus \pi_{n-1}$. This means that it cannot contain a point of $\pi_{n-1} \setminus \pi'_{n-2}$—the line joining such a point and a point of $N \cap (\pi'_n \setminus \pi_{n-1})$ would contain a point of B. Now $N \cap \pi_{n+1} = P\pi_b^*$ for some point $P \in \pi'_n \setminus \pi'_{n-2}$ and some $\pi_b^* \subseteq \pi'_{n-2}$, $-1 \leq b \leq n-2$. As in the first case, $N \subseteq (P\pi_b^*)^{\perp}$. Also $\pi_n = \langle Q, \pi_{n-1}\rangle \subseteq (P\pi_b^*)^{\perp}$, since $P\pi_b^* \subseteq \pi_{n-1}^{\perp}$ and $P\pi_b^* \subseteq Q^{\perp}$. But $\dim(N) = n$, $\dim(\pi_n) = n$ and $\dim((P\pi_b^*)^{\perp}) = 2n - b - 1$. Hence, $\dim(N \cap \pi_n) \geq b+1$. However, $N \cap \pi_n = N \cap \pi_{n-1}$ since $N \cap B = \emptyset$. So $N \cap \pi_n = N \cap \pi_{n-1} = N \cap \pi'_{n-2} = \pi_b^*$, which has dimension b, a contradiction.

Case 3. Assume $\mathbf{a = 1}$. In this case, N contains a line of $\pi'_n \setminus \pi_{n-1}$. As in the second case, it cannot contain a point of $\pi_{n-1} \setminus \pi'_{n-2}$. Now $N \cap \pi_{n+1} = l\pi_b^*$ for some line $l \subseteq \pi'_n \setminus \pi'_{n-2}$ and some $\pi_b^* \subseteq \pi'_{n-2}$, $-1 \leq b \leq n-3$. Note that $b \neq n-2$ since π'_n is not totally singular. As in the second case, $N \subseteq \langle l, \pi_b^*\rangle^{\perp}$. Also, $\pi_n \subseteq \langle l, \pi_b^*\rangle^{\perp}$. But $\dim(N) = n$, $\dim(\pi_n) = n$ and $\dim(\langle l, \pi_b^*\rangle^{\perp}) = 2n - b - 2$. Hence, $\dim(N \cap \pi_n) \geq b+2$. However, $N \cap \pi_n = N \cap \pi_{n-1} = N \cap \pi'_{n-2} = \pi_b^*$, which has dimension b, a contradiction.

It is easily checked that $|B| = q^{n+1} + q^n - q^{n-1}$. $\square$

Remark 5.5.3 1. We are now investigating whether, in the case $q = 3$, this blocking set is the smallest blocking set of $\mathrm{W}_{2n+1}(q)$.

2. For q even, smaller blocking sets of $\mathrm{W}_{2n+1}(q)$ exist. For $n = 1$, $\mathrm{W}_{2n+1}(q)$, q even, has ovoids, see Table 5.2. For $n > 1$, q even, the smallest blocking sets of $\mathrm{W}_{2n+1}(q)$ are characterised by Metsch [86], see Theorem 5.5.7; they have size $q^{n+1} + q^{n-1}$.

3. Also the dual of Remark 4.7.7 holds: the minimal blocking sets of size 12 of $\mathrm{W}_3(3)$ are always the known example. Using the notation of generalised quadrangles, they are a set $P^\perp \setminus \{P\}$ for some point P of $\mathrm{W}_3(3)$.

5.5.2 More bounds on the sizes

In this subsection, known results on (the size of) the smallest blocking sets of finite classical polar spaces are collected. Remember that Table 5.2 gives an overview of the known (non)existence results on ovoids of finite classical polar spaces. The results below consider the cases where the polar space has no ovoid.

Theorem 5.5.4 (Eisfeld et al. [49])

1. *If B is a blocking set of $\mathrm{W}_3(q)$, q odd, then $|B| > q^2 + 1 + (q-1)/3$.*

2. *Suppose B is a blocking set of $\mathrm{H}(4, q^2)$. If $q = 3$, then $|B| \geq q^5 + 3$; if $q = 4$, then $|B| \geq q^5 + 4$; if $q \geq 5$, then $|B| \geq q^5 + 5$.*

Using Theorem 5.3.2, the following corollary is obtained.

Corollary 5.5.5 1. *If B is a blocking set of $\mathrm{W}_{2n+1}(q)$, q odd, $n \geq 1$, then $|B| > q^{n+1} + 1 + (q-1)^n/3$.*

2. *Suppose B is a blocking set of $\mathrm{H}(2n, q^2)$, $n \geq 2$.*

- *If $q = 2$, then $|B| \geq q^{2n+1} + 1 + (q^2-1)^{n-2}$.*
- *If $q = 3$, then $|B| \geq q^{2n+1} + 1 + 2(q^2-1)^{n-2}$.*
- *If $q = 4$, then $|B| \geq q^{2n+1} + 1 + 3(q^2-1)^{n-2}$.*
- *If $q \geq 5$, then $|B| \geq q^{2n+1} + 1 + 4(q^2-1)^{n-2}$.*

Remember that the smallest blocking sets of $\mathrm{W}_3(3)$ were determined in Theorem 5.5.1.

The following bounds are sharp.

Theorem 5.5.6 (Metsch [85]) *If B is a blocking set of $Q = \mathrm{Q}^-(2n+1, q)$, $n \geq 2$, then $|B| \geq q^{n+1} + q^{n-1}$. Let $\perp$ denote the associated polarity in $\mathrm{PG}(2n+1, q)$. Then equality is reached if and only if $B = (U^\perp \setminus U) \cap Q$ for a subspace U of dimension $n-2$ with $U \subseteq Q$.*

Theorem 5.5.7 (Metsch [86])

1. *If B is a blocking set of $\mathrm{W}_{2n+1}(q)$, $n \geq 1$, q even, then $|B| \geq q^{n+1} + q^{n-1}$. Equality is reached if and only if B consists of the points outside the vertex of a cone with an $(n-2)$-dimensional vertex over an ovoid in a $\mathrm{W}_3(q)$.*

2. *If B is a blocking set of $\mathrm{Q}(2n, q)$, $n \geq 2$, q even, then $|B| \geq q^n + q^{n-2}$. Equality is reached if and only if B consists of the points outside the vertex of a cone with an $(n-3)$-dimensional vertex over an ovoid in a $\mathrm{Q}(4, q)$.*

Remark 5.5.8 The case $n = 3$ of Theorem 5.5.7.2 was independently proved by De Beule and Storme [38].

Theorem 5.5.9 (De Beule and Storme [37])

1. *Suppose that either $q = 5$, $q = 7$, or $q = n = 3$. If B is a minimal blocking set of $\mathrm{Q}(2n, q)$, $n \geq 3$, then either $|B| \geq q^n + q^{n-2}$ or $n = q = 3$ and B is an ovoid of $\mathrm{Q}(6, 3)$. In the former case, equality is reached if and only if B consists of the points outside the vertex of a cone with an $(n-3)$-dimensional vertex over an ovoid in a $\mathrm{Q}(4, q)$.*

2. *If B is a blocking set of $\mathrm{Q}(2n, 3)$, $n \geq 4$, then $|B| \geq q^n + q^{n-3}$. Equality is reached if and only if B consists of the points outside the vertex of a cone with an $(n-4)$-dimensional vertex over an ovoid in a $\mathrm{Q}(6, 3)$.*

Chapter 6

Cameron-Liebler line classes

Cameron-Liebler line classes were introduced by Cameron and Liebler [32] in an attempt to classify collineation groups of $\mathrm{PG}(n,q)$ that have equally many point orbits and line orbits. In their paper, they conjectured which groups these are. It is now known (T. Penttila, private communication, 2002) that the conjecture is true, but there is no classification yet of Cameron-Liebler line classes. In this chapter, some new nonexistence results are presented. Except for those in Section 6.7, they are collected in *P. Govaerts and L. Storme, On Cameron-Liebler line classes*, which is submitted to *Advances in Geometry.*

6.1 Introduction

Definition 6.1.1 A *Cameron-Liebler line class* is a set of lines in $\mathrm{PG}(3,q)$ that intersects every spread of $\mathrm{PG}(3,q)$ in the same number of lines.

Following Penttila [92], a *clique* in $\mathrm{PG}(3,q)$ is either the set of all lines through a point P, denoted by $\mathsf{star}(P)$, or dually the set of all lines in a plane π, denoted by $\mathsf{line}(\pi)$. The planar pencil of lines in a plane π through a point P is denoted by $\mathsf{pen}(P,\pi)$.

Cameron-Liebler line classes have many interesting intersection properties, several of which define them.

Definition 6.1.2 (Cameron and Liebler [32], Penttila [92]) Let $\mathcal{L}$ be a set of lines in $\mathrm{PG}(3,q)$ and let $\chi_{\mathcal{L}}$ be its characteristic function. Then $\mathcal{L}$ is called a *Cameron-Liebler line class* if one of the following equivalent conditions is satisfied.

1. There exists an integer x such that $|\mathcal{L}\cap\mathcal{S}| = x$ for all spreads $\mathcal{S}$.

2. There exists an integer x such that all regular spreads $\mathcal{S}$ contain exactly x lines of $\mathcal{L}$.

3. For every regulus $\mathcal{R}$ in $\mathrm{PG}(3,q)$, $|\mathcal{R} \cap \mathcal{L}| = |\mathcal{R}^{\mathrm{opp}} \cap \mathcal{L}|$.

4. There exists an integer x such that for every incident point-plane pair (P, π)
$$|\mathsf{star}(P) \cap \mathcal{L}| + |\mathsf{line}(\pi) \cap \mathcal{L}| = x + (q+1)|\mathsf{pen}(P,\pi) \cap \mathcal{L}|. \tag{6.1}$$

5. There exists an integer x such that for every line l of $\mathrm{PG}(3,q)$
$$|\{m \in \mathcal{L} : m \text{ meets } l, m \neq l\}| = (q+1)x + (q^2-1)\chi_{\mathcal{L}}(l). \tag{6.2}$$

6. There exists an integer x such that for every two skew lines l and m of $\mathrm{PG}(3,q)$
$$|\{n \in \mathcal{L} : n \text{ meets both } l \text{ and } m\}| = x + q(\chi_{\mathcal{L}}(l) + \chi_{\mathcal{L}}(m)). \tag{6.3}$$

It follows from the proof of the equivalence of these properties that the number x in each of these statements is the same. It is called the *parameter* of the Cameron-Liebler line class. Note that the first definition implies that $x \in \{0, 1, 2, \ldots, q^2+1\}$. Cameron and Liebler [32] showed that a Cameron-Liebler line class of parameter x consists of $x(q^2+q+1)$ lines and that the only Cameron-Liebler line classes for $x = 1$ are the cliques, and for $x = 2$ the unions of two disjoint cliques. They also noticed that the complement of a Cameron-Liebler line class with parameter x is a Cameron-Liebler line class with parameter q^2+1-x. So, it suffices to study Cameron-Liebler line classes with parameter $x \leq \lfloor (q^2+1)/2 \rfloor$. Thus, the case $q = 2$ was immediately solved. In their paper, Cameron and Liebler conjectured that no other Cameron-Liebler line classes exist.

Penttila [92] shows that for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $x = 3$ or $x = 4$, with possible exception of the cases $(x, q) \in \{(4,3), (4,4)\}$. Bruen and Drudge [28] prove the nonexistence of Cameron-Liebler line classes with parameter $2 < x \leq \sqrt{q}$. Drudge [43] excludes the existence of a Cameron-Liebler line class with parameter $x = 4$ in $\mathrm{PG}(3,3)$, and proves that for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $2 < x < \epsilon_q$, where $q + \epsilon_q$ denotes the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2,q)$, see Section 6.2. He also gives a counterexample to the conjecture of Cameron and Liebler: a Cameron-Liebler line class with parameter $x = 5$ in $\mathrm{PG}(3,3)$, in this way settling the

$q = p^h$, p prime	$0 \leq x \leq (q^2+1)/2$	Existence	References
any q	$x = 0$	yes	
	$x = 1$	yes	[32]
	$x = 2$	yes	[32]
$q > 2$	$x = 3$	no	[92]
	$3 < x < 2\epsilon_q - 2$	no	Thm 6.4.5
$q > 4$	$x = 4$	no	[92]
q odd	$x = (q^2+1)/2$	yes	[43, 29]
$q = 3$	$x = 4$	no	[43]
$q = 4$	$x = 4$	no	Thm 6.7.6
q square	$3 < x \leq \min(\epsilon'_q - 1, q^{3/4})$	no	Thm 6.5.1
q cube not square $p \geq 7$	$3 < x \leq \min(\epsilon''_q - 1, q^{5/6})$	no	Thm 6.6.1
q cube and square $p \geq 7$	$3 < x \leq \min(\epsilon''_q - 1, q^{3/4})$	no	Thm 6.6.3

Table 6.1: (Non)existence of Cameron-Liebler line classes with parameter x in $\mathrm{PG}(3, q)$

case $q = 3$. Bruen and Drudge [29] then construct a Cameron-Liebler line class with parameter $x = (q^2+1)/2$ for any odd q.

In this chapter, new bounds on x for nonexistence of Cameron-Liebler line classes are obtained. Theorem 6.4.5 gives a new bound for general $q \neq 2$, while Theorem 6.5.1, respectively Theorems 6.6.1 and 6.6.3, improves upon it for q square, respectively $q = p^{3h}$, $p \geq 7$ prime, $h \geq 1$. Finally, Theorem 6.7.6 proves the nonexistence of Cameron-Liebler line classes with parameter 4 in $\mathrm{PG}(3, 4)$. In Table 6.1, an overview is given of (non)existence results for Cameron-Liebler line classes.

These theorems will be proved by studying how the lines of the Cameron-Liebler line class are distributed among the cliques of $\mathrm{PG}(3, q)$. In the proofs, these cliques will be assumed to be of the form $\mathsf{star}(P)$ for some point P, but the dual arguments show that the considered properties also hold for cliques of the form $\mathsf{line}(\pi)$ for a plane π.

To study the lines of the Cameron-Liebler line class in a clique, Drudge's approach [43] is followed. A clique $\mathcal{C}$ and its lines correspond to a projective plane and its points in the following way. If $\mathcal{C} = \mathsf{star}(P)$, then it suffices to take the quotient space with respect to P. If $\mathcal{C} = \mathsf{line}(\pi)$, then the dual plane can be considered. In this way, the lines of the line class in a clique correspond to a set of points in a plane.

6.2 Cameron-Liebler line classes and blocking sets

The following two lemmas show where (multiple) blocking sets show up in the study of Cameron-Liebler line classes.

Lemma 6.2.1 (Drudge [43]) *Let $\mathcal{L}$ be a Cameron-Liebler line class with parameter x. If $\mathcal{C}$ is a clique satisfying $x < |\mathcal{C} \cap \mathcal{L}| \leq q + x$, then $\mathcal{C} \cap \mathcal{L}$ forms a blocking set B in $\mathcal{C}$. If there exist no Cameron-Liebler line classes with parameter $x - 1$, then B is nontrivial.*

Lemma 6.2.2 *Let $\mathcal{L}$ be a Cameron-Liebler line class with parameter x. If $\mathcal{C}$ is a clique satisfying $x + \alpha(q+1) < |\mathcal{C} \cap \mathcal{L}|$, then $\mathcal{C} \cap \mathcal{L}$ forms an $(\alpha+1)$-fold blocking set in $\mathcal{C}$.*

Proof Suppose $\mathcal{C} = \mathsf{star}(P)$ is a clique satisfying $x + \alpha(q+1) < |\mathcal{C} \cap \mathcal{L}|$. Let π be any plane through P. By (6.1) and the fact that $|\mathsf{line}(\pi) \cap \mathcal{L}|$ is at least zero, it can be concluded that $|\mathsf{pen}(P, \pi) \cap \mathcal{L}|$ is greater than α. □

6.3 Two lemmas

Lemma 6.3.1 *Let $\mathcal{L}$ be a Cameron-Liebler line class with parameter $x < q^2 + 1$. Then there exists a clique containing at most x lines of $\mathcal{L}$.*

Proof Let l be a line not in $\mathcal{L}$. By (6.2), there are $(q+1)x$ lines of $\mathcal{L}$ meeting l. This implies that there exists a point P on l that satisfies $|\mathsf{star}(P) \cap \mathcal{L}| \leq x$. □

Lemma 6.3.2 *If $\mathcal{L}$ is a Cameron-Liebler line class with parameter $0 < x \leq q$, then there exists a clique $\mathcal{C}$ satisfying $x < |\mathcal{C} \cap \mathcal{L}| \leq q + x$.*

Proof Suppose that $\mathcal{L}$ is a Cameron-Liebler line class with parameter $0 < x \leq q$ and that there exists no clique $\mathcal{C}$ satisfying $x < |\mathcal{C} \cap \mathcal{L}| \leq q + x$.

Suppose $\mathcal{C} = \mathsf{star}(P)$ is a clique satisfying $0 < |\mathcal{C} \cap \mathcal{L}| \leq x$. Then there exists a plane π through P containing exactly one line of $\mathcal{C} \cap \mathcal{L}$. By (6.1), $q + 1 \leq |\mathsf{line}(\pi) \cap \mathcal{L}| < q + x$, a contradiction. Dually, there exists no plane π satisfying $0 < |\mathsf{line}(\pi) \cap \mathcal{L}| \leq x$.

Suppose $\mathcal{C} = \mathsf{star}(P)$ is a clique satisfying $|\mathcal{C} \cap \mathcal{L}| = 0$. Then every plane π through P satisfies $|\mathsf{pen}(P, \pi) \cap \mathcal{L}| = 0$. By (6.1), $|\mathsf{line}(\pi) \cap \mathcal{L}| = x$, a contradiction with the preceding paragraph.

The previous observations show that there exist no cliques containing at most x lines of $\mathcal{L}$, a contradiction by Lemma 6.3.1. $\square$

6.4 The general case

In this section, assume that $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{PG}(3,q)$, $q > 2$, with parameter $x \leq q$, and that no Cameron-Liebler line classes with parameter $x-1$ exist. Recall that Penttila [92] proves that for $q > 2$, no Cameron-Liebler line classes with parameter 3 exist. As in the previous chapters, for $q > 2$, ϵ_q is defined by the fact that $q + \epsilon_q$ denotes the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2,q)$.

Lemma 6.4.1 *There exists no clique $\mathcal{C}$ satisfying $x < |\mathcal{C} \cap \mathcal{L}| \leq q + \min(\epsilon_q - 1, x)$.*

Proof Immediate from Lemma 6.2.1 and the definition of ϵ_q. $\square$

Corollary 6.4.2 (see also Drudge [43]) *There exist no Cameron-Liebler line classes with parameter $2 < x \leq \epsilon_q - 1$.*

Proof In this case Lemma 6.4.1 contradicts Lemma 6.3.2. $\square$

For the remainder of this section, assume that $x \geq \epsilon_q$.

Lemma 6.4.3 *There exists no clique $\mathcal{C}$ satisfying $x - \epsilon_q + 1 < |\mathcal{C} \cap \mathcal{L}| < q + 1$.*

Proof If $\mathcal{C} = \mathsf{star}(P)$ were a clique satisfying $x - \epsilon_q + 1 < |\mathcal{C} \cap \mathcal{L}| < q + 1$, then there would exist a plane π through P for which $|\mathsf{pen}(P, \pi) \cap \mathcal{L}| = 1$. By (6.1), this plane satisfies $x < |\mathsf{line}(\pi) \cap \mathcal{L}| \leq q + \epsilon_q - 1$, a contradiction by Lemma 6.4.1. $\square$

Lemma 6.4.4 *There exists no clique $\mathcal{C}$ satisfying $0 \leq |\mathcal{C} \cap \mathcal{L}| < \epsilon_q - 1$.*

Proof If $\mathcal{C} = \mathsf{star}(P)$ were a clique satisfying $0 \leq |\mathcal{C} \cap \mathcal{L}| < \epsilon_q - 1$, then there would exist a plane π through P for which $|\mathsf{pen}(P, \pi) \cap \mathcal{L}| = 0$. By (6.1), this plane satisfies $x - \epsilon_1 + 1 < |\mathsf{line}(\pi) \cap \mathcal{L}| \leq x$, a contradiction by Lemma 6.4.3. $\square$

Theorem 6.4.5 *In $\mathrm{PG}(3,q)$, $q > 2$, there exist no Cameron-Liebler line classes with parameter $2 < x < 2\epsilon_q - 2$.*

Proof If $x < 2\epsilon_q - 2$, then the intervals of Lemmas 6.4.3 and 6.4.4 partially overlap, implying that there exists no clique containing less than $q + 1$ lines of $\mathcal{L}$. This is contradictory to Lemma 6.3.1. $\square$

Corollary 6.4.6 *In* PG$(3,q)$*, q prime, $q>2$, there exist no Cameron-Liebler line classes with parameter $2<x\leq q$.*

Proof Use Theorem 1.2.3.1. □

6.5 Improvements for q square

Theorem 6.5.1 *In* PG$(3,q)$*, q square, there exist no Cameron-Liebler line classes with parameter $2<x\leq \min(\epsilon'_q-1, q^{3/4})$, where $q+\epsilon'_q$ denotes the size of the smallest nontrivial blocking sets in* PG$(2,q)$ *not containing a Baer subplane.*

Proof Suppose that $\mathcal{L}$ is a Cameron-Liebler line class with parameter $2<x\leq \min(\epsilon'_q-1, q^{3/4})$, and assume that no Cameron-Liebler line classes with parameter $x-1$ exist.

Suppose $\mathcal{C}=\mathsf{star}(P)$ is a clique satisfying $x<|\mathcal{C}\cap\mathcal{L}|\leq q+x$. By Lemma 6.2.1 and the restriction $x\leq\epsilon'_q-1$, in the plane corresponding to $\mathcal{C}$, $\mathcal{C}\cap\mathcal{L}$ contains a Baer subplane π_B. Since there are at most $x-\sqrt{q}-1$ points of $\mathcal{C}\cap\mathcal{L}$ outside π_B, there exists a $(\sqrt{q}+1)$-secant to $\mathcal{C}\cap\mathcal{L}$. Denote the corresponding plane through P by π. Since $|\mathsf{pen}(P,\pi)\cap\mathcal{L}|=\sqrt{q}+1$, it follows from (6.1) that $q\sqrt{q}+\sqrt{q}+1\leq|\mathsf{line}(\pi)\cap\mathcal{L}|\leq q\sqrt{q}+x$. By Lemma 6.2.2, $\mathsf{line}(\pi)\cap\mathcal{L}$ is a $\sqrt{q}$-fold blocking set in $\mathsf{line}(\pi)$. Comparing the upper bound on $|\mathsf{line}(\pi)\cap\mathcal{L}|$ with the known lower bounds for the size of multiple blocking sets from Theorem 1.2.10 yields a contradiction.

So, in contradiction with Lemma 6.3.2, there exists no clique $\mathcal{C}$ satisfying $x<|\mathcal{C}\cap\mathcal{L}|\leq q+x$. □

Corollary 6.5.2 *Let q be a square, $q=p^h$, p prime.*

1. *If $q>16$, then there exist no Cameron-Liebler line classes in* PG$(3,q)$ *with parameter $2<x\leq c_pq^{2/3}$, where c_p equals $2^{-1/3}$ when $p\in\{2,3\}$ and 1 when $p\geq 5$.*

2. *If $p>3$ and $h=2$, then there exist no Cameron-Liebler line classes in* PG$(3,q)$ *with parameter $2<x\leq q^{3/4}$.*

Proof Immediate by Theorems 6.5.1 and 1.2.4. □

6.6 Improvements for $q=q_0^3$

Theorem 6.6.1 *Let $q=q_0^3=p^{3h_0}$, $p\geq 7$ prime, $h_0\geq 1$ odd, and let $q+\epsilon''_q$ denote the size of the smallest nontrivial blocking sets in* PG$(2,q)$

containing neither a minimal blocking set of size $q + q_0^2 + 1$*, nor one of size* $q + q_0^2 + q_0 + 1$*. In* $\mathrm{PG}(3, q)$*, there exist no Cameron-Liebler line classes with parameter* $2 < x \leq \min(\epsilon''_q - 1, q^{5/6})$*.*

Proof Suppose that $\mathcal{L}$ is a Cameron-Liebler line class with parameter $2 < x \leq \min(\epsilon''_q - 1, q^{5/6})$, and assume that no Cameron-Liebler line classes with parameter $x - 1$ exist.

Suppose $\mathcal{C} = \mathsf{star}(P)$ is a clique satisfying $x < |\mathcal{C} \cap \mathcal{L}| \leq q + x$. By Lemma 6.2.1 and the restriction $x \leq \epsilon''_q - 1$, in the plane corresponding to $\mathcal{C}$, $\mathcal{C} \cap \mathcal{L}$ contains either a minimal blocking set of size $q + q_0^2 + 1$ or one of size $q + q_0^2 + q_0 + 1$. In both cases, $\mathcal{C} \cap \mathcal{L}$ has a $(q_0^2 + 1 + a)$-secant for some $0 \leq a \leq x - q_0^2 - 1$. Let π be the plane through P defined by this secant. By (6.1), it satisfies $(q+1)(q_0^2 + a) + 1 \leq |\mathsf{line}(\pi) \cap \mathcal{L}| < x + q_0^2 q + aq + a + 1$. By Lemma 6.2.2, $\mathsf{line}(\pi) \cap \mathcal{L}$ forms a $(q_0^2 + a)$-fold blocking set in $\mathsf{line}(\pi)$. However, comparing the upper bound for $|\mathsf{line}(\pi) \cap \mathcal{L}|$ with the known lower bounds for the size of multiple blocking sets from Theorem 1.2.10 yields a contradiction.

So, in contradiction with Lemma 6.3.2, there exists no clique $\mathcal{C}$ satisfying $x < |\mathcal{C} \cap \mathcal{L}| \leq q + x$. □

Corollary 6.6.2 *Let* $q = p^3$*,* $p \geq 7$ *prime. There exist no Cameron-Liebler line classes in* $\mathrm{PG}(3, p^3)$ *with parameter* $2 < x \leq q^{5/6}$*.*

Proof In this case $\epsilon''_q = (q + 3)/2$, see Theorem 1.2.5. □

Theorem 6.6.3 *Let* $q = q_0^3 = p^{3h_0}$*,* $p \geq 7$ *prime,* $h_0 > 1$ *even, and let* $q + \epsilon''_q$ *denote the size of the smallest nontrivial blocking sets in* $\mathrm{PG}(2, q)$ *containing neither a Baer subplane, nor a minimal blocking set of size* $q + q_0^2 + 1$*, nor one of size* $q + q_0^2 + q_0 + 1$*. In* $\mathrm{PG}(3, q)$*, there exist no Cameron-Liebler line classes with parameter* $2 < x \leq \min(\epsilon''_q - 1, q^{3/4})$*.*

Proof A combination of the proofs of Theorems 6.5.1 and 6.6.1 yields this result. □

Corollary 6.6.4 *Let* $q = p^6$*,* $p \geq 7$ *prime. There exist no Cameron-Liebler line classes in* $\mathrm{PG}(3, q)$ *with parameter* $2 < x \leq q^{3/4}$*.*

Proof By Theorem 6.6.3, it suffices to show that $\epsilon''_q - 1 \geq q^{3/4}$. Suppose that this is not the case, i.e., suppose that there exists a minimal nontrivial blocking set different from the three enumerated in Theorem 6.6.3 of size smaller than $q + 1 + q^{3/4}$. A result of Polverino and Storme [98] says that the exponent e of this blocking set must be 1. But a small minimal blocking set with exponent $e = 1$ has size at least $q + 1 + (q + p)/(p + 1)$, see Theorem 1.2.6. This number is greater than $q + 1 + q^{3/4}$, a contradiction. □

6.7 The smallest open case

The smallest open case is the case of a Cameron-Liebler line class with parameter $x = 4$ in $\mathrm{PG}(3,4)$. In this section, it is proved that no such Cameron-Liebler line classes exist.

When knowing the number of lines of a Cameron-Liebler line class $\mathcal{L}$ in a given clique, the following lemma gives severe restrictions on the possible intersections of other cliques with $\mathcal{L}$.

Lemma 6.7.1 *Suppose $\mathcal{L}$ is a Cameron-Liebler line class with parameter x. Then there exists an integer $0 \leq \alpha \leq x$ such that there exists a point through which there are exactly α lines of $\mathcal{C}$ and such that*

1. *for each point P: $|\mathsf{star}(P) \cap \mathcal{L}| \equiv \alpha \pmod{q+1}$, and*
2. *for each plane π: $|\mathsf{line}(\pi) \cap \mathcal{L}| \equiv x - \alpha \pmod{q+1}$.*

Proof The proof of Lemma 6.3.1 shows that there exists a point P such that $|\mathsf{star}(P) \cap \mathcal{L}| \leq x$. Let $\alpha = |\mathsf{star}(P) \cap \mathcal{L}|$. Equation (6.1) shows that each plane π containing P satisfies $|\mathsf{line}(\pi) \cap \mathcal{L}| \equiv x - \alpha \pmod{q+1}$. Again applying (6.1), now on the planes π through P and the points contained in them shows that for each point Q, $|\mathsf{star}(Q) \cap \mathcal{L}| \equiv \alpha \pmod{q+1}$. A final application of (6.1) proves the lemma for all planes. □

Remark 6.7.2 The preceding lemma shows that for Cameron-Liebler line classes, some sort of "$\mathrm{mod}(q+1)$ property" is valid, similar to the 1 mod p and t mod p results for minimal 1-fold and t-fold blocking sets in $\mathrm{PG}(2,q)$, $q = p^h$, p prime, see Subsections 1.2.1 and 1.2.3, and to the 1 mod p results for ovoids on $\mathrm{Q}(4,q)$ and $\mathrm{Q}(6,q)$, see Section 5.2. Of course, the exact value for α is missing in Lemma 6.7.1.

In the proof of the nonexistence of Cameron-Liebler line classes with parameter $x = 4$ in $\mathrm{PG}(3,4)$, intersection properties of double blocking sets of size 12 in $\mathrm{PG}(2,4)$ will be used. These properties are an easy corollary of the following theorem, which is proved in Section A.2.

Theorem 6.7.3 *Up to isomorphism, there are exactly three double blocking sets of size 12 in $\mathrm{PG}(2,4)$. If B is such a double blocking set, then either*

1. *B consists of the set of points of three nonconcurrent lines, or*
2. *there exist two lines l and m intersecting in a point P such that B consists of the set of points on l and m and three noncollinear further points, one on each of the three remaining lines through P, or*

3. *there exist three lines l_1, l_2 and l_3 through a point P and a fourth line l not through P such that B consists of the points of $l_i \setminus l$, $i = 1, 2, 3$, and the two points of l not on any of the lines l_i, $i \in \{1, 2, 3\}$.*

Proof See Section A.2. □

Corollary 6.7.4 *Suppose that B is a 2-fold blocking set of size 12 in $\mathrm{PG}(2, 4)$. Then, using the numbering from the theorem above, either B is of type 1 and has nine 2-secants, nine 3-secants and three 5-secants, or B is of type 2 and has ten 2-secants, six 3-secants, three 4-secants and two 5-secants, or B is of type 3 and has twelve 2-secants and nine 4-secants.*

Remark 6.7.5 In [78], Laskar and Sherk define the type of a double blocking set as follows. If B is a double blocking set in $\mathrm{PG}(2, q)$ and B has τ_i i-secants, $i = 0, 1, \ldots, q + 1$, then the *type* of B is $(|B|; \tau_2, \tau_3, \ldots, \tau_{q+1})$. In the same article, they determine all possible types of double blocking sets in $\mathrm{PG}(2, q)$, $q \leq 4$. Hence, the intersection properties from Corollary 6.7.4 above were already proved in that paper.

Theorem 6.7.6 *There exist no Cameron-Liebler line classes with parameter $x = 4$ in $\mathrm{PG}(3, 4)$.*

Proof Suppose $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{PG}(3, 4)$ with parameter $x = 4$. Let $\mathcal{C}$ be a clique. Note that $\epsilon_q = 3$, since a Baer subplane in $\mathrm{PG}(2, q = 4)$ has size $q + 3$. Hence, by Lemmas 6.4.3 and 6.4.4, $|\mathcal{C} \cap \mathcal{L}| \notin \{0, 1, 3, 4\}$. By Lemma 6.7.1, $|\mathcal{C} \cap \mathcal{L}| \equiv 2 \pmod 5$ for each clique $\mathcal{C}$. Hence $|\mathcal{C} \cap \mathcal{L}| \in \{2, 7, 12, 17\}$ for each clique $\mathcal{C}$.

Suppose there exists a clique $\mathcal{C}$ satisfying $|\mathcal{C} \cap \mathcal{L}| = 17$. Assume $\mathcal{C} = \mathsf{line}(\pi)$ for some plane π. Then, for each point P in π:

$$17 + |\mathsf{star}(P) \cap \mathcal{L}| = 4 + 5|\mathsf{pen}(P, \pi) \cap \mathcal{L}|. \tag{6.4}$$

Therefore, $|\mathsf{pen}(P, \pi) \cap \mathcal{L}| \geq 3$ for each $P \in \pi$. In π, exactly four lines do not belong to $\mathcal{C}$. Take a point Q on the intersection of two (or more) such lines. Then $|\mathsf{pen}(Q, \pi) \cap \mathcal{L}| \leq 3$, hence $|\mathsf{pen}(Q, \pi) \cap \mathcal{L}| = 3$. For this point Q, (6.4) yields $|\mathsf{star}(Q) \cap \mathcal{L}| = 2 < 3 = |\mathsf{pen}(Q, \pi) \cap \mathcal{L}|$, a contradiction.

It can be concluded that $|\mathcal{C} \cap \mathcal{L}| \in \{2, 7, 12\}$ for each clique $\mathcal{C}$. Remember, from the end of Section 6.1, that a clique and its lines can be identified with a projective plane and its points. If $|\mathcal{C} \cap \mathcal{L}| = 2$, then $\mathcal{C} \cap \mathcal{L}$ is a set of two points in $\mathcal{C}$. If $|\mathcal{C} \cap \mathcal{L}| = 7$, then $\mathcal{C} \cap \mathcal{L}$ is a nontrivial blocking set of size seven, hence a Baer subplane in $\mathcal{C}$. If $|\mathcal{C} \cap \mathcal{L}| = 12$, then $\mathcal{C} \cap \mathcal{L}$ is a double blocking set of size 12 in $\mathcal{C}$.

In the last case, $\mathcal{C} \cap \mathcal{L}$ cannot contain a line. Indeed, suppose for instance that in this case $\mathcal{C} = \mathsf{star}(P)$ contains a line ℓ. This line ℓ is a line in the quotient geometry of P and the following identities hold: $|\mathsf{pen}(P, \langle P, \ell\rangle) \cap \mathcal{L}| = 5$ and $|\mathsf{star}(P) \cap \mathcal{L}| = 12$. Hence, by (6.1), $|\mathsf{line}(\langle P, \ell\rangle) \cap \mathcal{C}| = 17$, a contradiction. So, if $|\mathcal{C} \cap \mathcal{L}| = 12$, then $\mathcal{C} \cap \mathcal{L}$ must be a blocking set of the third type of Theorem 6.7.3.

Let P be a point such that $|\mathsf{star}(P) \cap \mathcal{L}| = \alpha = 2$, see Lemma 6.7.1. Let π be a plane through P that contains exactly one line of $\mathcal{L}$ through P. Then, by (6.1), $|\mathsf{line}(\pi) \cap \mathcal{L}| = 7$ and $\mathsf{line}(\pi) \cap \mathcal{L}$ forms a dual Baer subplane in $\mathsf{line}(\pi)$. Now take a point P' in π that lies on three lines of $\pi \cap \mathcal{L}$. Then $|\mathsf{pen}(P', \pi) \cap \mathcal{L}| = 3$, and $|\mathsf{star}(P') \cap \mathcal{L}| = 12$. Hence, the double blocking set in the quotient space with respect to P' contains a 3-secant, $\mathsf{pen}(P', \pi)$. But this double blocking set is of the third type of Theorem 6.7.3, which has no 3-secants, see Corollary 6.7.4. A contradiction is obtained. $\square$

Appendix A

Two results on blocking sets

Two theorems are proved that were used in this thesis.

A.1 Small blocking sets in $\mathrm{PG}(n, q)$

In chapter 3, a result was needed that shows that "small" blocking sets in $\mathrm{PG}(n, q)$, $q = p^h$ square, contain a planar blocking set. Such a theorem exists, see Theorem 1.2.16, but unfortunately it excludes the cases where $p = 2$ or $p = 3$. Using basically the same techniques as in Storme and Weiner [105], the result can be proved for any square $q \geq 16$, when the restriction on the size of the blocking set is strengthened. This proof can be found below.

Lemma A.1.1 **([105])** *Let B be a blocking set in $\mathrm{PG}(n, q)$, $n \geq 3$.*

1. *If P is a point not in B and H is a hyperplane not containing P, then the projection of B from P into H is a blocking set in H.*

2. *If π is a plane intersecting B in more than $|B| - q$ points, then $B \cap \pi$ is a blocking set in π. So, B contains a planar blocking set.*

3. *Suppose $|B| \leq 2q$. If the projection B' of B from a point $P \notin B$ into a hyperplane H not containing P contains a line, then B contains a blocking set in a plane of $\mathrm{PG}(n, q)$.*

Lemma A.1.2 **([105])** *Let B be a blocking set in $\mathrm{PG}(3, q)$, q square, of smaller size than the second smallest nontrivial blocking sets in $\mathrm{PG}(2, q)$ and suppose B does not contain a planar blocking set. Project B from a point $P_1 \notin B$ onto a plane π not containing P_1. Then the projection of B contains a Baer subplane $\pi_{B,1}$. Let $S \in B$ be a point whose projection S' belongs to*

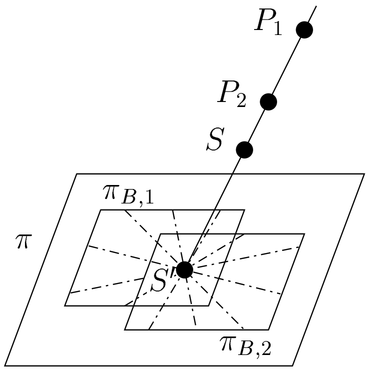

Figure A.1: Projecting the blocking set on a plane

$\pi_{B,1}$. *Now project B from a second point $P_2 \in P_1S$, $P_2 \notin \{P_1, S, S'\}$, $P_2 \notin B$, onto π. Then also this projection of B contains a Baer subplane $\pi_{B,2}$, and the following hold.*

1. *The point S' belongs to $\pi_{B,2}$, and the dual Baer subline of $\pi_{B,1}$ through S' coincides with the dual Baer subline of $\pi_{B,2}$ through S'. This configuration is depicted in Figure A.1.*

2. *The Baer cones $P_1\pi_{B,1}$ and $P_2\pi_{B,2}$ intersect in the union of a Baer subspace $D_{12} = \mathrm{PG}(3, \sqrt{q})$ and the line P_1P_2. The points P_1 and P_2 belong to D_{12}.*

Theorem A.1.3 *If B is a blocking set in $\mathrm{PG}(3, q)$, $q \geq 16$ square, with $|B| < q + c_p q^{2/3}$, then B contains a planar blocking set.*

Proof Suppose that B is a blocking set in $\mathrm{PG}(3, q)$ that satisfies the assumptions of the theorem, but that does not contain a planar blocking set. A contradiction will be obtained.

Consider the configuration of Lemma A.1.2 with the extra requirement that S is chosen in such a way that it is the unique point of B on the line P_1S, see Figure A.1. Denote the cones from the lemma by C_1 and C_2. Then $C_i \cap B \geq q + \sqrt{q} + 1$. Hence, $|C_1 \cap C_2 \cap B| \geq |C_1 \cap B| + |C_2 \cap B| - |B| \geq q - q^{2/3} + 2\sqrt{q} + 2$. Since C_1 and C_2 intersect in $D_{12} \cup P_1P_2$ and S is the unique point of B on P_1P_2,

$$|D_{12} \cap B| \geq q - q^{2/3} + 2\sqrt{q} + 1. \tag{A.1}$$

Let P_3 be a point on P_1P_2, $P_3 \notin D_{12}$, $P_3 \notin B$ and $P_3 \notin \pi$. Then P_3 defines a Baer cone $C_3 = P_3\pi_{B,3}$ that intersects C_1 in a Baer subspace $D_{13} =$

PG(3, $\sqrt{q}$) union the line $P_1P_3 = P_1P_2$. Since $P_3 \in D_{13}$, it follows that $D_{12} \neq D_{13}$. As before,

$$|D_{13} \cap B| \geq q - q^{2/3} + 2\sqrt{q} + 1. \tag{A.2}$$

From (A.1) and (A.2), it can be concluded that

$$|D_{12} \cap D_{13} \cap B| \geq q - 3q^{2/3} + 4\sqrt{q} + 2.$$

For $q \geq 16$, this number is greater than $2(\sqrt{q}+1)$. It is known, see Sved [106], that two distinct Baer 3-spaces intersecting in more than $2(\sqrt{q}+1)$ points, intersect either in a Baer subplane, or in a Baer subplane union a point. Hence, the points of $D_{12} \cap D_{13} \cap B$, except for at most one, lie in a plane. Thus there exists a plane in PG$(3, q)$ containing at least $q - 3q^{2/3} + 4\sqrt{q} + 1$ points of B. One easily checks that, for $q \geq 16$, this number is greater than $q^{2/3}$. By Lemma A.1.2, B contains a planar blocking set, a contradiction. □

Theorem A.1.4 *If B is a blocking set in* PG(n, q)*, $n \geq 3$, $q \geq 16$ square, with $|B| < q + c_p q^{2/3}$, then B contains a planar blocking set.*

Proof (cf. [105]) The previous theorem is the case where $n = 3$. So, let $n \geq 4$, and assume the theorem holds for all $n' < n$.

Project B from a point $P \notin B$ onto a hyperplane H not containing P. This projection contains a planar blocking set B_1. If B_1 contains a line, then Lemma A.1.1 shows that B contains a planar blocking set. So, assume B_1 does not contain a line. Then $|PB_1 \cap B| \geq q + \sqrt{q} + 1$. Let H' be a hyperplane containing PB_1, and project B onto H' from a point outside H'. The projection contains at least $q + \sqrt{q} + 1$ and less than $q + c_p q^{2/3}$ points. Also this projection contains a planar blocking set B_2. Assume—for the same reason as above—that B_2 does not contain a line. Then $|B_2 \cap (PB_1 \cap B)| \geq 2(q + \sqrt{q} + 1) - q - c_p q^{2/3} \geq q - q^{2/3} + 2\sqrt{q} + 2$. Hence $|B_2 \cap B| > q - q^{2/3} \geq q^{2/3} \geq |B| - q$. By Lemma A.1.1, the plane π containing B_2 intersects B in a blocking set. □

A.2 Double blocking sets in PG(2, 4)

In this section, the smallest double blocking sets in PG$(2, 4)$ are classified.

Theorem A.2.1 *Suppose B is a 2-fold blocking set of size 12 in* PG$(2, 4)$*. Then either*

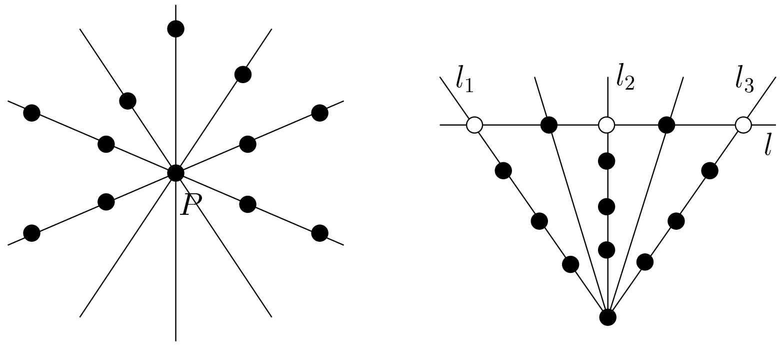

Figure A.2: The double blocking 12-sets in $\mathrm{PG}(2,4)$

1. *B contains two lines intersecting in a point P, and one further point on each of the three remaining lines through P, or*

2. *there exist three collinear lines l_1, l_2 and l_3 through a point P and a fourth line l not through P such that B consists of the points of $l_i \setminus l$, $i = 1, 2, 3$, and the two points of l not on any of the lines l_i, $i \in \{1, 2, 3\}$.*

These blocking sets are depicted in Figure A.2.

Proof Suppose B is a double blocking set of size 12 in $\mathrm{PG}(2,4)$.

Case 1. B contains a line l. As shown below, in this case, B contains a second line l'.

Consider a point P in $B \setminus l$. The 11 points of $B \setminus \{P\}$ lie on the five lines through P. So, there exists a line m through P containing at least four points of B. If m is contained in B, then take $l' = m$. So, suppose m contains a point $P' \notin B$.

Let Q be the point on the intersection of l and m, and let l_1, l_2 and l_3 be the lines through Q different from l and m. Let m_1, m_2, m_3 and m_4 be the lines different from m through P'. Then on each line m_i, there is exactly one more point of B. On one of the lines l_i, there are two more points of B, and the other two lines l_i contain one further point of B. Let l_1 be the line containing two more points of B and let the lines m_i containing these points be m_3 and m_4. Name the lines m_1, m_2, l_2 and l_3 in such a way that the remaining points of B are $m_1 \cap l_2$ and $m_2 \cap l_3$. Call the points P_3, P_4, Q_3, Q_4, R_1, R_2, S_1 and S_2 as indicated in Figure A.3. Consider the line P_4Q_3. It intersects l_2 in a point. This point is either R_1 or R_2. If it is R_1, then P_4Q_3 must also contain S_2 and a point of $m \setminus \{P'\}$. In this case, set $l' = P_4Q_3$. So, suppose P_4Q_3 does not contain R_1.

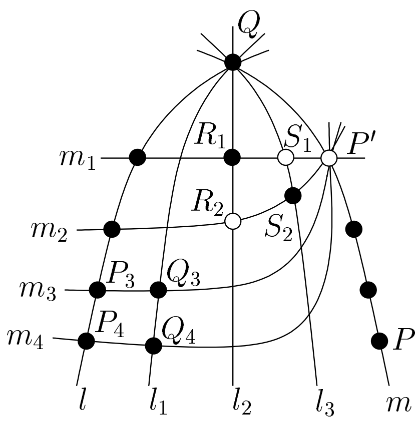

Figure A.3: Double blocking 12-set containing a line: notations

In that case, it contains R_2 and S_1. But then consider the line P_3Q_4. If it would contain R_2, then it also would contain S_1, and P_3Q_4 would equal P_4Q_3, a contradiction. Therefore P_3Q_4 contains R_1 and S_2. Hence, it is contained in B.

In any case, B contains a second line l', such that B is of type 1 in the statement of the lemma.

Case 2. B contains no line. Consider a point $R \in B$. Since B contains 11 points different from R, there exists a line n through R containing at least four points of B. Since B contains no lines, n contains exactly four points of B. Let S be the point on n that does not lie in B. All lines through S different from n contain exactly two points of B. Let R' be a point of B not on n. As above, there exists a line n' through R' containing exactly four points of B. This line intersects n in a point different from S.

From the reasoning above, it follows that there exists a point $O \in B$ that lies on two lines l and m that contain exactly four points of B. Denote the point on l (respectively m) not in B by P (respectively Q). The lines through P (respectively Q) different from PQ and l (respectively PQ and m) are denoted by l_1, l_2, and l_3 (respectively m_1, m_2, and m_3). Clearly, PQ must contain two further points of B, and the lines l_i and m_j, $i,j = 1,2,3$, each one more point of B. Let A, B and C be the three extra points of B on the lines l_i and m_j. Name these points and lines in such a way that A (respectively B, C) lies on l_1 and m_1 (respectively l_2 and m_2, l_3 and m_3), see Figure A.4.

Case 2.1. A, B and C are not collinear. In this case, none of the lines

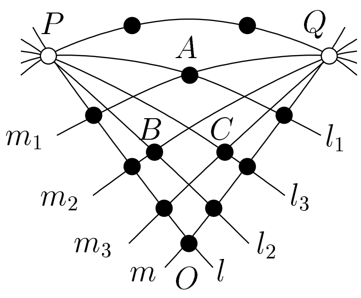

Figure A.4: Double blocking 12-set containing no line: notations

AB, BC and AC can contain O. Indeed, suppose for example that AB would contain O. Then AB would intersect m_3 in a point. This point could not lie on l nor on PQ, but also neither on l_1 nor on l_2. Hence it would lie on l_3. But then this point would be C, a contradiction. So, all three lines AB, BC and AC contain a point of $l \setminus \{O, P\}$ and a point of $m \setminus \{O, Q\}$. But they also contain a point of $PQ \setminus \{P, Q\}$ and no two of them contain the same point of $PQ \setminus \{P, Q\}$. Since $PQ \setminus \{P, Q\}$ contains only three points and two of them belong to B, two of the three lines AB, BC and AC are contained in B, a contradiction.

Case 2.2. A, B and C are collinear. Consider the line AB (which equals the line AC). If it would not contain O, then it would contain a point of $l \setminus \{O, P\}$ and a point of $m \setminus \{O, Q\}$, so that it would be contained in B, a contradiction. Thus the line AB contains O and intersects PQ in a point that is not contained in B. The remaining two points of PQ lie in B. Hence, B is of the second type in the statement of the lemma.

This concludes the proof. □

Corollary A.2.2 *Up to isomorphism, there are exactly three double blocking sets of size* 12 *in* PG(2, 4). *If B is such a blocking set, then either*

1. *B consists of the set of points of three nonconcurrent lines, or*

2. *there exist two lines l and m intersecting in a point P such that B consists of the set of points on l and m and three noncollinear further points, one on each of the three remaining lines through P, or*

3. *there exist three concurrent lines l_1, l_2 and l_3 through a point P and a fourth line l not through P such that B consists of the points of $l_i \setminus l$, $i = 1, 2, 3$, and the two points of l not on any of the lines l_i, $i \in \{1, 2, 3\}$.*

Remark A.2.3 The complement of a unital, see page 4, in PG$(2,4)$ is a double blocking set of size 12 with only 2- and 4-secants. Hence, it is the third type of blocking set from Corollary A.2.2. Since there is, up to isomorphism, only one blocking set with these intersection properties, also the unital is unique: it is the Hermitian curve. Note that this is not the shortest way to prove the uniqueness of the unital in PG$(2,4)$; it can be constructed from scratch using only the definition of a unital.

Appendix B

Mutually orthogonal Latin squares

Results on partial spreads in $\mathrm{PG}(3, q)$ and ovoids of the hyperbolic quadric are used to construct maximal sets of mutually orthogonal Latin squares.

This chapter diverts somewhat from the topic of the thesis, since the link between blocking sets and the sets of orthogonal squares is not clear. Still, the objects used to construct these sets are related to blocking sets, as seen in the previous chapters.

The results from this chapter are collected in *P. Govaerts, D. Jungnickel, L. Storme and J. A. Thas, Some new maximal sets of mutually orthogonal Latin squares* [53] which is to appear in *Designs, Codes and Cryptography.*

B.1 Introduction

The problem considered here is the determination of the pairs (s, t) for which a maximal set of t mutually orthogonal Latin squares of order s exist. This problem is, for instance, discussed in Beth et al. [10, Chapter X] and in [33, Section IV.27].

Two $s \times s$-matrices, simply called *squares* of *order* s, $A = (a_{ij})$ and $B = (b_{ij})$, with entries in a set S of size s are called *orthogonal* if the mapping $e : (i, j) \mapsto (a_{ij}, b_{ij})$ from $\{1, \ldots, s\}^2$ to S^2 is bijective.

A square $A = (a_{ij})$ of order s with entries in the set S is called a *Latin square* if the mappings $r_i : j \mapsto a_{ij}$ from $\{1, \ldots, s\}$ to S are bijective for each $i \in \{1, \ldots, s\}$ and the mappings $c_j : i \mapsto a_{ij}$ from $\{1, \ldots, s\}$ to S are bijective for each $j \in \{1, \ldots, s\}$, i.e., if each row and each column of A contains all elements of S.

A set of t mutually orthogonal Latin squares of order s, briefly denoted

affine $S_r(1, s; s^2)$
$\updownarrow$
$(s, r; 1)$-net
$\updownarrow$
$\mathrm{TD}[r, s]$
$\updownarrow$
$\mathrm{OA}(s, r)$
$\updownarrow$
$(r-2)$ MOLS(s)

Table B.1: MOLS and related combinatorial objects
Existence of one of these objects implies the existence of all the other ones. Some of these implications are described in Section B.1; for the others see, e.g., [9].

by t MOLS(s), is called *maximal* and denoted by t MAXMOLS(s) if no Latin square of order s exists that is orthogonal to all of them.

In the remainder of this introductory section, it is described how partial congruence partitions yield, via an intermediate step of translation nets, MOLS, and how transversal-free translation nets yield MAXMOLS. This approach will be applied in Section B.2 to construct MAXMOLS(16). In Section B.3, ovoids of the hyperbolic quadric will be used to construct infinite sets of MAXMOLS.

Sets of MOLS are closely connected to many other combinatorial objects, such as affine designs, nets, transversal designs, orthogonal arrays, see Table B.1. More information on these objects can be found in Beth et al. [9], where also several of the structures discussed below are treated in a more general way.

Let S be a set of cardinality s. An *orthogonal array of order s, degree r and index* 1 *(on S)*, briefly an $\mathrm{OA}(s, r)$, is an $r \times s^2$-matrix with entries from S such that each $2 \times s^2$-submatrix contains every possible 2×1-column vector exactly once.

Let A be an $\mathrm{OA}(s, r)$, with $r \geq 3$, on the set $S = \{\alpha_1, \ldots, \alpha_s\}$. Consider the first two rows of A. In these rows each 2×1-matrix with entries in S occurs exactly once, such that the s^2 columns of A can be arranged as follows:

$$A = \left(\begin{array}{cccc|cccc|c|cccc} \alpha_1 & \alpha_1 & \ldots & \alpha_1 & \alpha_2 & \alpha_2 & \ldots & \alpha_2 & \ldots & \alpha_s & \alpha_s & \ldots & \alpha_s \\ \alpha_1 & \alpha_2 & \ldots & \alpha_s & \alpha_1 & \alpha_2 & \ldots & \alpha_s & \ldots & \alpha_1 & \alpha_2 & \ldots & \alpha_s \\ * & * & \ldots & * & * & * & \ldots & * & \ldots & * & * & \ldots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \end{array} \right).$$

Now define the matrices A_μ, $\mu = 1, \ldots, r$, whose entries consist of the entries of a row of $A = (a_{ij})$ as follows:

$$A_\mu = \begin{pmatrix} a_{\mu,1} & a_{\mu,2} & \cdots & a_{\mu,s} \\ a_{\mu,s+1} & a_{\mu,s+2} & \cdots & a_{\mu,2s} \\ \vdots & \vdots & & \vdots \\ a_{\mu,s^2-s+1} & a_{\mu,s^2-s+2} & \cdots & a_{\mu,s^2} \end{pmatrix}.$$

Then, for all $\mu \neq \nu$, the matrices A_μ and A_ν are orthogonal. And, for $\mu \geq 3$, the matrices A_μ are Latin squares. Hence, an $\mathrm{OA}(s,r)$ with $r \geq 3$ yields a set of $r-2$ MOLS(s). Applying this construction the other way round, a set of $r-2$ MOLS(s) gives an $\mathrm{OA}(s,r)$.

The following notation will *only* be used to state the subsequent definition. If $\mathsf{D} = (\mathsf{P}, \mathsf{B}, \mathtt{I})$ is an incidence structure and $L = \{l_1, l_2, \ldots, l_k\}$ is a set of elements of B, then $(l_1, l_2, \ldots, l_k)$ denotes the set of all elements of P that are incident with all elements of L.

An $(s,r;1)$-*net*, $r \geq 3$, is an incidence structure $\mathsf{D} = (\mathsf{P}, \mathsf{B}, \mathtt{I})$ satisfying

1. the relation $\sim$ on B, with $l \sim m$ if $l = m$ or if there exists no point P such that $l \;\mathtt{I}\; P \;\mathtt{I}\; m$, is an equivalence relation which has r equivalence classes; these classes are called *parallel classes*;

2. for all $l, m \in \mathsf{B}$, if $l \not\sim m$, then there exists a unique point P such that $l \;\mathtt{I}\; P \;\mathtt{I}\; m$;

3. every point lies in an element of each parallel class.

If these properties hold, then there exists an integer s for which the following properties hold:

- the number of lines in a parallel class equals s;
- the number of lines through a point equals r;
- each line contains s points;
- there are s^2 points;
- there are rs lines.

Let D be an $(s,r;1)$-net and $S = \{\alpha_1, \ldots, \alpha_s\}$ a set of cardinality s. Label in each parallel class the lines in it arbitrarily with the elements of S. Define a matrix A whose rows are indexed by the parallel classes and whose columns are indexed by the points of D as follows. The $(\mathcal{P}, P)$-entry of A is α_i if and only if the unique line of the parallel class $\mathcal{P}$ passing through the point P has label α_i. It is easy to check that A is an OA(s, r). Hence, every $(s, r; 1)$-net yields an OA(s, r).

Conversely, let $A = (a_{ij})$ be an OA(s, r) on a set $S = \{\alpha_1, \ldots, \alpha_s\}$. Define an incidence structure D with parallel classes as follows. The parallel classes are the rows of A and are denoted by $\mathcal{P}_1, \ldots, \mathcal{P}_r$. The points are the columns of A and are denoted by $P_1, \ldots, P_{s^2}$. The lines are the pairs $(\mathcal{P}_i, \alpha_j)$, where $\mathcal{P}_i$ is a parallel class and $\alpha_j \in S$. A line $(\mathcal{P}_i, \alpha_j)$ is incident with a point P_k if and only if $a_{ik} = \alpha_j$. It is easy to check that D is an $(s, r; 1)$-net.

Let D be an $(s, r; 1)$-net. A *partial transversal* of D is a subset T of the point set of D satisfying $|\mathsf{T} \cap l| \leq 1$ for each line l of D. It is called a *transversal* if it has size s. The net D is called *maximal* if it cannot be embedded into an $(s, r + 1; 1)$-net, or, equivalently, if D has no set of s pairwise disjoint transversals.

A *transversal* of a Latin square $A = (a_{ij})$ of order s with entries from the set S is a set T of s cells of A, i.e., $T \subset \{1, \ldots, s\}^2$, such that the cells in T contain each element of S exactly once. Now suppose $A_1, \ldots, A_{r-2}$ are $r - 2$ MOLS(s) with entries from the set S. If $T_1, \ldots, T_s$ are s mutually disjoint sets of s cells, each of which is transversal to all squares A_l, $l = 1, \ldots, r - 2$, then there exists a Latin square $B = (b_{ij})$ of order s which is orthogonal to all squares A_l. Indeed, let f be any bijection from $\{1, \ldots, s\}$ to S, and let $b_{ij} = \alpha_k$ if and only if $(i, j) \in T_l$ and $f(l) = \alpha_k$. Then $B = (b_{ij})$ is such a Latin square. Conversely, if $B = (b_{ij})$ is a Latin square of order s which is orthogonal to all squares A_l, then the sets T_k, $k = 1, \ldots, s$, defined by $(i, j) \in T_k$ if and only if $f^{-1}(b_{ij}) = k$, is a set of s mutually disjoint transversals to the squares A_l. It follows that a set of MOLS(s) is a set of MAXMOLS(s) if and only if there exists no set of s mutually disjoint sets of entries that are transversals to each element of the set of MOLS.

From the constructions above, it is clear that if A is a Latin square of order s and D the corresponding $(s, 3; 1)$-net, then the transversals to A correspond bijectively to those of D. Even more, if D is an $(s, r; 1)$-net and $\{A_1, \ldots, A_{r-2}\}$ the corresponding set of MOLS, then the transversals of D correspond to those sets T of cells that are transversals for each square A_i, $i = 1, \ldots, r-2$. Hence, if an $(s, r; 1)$-net is maximal, then the corresponding set of MOLS is a set of MAXMOLS.

An *automorphism* of an incidence structure $(\mathsf{P}, \mathsf{B}, \mathtt{I})$ of points and lines is a bijection θ from $\mathsf{P} \cup \mathsf{B}$ to $\mathsf{P} \cup \mathsf{B}$ mapping points to points and lines to lines such that

$$\text{for all } P \in \mathsf{P}, \text{ for all } l \in \mathsf{B} : P \ \mathtt{I} \ l \Leftrightarrow P^{\theta} \ \mathtt{I} \ l^{\theta}.$$

A net D is called a *translation net* if it admits an automorphism group

$(s,r;1)$-PCP	
$\Downarrow$	transversal-free $(s,r;1)$-translation net
$(s,r;1)$-translation net	$\Downarrow$
$\Downarrow$	$(r-2)$ MAXMOLS(s)
$(r-2)$ MOLS(s)	

Table B.2: How to construct MOLS and MAXMOLS

G acting regularly on the point set of D and fixing every parallel class. The group G is called a *translation group* of D. In general, D may be a translation net for more than one translation group, and a group may be a translation group for more than one net.

Remark B.1.1 If a translation net D has a transversal T, then the net is not maximal. Indeed, taking translates of T, a set of s disjoint transversals is obtained.

Let G be a group of order s^2 and let $U_1, \ldots, U_r$ be subgroups of order s of G. Then $\mathsf{U} = \{U_1, \ldots, U_r\}$ is called an $(s,r;1)$-*partial congruence partition*, or shorter an $(s,r;1)$-PCP, in G if $|U_i \cap U_j| = 1$ for $i \neq j$, $i,j = 1, \ldots, r$.

Let $(G,+)$ be any finite group and U a set of subgroups of G. Then the incidence structure $\mathsf{D}(\mathsf{U})$ is defined as $\mathsf{D}(\mathsf{U}) = (G, \{U+g : U \in \mathsf{U}, g \in G\}, \in)$. An incidence structure is said to be *group constructible* if $\mathsf{D} \cong \mathsf{D}(\mathsf{U})$ for some such U. The elements of U are called the *components* of U.

Lemma B.1.2 *The group constructible nets are precisely the translation nets. Moreover* $\mathsf{D}(\mathsf{U})$ *is an* $(s,r;1)$*-translation net with translation group* G *if and only if* U *is an* $(s,r;1)$-PCP *in* G.

Proof See e.g. [9, p. 512]. $\square$

These observations are summarised in Table B.2.

B.2 MAXMOLS(16)

In this section, maximal partial spreads of size r in $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$ are used to construct transversal-free translation nets of degree $r+3$; this approach will give new examples of MAXMOLS(16).

According to the tables in [33] and some subsequent results of Drake et al. [42] and Bedford and Whitaker [7], MAXMOLS(16) are known for

$t \in \{1, 2, 3, 4, 11, 15\}$. By Bruck's completion theorem, they cannot exist for $t = 13$ and $t = 14$, cf. Beth et al. [10, Section X.7]. Using maximal partial spreads in $\mathrm{PG}(3, 4)$, Jungnickel and Storme [73] were recently able to construct sets of t MAXMOLS(16) for two previously undecided cases, namely for $t = 9$ and $t = 10$. Here, a similar approach—already suggested in [73]—is used to construct sets of t MAXMOLS(16) for the two further values $t = 7$ and $t = 8$, thus reducing the number of open cases to three. The remaining open cases are $t = 5$, $t = 6$, and $t = 12$.

Any r mutually skew lines in $\mathrm{PG}(3, q)$ may be viewed as a collection of r *pairwise disjoint* subgroups of order q^2 in the additive group of the vector space $V = V(4, q)$ (meaning, of course, that any two of these subgroups intersect trivially). This is a particular example of a partial congruence partition (PCP) and therefore leads to a translation net of order $s = q^2$ and degree r by taking the vectors in V as points and all the translates of the specified r subgroups as lines, see Section B.1. If the given partial spread is actually maximal, one may hope that the associated net is likewise maximal, resulting in $t = r - 2$ MAXMOLS(s), $s = q^2$. This approach has been used successfully by Jungnickel [71, 72]. However, in general, the associated net may well be extendable; as mentioned in Section B.1, this happens if and only if the net admits a transversal.

B.2.1 Partial spreads in $\mathrm{PG}(3, 4) \setminus \mathrm{PG}(3, 2)$

In what follows, let Σ_B denote the "natural" Baer subgeometry $\mathrm{PG}(3, 2)$ of $\Sigma = \mathrm{PG}(3, 4)$ which is coordinatised by the binary vectors in the vector space $V = V(4, 4)$. Denote the corresponding subgroup of order 16 of V by U, and write $\mathrm{GF}(4) = \{0, 1, \omega, \omega^2\}$. Then U, ωU and $\omega^2 U$ are three pairwise disjoint subgroups partitioning the quaternary vectors associated with the 15 points of Σ_B; hence they may be added to the r subgroups of V associated with any partial spread $\mathcal{S}'$ of r lines in $\Sigma \setminus \Sigma_B$ to give a PCP P with $r + 3$ components, and one may hope that the associated translation net is transversal-free (and hence the corresponding set of MOLS maximal) provided that $\mathcal{S}'$ is maximal.

As reported in Jungnickel and Storme [73], a computer search for maximal partial spreads in $\mathrm{PG}(3, 4) \setminus \mathrm{PG}(3, 2)$ based on the computer program of [35] for determining the spreads in $\mathrm{PG}(3, 4) \setminus \mathrm{PG}(3, 2)$ gave the following result.

Proposition B.2.1 *A maximal partial spread of r pairwise skew lines in* $\mathrm{PG}(3, 4) \setminus \mathrm{PG}(3, 2)$ *exists if and only if* $6 \leq r \leq 10$ *or* $r = 14$.

It turns out that every maximal partial spread of 6 or 7 pairwise skew lines in $\mathrm{PG}(3, 4) \setminus \mathrm{PG}(3, 2)$ gives rise to a transversal-free translation net of order

16 and degree 9 or 10, respectively, as explained above. This follows from an exhaustive computer search. To facilitate this search, this subsection provides some auxiliary theoretical results which allow to reduce the complexity of the search considerably. These results are appropriate modifications of similar results in [73].

In what follows, consider any fixed transversal T of the net D of degree $r+3$ associated with the PCP P coming from a given maximal partial spread $\mathcal{S}'$ of size $r \in \{6, 7\}$ in $\Sigma \setminus \Sigma_B$. Without loss of generality, also assume that T contains the origin $\bar{0}$.

The following simple but useful result is analogous to Lemma 3.3 of [73]. It concerns the *holes* of the maximal partial spread $\mathcal{S}'$, i.e., the points of $\Sigma \setminus \Sigma_B$ which are not covered by a line of $\mathcal{S}'$. For the remainder of this section, if $\bar{u}$ is a vector in $V(4, 4)$, then $\langle \bar{u} \rangle$ denotes the set $\{\bar{0}, \bar{u}, \omega\bar{u}, \omega^2\bar{u}\}$.

Lemma B.2.2 *The point $P(\bar{u})$ of Σ is a hole for every element $\bar{u} \in T \setminus \{\bar{0}\}$. Moreover, if $\bar{0}, \bar{u}, \bar{v}$ are three elements of T for which $P(\bar{u})$ and $P(\bar{v})$ are distinct points of Σ, then the "sum" $P(\bar{u} + \bar{v})$ of these two holes is likewise a hole.*

Proof If $P(\bar{u})$ would lie on a line of $\mathcal{S}'$ or in Σ_B, the corresponding subgroup U would intersect the transversal T in the distinct elements $\bar{0}$ and $\bar{u}$, a contradiction. Thus $P(\bar{u})$ is indeed a hole. Now let $P(\bar{u})$ and $P(\bar{v})$ be distinct points of Σ, and assume $\bar{0}, \bar{u}, \bar{v} \in \mathsf{T}$. Apply the first assertion to the transversal $\mathsf{T} + \bar{u}$ of D, noting that $\bar{u}$, $\bar{0}$ and $\bar{u} + \bar{v}$ are elements of $\mathsf{T} + \bar{u}$, to conclude that $P(\bar{u} + \bar{v})$ is indeed a hole. □

Let $\bar{u} \in \mathsf{T} \setminus \{\bar{0}\}$. Call a hole $P(\bar{u})$ of Σ, respectively a point $\bar{u}$ of T, *thin* if $\langle \bar{u} \rangle \cap \mathsf{T} = \{\bar{0}, \bar{u}\}$; *semifat* if $|\langle \bar{u} \rangle \cap \mathsf{T}| = 3$; and *fat* if $\langle \bar{u} \rangle \subset \mathsf{T}$. The major two theoretical steps consist of showing that T more or less "contains" thin points only. This corresponds to Proposition 3.4 in [73]. Indeed, the proof for the following first result proceeds exactly as in [73].

Proposition B.2.3 *There are no fat holes at all. Moreover, there exists at most one semifat hole.*

Proof See [73]. □

Proposition B.2.4 *Every point $\bar{u} \in \mathsf{T} \setminus \{\bar{0}\}$ is actually thin provided that $r = 7$. If there exists a semifat hole $P(\bar{u})$, $\bar{u} \in \mathsf{T} \setminus \{\bar{0}\}$, for the case $r = 6$, then $P(\bar{u})$ lies on* 13 *lines each of which contains precisely three further holes.*

Proof Assume the existence of a semifat point in T, say $\bar{0}, \bar{u}, \lambda\bar{u} \in \mathsf{T}$, where $\bar{u} \neq \bar{0}$ and $\lambda \notin \{0, 1\}$. As T has 16 elements, there are 13 vectors $\bar{v} \in \mathsf{T} \setminus \langle \bar{u} \rangle$.

By Lemma B.2.2, for each choice of $\bar{v}$, the points $P(\bar{u})$, $P(\bar{v})$, $P(\bar{u}+\bar{v})$ and $P(\lambda\bar{u}+\bar{v})$ are holes. By Proposition B.2.3, no point $P(\bar{v})$ can be semifat, and hence, in this way, $3 \cdot 13$ points distinct from $P(\bar{u})$ are obtained, all of which are holes.

If $r = 7$, then there are only 35 holes altogether, so that there must be holes occurring in two different ways, say $\bar{0}, \bar{u}, \lambda\bar{u}, \bar{v}, \bar{v}' \in \mathsf{T}$, where $\bar{v}'$ gives a hole on the line l through $P(\bar{u})$ and $P(\bar{v})$. As $P(\bar{v})$ is not semifat and as l cannot consist of holes only, it follows that

$$P(\bar{v}') \neq P(\bar{u}), P(\bar{v}), P(\lambda^2\bar{u}+\bar{v}).$$

Without loss of generality, assume $P(\bar{v}') = P(\bar{u}+\bar{v})$ (otherwise replace $\bar{u}$ by $\bar{u}' = \lambda\bar{u}$). Now there are three possibilities to consider. If $\bar{v}' = \bar{u}+\bar{v}$, the transversal $\mathsf{T}+\lambda\bar{u}$ contains the elements $\bar{0}$ and $(\bar{u}+\bar{v})+\lambda\bar{u} = \lambda^2\bar{u}+\bar{v}$, contradicting the observation that $P(\lambda^2\bar{u}+\bar{v})$ cannot be a hole. The case $\bar{v}' = \lambda(\bar{u}+\bar{v})$ leads to the same contradiction by considering $\mathsf{T}+\bar{v}$ and noting that $P(\lambda(\bar{u}+\bar{v})+\bar{v}) = P(\lambda^2\bar{u}+\bar{v})$. Finally, the case $\bar{v}' = \lambda^2(\bar{u}+\bar{v})$ is excluded as before by considering $\mathsf{T}+\bar{u}$.

For $r = 6$, no contradiction is obtained by assuming the existence of a semifat point, as there will be altogether 40 holes in this case. But then the same reasoning as before immediately gives the structural restriction stated in the assertion—the 13 lines are the lines joining $P(\bar{u})$ to $P(\bar{v})$, with $\bar{v} \in \mathsf{T} \setminus \langle \bar{u} \rangle$. □

B.2.2 The computer searches

To perform the computer searches, the share package PG [36] for the computer algebra system GAP [52] was used. The implementation of these searches can be found on the website `http://cage.rug.ac.be/~pg/thesis/`. As already announced, they established the following result.

Theorem B.2.5 *Every maximal partial spread of* 6 *or* 7 *pairwise skew lines in* $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$ *gives rise to a transversal-free translation net of order* 16 *and degree* 9 *or* 10, *respectively.*

In order to establish Theorem B.2.5, the setup of the preceding section was used. In particular, the restrictions in Proposition B.2.4 considerably simplify the exhaustive search for a possible transversal T of the translation net D constructed from a maximal partial spread $\mathcal{S}'$ in $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$.

By Proposition B.2.4, T gives rise to fifteen thin points of Σ provided that $r = 7$. The computer searches of [35] and [73] show that there is, up to equivalence under $\mathrm{P\Gamma L}(4,4)$, only one maximal partial spread of size

$r = 6$ in $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$. It is a simple matter to check that the 40 holes determined by this maximal partial spread do not form a configuration as described in Proposition B.2.4; thus T gives rise to fifteen thin points of Σ also for $r = 6$.

There is, up to equivalence under $\mathrm{P\Gamma L}(4,4)$, exactly one maximal partial spread of size $r = 7$ in $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$. This maximal partial spread was checked, as was the one for $r = 6$. In both cases, the corresponding net turned out not to admit a transversal (containing 0 and fifteen thin points). This establishes Theorem B.2.5. As an immediate consequence, the desired new examples of MAXMOLS(16) are obtained.

Corollary B.2.6 *There exist t MAXMOLS(16) for $t = 7$ and $t = 8$.*

B.2.3 Two remarks

As explained in the preamble of Section B.2, any maximal partial spread of $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$ of size r yields a PCP P with $r + 3$ components in the additive group of the vector space $V = V(4,4)$. Of course, this group can be seen as the additive group of $V(8,2)$, and hence P can be considered as a partial 3-spread $\mathcal{T}'$ of size $r + 3$ in $\mathrm{PG}(7,2)$. In view of Theorem B.2.5, the associated translation net is transversal-free for $r \in \{6,7\}$; thus $\mathcal{T}'$ is maximal in these cases.

The existence of maximal partial 3-spreads of size 9 in $\mathrm{PG}(7,2)$ is known. Indeed, the hyperbolic quadric $\mathrm{Q}^+(7,2)$ in $\mathrm{PG}(7,2)$ has a spread consisting of nine 3-dimensional subspaces [44, 111, 112]. This spread of $\mathrm{Q}^+(7,2)$ is maximal considered as partial 3-spread of $\mathrm{PG}(7,2)$, since an arbitrary 3-dimensional space in $\mathrm{PG}(7,2)$ intersects a hyperbolic quadric non-trivially.

On the other hand, the existence of a maximal partial 3-spread in $\mathrm{PG}(7,2)$ of size 10 was not known. These observations are summarised in the following proposition.

Proposition B.2.7 *There exist maximal partial 3-spreads of sizes 9 and 10 in $\mathrm{PG}(7,2)$.*

The second remark concerns a failed attempt to find 12 MAXMOLS(16) by a similar approach. It is possible to find three pairwise disjoint Baer subgeometries in $\mathrm{PG}(3,4)$, actually even to partition $\mathrm{PG}(3,4)$ into three Baer subgeometries and eight lines. By a computer result of Penttila (Private communication, 2001), there are precisely two such partitions up to equivalence, see also Mellinger [82]. Motivated by this fact, we decided to look for maximal partial spreads in $\Sigma \setminus (\Sigma_B \cup \Sigma'_B \cup \Sigma''_B)$, where Σ_B, Σ'_B and Σ''_B are three pairwise disjoint Baer 3-spaces in $\Sigma = \mathrm{PG}(3,4)$. Clearly the first Baer subspace Σ_B may always be assumed to be the standard $\mathrm{PG}(3,2)$. Also Σ'_B can be chosen as a fixed Baer subspace skew to Σ_B, see [82]. So, the difference in the tuples $(\Sigma_B, \Sigma'_B, \Sigma''_B)$ that need to be investigated, occurs only in the third position; for the third Baer subspace Σ''_B there are precisely three choices.

Now there exist maximal partial spreads of five mutually skew lines in $\Sigma\backslash(\Sigma_B \cup \Sigma'_B \cup \Sigma''_B)$. Such a maximal partial spread $\mathcal{S}'$ gives rise to a translation net D of order 16 and degree 14, by extending the PCP associated with $\mathcal{S}'$ with nine new components, three for each of the Baer subspaces Σ_B, Σ'_B and Σ''_B (similar to the approach explained at the beginning of this section). One could hope that this would yield a transversal-free translation net D and hence a corresponding set of 12 MAXMOLS(16). Unfortunately, in all cases D turns out to have a transversal $\mathrm{PG}(3,2)$ (and thus to extend to an affine translation plane of order 16).

B.3 Infinite classes of MAXMOLS arising from spreads of $\mathrm{Q}^+(4n-1,q)$

(Non)existence results on spreads and ovoids of the hyperbolic quadrics, see Chapters 4 and 5, are used to construct infinite classes of $q^{2n-1}-1$ MAXMOLS(q^{2n}), for $n \geq 2$ and q a power of two, and for $n = 2$ and q a power of three. The first example arises for $q = 2$ and $n = 2$, giving 7 MAXMOLS(16) via a computer free method.

It is known, see Table 4.2, that for m even, the quadric $\mathrm{Q}^+(2m+1,q)$ has no spread.

Theorem B.3.1 *Suppose that* $\mathrm{Q}^+(4n-1,q)$ *has a spread and that* $\mathrm{Q}^+(4n+1,q)$ *does not have an ovoid. Then there exist* $q^{2n-1}-1$ MAXMOLS(q^{2n}).

Proof Start with a spread $\mathcal{S}$ of $\mathrm{Q}^+(4n-1,q)$ in $\mathrm{PG}(4n-1,q)$. Embed $\mathrm{PG}(4n-1,q)$ in $\mathrm{PG}(4n,q)$ and consider the net whose points are the affine points of $\mathrm{PG}(4n,q)$ and whose lines are the sets of affine points of $(2n)$-spaces in $\mathrm{PG}(4n,q)$ that intersect $\mathrm{PG}(4n-1,q)$ in an element of $\mathcal{S}$.

To this net, there corresponds a set of $q^{2n-1}-1$ MOLS(q^{2n}). It suffices to show that the net is transversal-free to prove that these MOLS are in fact MAXMOLS. Suppose, by way of contradiction, that it admits a transversal T. Then T consists of q^{2n} points of $\mathrm{PG}(4n,q) \setminus \mathrm{PG}(4n-1,q)$.

If P_1 and P_2 are points of T, then P_1P_2 intersects $\mathrm{PG}(4n-1,q)$ in a point outside $\mathrm{Q}^+(4n-1,q)$. Indeed, otherwise this line would intersect $\mathrm{PG}(4n-1,q)$

in a point of an element of $\mathcal{S}$, say S, and the transversal T would contain at least two points of the line $\langle P_1, S\rangle \setminus \mathrm{PG}(4n-1, q)$ of the net.

Now in the dual space of $\mathrm{PG}(4n, q)$, $\mathrm{PG}(4n-1, q)$ becomes a point P, the elements of $\mathcal{S}$ become $(2n)$-spaces through P, and $\mathrm{Q}^+(4n-1, q)$ becomes a cone with vertex P and base a quadric $\mathrm{Q}^+(4n-1, q)$. The point P_1, respectively P_2, becomes a $(4n-1)$-space π_1, respectively π_2, not through P, and the line P_1P_2 becomes a $(4n-2)$-space that intersects the cone in a nonsingular quadric $\mathrm{Q}(4n-2, q)$.

Embed the cone in a nonsingular $\mathrm{Q}^+(4n+1, q)$ in $\mathrm{PG}(4n+1, q)$ and apply the polarity of $\mathrm{Q}^+(4n+1, q)$. This polarity maps π_i onto a bisecant to $\mathrm{Q}^+(4n+1, q)$ through P, $i = 1, 2$. Call the second point of $\mathrm{Q}^+(4n+1, q)$ on this line P_i'. Then $\langle P_1', P_2', P\rangle$ intersects $\mathrm{Q}^+(4n+1, q)$ in a nonsingular conic, since π_1 and π_2 intersect in a space that has a nonsingular intersection with $\mathrm{Q}^+(4n+1, q)$.

Therefore the $q^{2n}+1$ points $P, P_1', P_2', \dots$ form an ovoid of $\mathrm{Q}^+(4n+1, q)$, a contradiction. $\square$

Corollary B.3.2 *There exist* $q^{2n-1}-1$ MAXMOLS(q^{2n}) *for* $n \geq 2$ *and* q *even, and for* $n = 2$ *and* q *a power of three.*

Proof For these values for n and q, it is known that $\mathrm{Q}^+(4n-1, q)$ has a spread, see Dye [44] and Thas [111, 112], and that $\mathrm{Q}^+(4n+1, q)$ does not have an ovoid, see Kantor [74] and Shult [102]. $\square$

Remark B.3.3 For $q = 2$ and $n = 2$, this corollary gives 7 MAXMOLS(16). Hence, in addition to Corollary B.2.6, also a computer free construction of 7 MAXMOLS(16) is presented.

Bijlage C

Nederlandstalige samenvatting

C.1 Introductie

In het eerste hoofdstuk wordt het onderzoek gesitueerd en worden enkele basisbegrippen uit de eindige projectieve meetkunde uitgelegd. Voor een uitvoerige bespreking van deze objecten wordt verwezen naar de boeken Hirscheld [68], Hirschfeld en Thas [70], en Payne en Thas [91].

De inhoud van dit proefschrift behoort tot het gebied van de eindige meetkunde. Meer specifiek worden enkele objecten in de n-dimensionale projectieve ruimte $\mathrm{PG}(n, q)$ over het eindige veld $\mathrm{GF}(q)$ van de orde q bestudeerd. De structuren die aan bod komen zijn alle verwant met zogenaamde blokkerende verzamelingen.

Een *blokkerende verzameling* B in $\mathrm{PG}(2, q)$ is een verzameling punten van $\mathrm{PG}(2, q)$ die een niet-ledige doorsnede heeft met elke rechte van $\mathrm{PG}(2, q)$. Elke rechte van $\mathrm{PG}(2, q)$ wordt als het ware "geblokkeerd" door die verzameling. Deze definitie kan veralgemeend worden, zowel naar de dimensie van de ruimte waarin gewerkt wordt, als naar de dimensie van de te blokkeren deelruimten, als naar het aantal punten dat hun moet blokkeren. Zo is een *k-voudige t-blokkerende verzameling in* $\mathrm{PG}(n, q)$ een verzameling punten in $\mathrm{PG}(n, q)$ die met elke $(n-t)$-dimensionale deelruimte van $\mathrm{PG}(n, q)$ minstens k punten gemeen heeft. Bijgevolg is een "blokkerende verzameling" een 1-voudige 1-blokkerende verzameling in $\mathrm{PG}(2, q)$.

Verscheidene van de bestudeerde objecten leven in *eindige klassieke polaire ruimten.* Deze zijn:

- $\mathrm{Q}^-(2n+1, q)$, $n \geq 2$, de polaire ruimte afkomstig van een niet-singuliere elliptische kwadriek in $\mathrm{PG}(2n+1, q)$;

- $\mathrm{Q}(2n, q)$, $n \geq 2$, de polaire ruimte afkomstig van een niet-singuliere kwadriek in $\mathrm{PG}(2n, q)$;

- $\mathrm{Q}^+(2n+1, q)$, $n \geq 1$, de polaire ruimte afkomstig van een niet-singuliere hyperbolische kwadriek in $\mathrm{PG}(2n+1, q)$;
- $\mathrm{H}(n, q^2)$, $n \geq 3$, de polaire ruimte afkomstig van een niet-singuliere Hermitische variëteit in $\mathrm{PG}(n, q^2)$;
- $\mathrm{W}_{2n+1}(q)$, $n \geq 1$, de polaire ruimte afkomstig van een niet-singuliere symplectische polariteit in $\mathrm{PG}(2n+1, q)$.

Op deze polaire ruimten liggen projectieve deelruimten. De projectieve deelruimten met maximale dimensie op een eindige klassieke polaire ruimte worden *generatoren* genoemd; zij hebben dimensie $k-1$, waarbij k de *rang* van de polaire ruimte genoemd wordt. In Tabel 1.1 worden de rang, het aantal punten en het aantal generatoren van de eindige klassieke polaire ruimten gegeven.

C.2 Minihypers

Minihypers, geïntroduceerd door Hamada en Tamari in [64], werden ingevoerd wegens hun belang voor de codeertheorie. Doorgaans worden welbepaalde klassen van minihypers bestudeerd wegens hun relatie met lineaire codes die de Griesmer grens bereiken. Dergelijke codes zijn belangrijk, aangezien zij optimaal zijn in dat opzicht dat, voor gegeven dimensie en minimum afstand er geen lineaire codes met kleinere lengte bestaan.

In Hoofdstukken 2 en 3 worden dergelijke minihypers bestudeerd. De classificatieresultaten van minihypers die in deze hoofdstukken bekomen worden, kunnen onmiddellijk vertaald worden naar classificatieresultaten van lineaire codes die de Griesmer grens bereiken. Dit is echter niet de hoofdreden voor de studie die in deze hoofdstukken geleverd wordt. Zoals uit de daaropvolgende Hoofdstukken 4 en 5 blijkt, hebben de classificaties van minihypers tal van toepassingen in de theorie van de eindige meetkunde. Zo worden zij gebruikt om nieuwe resultaten op het gebied van partiële spreads en bedekkingen van eindige projectieve ruimten te bekomen, alsook om nieuwe stellingen betreffende partiële spreads, bedekkingen en partiële ovoïden van eindige klassieke polaire ruimten te bewijzen.

De resultaten uit Hoofdstuk 2 werden gepubliceerd in *Designs, Codes and Cryptography* in *P. Govaerts en L. Storme, On a particular class of minihypers and its applications. I. The result for general* q [56], terwijl die uit Hoofdstuk 3 verschenen in *Journal of Combinatorial Theory. Series A* in *P. Govaerts en L. Storme, On a particular class of minihypers and its applications. II. Improvements for* q *square* [55].

Notatie C.2.1 Zij $\mathrm{PG}^{(k)}(n,q)$ de verzameling van alle k-dimensionale deelruimten van $\mathrm{PG}(n,q)$.

Definitie C.2.2 Een $\{f,m;n,q\}$-*minihyper* is een paar (F,w), met F een deelverzameling van de puntenverzameling van $\mathrm{PG}(n,q)$, en w een gewichtsfunctie $w : \mathrm{PG}(n,q) \to \mathbb{N} : P \mapsto w(P)$, die voldoet aan

1. $w(P) > 0 \Leftrightarrow P \in F$,
2. $\sum_{P\in F} w(P) = f$, en
3. $\min\{\sum_{P\in H} w(P) : H \in \mathrm{PG}^{(n-1)}(n,q)\} = m$.

Vaak worden minihypers zonder gewichten bestudeerd. Dit zijn minihypers waarbij de gewichtsfunctie w de verzameling $\{0,1\}$ als beeldverzameling heeft. In dit geval kan de minihyper (F,w) geïdentificeerd worden met de verzameling F en wordt zij eenvoudigweg aangeduid met F en een *gewichtloze* minihyper genoemd.

Het is niet moeilijk in te zien dat minihypers zeer nauw verwant zijn met blokkerende verzamelingen. Inderdaad, gewichtloze minihypers zijn niets anders dan meervoudige blokkerende verzamelingen. Immers, uit de definitie volgt dat een $\{f,m;n,q\}$-minihyper F een m-voudige 1-blokkerende verzameling is. Omgekeerd is elke m-voudige 1-blokkerende verzameling B in $\mathrm{PG}(n,q)$ die geen $(m+1)$-voudige blokkerende verzameling is, een $\{|B|,m;n,q\}$-minihyper.

Er werd reeds vermeld dat minihypers verwant zijn met lineaire codes die de Griesmer grens bereiken. Dit verband wordt nauwkeuriger omschreven in de volgende stelling. Voor een introductie in de codeertheorie, zie Hill [66].

Notatie C.2.3 Als l een natuurlijk getal is, dan wordt v_l gebruikt om het getal $(q^l-1)/(q-1)$ aan te duiden. Bijgevolg is $|\mathrm{PG}(l-1,q)| = v_l$.

Stelling C.2.4 (Hamada [60]) *Zij q een priemmacht en zij k, θ en ζ_i, $i = 0,1,\ldots,k-2$, natuurlijke getallen waarvoor geldt dat $k \geq 3$, $\theta \geq 1$, $0 \leq \zeta_i \leq q-1$ en $(\zeta_0,\zeta_1,\ldots,\zeta_{k-2}) \neq \bar{0}$. Zij $d = \theta q^{k-1} - \sum_{i=0}^{k-2}\zeta_i q^i$. Er is een bijectief verband tussen de verzameling van alle niet-equivalente $[\tilde{n},k,d;q]$-codes die de Griesmer grens bereiken en de verzameling van alle*

$$\left\{\sum_{i=0}^{k-2}\zeta_i v_{i+1}, \sum_{i=1}^{k-2}\zeta_i v_i; k-1, q\right\}\text{-}minihypers\ (F,w)$$

waarvoor $w(P) \leq \theta$ voor elk punt P van $\mathrm{PG}(k-1,q)$.

Deze stelling leert dat, om alle $[\tilde{n}, k, d; q]$-codes die de Griesmer grens bereiken te classificeren voor gegeven k, d en q, het volstaat volgend probleem op te lossen.

Probleem C.2.5 Voor gegeven n, q en ζ_i, $i = 0, 1, \ldots, n-1$, classificeer alle

$$\left\{\sum_{i=0}^{n-1} \zeta_i v_{i+1}, \sum_{i=1}^{n-1} \zeta_i v_i; n, q\right\}\text{-minihypers } (F, w)$$

waarvoor $w(P) \leq \theta$ voor elk punt P van $\mathrm{PG}(n, q)$.

Dit toont het belang van minihypers voor de codeertheorie aan en verklaart waarom talrijke artikels verschenen waarin dergelijke minihypers worden bestudeerd.

Volgende stelling geeft twee classificatieresultaten voor minihypers voor algemene waarden van n, q en ζ_i, op voorwaarde dat $\sum \zeta_i$ niet te groot is. Merk hierbij op dat de beschouwde minihypers gewichtloos zijn.

Stelling C.2.6 *Zij F een $\{\sum_{i=0}^{s} \zeta_i v_{i+1}, \sum_{i=0}^{s} \zeta_i v_i; n, q\}$-minihyper.*

1. **(Hamada en Helleseth [62], Hamada en Maekawa [63])** *Indien $\sum_{i=0}^{s} \zeta_i \leq \sqrt{q}$, dan is F de unie van ζ_s s-ruimten, ζ_{s-1} $(s-1)$-ruimten, $\ldots$, ζ_0 punten van $\mathrm{PG}(n, q)$ die paarsgewijs disjunct zijn.*

2. **(Ferret en Storme [51])** *Indien $\sum_{i=0}^{s} \zeta_i < 2\sqrt{q}$, $q > q_0$, dan bestaat F uit de unie van ofwel*
(a) *ζ_s s-ruimten, ζ_{s-1} $(s-1)$-ruimten, $\ldots$, en ζ_0 punten, ofwel*
(b) *één deelmeetkunde $\mathrm{PG}(2l+1, \sqrt{q})$, voor een welbepaald natuurlijk getal l met $1 \leq l \leq s$, ζ_s s-ruimten, $\ldots$, ζ_{l+1} $(l+1)$-ruimten, $\zeta_l - \sqrt{q} - 1$ l-ruimten, ζ_{l-1} $(l-1)$-ruimten, $\ldots$, en ζ_0 punten, ofwel*
(c) *één deelmeetkunde $\mathrm{PG}(2l, \sqrt{q})$, voor een welbepaald natuurlijk getal l met $1 \leq l \leq s$, ζ_s s-ruimten, $\ldots$, ζ_{l+1} $(l+1)$-ruimten, $\zeta_l - 1$ l-ruimten, $\zeta_{l-1} - \sqrt{q}$ $(l-1)$-ruimten, $\ldots$, en ζ_0 punten.*
In alle drie de gevallen zijn de objecten paarsgewijs disjunct.

In Hoofdstukken 2 en 3 wordt een deelprobleem van Probleem C.2.5 behandeld.

Probleem C.2.7 Voor gegeven n, q, μ en δ, classificeer alle

$$\{\delta v_{\mu+1}, \delta v_\mu; n, q\}\text{-minihypers } (F, w)$$

waarvoor $w(P) \leq \theta$ voor elk punt P van $\mathrm{PG}(n, q)$.

Een *som* van t-ruimten is een gewichtsfunctie w : $\mathrm{PG}^{(t)}(n,q) \to \mathbb{N}$: $\pi_t \mapsto w(\pi_t)$. Dergelijke som induceert een gewichtsfunctie op deelruimten van kleinere dimensie. Zij π_r een deelruimte met dimensie $r < t$, dan is per definitie $w(\pi_r) = \sum_{\pi \in \mathrm{PG}^{(t)}(n,q), \pi \supset \pi_r} w(\pi)$. In het bijzonder is het gewicht van een punt de som van de gewichten van de t-ruimten die dat punt bevatten. In het geval w een afbeelding op $\{0,1\}$ is, kan de som w geïdentificeerd worden met de verzameling van deelruimten met gewicht 1.

Voorbeeld C.2.8 Beschouw δ μ-ruimten $W_1, W_2, \ldots, W_\delta$ in $\mathrm{PG}(n,q)$, met $1 \leq \delta \leq q-1$ en $1 \leq \mu \leq n-1$. Definieer voor elk punt P van $\mathrm{PG}(n,q)$ het gewicht $w(P)$ als het aantal μ-ruimten W uit $\{W_1, W_2, \ldots, W_\delta\}$ waarvoor $P \in W$. Zij F de verzameling van punten P van $\mathrm{PG}(n,q)$ waarvoor $w(P) \geq 1$. Dan is (F,w) een $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper.

Het hoofdresultaat van Hoofdstuk 2 toont aan dat onder zekere voorwaarden voor δ en μ ook het omgekeerde geldt. Om het resultaat te formuleren wordt volgende notatie ingevoerd.

Notatie C.2.9 Voor $q = 2$, zij $\epsilon_q = 2$. Voor $q > 2$, zij $q + \epsilon_q$ de grootte van de kleinste niet-triviale blokkerende verzameling in $\mathrm{PG}(2,q)$.

Stelling C.2.10 *Indien (F,w) een $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper is, met $0 \leq \delta < \epsilon_q$ en $\mu \leq n-1$, dan is w de gewichtsfunctie geïnduceerd op de punten van $\mathrm{PG}(n,q)$ door een som van δ μ-ruimten. Bovendien is deze som uniek.*

Hierbij dient opgemerkt te worden dat deze stelling reeds bewezen was onder de bijkomende veronderstellingen dat de beschouwde minihyper gewichtloos is en $\delta < \sqrt{q} + 1$, zie Stelling C.2.6.

Stelling C.2.10 is het zwakst in het geval dat q een kwadraat is. Immers, in dat geval zijn de kleinste niet-triviale blokkerende verzamelingen erg klein: zij hebben grootte $q + \sqrt{q} + 1$. Deze blokkerende verzamelingen hebben echter een bijzonder mooie structuur, en er bestaan verschillende resultaten aangaande de grootte van de op één na kleinste niet-triviale minimale blokkerende verzamelingen in $\mathrm{PG}(2,q)$, q een kwadraat. Deze twee zaken laten toe Stelling C.2.10 te verbeteren in het geval dat q een kwadraat is en onder de bijkomende voorwaarde dat de beschouwde minihyper gewichtloos is. Zo een verbetering wordt in Hoofdstuk 3 bekomen.

In dat hoofdstuk is het dus de bedoeling

$$\{\delta v_{\mu+1}, \delta v_\mu; n, q\}\text{-minihypers } F$$

in $\mathrm{PG}(n,q)$, q een kwadraat, te classificeren voor alle $\delta \leq \alpha$ voor een bepaald natuurlijk getal α. Het spreekt voor zich dat het de bedoeling is een classificatie te bekomen voor zo groot mogelijke α. Indien $\delta \leq \sqrt{q}$, dan stellen

Stellingen C.2.6 en C.2.10 dat zo een minihyper bestaat uit een disjuncte unie van μ-ruimten. Voor grotere δ bestaan er andere voorbeelden.

Voorbeeld C.2.11 Zij q een kwadraat, $1 \leq \delta \leq q-1$, $1 \leq \mu \leq (n-1)/2$ en $k \leq \delta/(\sqrt{q}+1)$. Beschouw k onderling disjuncte deelmeetkunden $\mathrm{PG}(2\mu+1,\sqrt{q})$ in $\mathrm{PG}(n,q)$, noteer deze met $D_1, D_2, \ldots, D_k$, en $l = \delta - k(\sqrt{q}+1)$ onderling disjuncte μ-ruimten $W_1, W_2, \ldots, W_l$ in $\mathrm{PG}(n,q)$ die scheef zijn aan $\cup_{i=1}^{k} D_i$. Dan is $F = (\cup_{i=1}^{k} D_i) \cup (\cup_{i=1}^{l} W_i)$ een $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper, aangezien $|\mathrm{PG}(2\mu+1,\sqrt{q})| = (\sqrt{q}+1)v_{\mu+1}$ en een hypervlak steeds D_i snijdt in een deelmeetkunde $\mathrm{PG}(s,\sqrt{q})$ voor een welbepaalde $s \in \{2\mu-1, 2\mu, 2\mu+1\}$.

Het hoofdresultaat van Hoofdstuk 3 toont aan dat, indien δ niet te groot is, ook het omgekeerde geldt.

Stelling C.2.12 *Een $\{\delta v_{\mu+1}, \delta v_\mu; n, q\}$-minihyper F, $q > 16$ een kwadraat, $\delta < q^{5/8}/\sqrt{2}+1$, $\mu \geq 1$, is op unieke wijze te schrijven als een unie van paarsgewijs disjuncte μ-ruimten en deelmeetkunden* $\mathrm{PG}(2\mu+1,\sqrt{q})$.

C.3 Partiële spreads en bedekkingen

In Hoofdstuk 4 worden (niet-)existentieresultaten voor partiële t-spreads en t-bedekkingen in eindige projectieve en polaire ruimten bekomen. Sommige daarvan zijn toepassingen van de classificatieresultaten van minihypers, zie Stellingen C.2.10 en C.2.12. Deze werden gepubliceerd in *European Journal of Combinatorics* in *P. Govaerts, L. Storme en H. Van Maldeghem, On a particular class of minihypers and its applications. III. Applications* [57].

Een *partiële t-spread* van $\mathrm{PG}(n,q)$ is een verzameling van onderling scheve t-ruimten in $\mathrm{PG}(n,q)$. Een *t-bedekking* $\mathcal{C}$ van $\mathrm{PG}(n,q)$ is een verzameling van t-ruimten van $\mathrm{PG}(n,q)$ zo dat elk punt van $\mathrm{PG}(n,q)$ in ten minste één element van $\mathcal{C}$ ligt. Een *t-spread* van $\mathrm{PG}(n,q)$ is een verzameling van t-ruimten in $\mathrm{PG}(n,q)$ die de puntenverzameling van $\mathrm{PG}(n,q)$ partitioneert. Bovenstaande definities kunnen eenvoudigweg overgenomen worden voor eindige klassieke polaire ruimten door "$\mathrm{PG}(n,q)$" te vervangen door "een eindige klassieke polaire ruimte $\mathcal{P}$".

Indien n oneven is en $t = (n+1)/2$, dan wordt een (partiële) t-spread, respectievelijk t-bedekking, van $\mathrm{PG}(n,q)$ eenvoudigweg een *(partiële) spread*, respectievelijk *bedekking*, van $\mathrm{PG}(n,q)$ genoemd.

Indien $t+1$ de rang is van de eindige klassieke polaire ruimte $\mathcal{P}$, dan wordt een (partiële) t-spread, respectievelijk t-bedekking, van $\mathcal{S}$ eenvoudigweg een *(partiële) spread*, respectievelijk *bedekking*, van $\mathcal{P}$ genoemd.

Bij het bestuderen van deze structuren duiken volgende vragen op natuurlijke wijze op.

Vraag C.3.1 Wanneer bestaan t-spreads in $\mathrm{PG}(n, q)$ of $\mathcal{P}$?

Vraag C.3.2 Indien t-spreads niet bestaan, wat is de grootte van de verzamelingen die "het meest op hen lijken"? Dat is, wat is de grootte van de grootste partiële t-spreads en de grootte van de kleinste t-bedekkingen van $\mathrm{PG}(n, q)$ of $\mathcal{P}$?

Reeds een stapje verder kan men zich het volgende afvragen.

Vraag C.3.3 Wat zijn de mogelijke groottes van maximale partiële t-spreads en minimale t-bedekkingen in $\mathrm{PG}(n, q)$ of $\mathcal{P}$?

Wat betreft Vraag C.3.1 kan volgende opmerking gemaakt worden. Indien $\mathrm{PG}(n, q)$ of $\mathcal{P}$ een t-spread heeft, dan is $|\mathrm{PG}(t, q)|$ een deler van $|\mathrm{PG}(n, q)|$ of $|\mathcal{P}|$.

In het geval van de projectieve ruimten, is het welbekend dat deze delingsvoorwaarde equivalent is met de voorwaarde dat $t + 1$ een deler is van $n + 1$. Meer zelfs, deze nodige voorwaarde is voldoende: $\mathrm{PG}(n, q)$ heeft een t-spread als en slechts als $t + 1$ een deler is van $n + 1$.

In het geval van polaire ruimten worden equivalente delingsvoorwaarden die gemakkelijker hanteerbaar zijn opgesteld in Stelling C.3.15. In het bijzondere geval van spreads van de polaire ruimten gaan deze delingsvoorwaarden steeds op en wordt de grootte van een hypothetische spread gegeven in Tabel 4.1. In dit geval echter, is de nodige voorwaarde (die voldaan is) niet voldoende om het bestaan van een spread te garanderen. In Tabel 4.2 wordt een overzicht gegeven van de gekende (niet-)existentieresultaten voor spreads van eindige klassieke polaire ruimten.

In Hoofdstuk 4 wordt voornamelijk naar Vraag C.3.3 gekeken in het geval dat de delingsvoorwaarde voor het bestaan van een t-spread van de projectieve ruimte of eindige klassieke polaire ruimte vervuld is.

C.3.1 Partiële t-spreads in projectieve ruimten

Volgende stelling geeft het verband tussen partiële t-spreads en minihypers.

Notatie C.3.4 Indien a en b twee gehele getallen zijn, dan wordt "a deelt b" kortweg voorgesteld door "$a|b$".

Stelling C.3.5 *Zij $\mathcal{S}'$ een partiële t-spread van* $\mathrm{PG}(n,q)$*,* $(t+1)|(n+1)$*, met deficiëntie $\delta < q$ en zij F de verzameling gaten van $\mathcal{S}$. Dan is F een $\{\delta v_{t+1}, \delta v_t; n, q\}$-minihyper.*

Bijgevolg kunnen de resultaten over minihypers uit Hoofdstukken 2 en 3 gebruikt worden om enkele observaties aangaande de structuur van partiële t-spreads te maken.

Stelling C.3.6 *Onderstel dat* $(t+1)|(n+1)$ *en zij $\mathcal{S}'$ een maximale partiële t-spread met deficiëntie δ in* $\mathrm{PG}(n,q)$*.*

1. *Als $\delta > 0$, dan is $\delta \geq \epsilon_q$.*

2. *Als $q > 16$ een kwadraat is en als $\delta < q^{5/8}/\sqrt{2}+1$, dan is $\delta \equiv 0 \pmod{\sqrt{q}+1}$ en is de verzameling gaten een disjuncte unie van deelmeetkunden* $\mathrm{PG}(2t+1,\sqrt{q})$*. Als bovendien $\delta > 0$ en $n < \sqrt{q}+1$, dan is $\delta \geq 2(\sqrt{q}+1)$.*

Deze resultaten zijn veralgemeningen van de Metsch-Storme resultaten [87] over maximale partiële 1-spreads in $\mathrm{PG}(3,q)$ en maximale partiële t-spreads in $\mathrm{PG}(2t+1,q)$. Zij zijn veralgemeningen in die zin dat ze geldig zijn voor partiële t-spreads in $\mathrm{PG}(n,q)$ voor elke t en n waarvoor $(t+1)|(n+1)$. Voor $n = 2t+1$ echter, zijn de grenzen op δ uit de stellingen van Metsch en Storme beter.

C.3.2 t-Bedekkingen in projectieve ruimten

Eerst worden de resultaten op minihypers uit Hoofdstuk 2 toegepast op t-bedekkingen van $\mathrm{PG}(n,q)$ om een karakterisering van de verzameling meervoudige punten te bekomen.

Vervolgens worden kleine rechtenbedekkingen van $\mathrm{PG}(4,q)$ bestudeerd.

Toepassingen van de minihyperresultaten

Volgende stelling toont aan dat de meervoudige punten van een t-bedekking op een bijzondere wijze verdeeld zijn over de projectieve ruimte.

Stelling C.3.7 *Zij $\mathcal{C}$ een t-bedekking van* $\mathrm{PG}(n,q)$*,* $(t+1)|(n+1)$*, met exces $\varepsilon < q$. Zij F de verzameling van meervoudige punten van $\mathcal{C}$ en zij* $w(P) = \mathrm{surplus}(P)$ *voor $P \in$* $\mathrm{PG}(n,q)$*. Dan is (F,w) een $\{\varepsilon v_{t+1}, \varepsilon v_t; n, q\}$-minihyper.*

Bijgevolg kunnen de resultaten over minihypers uit Hoofdstuk 2 gebruikt worden om enkele observaties aangaande de structuur van t-bedekkingen te maken.

Stelling C.3.8 *Onderstel dat* $(t+1)|(n+1)$ *en dat* $\varepsilon < \epsilon_q$. *Als* $\mathcal{C}$ *een* t-*bedekking is van* $\mathrm{PG}(n,q)$ *met exces* ε, *dan vormen de meervoudige punten een som van* ε t-*ruimten.*

Opmerking C.3.9 Deze stelling was reeds bewezen in het bijzondere geval van rechtenbedekkingen van $\mathrm{PG}(3,q)$ door Blokhuis et al. in [21].

Tevens worden voorbeelden van minimale rechtenbedekkingen (uit [21]) in $\mathrm{PG}(3,q)$ met exces ε voor elke $\varepsilon \in \{0,1,\ldots,q-1\} \cup \{0,2,4,\ldots,2q\}$ gebruikt om minimale $(2n-1)$-bedekkingen van $\mathrm{PG}(4n-1,q)$ te bekomen.

Kleine rechtenbedekkingen van $\mathrm{PG}(4,q)$

Het is geweten dat een rechtenbedekking van $\mathrm{PG}(4,q)$ uit ten minste q^3+q+1 rechten bestaat, in welk geval er een punt P bestaat dat exact $q+1$ maal bedekt wordt, terwijl alle andere punten juist één maal bedekt worden.

Wat gebeurt er als een bedekking één rechte meer bevat? Volgende stelling toont hoe de meervoudige punten gedistribueerd zijn over $\mathrm{PG}(4,q)$.

Stelling C.3.10 *Onderstel dat* $\mathcal{C}$ *een rechtenbedekking van* $\mathrm{PG}(4,q)$ *is van grootte* q^3+q+2. *Dan zijn de meervoudige punten van* $\mathrm{PG}(4,q)$ *op één van de volgende manieren verspreid over* $\mathrm{PG}(4,q)$.

1. *Er bestaat een punt* P *met surplus* q *en er bestaat een rechte scheef aan* P *waarop elk punt surplus* 1 *heeft.*

2. *Er bestaat een punt* P *met surplus* $q+1$ *en een rechte door* P *waarop elk punt verschillend van* P *surplus* 1 *heeft.*

3. *De orde van het veld waarover gewerkt wordt is* 2, *dat is* $q=2$ *en er zijn vijf punten met surplus* 1 *die een elliptische kwadriek* $\mathrm{Q}^-(3,q)$ *in een hypervlak vormen.*

Opmerking C.3.11

1. Er bestaan rechtenbedekkingen van $\mathrm{PG}(4,q)$ waarvan de meervoudige punten verdeeld zijn over $\mathrm{PG}(4,q)$ als in gevallen 1 en 2 van voorgaande stelling. Het volstaat een rechtenbedekking van $\mathrm{PG}(4,q)$ met minimale kardinaliteit q^3+q+1 te nemen en er een rechte scheef aan, respectievelijk door, het unieke meervoudige punt aan toe te voegen. Zo een bedekking is echter niet minimaal. Verderop worden minimale voorbeelden geconstrueerd.

2. Aangezien in geval 3 van voorgaande stelling de verzameling van meervoudige punten geen rechte bevat, zal een rechtenbedekking van

grootte 12 in $\mathrm{PG}(4,2)$ met een dergelijke verzameling van meervoudige punten steeds minimaal zijn. Een computerzoektocht, gebruik makende van het pakket PG [36] voor het computeralgebrasysteem GAP [52], toont aan dat dergelijke bedekkingen wel degelijk bestaan. De implementatie van deze zoektocht kan gevonden worden op de webpagina `http://cage.rug.ac.be/~pg/thesis/`.

Startende van een reguliere vlakkenspread van $\mathrm{PG}(5,q)$ kunnen minimale bedekkingen van $\mathrm{PG}(4,q)$ geconstrueerd worden. Onderstel dat $\mathrm{PG}(4,q)$ ingebed is in $\mathrm{PG}(5,q)$ en beschouw een reguliere vlakkenspread van $\mathrm{PG}(5,q)$. Door de doorsnijding van de reguli van de vlakkenspread met $\mathrm{PG}(4,q)$ te bekijken en dan één of meerdere rechtenreguli in deze doorsnede te "switchen", dat wil zeggen de rechtenregulus te vervangen door zijn tegengestelde rechtenregulus, kunnen minimale rechtenbedekkingen van $\mathrm{PG}(4,q)$ van verschillende groottes bekomen worden.

Stelling C.3.12 *Er bestaan minimale rechtenbedekkingen met grootte $q^3+q+1+\alpha$ in $\mathrm{PG}(4,q)$ voor alle α uit $\{0,1,\ldots,q^2\}$.*

C.3.3 t-Spreads in eindige klassieke polaire ruimten

Indien een eindige klassieke polaire ruimte $\mathcal{P}$ een t-spread heeft, dan is $|\mathrm{PG}(t,q)|$ een deler van $|\mathcal{P}|$. Daar deze delingsvoorwaarde niet erg overzichtelijk is, wordt een equivalente maar veel eenvoudigere voorwaarde opgesteld. Om deze vereenvoudiging door te voeren, wordt eerst een hulpstelling bewezen.

Notatie C.3.13 De grootste gemene deler van twee natuurlijke getallen a en b, niet beide nul, wordt met (a,b) genoteerd.

Lemma C.3.14 *Onderstel dat a en b natuurlijke getallen zijn met $a+b\geq 1$. Dan is*

$$(q^a+1,q^b-1)=\begin{cases} q^{(a,b)}+1 & \text{als } a/(a,b) \text{ oneven is en } b/(a,b) \text{ even is,}\\ d & \text{in de overige gevallen, met } \begin{cases} d=1 & \text{als } q \text{ even is,}\\ d=2 & \text{als } q \text{ oneven is.}\end{cases}\end{cases}$$

Hiervan gebruik makend, wordt volgende stelling bewezen.

Stelling C.3.15 *Onderstel dat $\mathcal{P}$ een eindige klassieke polaire ruimte is die een t-spread heeft. Indien $\mathcal{P}$*

1. *een symplectische ruimte $\mathrm{W}_{2n+1}(q)$ is, dan $(t+1)|(2n+2)$;*

2. *een parabolische kwadriek $\mathrm{Q}(2n,q)$ is, dan $(t+1)|(2n)$;*

3. *een hyperbolische kwadriek* $\mathrm{Q}^+(2n+1,q)$ *is, dan* $(t+1)|(n+1)$*;*

4. *een elliptische kwadriek* $\mathrm{Q}^-(2n+1,q)$ *is, dan* $(t+1)|n$*;*

5. *een Hermitische variëteit* $\mathrm{H}(2n,q^2)$ *is, dan* $(t+1)|n$*;*

6. *een Hermitische variëteit* $\mathrm{H}(2n+1,q^2)$ *is, dan* $(t+1)|(n+1)$*.*

Deze stelling heeft een aantal interessante gevolgen.

Gevolg C.3.16

1. *Als* $\mathcal{P} = \mathrm{W}_{2n+1}(q)$ *en* t *even is, dan heeft* $\mathcal{P}$ *een* t*-spread als en slechts als* $(t+1)|(2n+2)$*.*

2. *Als* t *even is en* $\mathcal{P} = \mathrm{Q}(2n,q)$ *en ofwel* $n \geq 2$ *en* q *even is, ofwel* $n=3$ *en* q *een oneven priem is, ofwel* $n=3$, q *oneven is en* $q \equiv 0$ *of* $2 \pmod 3$*, dan heeft* $\mathcal{P}$ *een* t*-spread als en slechts als* $(t+1)|(2n)$*.*

3. *Als* $\mathcal{P} = \mathrm{Q}^+(2n+1,q)$ *en ofwel* $n=1$*, ofwel* $n = 2n'+1$, $n' \geq 1$ *en* q *even is, ofwel* $n=3$ *en* q *oneven en priem is, ofwel* $n=3$, q *oneven is en* $q \equiv 0$ *of* $2 \pmod 3$*, dan heeft* $\mathcal{P}$ *een* t*-spread als en slechts als* $(t+1)|(n+1)$*.*

4. *Als* $\mathcal{P} = \mathrm{Q}^-(2n+1,q)$ *en ofwel* $n=2$*, ofwel* $n \geq 2$ *en* q *even is, dan heeft* $\mathcal{P}$ *een* t*-spread als en slechts als* $(t+1)|n$*.*

C.3.4 Partiële t-spreads in polaire ruimten

Net zoals in het projectieve geval zijn de gaten van een partiële t-spread in een eindige klassieke polaire ruimte op een bijzondere wijze verdeeld.

Stelling C.3.17 *Zij* $\mathcal{P}$ *een klassieke polaire ruimte in* $\mathrm{PG}(n,q)$ *wiens grootte een* t*-spread toelaat, i.e., die voldoet aan Stelling C.3.15. Als* $\mathcal{S}'$ *een partiële* t*-spread van* $\mathcal{P}$ *is met deficiëntie* $\delta < q$*, dan vormt de verzameling* F *van gaten een* $\{\delta v_{t+1}, \delta v_t; n, q\}$*-minihyper.*

Weerom kunnen de resultaten van Hoofdstukken 2 en 3 toegepast worden.

Gevolg C.3.18 *Zij* $\mathcal{P}$ *een eindige klassieke polaire ruimte in* $\mathrm{PG}(n,q)$ *wiens grootte een* t*-spread toelaat. Onderstel dat* q *even is als* $\mathcal{P} = \mathrm{W}_n(q)$*.*

1. *Elke partiële* t*-spread* $\mathcal{S}'$ *met deficiëntie* $\delta < \epsilon_q$ *van* $\mathcal{P}$ *kan uitgebreid worden tot een* t*-spread van* $\mathcal{P}$*.*

2. *Onderstel dat* $q > 16$ *een kwadraat is en* $\delta < q^{5/8}/\sqrt{2}+1$*. Als* $\mathcal{S}'$ *een maximale partiële* t*-spread van* $\mathcal{P}$ *met deficiëntie* δ *is, dan vormt de verzameling gaten een unie van paarsgewijs disjuncte deelmeetkunden* $\mathrm{PG}(2t+1, \sqrt{q})$*.*

Gebruik makend van het gewichtsargument van Blokhuis en Metsch [19] kan volgend gevolg bewezen worden.

Gevolg C.3.19 *Als* $q > 16$ *een kwadraat is,* $n \leq \sqrt{q}$*,* $(t+1)|(2n+2)$*, en* $\mathcal{S}'$ *een maximale partiële* t*-spread is van* $\mathrm{W}_{2n+1}(q)$*,* q *even, met deficiëntie* $\delta < q^{5/8}/\sqrt{2}+1$*, dan is* $\delta = k(\sqrt{q}+1)$ *voor een welbepaalde* $k \geq 2$ *en vormt de verzameling gaten een unie van* k *paarsgewijs disjuncte deelmeetkunden* $\mathrm{PG}(2t+1, \sqrt{q})$*.*

Gevolg C.3.18 behandelt het geval $\mathcal{P} = \mathrm{W}_n(q)$, q oneven, niet. Dit geval wordt apart behandeld en een resultaat dat sterk gelijkt op het resultaat voor partiële ovoïden van de veralgemeende zeshoek $\mathrm{H}(q)$, zie Stelling C.4.9, wordt bekomen.

Gevolg C.3.20 *Zij* $\mathcal{S}'$ *een maximale partiële* n*-spread van* $\mathrm{W}_{2n+1}(q)$*,* q *oneven, met deficiëntie* δ*. Onderstel ofwel dat* $\delta < \epsilon_q$*, ofwel dat* $q > 16$ *een kwadraat is en* $\delta < q^{5/8}/\sqrt{2}+1$*. Dan is* δ *even.*

C.3.5 t-Bedekkingen in polaire ruimten

Eerst worden de resultaten over minihypers uit Hoofdstuk 2 toegepast op t-bedekkingen van eindige klassieke polaire ruimten. Vervolgens wordt de uniciteit van de kleinste rechtenbedekking van $\mathrm{Q}(4,3)$ bewezen.

Toepassing van de resultaten over minihypers

Nogmaals vinden we een bijzondere structuur terug bij de verdeling van de meervoudige punten van een t-bedekking.

Stelling C.3.21 *Zij* $\mathcal{P}$ *een eindige klassieke polaire ruimte in* $\mathrm{PG}(n,q)$ *wiens grootte een* t*-spread toelaat. Als* $\mathcal{C}$ *een* t*-bedekking is van* $\mathcal{P}$ *met exces* $\varepsilon < q$*, dan definieert de gewichtsfunctie* $w(P) = \mathrm{surplus}(P)$ *voor* $P \in \mathcal{P}$ *en* $w(P) = 0$ *voor* $P \notin \mathcal{P}$ *een* $\{\varepsilon v_{t+1}, \varepsilon v_t; n, q\}$*-minihyper* (F, w)*, met* F *de verzameling van de meervoudige punten van* $\mathcal{C}$*.*

De resultaten uit Hoofdstuk 2 kunnen toegepast worden.

Gevolg C.3.22 *Zij $\mathcal{P}$ een eindige klassieke polaire ruimte in* $\mathrm{PG}(n,q)$ *wiens grootte een t-spread toelaat. Als $\mathcal{C}$ een t-bedekking is van $\mathcal{P}$ met exces $\varepsilon < \epsilon_q$, dan is de functie* surplus *de gewichtsfunctie geïnduceerd op de puntenverzameling van $\mathcal{P}$ door een som van δ t-ruimten.*

Opmerking C.3.23 Dit gevolg was reeds bewezen door Eisfeld et al. [49] in het bijzondere geval dat $\mathcal{P}$ een eindige klassieke veralgemeende vierhoek is en $\mathcal{C}$ een rechtenbedekking is van $\mathcal{P}$.

De kleinste bedekking van $\mathrm{Q}(4,3)$

Er wordt bewezen dat het gekende voorbeeld van een kleinste bedekking van $\mathrm{Q}(4,3)$ uniek is.

Stelling C.3.24 *Een bedekking $\mathcal{C}$ van* $\mathrm{Q}(4,3)$ *heeft ten minste grootte* 11. *Indien $\mathcal{C}$ een bedekking is van* $\mathrm{Q}(4,3)$ *bestaande uit* 11 *rechten, dan bestaan er twee disjuncte rechten l en m op* $\mathrm{Q}(4,3)$ *zodat $\mathcal{C}$ bestaat uit*

- *de rechten verschillend van l die l snijden maar niet m, en*
- *de rechten verschillend van l uit de regulus op* $\mathrm{Q}(4,3)$ *die zowel l als m bevat.*

C.4 Partiële ovoïden en blokkerende verzamelingen

In Hoofdstuk 5 worden verschillende resultaten over partiële ovoïden en blokkerende verzamelingen bewezen. Sommige daarvan zijn toepassingen van de classificatiestellingen over minihypers uit Hoofdstukken 2 en 3. Deze werden gepubliceerd in *European Journal of Combinatorics* in *P. Govaerts, L. Storme en H. Van Maldeghem, On a particular class of minihypers and its applications. III. Applications* [57].

Zij $\mathcal{P}_m$ een eindige klassieke polaire ruimte van rang k in $\mathrm{PG}(m,q^*)$, met $q^* = q^2$ indien $\mathcal{P}_m$ een Hermitische variëteit is en $q^* = q$ in de andere gevallen. Een *partiële ovoïde* O' van $\mathcal{P}_m$ is een verzameling punten van $\mathcal{P}_m$ zodat geen enkele generator van $\mathcal{P}_m$ meer dan één punt van O' bevat. Een *blokkerende verzameling* B van $\mathcal{P}_m$ is een verzameling punten van $\mathcal{P}_m$ die een niet-ledige doorsnede heeft met elke generator van $\mathcal{P}_m$. Een verzameling O van punten van $\mathcal{P}_m$ die zowel een partiële ovoïde als een blokkerende verzameling is, wordt een *ovoïde* van $\mathcal{P}_m$ genoemd. De grootte van een hypothetische ovoïde van $\mathcal{P}_m$ wordt hier aangeduid met $o(\mathcal{P}_m)$ en wordt gegeven in Tabel 5.1. In

Tabel 5.2 wordt een overzicht gegeven van die gevallen waar het gekend is of $\mathcal{P}_m$ al dan niet een ovoïde bezit.

De *deficiëntie* van een partiële ovoïde is het aantal elementen dat ze minder heeft dan een ovoïde, terwijl het *exces* van een blokkerende verzameling het aantal elementen is dat ze meer heeft dan een ovoïde.

C.4.1 Ovoïden op $\mathrm{Q}(6, q)$

In de theorie van minimale t-voudige blokkerende verzamelingen spelen zogenaamde $t \bmod p$ resultaten een belangrijke rol. Dergelijke resultaten beschrijven "hoe" een deelruimte een minimale t-voudige blokkerende verzameling snijdt: in $t \pmod p$ punten. Zij zorgen ervoor dat de blokkerende verzamelingen meer handelbaar worden en maakten reeds verscheidene classificatieresultaten mogelijk.

Het is te verwachten dat vergelijkbare resultaten voor andere objecten even nuttig zullen blijken. Ball [5] bewees een $1 \bmod p$ resultaat voor ovoïden van de kwadriek $\mathrm{Q}(4, q)$.

Stelling C.4.1 (Ball [5]) *Zij O een ovoïde van $\mathrm{Q}(4, q)$, $q = p^h$, p priem. Elke elliptische kwadriek $\mathrm{Q}^-(3, q)$ op $\mathrm{Q}(4, q)$ snijdt O in $1 \pmod p$ punten.*

Om volgende stelling te bewijzen werd gebruik gemaakt van de voorgaande.

Stelling C.4.2 *Een ovoïde O van $\mathrm{Q}(6, q)$, $q = p^h$, p priem, snijdt elke elliptische kwadriek $\mathrm{Q}^-(5, q)$ op $\mathrm{Q}(6, q)$ in $1 \pmod p$ punten.*

Opmerking C.4.3 Het is niet nodig een gelijkaardig resultaat voor parabolische kwadrieken in hoger-dimensionale ruimten te bewijzen, aangezien dergelijke kwadrieken geen ovoïden hebben, zie Tabel 5.2.

C.4.2 Inductieve stellingen

Inductieve stellingen worden bewezen die toelaten ondergrenzen op de grootte van blokkerende verzamelingen, respectievelijk bovengrenzen op de grootte van partiële ovoïden, van eindige klassieke polaire ruimten in een gegeven dimensie om te zetten naar ondergrenzen op de grootte van blokkerende verzamelingen, respectievelijk bovengrenzen op de grootte van partiële ovoïden, van eindige klassieke polaire ruimten in hogere dimensies. Hiertoe wordt een dubbeltelling uitgevoerd die werkt voor alle eindige klassieke polaire ruimten en zowel voor blokkerende verzamelingen als voor partiële ovoïden.

Stelling C.4.4 *Onderstel dat $\mathcal{P}_m$ een eindige klassieke polaire ruimte is en dat, indien $\mathcal{P}_m = \mathrm{H}(2n, q^2)$, dan $n \geq 2$; indien $\mathcal{P}_m = \mathrm{H}(2n+1, q^2)$, dan $n \geq 1$; indien $\mathcal{P}_m = \mathrm{Q}^-(2n+1, q)$, dan $n \geq 2$; indien $\mathcal{P}_m = \mathrm{Q}(2n, q)$, dan $n \geq 2$; indien $\mathcal{P}_m = \mathrm{Q}^+(2n+1, q)$, dan $n \geq 3$; indien $\mathcal{P}_m = \mathrm{W}_{2n+1}(q)$, dan $n \geq 1$. Als elke blokkerende verzameling van $\mathcal{P}_m$ ten minste grootte $o(\mathcal{P}_m) + \varepsilon$ heeft, dan heeft voor elke $i \geq 0$ elke blokkerende verzameling van $\mathcal{P}_{m+2i}$ ten minste grootte $o(\mathcal{P}_{m+2i}) + (q^* - 1)^i \varepsilon$, waarbij $q^* = q^2$ indien $\mathcal{P}_m$ een Hermitische variëteit is en $q^* = q$ in de andere gevallen.*

Stelling C.4.5 *Onderstel dat $\mathcal{P}_m$ een eindige klassieke polaire ruimte is en dat, indien $\mathcal{P}_m = \mathrm{H}(2n, q^2)$, dan $n \geq 2$; indien $\mathcal{P}_m = \mathrm{H}(2n+1, q^2)$, dan $n \geq 1$; indien $\mathcal{P}_m = \mathrm{Q}^-(2n+1, q)$, dan $n \geq 2$; indien $\mathcal{P}_m = \mathrm{Q}(2n, q)$, dan $n \geq 2$; indien $\mathcal{P}_m = \mathrm{Q}^+(2n+1, q)$, dan $n \geq 3$; indien $\mathcal{P}_m = \mathrm{W}_{2n+1}(q)$, dan $n \geq 1$. Als elke partiële ovoïde van $\mathcal{P}_m$ ten hoogste grootte $o(\mathcal{P}_m) - \delta$ heeft, dan heeft voor elke $i \geq 0$ elke partiële ovoïde van $\mathcal{P}_{m+2i}$ ten hoogste grootte $o(\mathcal{P}_{m+2i}) - (q^* - 1)^i \delta$, waarbij $q^* = q^2$ indien $\mathcal{P}_m$ een Hermitische variëteit is en $q^* = q$ in de andere gevallen.*

Ook Klein bewees een stelling die toelaat grenzen voor partiële ovoïden om te zettten naar grenzen voor partiële ovoïden in hoger-dimensionale ruimten.

Stelling C.4.6 (Klein [77]) *Zij $O(\mathcal{P})$ de grootte van de grootste partiële ovoïden van de eindige klassieke polaire ruimte $\mathcal{P}$.*

(i) *Voor $n \geq 3$, $O(\mathrm{Q}^-(2n+1, q)) \leq \frac{q^n+1}{q^{n-1}+1}(O(\mathrm{Q}^-(2n-1, q)) - 2) + 2$.*

(ii) *Voor $n \geq 2$, $O(\mathrm{W}_{2n+1}(q)) \leq \frac{q^{2n}-1}{q^{2n-1}-1}(O(\mathrm{W}_{2n-1}(q)) - 2) + 2$.*

C.4.3 Partiële ovoïden

Partiële ovoïden van $\mathrm{H}(4, q^2)$

Resultaten over veralgemeende vierhoeken, zie Payne en Thas [91], worden aangewend om bovengrenzen op de grootte van partiële ovoïden van $\mathrm{H}(4, q^2)$ te bekomen.

Als eerste stap wordt bewezen dat $\mathrm{H}(4, q^2)$ geen maximale partiële ovoïde van grootte $q^5 - q$ heeft.

Stelling C.4.7 $\mathrm{H}(4, q^2)$ *heeft geen maximale partiële ovoïde van grootte $q^5 - q$.*

Deze stelling werd onafhankelijk bewezen door K. Thas in [119].

Gebruik makend van een uitbreidbaarheidsresultaat voor partiële ovoïden van $\mathrm{H}(3, q^2)$ dat een eenvoudig gevolg is van de stellingen uit het voorgaande hoofdstuk, wordt dan een substantiële verbetering aangebracht op de gekende bovengrens voor de grootte van partiële ovoïden van $\mathrm{H}(4, q^2)$.

Stelling C.4.8 *Indien O' een partiële ovoïde van $\mathrm{H}(4, q^2)$ is, dan is $|O'| < q^5 - (4q-1)/3$.*

Partiële ovoïden van de split Cayley hexagon

Voor informatie over veralgemeende zeshoeken, zie Van Maldeghem [123].

Gebruik makend van Gevolg C.3.18 wordt aangetoond dat een partiële ovoïde van de split Cayley hexagon $\mathrm{H}(q)$ met kleine deficiëntie een even deficiëntie heeft. Om dit aan te tonen wordt gebruik gemaakt van het feit dat $\mathrm{H}(q)$ bestaat uit punten en rechten van de kwadriek $\mathrm{Q}(6, q)$ en van het feit dat de rechten van $\mathrm{H}(q)$ door een punt van $\mathrm{H}(q)$ een vlak vormen op de kwadriek $\mathrm{Q}(6, q)$.

Stelling C.4.9 *Indien de deficiëntie δ van een maximale partiële ovoïde van $\mathrm{H}(q)$ kleiner is dan ϵ_q, of indien q een kwadraat is en δ kleiner is dan $q^{5/8}/\sqrt{2}+1$, dan is δ even.*

Gevolg C.4.10 *Een partiële ovoïde van $\mathrm{H}(q)$, q even, bevat ten hoogste q^3-1 punten.*

Er wordt aangetoond dat voor $q = 2$ deze grens scherp is door, gebruik makend van de voorstelling van Van Maldeghem [124] voor $\mathrm{H}(2)$, een partiële ovoïde van $\mathrm{H}(2)$ met grootte 7 te construeren.

Meer grenzen

In Subsectie 5.4.3 worden gekende resultaten aangaande de grootte van de grootste partiële ovoïden van eindige klassieke polaire ruimten verzameld. Tevens worden Stellingen C.4.5 en C.4.6 gebruikt om gekende grenzen op te tillen naar hogere dimensies. Bovendien worden de nieuw-bekomen resultaten vergeleken met de Blokhuis-Moorhouse grenzen.

C.4.4 Blokkerende verzamelingen

Een kleine blokkerende verzameling op $\mathrm{W}_{2n+1}(q)$

Dualiseren van Stelling C.3.24 levert volgend resultaat op.

Stelling C.4.11 *Een blokkerende verzameling* B *van* $\mathrm{W}_3(3)$ *heeft ten minste grootte* 11*. Indien* $|B| = 11$*, dan bestaan er twee punten* P *en* Q *op* $\mathrm{W}_3(3)$*,* $P \not\sim Q$*, zodat* B *bestaat uit:*

- *de punten van* $P^\perp \setminus (Q^\perp \cup \{P\})$*, en*
- *de punten van* $\{P, Q\}^{\perp\perp} \setminus \{P\}$*.*

Deze constructie wordt veralgemeend tot een constructie voor willekeurige n en q.

Stelling C.4.12 *De symplectische ruimte* $\mathrm{W}_{2n+1}(q)$ *heeft een blokkerende verzameling met grootte* $q^{n+1} + q^n - q^{n-1}$*.*

Opmerking C.4.13

1. Op dit moment wordt nagegaan of, in het geval $q = 3$, deze blokkerende verzameling de kleinste blokkerende verzameling van $\mathrm{W}_{2n+1}(q)$ is.
2. Het is geweten, zie Metsch [86], dat voor q even er kleinere blokkerende verzamelingen van $\mathrm{W}_{2n+1}(q)$ bestaan.

Meer grenzen

In Subsectie 5.5.2 worden gekende resultaten aangaande de grootte van de kleinste blokkerende verzamelingen van eindige klassieke polaire ruimten verzameld.

C.5 Cameron-Liebler rechtenverzamelingen

Cameron-Liebler rechtenverzamelingen werden door Cameron en Liebler [32] ingevoerd in een poging de collineatiegroepen van $\mathrm{PG}(n, q)$ te bepalen die evenveel banen op de puntenverzameling als op de rechtenverzameling hebben. In hun artikel voorspelden ze welke groepen dit zouden zijn en het is nu geweten (T. Penttila, mondelinge communicatie, 2002) dat hun vermoeden juist is. Er bestaat echter nog geen classificatie van de Cameron-Liebler rechtenverzamelingen. In Hoofdstuk 6 worden enkele nieuwe niet-existentie resultaten bewezen. Op Stelling C.5.12 na, werden deze verzameld in het manuscript *P. Govaerts en L. Storme, On Cameron-Liebler line classes*, dat opgestuurd is ter publicatie in *Advances in Geometry.*

Definitie C.5.1 Een *Cameron-Liebler rechtenverzameling* is een verzameling rechten in $\mathrm{PG}(3, q)$ die elke spread van $\mathrm{PG}(3, q)$ in een vast aantal x van rechten snijdt.

Penttila [92] definieert een *clique* in $\mathrm{PG}(3,q)$ als zijnde een verzameling van rechten, ofwel van de vorm $\mathsf{star}(P)$, dit is, alle rechten door een punt P van $\mathrm{PG}(3,q)$, ofwel van de vorm $\mathsf{line}(\pi)$, dit is, alle rechten in een vlak π van $\mathrm{PG}(3,q)$. De waaier van rechten in een vlak π door een punt P wordt genoteerd met $\mathsf{pen}(P,\pi)$.

Cameron-Liebler rechtenverzamelingen hebben verschillende interessante intersectie-eigenschappen. Verschillende van deze definiëren hen. Hieronder worden er twee vermeld.

Eigenschap C.5.2 (Cameron en Liebler [32], Penttila [92]) Zij $\mathcal{L}$ een verzameling rechten in $\mathrm{PG}(3,q)$ en zij $\chi_{\mathcal{L}}$ haar karakteristieke functie. Dan is $\mathcal{L}$ een *Cameron-Liebler rechtenverzameling* als en slechts als één van de volgende equivalente voorwaarden vervuld is.

1. Er bestaat een geheel getal x zodat voor elk incident punt-rechte paar (P,π) volgende gelijkheid geldt:

$$|\mathsf{star}(P)\cap\mathcal{L}| + |\mathsf{line}(\pi)\cap\mathcal{L}| = x + (q+1)|\mathsf{pen}(P,\pi)\cap\mathcal{L}|. \tag{C.1}$$

2. Er bestaat een geheel getal x zodat voor elke rechte l van $\mathrm{PG}(3,q)$

$$|\{m\in\mathcal{L} : m \text{ snijdt } l, m\neq l\}| = (q+1)x + (q^2-1)\chi_{\mathcal{L}}(l). \tag{C.2}$$

Het geheel getal x uit elk van deze eigenschappen is hetzelfde en is gelijk aan het getal x uit Definitie C.5.1. Het wordt de *parameter* van de Cameron-Liebler rechtenverzameling genoemd. Uit Definitie C.5.1 volgt dat $x \in \{0,1,2,\ldots,q^2+1\}$. Cameron en Liebler [32] bewezen dat een Cameron-Liebler rechtenverzameling met parameter x bestaat uit $x(q^2+q+1)$ rechten en dat de enige Cameron-Liebler rechtenverzamelingen voor $x=1$ de cliques zijn en voor $x=2$ de unies van twee disjuncte cliques. Zij merkten ook op dat het complement van een Cameron-Liebler rechtenverzameling met parameter x een Cameron-Liebler rechtenverzameling is met parameter q^2+1-x. Het volstaat dus Cameron-Liebler rechtenverzamelingen met parameter $x \leq \lfloor (q^2+1)/2 \rfloor$ te bestuderen. Bijgevolg was het geval $q=2$ onmiddellijk opgelost. In hun artikel formuleerden Cameron en Liebler hun vermoeden dat er geen andere Cameron-Liebler rechtenverzamelingen bestaan.

Penttila [92] toont aan dat voor $q\neq 2$ er geen Cameron-Liebler rechtenverzamelingen met $x=3$ of $x=4$ bestaan, met eventuele uitzonderingen voor $(x,q)\in\{(4,3),(4,4)\}$. Bruen en Drudge [28] bewijzen het niet-bestaan van Cameron-Liebler rechtenverzamelingen met parameter $2 < x \leq \sqrt{q}$. Drudge [43] voegt hier het niet-bestaan van Cameron-Liebler rechtenverzamelingen met parameter $x=4$ in $\mathrm{PG}(3,3)$ aan toe en bewijst dat voor $q\neq 2$

er geen Cameron-Liebler rechtenverzamelingen met parameter $2 < x < \epsilon_q$ bestaan, met $q+\epsilon_q$ de grootte van de kleinste niet-triviale blokkerende verzamelingen in $\mathrm{PG}(2,q)$. Hij geeft ook een tegenvoorbeeld voor het vermoeden van Cameron en Liebler: een Cameron-Liebler rechtenverzameling met parameter $x = 5$ in $\mathrm{PG}(3,3)$, hiermede het geval $q = 3$ afsluitend. Bruen en Drudge [29] construeren een Cameron-Liebler rechtenverzameling met parameter $x = (q^2+1)/2$ voor elke oneven priemmacht q.

In Hoofdstuk 6 worden nieuwe grenzen op x voor het niet-bestaan van Cameron-Liebler rechtenverzamelingen bewezen. In Tabel 6.1 wordt een overzicht gegeven van de (niet-)existentieresultaten (inclusief de nieuwe) voor Cameron-Liebler rechtenverzamelingen.

De nieuwe grenzen worden bekomen door te bestuderen hoe de rechten van de Cameron-Liebler rechtenverzameling verdeeld zijn over de cliques van $\mathrm{PG}(3,q)$. Hierbij wordt de benadering van Drudge [43] gevolgd. Een clique $\mathcal{C}$ en zijn rechten corresponderen op volgende wijze met een projectief vlak en haar rechten. Indien $\mathcal{C} = \mathsf{star}(P)$, dan volstaat het de quotiëntruimte van P te nemen. Indien $\mathcal{C} = \mathsf{line}(\pi)$, dan kan het duale vlak genomen worden. Op deze manier corresponderen de rechten van een rechtenverzameling in een clique met een verzameling punten in een vlak.

Volgende twee lemma's tonen hoe (meervoudige) blokkerende verzamelingen opduiken in de studie van Cameron-Liebler rechtenverzamelingen.

Lemma C.5.3 (Drudge [43]) *Zij $\mathcal{L}$ een Cameron-Liebler rechtenverzameling met parameter x. Indien $\mathcal{C}$ een clique is waarvoor $x < |\mathcal{C}\cap\mathcal{L}| \leq q+x$, dan vormt $\mathcal{C}\cap\mathcal{L}$ een blokkerende verzameling B in $\mathcal{C}$. Indien er geen Cameron-Liebler rechtenverzamelingen met parameter $x-1$ bestaan, dan is B niet triviaal.*

Dit lemma wordt veralgemeend naar meervoudige blokkerende verzamelingen.

Lemma C.5.4 *Zij $\mathcal{L}$ een Cameron-Liebler rechtenverzameling met parameter x. Indien $\mathcal{C}$ een clique is waarvoor $x + \alpha(q+1) < |\mathcal{C}\cap\mathcal{L}|$, dan vormt $\mathcal{C}\cap\mathcal{L}$ een $(\alpha+1)$-voudige blokkerende verzameling B in $\mathcal{C}$.*

Gebruik makend van gekende resultaten over (meervoudige) blokkerende verzamelingen worden vervolgens tegenstrijdigheden opgezocht om het niet-bestaan van Cameron-Liebler rechtenverzamelingen met bepaalde parameters x aan te tonen.

Voor het algemene geval (geen restricties op q) wordt volgende stelling bekomen.

Stelling C.5.5 *In* PG$(3,q)$*,* $q>2$*, bestaan er geen Cameron-Liebler rechtenverzamelingen met parameter* $2<x<2\epsilon_q-2$*.*

Gevolg C.5.6 *In* PG$(3,q)$*,* q *een priem,* $q>2$*, bestaan er geen Cameron-Liebler rechtenverzamelingen met parameter* $2<x\leq q$*.*

Onder extra voorwaarden voor q worden verbeteringen op deze algemene stelling bekomen.

Stelling C.5.7 *In* PG$(3,q)$*,* q *een kwadraat, bestaan er geen Cameron-Liebler rechtenverzamelingen met parameter* $2<x\leq\min(\epsilon'_q-1,q^{3/4})$*, waarbij* $q+\epsilon'_q$ *de grootte van de kleinste niet-triviale blokkerende verzamelingen in* PG$(2,q)$ *aanduidt die geen Baer deelvlak bevatten.*

Stelling C.5.8 *Zij* $q=q_0^3=p^{3h_0}$*,* $p\geq 7$ *priem,* $h_0\geq 1$ *oneven, en zij* $q+\epsilon''_q$ *de grootte van de kleinste niet-triviale blokkerende verzamelingen in* PG$(2,q)$ *die noch een minimale blokkerende verzameling van grootte* $q+q_0^2+1$*, noch één van grootte* $q+q_0^2+q_0+1$*, bevatten. In* PG$(3,q)$ *bestaan er geen Cameron-Liebler rechtenverzamelingen met parameter* $2<x\leq\min(\epsilon''_q-1,q^{5/6})$*.*

Stelling C.5.9 *Zij* $q=q_0^3=p^{3h_0}$*,* $p\geq 7$ *priem,* $h_0>1$ *even, en zij* $q+\epsilon''_q$ *de grootte van de kleinste niet-triviale blokkerende verzamelingen in* PG$(2,q)$ *die noch een Baer deelvlak, noch een minimale blokkerende verzameling van grootte* $q+q_0^2+1$*, noch één van grootte* $q+q_0^2+q_0+1$*, bevatten. In* PG$(3,q)$ *bestaan er geen Cameron-Liebler rechtenverzamelingen met parameter* $2<x\leq\min(\epsilon''_q-1,q^{3/4})$*.*

Tevens wordt het kleinste open geval, het al dan niet bestaan van Cameron-Liebler rechtenverzamelingen met parameter 4 in PG$(3,4)$ opgelost. Hiertoe wordt een lemma bewezen dat strenge restricties geeft op het mogelijk aantal rechten in een clique.

Lemma C.5.10 *Zij* $\mathcal{L}$ *een Cameron-Liebler rechtenverzameling met parameter* x*. Er bestaat een natuurlijk getal* $0\leq\alpha\leq x$ *zodat er een punt bestaat waardoor er precies* α *rechten van* $\mathcal{C}$ *gaan en zodat*

1. *voor elk punt* P*:* $|\mathsf{star}(P)\cap\mathcal{L}|\equiv\alpha\pmod{q+1}$*, en*

2. *voor elk vlak* π*:* $|\mathsf{line}(\pi)\cap\mathcal{L}|\equiv x-\alpha\pmod{q+1}$*.*

Opmerking C.5.11 Dit lemma toont aan dat ook voor Cameron-Liebler rechtenverzamelingen, er een soort "mod$(q+1)$ eigenschap" geldt, gelijkend op de 1 mod p en t mod p resultaten voor minimale 1-voudige en t-voudige blokkerende verzamelingen in $\mathrm{PG}(2,q)$, $q=p^h$, p priem, en de 1 mod p resultaten voor ovoïden van $\mathrm{Q}(4,q)$ en $\mathrm{Q}(6,q)$, $q=p^h$, p priem, zie Stellingen C.4.1 en C.4.2. In het geval van de Cameron-Liebler rechtenverzamelingen echter, ontbreekt de exacte waarde voor α in Lemma C.5.10.

Tevens wordt een classificatieresultaat, Stelling C.6.2, voor tweevoudig blokkerende verzamelingen, dat bewezen wordt in Bijlage A, aangewend om volgende stelling te bekomen.

Stelling C.5.12 *Er bestaan geen Cameron-Liebler rechtenverzamelingen met parameter* $x=4$ *in* $\mathrm{PG}(3,4)$.

C.6 Twee resultaten over blokkerende verzamelingen

In Bijlage A worden twee stellingen over blokkerende verzamelingen bewezen die op andere plaatsen in de thesis gebruikt werden.

C.6.1 Kleine blokkerende verzamelingen in $\mathrm{PG}(n,q)$

Om Stelling C.2.12 te bewijzen, was een resultaat nodig dat aantoont dat "kleine" blokkerende verzamelingen in $\mathrm{PG}(n,q)$, $q=p^h$ een kwadraat, een blokkerende verzameling in een vlak bevatten. Een dergelijke stelling bestaat, zie Storme en Weiner [105], maar jammer genoeg wordt het in [105] niet bewezen voor de gevallen $p=2$ of $p=3$. Gebruik makend van de technieken uit [105], wordt in de appendix het resultaat uit [105] bewezen voor elk kwadraat $q\geq 16$, evenwel onder een strengere voorwaarde op de grootte van de blokkerende verzamelingen. De stelling die bekomen wordt luidt als volgt.

Notatie C.6.1 Zij p een priem. Dan is $c_p=2^{-1/3}$ als $p\in\{2,3\}$ en $c_p=1$ als $p>3$.

Stelling C.6.2 *Als B een blokkerende verzameling is in* $\mathrm{PG}(n,q)$, $n\geq 3$, $q\geq 16$ *een kwadraat,* $q=p^h$, *p priem, met* $|B|<q+c_pq^{2/3}$, *dan bevat B een blokkerende verzameling in een vlak.*

C.6.2 De kleinste tweevoudig blokkerende verzamelingen in $\mathrm{PG}(2,4)$

In het bewijs van Stelling C.5.12 wordt de classificatie van de tweevoudige blokkerende verzamelingen van grootte 12 in $\mathrm{PG}(2,4)$ gebruikt. Deze classificatie wordt uitgevoerd in Sectie A.2 en levert volgend resultaat op.

Stelling C.6.3 *Onderstel dat B een 2-voudige blokkerende verzameling van grootte* 12 *in* PG(2, 4) *is. Dan is één van de volgende twee mogelijkheden vervuld.*

1. *De verzameling B bevat de unie van de puntenverzamelingen van twee rechten die snijden in een punt P en drie punten buiten deze unie, één op elk van de drie resterende rechten door P.*

2. *Er bestaan drie rechten l_1, l_2 en l_3 door een punt P en een vierde rechte l niet door P zodat B bestaat uit de punten van $l_i \setminus l$, $i = 1, 2, 3$, en de twee punten van l op geen van de rechten l_i, $i \in \{1, 2, 3\}$.*

Deze blokkerende verzamelingen worden weergegeven in Figuur A.2.

Gevolg C.6.4 *Op isomorfisme na zijn er exact drie 2-voudige blokkerende verzamelingen van grootte* 12 *in* PG(2, 4)*. Indien B zo een blokkerende verzameling is, dan behoort B tot één van de volgende drie types.*

1. *De verzameling B bestaat uit de unie van de puntenverzamelingen van drie niet-concurrente rechten.*

2. *Er bestaan twee rechten l en m die snijden in een punt P zodat B bestaat uit de unie van de puntenverzamelingen van l en m en drie verdere niet-collineaire punten, één op elk van de drie resterende rechten door P.*

3. *Er bestaan drie concurrente rechten l_1, l_2 en l_3 door een punt P en een vierde rechte l niet door P zodat B bestaat uit de punten van $l_i \setminus l$, $i = 1, 2, 3$, en de twee punten van l die op geen van de rechten l_i, $i \in \{1, 2, 3\}$, gelegen zijn. In dit geval is B het complement van de kleinste unitaal, de Hermitische kromme in* PG(2, 4)*.*

C.7 Maximale verzamelingen van onderling orthogonale Latijnse vierkanten

In Bijlage B worden resultaten over partiële spreads in PG$(3, q)$ en ovoïden van de hyperbolische kwadrieken gebruikt om maximale verzamelingen van onderling orthogonale Latijnse vierkanten te construeren.

De problemen die in Bijlage B bestudeerd worden, liggen eerder aan de rand van het onderzoeksonderwerp van deze thesis, aangezien het verband tussen blokkerende verzamelingen en verzamelingen orthogonale Latijnse vierkanten niet onmiddellijk duidelijk is. De objecten die gebruikt worden om de verzamelingen Latijnse vierkanten te construeren zijn echter wel verwant aan blokkerende verzamelingen, zoals uit voorgaande secties duidelijk moge zijn.

De resultaten uit Bijlage B werden verzameld in het artikel *P. Govaerts, D. Jungnickel, L. Storme en J. A. Thas, Some new maximal sets of mutually orthogonal Latin squares* [53] dat zal verschijnen in *Designs, Codes and Cryptography.*

Het probleem dat behandeld wordt, is het bepalen van de paren (s, t) waarvoor er een maximale verzameling van t onderling orthogonale Latijnse vierkanten van de orde s bestaat. Meer informatie over dit probleem kan gevonden worden in [10, Hoofdstuk X] en in [33, Sectie IV.27].

Twee $s \times s$-matrices, *vierkanten* van de *orde* s genoemd, $A = (a_{ij})$ en $B = (b_{ij})$, met elementen uit de verzameling S van grootte s worden *orthogonaal* genoemd indien de afbeelding $e : (i, j) \mapsto (a_{ij}, b_{ij})$ van $\{1, \dots, s\}^2$ naar S^2 bijectief is.

Een vierkant $A = (a_{ij})$ van de orde s met elementen uit de verzameling S van grootte s wordt een *Latijns vierkant* genoemd indien de afbeeldingen $r_i : j \mapsto a_{ij}$ van $\{1, \dots, s\}$ naar S bijectief zijn voor elke $i \in \{1, \dots, s\}$ en de afbeeldingen $c_j : i \mapsto a_{ij}$ van $\{1, \dots, s\}$ naar S bijectief zijn voor elke $j \in \{1, \dots, s\}$, i.e., indien elke rij en elke kolom van A alle elementen van S bevat.

Een verzameling van t onderling orthogonale Latijnse vierkanten van de orde s, ook t MOLS(s) genoemd, wordt *maximaal* genoemd en genoteerd met t MAXMOLS(s) indien er geen Latijns vierkant van de orde s bestaat dat orthogonaal is aan elk vierkant uit de verzameling.

Twee manieren om MAXMOLS te construeren worden toegepast om nieuwe voorbeelden te vinden.

De eerste constructie maakt gebruik van maximale partiële spreads van grootte r in $\mathrm{PG}(3, 4) \setminus \mathrm{PG}(3, 2)$ om translatienetten van graad $r + 3$ die geen transversalen hebben te bekomen; deze benadering levert nieuwe voorbeelden van MAXMOLS(16).

Elke r onderling scheve rechten in $\mathrm{PG}(3, q)$ kunnen beschouwd worden als een verzameling *onderling scheve* deelgroepen van de orde q^2 in de additieve groep van de vectorruimte $V = V(4, q)$ (hiermee wordt bedoeld dat elke twee van deze deelgroepen een triviale doorsnede hebben). Dit is een bijzonder geval van een *partiële congruentie partitie* (PCP) en levert bijgevolg een translatienet van de orde $s = q^2$ en graad r door de vectoren in V als punten te nemen en alle getranslateerden van de r deelgroepen als rechten te nemen. Indien de partiële spread waarvan vertrokken wordt maximaal is, kan men hopen dat het geassocieerde net ook maximaal is en bijgevolg $t = r - 2$ MAXMOLS(s), $s = q^2$, oplevert. Deze benadering werd succesvol toegepast door Jungnickel [71, 72]. In het algemene geval is het echter mogelijk dat het geassocieerde net uitbreidbaar is; dit is het geval als en slechts als het net een transversaal heeft.

Zoals vermeld in Jungnickel en Storme [73], leverde een computerzoektocht naar maximale partiële spreads van $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$ het volgende resultaat.

Propositie C.7.1 (Jungnickel en Storme [73]) *Een maximale partiële spread van grootte r in $\mathrm{PG}(3,4)\setminus\mathrm{PG}(3,2)$ bestaat als en slechts als $6 \leq r \leq 10$ of $r = 14$.*

Nieuwe zoektochten tonen aan dat elke maximale partiële spread van grootte 6 of 7 in $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$ een translatienet oplevert van de orde 16 en graad 9 of 10, respectievelijk. Om de exhaustieve zoektochten efficiënt uit te voeren worden verschillende hulpstellingen bewezen die nagaan hoe een eventuele transversaal aan de bekomen netten er uit zou moeten zien. Deze hulpstellingen zijn aangepaste versies van gelijkaardige resultaten uit [73].

De computerzoektochten werden uitgevoerd met behulp van het pakket PG [36] voor het computeralgebrasysteem GAP [52]. Hun implementatie kan gevonden worden op de webpagina `http://cage.rug.ac.be/~pg/thesis/`. Zij leveren het volgende resultaat op.

Stelling C.7.2 *Elke maximale partiële spread van 6 of 7 rechten in $\mathrm{PG}(3,4)\setminus \mathrm{PG}(3,2)$ levert een translatienet van de orde 16 en graad 9 of 10, respectievelijk, dat geen transversalen heeft.*

Een onmiddellijk gevolg is het bestaan van de gewenste MAXMOLS(16).

Gevolg C.7.3 *Er bestaan t MAXMOLS(16) voor $t = 7$ en $t = 8$.*

Hier kan nog vermeld worden dat de geconstrueerde maximale partiële spreads in $\mathrm{PG}(3,4) \setminus \mathrm{PG}(3,2)$ maximale partiële 3-spreads in $\mathrm{PG}(7,2)$ opleveren.

Propositie C.7.4 *Er bestaan maximale partiële 3-spreads van grootte 9 en 10 in $\mathrm{PG}(7,2)$.*

Opmerking C.7.5 Met behulp van gelijkaardige technieken als die hierboven beschreven werd gepoogd 12 MAXMOLS(16) te construeren. Het was de bedoeling de maximale partiële spreads van de grootte vijf te vinden in $\mathrm{PG}(3,4) \setminus \Sigma_B \cup \Sigma'_B \cup \Sigma''_B$, met Σ_B, Σ'_B en Σ''_B drie onderling scheve Baer 3-deelmeetkunden $\mathrm{PG}(3,2)$, en te controleren of deze translatienetten opleveren die geen transversalen hebben. Jammer genoeg bleek dat elk dergelijk translatienet een transversale Baer 3-deelmeetkunde heeft.

Een tweede constructiemethode levert oneindige klassen van MAXMOLS. Bij deze constructie worden (niet-)existentieresultaten voor spreads en ovoïden van de hyperbolische kwadrieken gebruikt om $q^{2n-1}-1$ MAXMOLS(q^{2n}) te bekomen, dit voor $n \geq 2$ en q een macht van twee, en voor $n=2$ en q een macht van drie. Het eerste voorbeeld doet zich voor voor $q=2$ en $n=2$, en levert 7 MAXMOLS(16) via een computervrije methode. De exacte formulering luidt als volgt.

Stelling C.7.6 *Onderstel dat* $\mathrm{Q}^+(4n-1,q)$ *een spread heeft en dat* $\mathrm{Q}^+(4n+1,q)$ *geen ovoïde heeft. Dan bestaan er* $q^{2n-1}-1$ MAXMOLS(q^{2n}).

Gevolg C.7.7 *Er bestaan* $q^{2n-1}-1$ MAXMOLS(q^{2n}) *voor* $n \geq 2$ *en* q *even, en voor* $n=2$ *en* q *een macht van drie.*

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Index