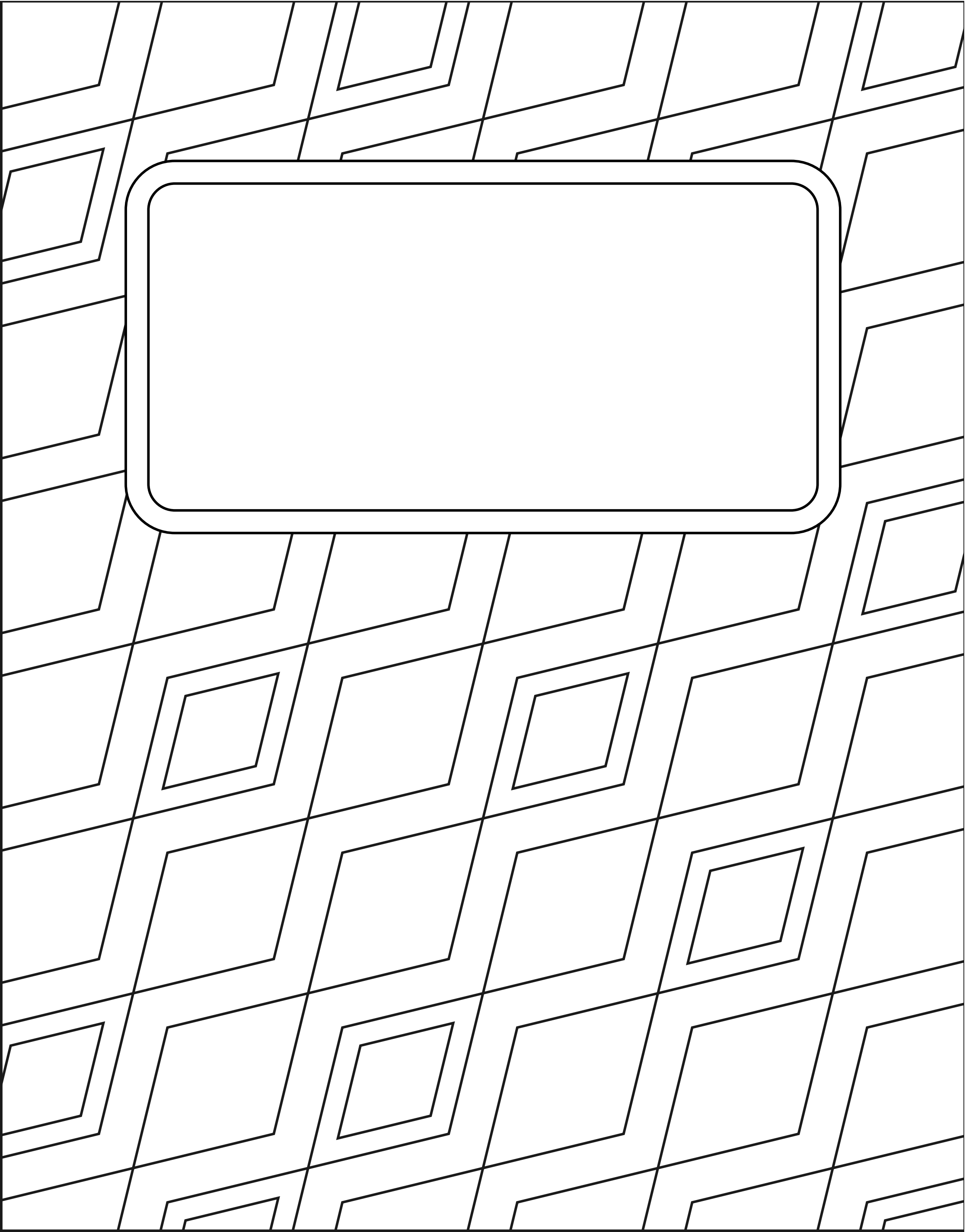
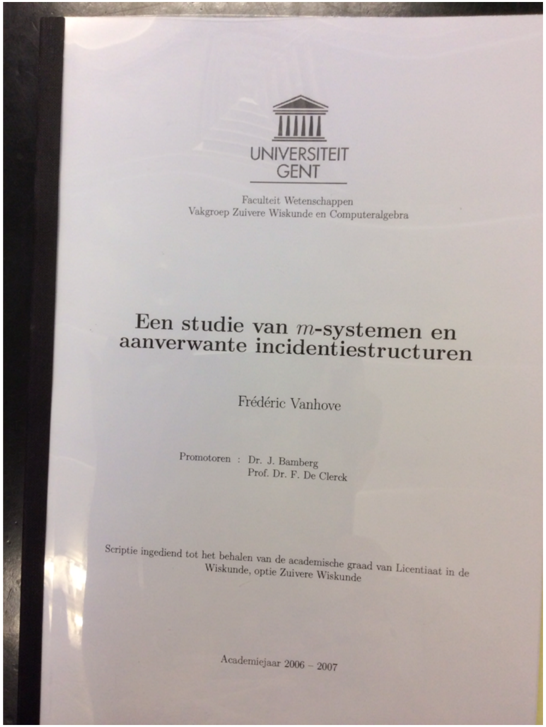


The brilliant career of Frédéric Vanhove

John Bamberg, The University of Western Australia.

28/02/2014.

Masters (2006 - 2007)



"A study of m-systems and related incidence structures"

- Original proof of the connection between two-character sets and strongly regular graphs
- Showed that 'field reduction' and linear representation commute.
- Showed that sometimes m-systems give rise to $(0,\alpha)$-geometries, giving insight into non-existence results on m-systems.

Frédéric, Frank, & I

From: **Frédéric Vanhove** frederic.vanhove.wetteren@pandora.be
Subject: latest version
Date: 2 April 2007 7:55 am
To: John Bamberg bamberg@cage.ugent.be, Frank De Clerck fdc@cage.ugent.be

Hello,

this is the latest version.
In case that is alright, I might upload a newer version today (since a lot of work is being done at the time)

Among the newer things are :

1.3.1 on page six
1.3.3. on page seven (this is a new approach I came up with, inspired by mister Bamberg's idea of using k-ovoids)
1.4.1 on page 11

1.7.(page 14) I spent a lot of time here thinking about subtleties here

The problem with the Q(4,q) and the nucleus etc.. has been solved but hasn't been included yet.
I do have a question about generalized linear representations. What do pairs of points such that the line meets a certain fixed point p at infinity have in common? I mean, does this have an intrinsic geometric meaning?

Greetings and thanks,

Frédéric Vanhove

From: **Frédéric Vanhove** frederic.vanhove.wetteren@pandora.be
Subject: Re: latest version
Date: 3 April 2007 8:06 am
To: bamberg@cage.ugent.be, Frank De Clerck fdc@cage.ugent.be

Hello,

> Thanks Frederic,
>
> It's getting there... but I guess you still have a lot to do yet. So perhaps we
> should wait for the next version? I had a quick skim through, and there's still
> a lot there that has been untouched.

Yes, there is still a lot that I know that I have to do. The calculations are quite time consuming at times. I'd expected it to go faster.

The professor has suggested that a proof of that theorem with the graphs (from Calderbank and Kantor) could be included. I spent some time thinking about that, and perhaps you could take a look at that graphtheorem.pdf file?

I also included more on hyperovals, ovoids and ovals.. One of my reasons for this is that I am talking about pseudo ovals and ovoids which is in fact a generalisation.
I wonder if it's wise to prove everything, but as one of my sources I'm using
http://cage.ugent.be/~fdc/intensivecourse2/brown_2.pdf

Greetings and thanks,
Frédéric Vanhove

PhD.

- FWO-aspirant (2007-2011)

  "A study of m-systems in finite projective spaces and related incidence geometries".

  Supervisors: Frank De Clerck & John Bamberg

PhD; the first year

- Attempted to generalise the "two-graph" argument in Gunawardena–Moorhouse (1997) to m-systems

Ovoid of $Q(8,q)$

Regular two-graph

vertices: points of ovoid $\mathcal{O}$

triples: $\langle p_1, p_2\rangle\langle p_2, p_3\rangle\langle p_3, p_1\rangle$ is non-square

Complementary two-graph has a multiplicity $\mu = q^2 - \frac{q^2-1}{q^2+1} \notin \mathbb{Z}$

Two-Graphs and Ovoids in Polar Spaces

Theorem 1. *[2] Let Γ be a graph on v vertices, such that the associated two-graph is regular of degree k. Then the $(0,-1,+1)$-adjacency matrix A satisfies $A^2-(2k-(v-2))A+(1-v)=0$, and the two eigenvalues ρ_1,ρ_2 satisfy the equation $x^2+(2k-(v-2))x+(1-v)I=0$. The multiplicities m_1 and m_2 are given by :*

$$m_1=\frac{v\rho_2}{\rho_2-\rho_1} \text{ and } m_2=\frac{v\rho_1}{\rho_1-\rho_2}.$$

1 $W(2n+1,q), q\equiv 1 \bmod 4, Q^-(2n+1,q), q$ odd

Let f be the non-singular symplectic form (in the symplectic case), or symmetric form (in the elliptic case), defining the polar space. Suppose $\mathcal{O}$ is an ovoid of the polar space. For every point p in $\mathcal{O}$, let v_p be a vector representing it. Let Δ be the following two-graph: the vertices are the points of $\mathcal{O}$, and a triple (p_1,p_2,p_3) is in Δ if and only if $f(v_{p_1},v_{p_2})f(v_{p_2},v_{p_3})f(v_{p_3},v_{p_1})$ is non-square. One can check that this is well-defined and indeed a two-graph by noting that this is the two-graph defined by the graph with $\mathcal{O}$ as vertices, and such that (p_1,p_2) are adjacent if and only if $f(v_{p_1}),f(v_{p_2})$ is non-square.

Theorem 2. *The two-graph Δ is regular of degree $\frac{(q-1)(q^n+1)}{2}$.*

Proof. Let a and b be two different points in $\mathcal{O}$. Every other point on the line ab is uniquely represented by v_a-tv_b for some $t\neq 0$. If c is a third point in $\mathcal{O}$, then the unique point in $ab\cap <c>^\perp$ is $<u-tv>$,with $t=f(u,w)/f(v,w)$. The triple $\{a,b,c\}$ will thus be in Δ if and only if $t=f(v_a,v_b)\epsilon^2$ for some $\epsilon\neq 0$. This leaves $(q-1)/2$ possibilities for t. For every fixed t, $<v_a-tv_b>^\perp$ will contain exactly q^n+1 elements of $\mathcal{O}$ [Theorem 6, [3]]. Consequently, there are $(q-1)(q^n+1)/2$ points c such that $\{a,b,c\}$ is in Δ. $\square$

The eigenvalues of the adjacency matrix are:

$$\rho_1=-q,\ \rho_2=q^n.$$

The corresponding multiplicities are:

$$m_1=(q^{n+1}+1-q^2)+\frac{q^2-1}{q^{n-1}+1},\ m_2=q^2-\frac{q^2-1}{q^{n-1}+1}.$$

Neither of these can be integers if $n\geq 2$.

2 Doubly transitive two-graphs

We consider three infinite families of doubly transitive two-graphs (see [1] for instance):

Type	Notation	v	k
Paley	$\mathcal{P}(q)$, $q \equiv 1 \bmod 4$	$q+1$	$(q-1)/2$
Hermitian	$\mathcal{H}(q)$,q odd	q^3+1	$(q-1)(q^2+1)/2$
Ree	$\mathcal{R}(q)$,q odd	q^3+1	$(q-1)(q^2+1)/2$

- Ovoids of $Q^-(3,q)$, q odd, exist of course, and the resulting two-graph is regular, with $v = q^2+1$ and $k = (q^2-1)/2$. It is the two-transitive Paley graph $\mathcal{P}(q^2)$.
- An ovoid of $Q^-(5,q)$, q odd, would give us a regular two-graph with $v = q^3+1$ and $k = (q-1)(q^2+1)/2$. Even though ovoids of $Q^-(5,q)$ don't exist, two-graphs with these parameters always exist, since $\mathcal{H}(q)$ and $\mathcal{R}(q)$ have the same parameters.
- An ovoid of $W(3,q)$, q odd, would result in a regular two-graph with $v = q^2+1$ and $k = (q^2-1)/2$. Even though ovoids of $W(3,q)$ cannot exist if q is odd, these parameters are always possible, because the Paley two-graph $\mathcal{P}(q^2)$ exhibits them.
- Ovoids of $W(5,q)$,q odd, would give us a regular two-graph with $v = q^3+1$ and $k = (q^2-1)/2$. Even though $W(5,q)$ doesn't have ovoids for any q, these parameters are possible for all odd prime powers q, since $\mathcal{H}(q)$ and $\mathcal{R}(q)$ have them as well.

References

[1] E. Kuijken. *A study of incidence structures and codes related to regular two-graph.* PhD thesis, Ghent University, 2003.

[2] J. J. Seidel. *Geometry and combinatorics.* Academic Press Inc., Boston, MA, 1991. Selected works of J. J. Seidel, Edited and with a preface by D. G. Corneil and R. Mathon.

[3] E. E. Shult and J. A. Thas. m-systems of polar spaces. *J. Combin. Theory Ser. A*, 68(1):184–204, 1994.

Work with T. Penttila (2012)

From: **Tim Penttila** penttila86@msn.com
Subject: news
Date: 2 June 2012 10:17 pm
To: John Bamberg john.bamberg@uwa.edu.au

John,

Spreads of H(4,q^2) have bitten me again. Frederic and I have shown that each spread of H(4,q^2), q odd, gives a regular two-graph. But unfortunately, that regular two-graph passes all known existence conditions. So we have no nonexistence results whatsoever.

Tim

- "Generalised m-systems":

– polarity ρ of $PG(n,q)$
– set $\mathcal{M}$ of mutually disjoint, pairwise opposite m-subspaces, such that $(\exists k)(\forall \pi \in \mathcal{M})$ $\dim(\pi \cap \pi^\rho) = k$

$k = m \longrightarrow$ partial m-system
$k = -1 \longrightarrow$ partial perp-system

Theorem 2. *Let $\mathcal{M}$ be a partial generalised m-system in $\mathsf{PG}(s,q)$, such that for every element π_i, the intersection $\pi_i^\perp \cap \pi_i$ is a k-space. Then the following equality holds:*

$$\left(q^{s-m} - q^{k+1} + (|\mathcal{M}|-1)(q^{s-2m-1}-1)\right)\left(q^{s+1} - 1 - |\mathcal{M}|(q^{m+1}-1)\right) - |\mathcal{M}|(q^{s-m} - q^{k+1})^2 \geq 0,$$

where equality holds if and only if there is a fixed intersection number for hyperplanes $p^\perp, p \notin \tilde{\mathcal{M}}$.

Opposite: $\pi \cap \sigma^\perp = \emptyset$

It turns out that when I demand $s > 2m+1$, the discriminant doesn't even have rational roots. I did not find a proof for this though. One can however rewrite the quadratic equation in $|\mathcal{M}|$ as follows:

$$-BD|\mathcal{M}|^2 + (BC - (A-B)D - A^2)|\mathcal{M}| + (A-B)C = 0, \qquad (4)$$

with:

$$\begin{aligned} A &= q^{s-m} - q^{k+1} \\ B &= q^{s-2m-1} - 1 \\ C &= q^{s+1} - 1 \\ D &= q^{m+1} - 1. \end{aligned}$$

For I moment I thought I could prove that equations like (4) can never have a discriminant with a rational root, but even when using restrictions, yielded by the conditions implied by the problem (like $A > B$), I still keep finding solutions, so that approach is not useful.

Change of focus

- Summer 2008, Luke Bayens is visiting
- I propose Frédéric, Luke, and I study "intriguing sets" of lines of polar spaces
  - We end up using the language of association schemes.
- We get some nice observations but Luke loses interest.
- Frédéric toils and toils ....
  - Reads Delsarte, Stanton, Eisfeld
  - He finds mistakes in the literature, new proofs, and new perspectives.

Design orthogonality

LP-bounds

Finite geometry

Technique

Geometry $\longrightarrow$ Binary Relations $\longrightarrow$ Association Scheme

Lines of a polar space

equality

Spans a t.i. plane

spans a non-t.i. plane

spans a t.i. solid

spans a degenerate solid

Opposite

positive semidefiniteness

$$\mathbb{C}\mathcal{L} = \langle \mathbb{1} \rangle \perp V^1 \perp V^2 \perp V^3 \perp V^4 \perp V^5$$

Partial spreads of Hermitian spaces

- Thas (1992): $H(2n+1, q^2)$ does not have spreads
- De Beule, Metsch (2007): The maximum size of a partial spread of $H(5, q^2)$ is q^3+1.
  - Frédéric: $|\text{partial spread}| \leq q^{2n+1}+1$ in $H(4n+1, q^2)$
- Partial spread of $H(2n-1, q^2) \longleftrightarrow$ Partial spread set of $n \times n$ Hermitian matrices over $\mathbb{F}_{q^2}$.

Constant rank-distance sets of hermitian matrices and partial spreads in hermitian polar spaces, to appear in Elec. J. Combin. With R. Gow, M. Lavrauw, J. Sheekey.

A geometric proof of the upper bound on the size of partial spreads in H(4n+1, q^2), Adv. Math. Commun. 2011

The maximum size of a partial spread in H(4n + 1, q^2) is q^{2n+1} + 1, Elec. J. Combin. 2009

Dual polar spaces

Subconfigurations yielding "regularity": designs, antidesigns, completely regular codes.

partial spread of $H(2d-1, q^2)$ size q^d+1	antidesign, 1-regular code, $d=3 \Rightarrow$ c.r.
spread of $Q(2d, q)$ or $W(2d-1, q)$ d odd	1-design, 2-regular code, $d \in \{3,5\} \Rightarrow$ c.r.
$Q^+ \subseteq Q$, $H_{2d-1} \subseteq H_{2d}$, $Q \subseteq Q^-$	1-antidesign, c.r.
$G_2(q) \subseteq Q(6, q)$	1-design, 2-antidesign, c.r.

Antidesigns and regularity of partial spreads in dual polar graphs, J. Combin. Designs (2011)

Regular near polygons

Regular near 2d-gon $\mathcal{S}$ $\longrightarrow$ distance regular graph (a_i, b_i, c_i)

parameters $(s, t_2, t_3, \ldots, t_{d-1}, t)$

E.g., Generalised 2d-gon, dual polar spaces

Higman inequality for generalised quadrangles of order (s,t): $\sqrt{s} \leq t \leq s^2$

Theorem: $$c_i \leq \frac{s^{2i}-1}{s^2-1}. \quad \forall i \in \{1, \ldots, d\}.$$

Corollary $(i=d)$: $$t+1 \leq \frac{s^{2d}-1}{s^2-1}.$$

Theorem: If $\exists j \in \{2, \ldots, d\}$ s.t. $c_j = \frac{s^{2j}-1}{s^2-1}$, then m-ovoids $(0 < m < s+1)$ can ONLY EXIST when $m = \frac{s+1}{2}$.

A Higman inequality for regular near polygons, JAC (2011)

$$\frac{(s^i - 1)(c_{i-1} - s^{i-2})}{s^{i-2} - 1} \leq c_i \leq \frac{(s^i + 1)(c_{i-1} + s^{i-2})}{s^{i-2} + 1}$$

$\forall i \in \{3, 4, \ldots, d\}$.

EXTENDS RESULTS OF NEUMAIER (1990)
MATHON (unpublished)

- Characterises equality in the bound
- $DQ(2d, q)$, $DW(2d-1, q)$, $d \geq 3$, have no 1-ovoids.
- Generalised hexagon of order (s, s^3), $s \geq 2$, have no 1-ovoids.
- $GH(s, t')$ cannot have maximal GHs (as full proper subgeos)

Inequalities for regular near polygons, with applications to m-ovoids, JCTA 2013, with De Bruyn.

Erdős-Ko-Rado sets

Table 1

Polar space	Maximum size	Classification
$Q^-(2n+1,q)$	$(q^2+1)\cdots(q^n+1)$	p.-p., Theorem 15
$Q(4n,q)$	$(q+1)\cdots(q^{2n-1}+1)$	p.-p., Theorem 15
$Q(4n+2,q)$, $n \geqslant 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $Q^+(4n+1,q)$, Theorem 23
$Q(6,q)$	$(q+1)(q^2+1)$	p.-p., Latins $Q^+(5,q)$, base, Theorem 23
$Q^+(4n+1,q)$	$(q+1)\cdots(q^{2n}+1)$	one system, Theorem 16
Latins $Q^+(4n+3,q)$, $n \geqslant 2$	$(q+1)\cdots(q^{2n}+1)$	p.-p., Theorem 21
Latins $Q^+(7,q)$	$(q+1)(q^2+1)$	p.-p., meeting Greek in plane, Theorem 22
$W(4n+1,q)$, $n \geqslant 2$, q odd	$(q+1)\cdots(q^{2n}+1)$	p.-p., Theorem 39
$W(4n+1,q)$, $n \geqslant 2$, q even	$(q+1)\cdots(q^{2n}+1)$	p.-p., Latins $Q^+(4n+1,q)$, Theorem 24
$W(5,q)$, q odd	$(q+1)(q^2+1)$	p.-p., base, Theorem 40
$W(5,q)$, q even	$(q+1)(q^2+1)$	p.-p., base, Latins $Q^+(5,q)$, Theorem 24
$W(4n+3,q)$	$(q+1)\cdots(q^{2n+1}+1)$	p.-p., Theorem 15
$H(2n,q^2)$	$(q^3+1)(q^5+1)\cdots(q^{2n-1}+1)$	p.-p., Theorem 15
$H(4n+3,q^2)$	$(q+1)(q^3+1)\cdots(q^{4n+1}+1)$	p.-p., Theorem 15
$H(4n+1,q^2)$, $n \geqslant 2$	$< \lvert\Omega\rvert/(q^{2n+1}+1)$	?, Theorem 42
$H(5,q^2)$	$q(q^4+q^2+1)+1$	base, Theorem 45

Frédéric's Open Problems

1. Are there any t-$(n,k,1;q)$-designs with $2 \le t < k < n$?
2. What is the max size of a partial spread of $H(2d-1,q^2)$, d even?

   F. Ihringer: $q^{2d-1} - q\,\frac{q^{2d-2}-1}{q+1}$.
3. Can $Q(2d,q)$ have spreads for $d \ge 5$, q odd?
4. What is the max size of a set of pairwise non-trivially intersecting maximals of $H(2d-1,q^2)$ for odd $d \ge 5$?
5. Can $Q(2d,q)$ or $W(2d-1,q)$, $d = 2^m - 1$, $m \ge 3$, have a perfect 1-code of maximals?
6. In a polar space with rank ≥ 3, are there any non-trivial combinatorial designs of maximals with respect to t-spaces ($t \ge 2$)?
7. Do there exist $\frac{q+1}{2}$-ovoids of $DH(2d-1,q^2)$, q odd, $d \ge 3$?
8. Are all drg's with classical parameters $(d,b,\alpha,\beta) = (d,-q,-(q+1)/2,-((-q)^d+1)/2)$, q odd, subgraphs of the dual polar graph on $H(2d-1,q^2)$?