Strong maximum principle for nonlinear cooperative elliptic systems with non-linear principal symbol

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Abstract

In this talk we consider validity of strong maximum principle for the nonlinear system

$$G^k(x, u_k, Du^k, D^2u^k) + \sum_{i=1}^{n} c_{ki}u^i = 0$$

for $x \in \Omega$ and $k = 1, ...N$. Here Ω is a bounded domain in R^n with C^1 smooth boundary $\partial\Omega$.

The principal symbols G^k are supposed uniformly elliptic ones. Furthermore, functions $G^k(x, s, p, q)$ are assumed non-decreasimg with respect to s and Lipschitz continuous with respect to p variable.

Functions $c_{ki}(x)$ are supposed continuous, $c_{ki}(x) \leq 0$ for $k \neq i$, and $\sum_{i=1}^{n} c_{ki}(x) \geq 0$.

The validity of strong interior maximum principle for the classical sub- and supersolutons of the nonlinear system above is shown, as well as the validity of strong boundary maximum principle for the same system.

Key words: Strong maximum principle, degenerate elliptic cooperative systems.

1 Introduction

Let Ω be a bounded domain in R^n with C^1 smooth boundary $\partial\Omega$. Let us consider in Ω the weakly coupled non-linear system

$$F^k(x, u^1(x), ...u^n(x), Du^k(x), D^2u^k(x)) = 0 \tag{1}$$

for $k = 1, ...N$ and $x \in \Omega$, where

$$F^k(x, u^1(x), ...u^n(x), Du^k(x), D^2u^k(x)) = G^k(x, u^k(x), Du^k(x), D^2u^k(x)) + \sum_{j=1}^{n} c_{kj}(x)u^j(x)$$

In the linear case we consider

$$L^k u^k = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a^k_{ij}(x)\frac{\partial u^k}{\partial x_j}\right) + \sum_{i=1}^{n} b^k_i(x)\frac{\partial u^k}{\partial x_i} + \sum_{l=1}^{N} m_{kl}(x)u^l + f^k(x) = 0$$

for $x \in \Omega$ and $k = 1, ...N$.

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Here $G^k(c, z^k, p^k, X^k) \in C(\Omega \times R \times R^n \times S^n)$, S^n denotes the set of all real symmetric matrices of order n, and $c_{kj}(x) \in C(\Omega)$ for $k, j = 1, ...N$

In the linear case:

$$a^k_{ij}(x) \in C^1(\overline{\Omega}), b^k_i(x), m_{kl}(x) \in C(\overline{\Omega}), f^k(x) \in C(\overline{\Omega})$$

for $i, j = 1, ...n$, $k, l = 1, ...N$.

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$$G^k(x, u^k(x), Du^k(x), D^2u^k(x)) + \sum_{j=1}^{n} c_{kj}(x)u^j(x) - 0$$

$$L^k u^k = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^k(x)\frac{\partial u^k}{\partial x_j}\right) + \sum_{i=1}^{n} b_i^k(x)\frac{\partial u^k}{\partial x_i} + \sum_{l=1}^{N} m_{kl}(x)u^l + f^k(x) = 0$$

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We suppose that (1) is a quasimonotone system, i.e.

$$c_{kj} \leq 0 \text{ for } k \neq j, \sum_{j=1}^{n} c_{kj}(x) \geq 0 \text{ in } \Omega \tag{2}$$

(see Ishii-Koike - [I-K]) as well as stronger condition

$$c_{kj} \leq 0 \text{ for } k \neq j, \sum_{j=1}^{n} c_{kj}(x) \geq \lambda > 0 \text{ in } \Omega \tag{3}$$

Condition (3) coincides with condition (A3) in [I-K] for weakly coupled system (1).

Moreover, the system (1) is degenerate elliptic one, i.e.

$$G^k(c, z^k, p^k, X^k) \leq G^k(c, z^k, p^k, Y^k) \text{ whenever } X^k \geq Y^k \tag{4}$$

and monotone increasing one w.r.t. z variable, i.e.

$$G^k(c, z^k, p^k, X^k) \geq G^k(c, y^k, p^k, X^k) \text{ whenever } z^k \geq y^k \tag{5}$$

for $k = 1, ...N$, $x \in \Omega$, $p^k \in R^n$, $X^k, Y^k \in S^n$.

In the linear case: The linear system is supposed uniformly elliptic and cooperative one in Ω, i.e. there is a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}^k(x)\xi_i\xi_j \geq \lambda |\xi|^2$$

for $x \in \overline{\Omega}$, $\xi = (\xi_1, ...\xi_n) \in R^n \setminus \{0\}$ and $k = 1, ...N$; and

$$m_{kl} \leq 0 \text{ for } k \neq j, \sum_{l=1}^{n} m_{kl}(x) \geq 0 \text{ in } \Omega$$

for $x \in \Omega$ and $k, l = 1, ...N$, $k \neq l$.

Moreover, we assume that the system (1) is irreducible

Definition: *We call system of differential equations (1) irreducible if for every integer $0 < s < N$ there is a point $x \in \Omega$ such that at least one of the coefficients $m_{kj}(x)$, $k = 1, ..., s$, $j = s+1, ...N$ is nonzero (up to reorder of the equations L^k). Otherwise the system is called reducible.*

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Since the principal symbols of system (1) are non-linear, the proper choice of functional spaces we work in is crucial. In this particular case we work in the class of viscosity solutions.

Let us recall the definition of viscosity sub- and super-solution to (1) (Definition 2,1, page 1997, [I-K]):

Definition 1: *Let $u = (u^1, ...u^N) : \overline{\Omega} \to R^N$ be a locally bounded function.*

(i) We call u a viscosity subsolution to (1) if whenever $\psi \in C^2(\Omega))$, $1 \leq k \leq N$ and $u^{k*} - \psi$ attains its local maximum at $x \in \Omega$, then

$$F^k_*(x, u^*(x), D\psi, D^2\psi)) \leq 0.$$

(ii) We call u a viscosity supersolution to (1) if whenever $\psi \in C^2(\Omega))$, $1 \leq k \leq N$ and $u^k_* - \psi$ attains its local minimum at $x \in \Omega$, then

$$F^{k*}(x, u_*(x), D\psi, D^2\psi)) \geq 0.$$

(iii) Finally, we call u a viscosity solution to (1) if it is both viscosity sub-and supersolution of (1).

Here

$$u^{k*} = limsup_{\epsilon \to 0}\{u^{k*}(y) || |x-y| < \epsilon, y \in \overline{\Omega}\}$$

and

$$u^k_* = liminf_{\epsilon \to 0}\{u^{k*}(y) || |x-y| < \epsilon, y \in \overline{\Omega}\}.$$

Note that u^{k*} and u^k_* are upper and lower semicontinuous functions, respectively, on $\overline{\Omega}$ with values in $R \cup \{\pm\infty\}$ and $u^k_* \leq u^k \leq u^{k*}$ in $\overline{\Omega}$.

In the linear case:

The vector function $u = (u^1, ...u^N)$, $u^k(x) \in C^2(\Omega) \bigcap C(\overline{\Omega})$, is a classical subsolution of the linear system if

$$L^k u^k \leq 0 \tag{6}$$

for $x \in \Omega$ and $k = 1, 2, ...N$.

If u satisfies the opposite inequality to (6) then u is a supersolution of (1).

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Some more definitions:

Definition 2: *The set ot upper semicontinuous functions $u = (u^1, ...u^N) : \overline{\Omega} \to R^N$ is named $USC(\Omega)$*

Definition 3: *If $sup_{\overline{\Omega}} u^k(x) = M_k$ then $M = max_{1 \leq k}\{M_k\}$ we call the absolute maximum of $u(x)$.*

2 Strong interior maximum principle

In the following theorem is formulated the strong interior maximum principle for viscosity subsolutions of the nonlinear, weakly coupled and cooperative system (1).

Theorem 1 *(Strong interior maximum principle) Suppose conditions (3)-(5) hold. If* $u(x) \in USC(\Omega)$, $u = (u^1, ...u^N)$, *is a viscosity subsolution to (1) and*

$$F^k(x, 0, 0) = G^k(x, 0, 0) \geq 0 \tag{7}$$

for $x \in \overline{\Omega}$ *and* $k = 1, 2, ...N$, *then* $u(x)$ *does not attain absolute positive maximum at an interior poin of* Ω.

In the linear case:

Theorem 2 *(Strong interior maximum principle) Let* $u = (u^1, ...u^N)$, $u^k(x) \in C^2(\Omega) \bigcap C(\overline{\Omega})$, $k = 1, 2, ...N$, *be a classical subsolution of (1) and* M_1 *be its absolute maximum. Suppose conditions (2*)-(5*), (7*) hold and*

$$f^k(x) \geq 0 \tag{8}$$

for $x \in \overline{\Omega}$ *and* $k = 1, 2, ...N$.

If M_1 *is attained at some interior point* $x_1 \in \Omega$ *and* $M_1 \geq 0$, *then* $u^k(x) \equiv M_1$ *and*

(i)
$$f^k(x) \equiv 0$$

for all $k = 1, 2, ...N$. *Moreover, if* $M_1 > 0$ *then*

(ii)
$$\sum_{l=1}^{N} m_{kl} = 0$$

for $k = 1, 2, ...N$.

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As a consequence of Theorem 1 we obtain the following comparison principle for viscosity sub-and supersolutions to (1) when one of them is a classical sub- or supersolution:

Theorem 3 *Suppose conditions (3)-(5) hold,* $u = (u^1, ...u^N)$ *and* $u(x) \in USC(\Omega)$ *is a viscosity subsolution to (1) and* $v(x)$, $v^k(x) \in C^2(\Omega) \cap C(\overline{\Omega})$, $k = 1, ...N$ *is a classical supersolution to (1). If* $u^k(x) \leq v^k(x)$ *for* $k = 1, ...N$ *and* $x \in \partial\Omega$, *then* $u^k(x) \leq v^k(x)$ *for* $x \in \partial\Omega$ *and* $k = 1, ...N$.

3 Strong boundary maximum principle

In this section we prove strong boundary maximum principle for the viscosity subsolutions of the nonlinear cooperative elliptic system (1).

Theorem 4 *(Strong boundary maximum principle) Suppose conditions (3)-(5) hold, Ω satisfies an interior sphere condition and $u(x) \in USC(\Omega)$,$u = (u^1, ...u^N)$ is a viscosity subsolution to (1). If $u(x)$ atteins an absolute positive maximum M at some boundary point $x_0 \in \partial\Omega$, i.e. $u^k(x_0) = M$ for some $1 \leq k \leq N$, then for every nontangential direction ρ pointionf into Ω, i.e. $(x_0, \rho) < 0$, the following inequality holds:*

$$lim_{t \to +0} \frac{u^k(x_0 + \rho t) - u^k(x_0)}{t} < 0 \tag{9}$$

for every $x \in \Omega$, $k = 1, 2, ...N$, and $t \in [0, M_0]$

In the linear case:

Theorem 5 *(Strong boundary maximum principle) Suppose $u = (u^1, ...u^N)$, $u^k(x) \in C^2(\Omega) \bigcap C^1(\overline{\Omega})$, $k = 1, 2, ...N$, is a subsolution of (1), conditions (2)-(5), (7) and (8) hold and $\partial\Omega$ satisfies an interior sphere condition. If the absolute maximum M_1 of u is attained for $u^{k_1}(x)$, $1 \leq k_1 \leq N$ at a boundary point $z \in \partial\Omega$ and $M_1 \geq 0$, then either*

$$u^k(x) \equiv M_1, f^k(x) \equiv 0 \tag{10}$$

for $x \in \overline{\Omega}$, $k = 1, 2, ...N$, and if $M_1 > 0$ then $\sum_{j=1}^{N} m_{kj} = 0$ for $k = 1, 2, ...N$,

or

$$\frac{\partial u^{k_1}}{\partial \nu}(z) > 0, \tag{11}$$

where ν is the unite outer normal to $\partial\Omega$ at the point z.

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