Applications of the spectral theory on the S-spectrum to fractional diffusion problems

Fabrizio Colombo
Politecnico di Milano, Italy

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Function theories and spectral theories

The following diagram illustrates the possible extensions:

$$
\begin{tikzcd}
Hol(U) \arrow[r, "FSQ"] \arrow[d] & Two\ classes\ of\ Hyperholomorphic\ functions \arrow[d] \\
Several\ complex\ variables \arrow[d] & S-spectrum\ and\ monogenic\ spectrum \arrow[d] \\
Taylor\ joint\ spectrum \arrow[d] & Hyperholomorphic\ spectral\ theories\ (HST) \arrow[d] \\
Complex\ spectral\ theory & Connections\ between\ (HST)
\end{tikzcd}
$$

where FSQ denotes the Fueter-Sce-Qian construction that will be explained in the sequel.

Hyperholomorphic Function theories and spectral theories

The Fueter-Sce-Qian mapping theorem.

$$Hol(U) \xrightarrow{F_1} SH(U) \xrightarrow{F_2=\Delta \quad (or \ F_2=\Delta^{(n-1)/2})} M(U)$$

- Holomorphic functions, Cauchy–Riemann $\partial_z = \partial_x + i\partial_y$
- Slice hyperholomorphicity set $\underline{x} := \sum_{j=1}^n x_j e_j$ and $|\underline{x}|^2 = \sum_{j=1}^n x_j^2$

$$\mathcal{G} = |\underline{x}|^2 \frac{\partial}{\partial x_0} + \underline{x} \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$$

- Cauchy- Fueter regularity (or Dirac)

$$\mathcal{D} = \partial_{x_0} + \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$$

$$\begin{tikzcd}
SH(U) \arrow[r, "F_2=\Delta"] \arrow[d, "Slice\ \ Cauchy\ \ Formula"'] & M(U) \arrow[d, "Monogenic\ \ Cauchy\ \ Formula"'] \\
S-spectrum \arrow[d] & monogenic\ spectrum \arrow[d] \\
S-Functional\ \ calculus \arrow[d] & Monogenic\ \ Functional\ \ Calculus \arrow[d] \\
H^\infty-Functional\ \ calculus & H^\infty-Monogenic\ \ Functional\ \ Calculus
\end{tikzcd}$$

- The quaternionic spectral theorem is based on the S-spectrum (D. Alpay, F.C., D.P.Kimsey 2016)

References

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- T. Qian, *Generalization of Fueter's result to $\mathbb{R}^{n+1}$*, Rend. Mat. Acc. Lincei, **8** (1997), 111–117.
- F.C., I. Sabadini e D.C. Struppa, *Michele Sce's Works in Hypercomplex Analysis. A Translation with Commentaries*, Birkhäuser, 2020.

The first step in the Fueter-Sce-Qian construction

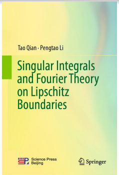
- The Fueter-Sce-Qian mapping theorem is a crucial result to understand holomorphicity in high dimension.
- It is possible to pass from one function theory to the other also with the Radon and Dual Radon transform.
  - F.C., R. Lavicka, I. Sabadini, V. Soucek, *The Radon transform between monogenic and generalized slice monogenic functions*, Mathematische Annalen, **363** (2015), no. 3-4, 733–752.
- The fist step in the Fueter-Sce-Qian construction generates slice hyperholomorphic functions and the spectral theory of the S-spectrum.
- Main applications of the spectral theory of the S-spectrum are:
  - The foundations of quaternionic quantum mechanics (Spectral Theorem),
  - Fractional powers of vector operators and fractional diffusion problems (H^∞-functional calculus).

Recent research directions in the slice hypercomplex setting

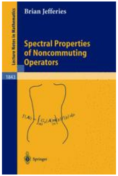
- Slice hyperholomorphic Schur analysis
  - F. C., D. Alpay, I. Sabadini, *Slice Hyperholomorphic Schur analysis*, Slice hyperholomorphic Schur analysis. Operator Theory: Advances and Applications, 256. Birkhäuser/Springer, Cham, 2016. xii+362.
- Perturbation of normal quaternionic operators (P. Cerejeiras, F.C., U. Kaehler, I. Sabadini) in Trans. Amer. Math. Soc. (2019)
- Volterra operators (P. Cerejeiras, F.C., U. Kaehler, I. Sabadini)
- Function spaces of slice hyperholomorphic functions and characteristic operator function
  - F. C., D. Alpay, I. Sabadini, *Quaternionic de Branges spaces and characteristic operator function*, SpringerBriefs in Mathematics. Springer 2020.
- Quaternionic spectral operators (J. Gantner) Memoir AMS 2020/21
- Composition of operators (Ren, Guangbin, Wang, Xieping) Slice regular composition operators, Complex Var. Elliptic Equ. 61 (2016), no. 5, 682–711

The second step in the Fueter-Sce-Qian construction

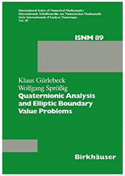
- The second step in the Fueter-Sce-Qian construction generates Fueter or monogenic functions and the spectral theory on the monogenic spectrum, harmonic analysis in high dimension. (see the work of McIntosh, Qian, and many others):
  - T. Qian, P. Li, *Singular integrals and Fourier theory on Lipschitz boundaries*, Science Press Beijing, Beijing; Springer, Singapore, 2019. xv+306 pp.
  - B. Jefferies, *Spectral properties of noncommuting operators*, Lecture Notes in Mathematics, 1843, Springer-Verlag, Berlin, 2004.
- Application in boundary value problems:
  - K. Gürlebeck, W. Sprössig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, International Series of Numerical Mathematics, 89. Birkhäuser Verlag, Basel, 1990, 253pp.



Tao Qian, Pengtao Li, *Singular Integrals and Fourier Theory on Lipschitz Boundaries*, Science Press Beijing, Beijing; Springer, Singapore, 2019. xv+306 pp.



B. Jefferies, *Spectral properties of noncommuting operators*, Lecture Notes in Mathematics, 1843, Springer-Verlag, Berlin, 2004.



K. Gürlebeck, W. Sprössig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, International Series of Numerical Mathematics, 89. Birkhäuser Verlag, Basel, 1990, 253pp.

Motivation for fractional operators

We observe that the heat equation is based on

- Fourier's law $\mathbf{q}(t,x) = -\mathcal{K}\nabla u(t,x), \quad t \geq 0, \quad x \in \mathbb{R}^3$, where u is the temperature, $\mathbf{q}$ is the heat flow and $\mathcal{K}$ is the thermal diffusivity, and
- conservation of energy $\partial_t u(t,x) + \operatorname{div} \mathbf{q}(t,x) = 0$.
- Their combination yields the heat equation

$$u_t(t,x) - \mathcal{K} \operatorname{div}(\nabla u(t,x)) = 0.$$

- Setting $\mathcal{K} = 1$, for sake of simplicity, and having at hand the fractional Laplacian we obtain the fractional evolution equation

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0.$$

The operator $(-\Delta)^\alpha$ is non local and takes into account global contributions of the heat propagation

Fractional Laplacian via Fourier transform

- The fractional Laplacian, can be defined in different ways, for example by the Fourier transform

$$(\mathcal{F}u)(\xi) = \int_{\mathbb{R}^n} u(x)e^{-2\pi i x\cdot\xi}\, dx, \quad (\mathcal{F}^{-1}u)(x) = \int_{\mathbb{R}^n} \hat{u}(\xi)e^{2\pi i x\cdot\xi}\, d\xi.$$

For $u \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions which is defined as

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) \ : \ \forall \beta, \gamma \in \mathbf{N}_0^n \ \sup_{x\in\mathbb{R}^n} |x^\beta \partial_x^\gamma u(x)| < \infty\},$$

we define $(-\Delta)^\alpha u(x) = \mathcal{F}^{-1}((2\pi|\xi|^{2\alpha}(\mathcal{F}u)(\xi)), \quad \text{for} \quad \alpha \in (0,1).$

Fractional Laplacian via semigroup approach

- A different approach considers the semigroup generated by $-\Delta$.

Balakrishnan's approach

- Balakrishnan obtains a construction for fractional powers of an operator A, in which it is not required that A generates a semigroup.
- He assumes that the linear operator A is closed with domain and range in a Banach space X. He proved that if any $\lambda > 0$ belongs to the resolvent set of A and there exists a positive constant M such that $\|\lambda(\lambda - A)^{-1}\| < M, \qquad \lambda > 0$, i.e. if $-A$ is a sectorial operator in today's terminology, then the fractional powers of $-A$ can be defined by the integral

$$(-A)^\alpha x = \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda - A)^{-1}(-A)x \, d\lambda, \quad x \in \mathcal{D}(A),$$

  for $\alpha \in (0, 1)$.
- This formula can be obtained as a particular case of the H^∞-functional calculus introduced by A. McIntosh

Part II: Motivation for (vector) fractional operators in Physics

- Fractional heat equation

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0$$

- New approach based on the spectral theory on the S-spectrum: replace

$$\nabla = e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3}, \quad e_1, e_2, e_3 \textit{ imaginary units}$$

by ∇^α but for more general operators

$$\tilde{\nabla}(t,x) = (e_1 a(x)\partial_{x_1} + e_2 b(x)\partial_{x_2} + e_3 c(x)\partial_{x_3})$$

to get

$$\partial_t u(t,x) + \mathrm{div}(\tilde{\nabla}(t,x))^\alpha u(t,x) = 0$$

- where $(\tilde{\nabla}(t,x))^\alpha$ is the Balakrishnan analogue

- In analogy with the complex case, we say that a linear operator, whose domain $\mathcal{D}(T) := \{v \in V \ : \ Tv \in V\}$, is closed if its graph is closed.

Definition

We define the S-resolvent set of a linear closed operator T as

$$\rho_S(T) := \{s \in \mathbb{H} \ : \ (T^2 - 2\,\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1} \in \mathcal{B}(V)\},$$

where

$$T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} : \mathcal{D}(T^2) \to V,$$

and the S-spectrum of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

Definition

Let T be a closed right linear operator on a two-sided quaternionic Banach space V and assume that $s \in \rho_S(T) \neq \emptyset$, then the operator

$$Q_s(T) := (T^2 - 2\,\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}$$

is called the pseudo-resolvent of T.
Let $T \in \mathcal{K}(V)$, where $\mathcal{K}(V)$ is the set of closed operators.

- The left S-resolvent operator is defined as

$$S_L^{-1}(s,T) := Q_s(T)\bar{s} - TQ_s(T), \qquad s \in \rho_S(T), \tag{1}$$

- and the right S-resolvent operator is defined as

$$S_R^{-1}(s,T) := -(T - \mathcal{I}\bar{s})Q_s(T), \qquad s \in \rho_S(T). \tag{2}$$

Formulations of the quaternionic functional calculus

- Let $U \subset \mathbb{H}$ be a suitable domain that contains the S-spectrum of T. We define the quaternionic functional calculus for left slice hyperholomorphic functions $f : U \to \mathbb{H}$ as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, T)\, ds_j\, f(s), \tag{3}$$

  where $ds_j = -ds j$;
- for right slice hyperholomorphic functions, we define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s, T). \tag{4}$$

- These definitions are well posed since the integrals depend neither on the open set U nor on the complex plane $\mathbb{C}_j$.

- D. Alpay, F.C., T. Qian, I. Sabadini, *The H^∞ functional calculus based on the S-spectrum for quaternionic operators and for n-tuples of noncommuting operators*, Journal of Functional Analysis, **271** (2016), 1544–1584.
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Fractional Evolution

Recall that

- u = temperature, $\mathbf{q}$ = heat flow, $k = 1$ thermal diffusivity

$$\mathbf{q} = -\nabla u \quad \text{(Fourier's law)}$$
$$\partial_t u + \operatorname{div} \mathbf{q} = 0 \quad \text{(Conservation of Energy)}$$

- Their combination yields the heat equation

$$\partial_t u - \Delta u = 0$$

- Alternative model: fractional heat equation

$$\partial_t u + (-\Delta)^\alpha u = 0$$

The main idea

- We identify
$$\mathbb{R}^3 \cong \{s \in \mathbb{H} : \mathrm{Re}(s) = 0\}$$
- We identify the gradient with the quaternionic nabla operator
$$\nabla = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \partial_{x_3} e_3$$
- We replace the gradient in Fourier's law
$$u_t - \mathrm{div}(\nabla^\alpha u) = 0.$$
- Modifies flow, keeps conservation of energy, if this strategy works it is applicable to a large class of operators, for instance
$$\widehat{\nabla} = a(x_1, x_2, x_3)\partial_{x_1} e_1 + b(x_1, x_2, x_3)\partial_{x_2} e_2 + c(x_1, x_2, x_3)\partial_{x_3} e_3.$$

Technical problems and Workaround

Theorem

Consider ∇ on $L^2(\mathbb{R}^3, \mathbb{H})$. Then

$$\sigma_S(\nabla) = \mathbb{R}$$

- ∇^α cannot be defined because s^α is not defined on $(-\infty, 0)$
- workaround: define ∇^α only on the subspace associated to $[0, \infty)$ via

$$P_\alpha(\nabla)u = \frac{1}{2\pi}\int_{-j\mathbb{R}} S_L^{-1}(s, \nabla)\, ds_j\, s^{\alpha-1}\nabla u$$

for $u : \mathbb{R}^3 \to \mathbb{R}$ sufficiently regular; corresponds to Balakrishnan approach (deduced here by the quaternionic H^∞-functional calculus).

A surprising relation

- We have

$$S_L^{-1}(-tj,\nabla) = (-tj+\nabla)\underbrace{(-t^2+\Delta)^{-1}}_{=R_{-t^2}(-\Delta)}$$

- Some computations yield

$$P_\alpha(\nabla)u = \frac{1}{2\pi}\int_{-j\mathbb{R}} S_L^{-1}(s,\nabla)\, ds_j\, s^{\alpha-1}\nabla u = \dots$$

$$= \underbrace{\frac{1}{2}\nabla(-\Delta)^{\frac{\alpha}{2}-1}\nabla u}_{\mathrm{Scal}P_\alpha(\nabla)u} + \underbrace{\frac{1}{2}(-\Delta)^{\frac{1}{2}}(-\Delta)^{\frac{\alpha}{2}-1}\nabla u}_{=\mathrm{Vec}P_\alpha(\nabla)u}.$$

- We observe

$$\mathrm{div}\mathrm{Vec}P_\alpha(\nabla)u = -\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}+\frac{1}{2}}u$$

Definition (The S-spectrum approach to fractional diffusion processes)

Suppose that $\Omega \subseteq \mathbb{R}^3$ is a suitable bounded or unbounded domain.

(1) Suppose we are given the initial-boundary value problem for non-homogeneous materials, for $(x,t) \in \Omega \times (0,T]$ we consider

(a) $$T := \mathbf{q}(x) = a_1(x_1)\partial_{x_1} e_1 + a_2(x_2)\partial_{x_2} e_2 + a_3(x_3)\partial_{x_3} e_3,$$

(b) $$\partial_t u(x,t) + \mathrm{div}\mathbf{q}(x,t)u(x,t) = 0, \quad +\text{initial–boundary conditions}$$

(2) The obtain the S-resolvent operator we need invertibility of

$$Q_s(T) := T^2 - 2\,\mathrm{Re}(s)\,T + |s|^2\mathcal{I}.$$

(3) Using the H^∞-functional calculus we get, for $\alpha \in (0,1)$:

$$P_\alpha(T)u = \frac{1}{2\pi}\int_{-j\mathbb{R}} S_L^{-1}(s,T)\,ds_j\,s^{\alpha-1}\,Tu$$

where $S_L^{-1}(s,T) := Q_s^{-1}(T)\overline{s} - T Q_s^{-1}(T)$

Definition (The S-spectrum approach to fractional diffusion processes)

Given $T := \mathbf{q}(x) = a_1(x_1)\partial_{x_1} e_1 + a_2(x_2)\partial_{x_2} e_2 + a_3(x_3)\partial_{x_3} e_3$ we have

$$Q_s(T) = T^2 + s_1^2\mathcal{I} = -(a_1(x_1)\partial_{x_1})^2 - (a_2(x_2)\partial_{x_2})^2 - (a_3(x_3)\partial_{x_3})^2 + s_1^2\mathcal{I}.$$

To get the S-resolvent operator for $s = js_1 \in \mathbb{H}$ we have to solve

$$\Big(- (a_1(x_1)\partial_{x_1})^2 - (a_2(x_2)\partial_{x_2})^2 - (a_3(x_3)\partial_{x_3})^2 + s_1^2\mathcal{I}\Big) X(x) = F(x),$$
$$X(x) = 0, \quad x \in \partial\Omega.$$

Given F we want to find existence and uniqueness on X in $H_0^1(\Omega;\mathbb{H})$ and show that $\exists C > 0$ such that

$$\|Q_s(T)^{-1}\|_{\mathcal{B}(L^2)} \leq C\frac{1}{s_1^2}, \qquad \|TQ_s(T)^{-1}\|_{\mathcal{B}(L^2)} \leq C\frac{1}{s_1}.$$

Theorem

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary. Let $a_\ell \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R})$ and $a_\ell(x_\ell) \geq m > 0$. Moreover, assume that

$$\inf_{x\in\Omega} \left|a_\ell(x_\ell)^2\right| - \frac{\sqrt{C_\Omega}}{2} \left\|\partial_{x_\ell} a_\ell(x_\ell)^2\right\|_\infty > 0, \qquad \ell = 1, 2, 3,$$

and

$$\frac{1}{2} - \frac{1}{2}\|\Phi\|_\infty^2 C_\Omega^2 C_a^2 > 0$$

where C_Ω is the Poincaré constant of Ω and

$$\Phi(x) := \sum_{\ell=1}^{3} e_\ell \partial_{x_\ell} a_\ell(x_\ell) \quad \textit{and} \quad C_a := \sup_{\substack{x\in\Omega \\ \ell=1,2,3}} \frac{1}{|a_\ell(x_\ell)|}.$$

Then any $s \in \mathbb{H} \setminus \{0\}$ with $\mathrm{Re}(s) = 0$ belongs to $\rho_S(T)$ and the S-resolvents.

Theorem

Moreover, $S_L^{-1}(s,T)$ *satisfy the estimate*

$$\left\|S_L^{-1}(s,T)\right\| \leq \frac{\Theta}{|s|} \qquad \textit{and} \qquad \left\|S_R^{-1}(s,T)\right\| \leq \frac{\Theta}{|s|}, \qquad \textit{if} \quad \operatorname{Re}(s)=0, \tag{5}$$

with a constant $\Theta > 0$ *that does not depend on* s *and for* $\alpha \in (0,1)$, *and for any* $v \in \operatorname{dom}(T)$, *the integral*

$$P_\alpha(T)v := \frac{1}{2\pi}\int_{-j\mathbb{R}} s^{\alpha-1}\, ds_j\, S_R^{-1}(s,T)Tv$$

converges absolutely in $L^2(\Omega,\mathbb{H})$.

Existence of the fractional powers with Robin-like boundary conditions

Let Ω be a bounded domain. Let T be the vector operators defined

$$T = \sum_{\ell=1}^{3} e_\ell a_\ell(x)\partial_{x_\ell}, \qquad x \in \overline{\Omega}, \tag{6}$$

and we suppose that the coefficient a_1, a_2, $a_3 : \overline{\Omega} \subset \mathbb{R}^3 \to \mathbb{R}$ of T are not necessarily nonconstant. Let $F : \Omega \to \mathbb{H}$ be a given function and denote by $u : \Omega \to \mathbb{H}$ the unknown function satisfying the boundary value problem:

$$\begin{cases} (T^2 - 2s_0 T + |s|^2\mathcal{I})u(x) = F(x), & x \in \Omega, \\ \sum_{\ell=1}^{3} a_\ell^2(x) n_\ell(x)\partial_{x_\ell} u(x) + a(x)u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{7}$$

where $a : \partial\Omega \to \mathbb{R}$ is a given function and $n = (n_1, n_2, n_3)$ is the outward unit normal vector to $\partial\Omega$.

The boundary operator of the spectral problem

$$\sum_{\ell=1}^{3} a_\ell^2(x) n_\ell(x)\partial_{x_\ell} u(x) + a(x)u(x) = 0, \quad x \in \partial\Omega$$

naturally arise in the definition of the bilinear form associated with the existence of the pseudo S-resolvent operator as a bounded linear operator, while the operator

$$n \cdot T(x) = \sum_{\ell=1}^{3} a_\ell(x) n_\ell(x)\partial_{x_\ell}$$

in associated with the boundary condition of the flux condition. The stationary heat equation for nonhomogeneous materials with Robin boundary conditions, for $v : \Omega \to \mathbb{R}$, is given by

$$\begin{cases} \mathrm{div}\, T(x)v(x) = 0, \quad x \in \Omega, \\ b(x)v(x) + \sum_{\ell=1}^{3} a_\ell(x) n_\ell(x)\partial_{x_\ell} v(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{8}$$

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This approach has several advantages

(I) It modifies the Fourier law but keeps the law of conservation of energy.

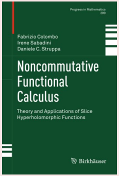
(II) It is applicable to a large class of operators that includes the gradient but also operators with variable coefficients

$$T = a(x_1, x_2, x_3)\partial_{x_1} e_1 + b(x_1, x_2, x_3)\partial_{x_2} e_2 + c(x_1, x_2, x_3)\partial_{x_3} e_3 \quad (*).$$

(III) The fractional powers of the operator T are more realistic for non homogeneous materials.

(IV) The fact that we keep the evolution equation in divergence form allows an immediate definition of the weak solution of the fractional evolution problem.

Books on spectral theory on the S-spectrum



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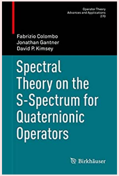

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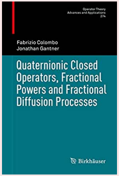

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