# Existence of strong traces for entropy solutions of degenerate parabolic equations


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# Main question

Under which conditions any *solution* $u : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ to

$$\partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \ ,$$

admits the strong trace at $t = 0$, i.e. does there exist $u_0 \in \mathrm{L}^\infty(\mathbb{R}^d)$ such that

$$\operatorname{ess\,lim}_{t \to 0^+} u(t, \cdot) = u_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)\,.$$

$\mathsf{div}_{\mathbf{x}} \mathsf{f}(u) \ldots$ convective term
$D^2_{\mathbf{x}} \cdot A(u) \ldots$ diffusive term

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Let us first consider the case $A = 0$.

# First order quasilinear equations

$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^d_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d)\,, \end{cases}$$

where $\mathsf{f} : \mathbb{R} \to \mathbb{R}^d$ (homogeneous) flux, $u : \mathbb{R}^d_+ \to \mathbb{R}$ unknown.

Classical solutions are too strong (we want allow discontinuities in $\mathbf{x}$)

<u>Weak solutions:</u> $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ s.t. $\mathsf{f}(u) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d_+; \mathbb{R}^d)$ and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$

$$\int_{\mathbb{R}^d_+} u\varphi_t + \mathsf{f}(u) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x}dt + \int_{\mathbb{R}^d} u_0 \varphi(0, \cdot)\, d\mathbf{x} = 0\,.$$

Even for smooth f's non-uniqueness:

$d = 1$, $f(\lambda) = \frac{\lambda^2}{2}$ (Burgers equation), $u_0(x) = \begin{cases} 0\,, & x < 0 \\ 1\,, & x \geq 0 \end{cases}$.

Both functions are a weak solution:

$$u_1(t,x) = \begin{cases} 0\,, & x < t/2 \\ 1\,, & x \geq t/2 \end{cases} \,, \qquad u_2(x) = \begin{cases} 0\,, & x < 0 \\ x/t\,, & 0 \leq x < t \\ 1\,, & x \geq t \end{cases} \text{(rarefaction wave)}$$

# Entropy solutions

$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^d_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

Entropy solutions: $u$ a weak solution and s.t. $\forall \eta \in \mathrm{C}(\mathbb{R})$ convex and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$,

$$\int_{\mathbb{R}^d_+} \eta(u)\varphi_t + \mathsf{f}^\eta(u) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x}dt + \int_{\mathbb{R}^d} \eta(u_0)\varphi(0,\cdot) \, d\mathbf{x} \geq 0 \,,$$

here $\mathsf{f}^\eta(\lambda) = \int_0^\lambda \mathsf{f}' \eta' \, ds$ is an entropy-flux.

- $\eta$ is called (mathematical) entropy ($-\eta$ corresponds to physical entropy)
- The above inequality is due to the fact that the physical entropy has a tendency to increase in time, i.e. the mathematical entropy decreases in time

# Entropy solutions

$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^d_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

Entropy solutions: (Kružkov) $u \in \mathrm{L}^\infty(\mathbb{R}^d_+)$ s.t. $\forall \lambda \in \mathbb{R}$ and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$,

$$\int_{\mathbb{R}^d_+} |u-\lambda|\varphi_t + \mathrm{sgn}(u-\lambda)(\mathsf{f}(u)-\mathsf{f}(\lambda)) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x} dt + \int_{\mathbb{R}^d} |u_0 - \lambda| \varphi(0,\cdot) \, d\mathbf{x} \geq 0 \,.$$

Kružkov (1970): existence and uniqueness of entropy solutions for smooth fluxes f.

Panov (2010): existence of entropy solutions for non-smooth fluxes

# Strong traces

$$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = 0 \quad \text{in} \quad \mathbb{R}^d_+ := \mathbb{R}^+ \times \mathbb{R}^d \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,. \end{cases}$$

$\forall \lambda \in \mathbb{R}$ and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$:

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# Strong traces

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$\forall \lambda \in \mathbb{R}$ and $\forall \varphi \in \mathrm{C}^\infty_c(\mathbb{R}^{1+d})$, $\varphi \geq 0$:

$$\int_{\mathbb{R}^d_+} |u-\lambda|\varphi_t + \operatorname{sgn}(u-\lambda)(\mathsf{f}(u)-\mathsf{f}(\lambda)) \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x} dt + \int_{\mathbb{R}^d} |u_0 - \lambda| \varphi(0,\cdot) \, d\mathbf{x} \geq 0 \,.$$

$\Longleftrightarrow$

$$\text{(a.e. } \lambda \in \mathbb{R}) \qquad \partial_t |u-\lambda| + \mathsf{div}_{\mathbf{x}} \Big( \operatorname{sgn}(u-\lambda)(\mathsf{f}(u)-\mathsf{f}(\lambda)) \Big) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d_+) \,,$$

$$\operatorname{ess\,lim}_{t \to 0^+} u(t,\cdot) = u_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \,.$$

# Strong traces

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$$\operatorname{ess\,lim}_{t \to 0^+} u(t, \cdot) = u_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \,. \quad \text{strong trace}$$

Vasseur (2001): existence of strong traces for entropy solutions for smooth fluxes f and with a non-degeneracy condition

Panov (2005): existence of strong traces for entropy solutions (without non-degeneracy conditions)

Neves, Panov, Silva (2018): existence of strong traces for entropy solutions for heterogeneous fluxes f and with a non-degeneracy condition

# Degenerate parabolic equation

$$
\text{(DP)} \qquad \begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+ \,, \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \,, \end{cases} \qquad \Big(\mathsf{div}_{\mathbf{x}} (A'(u) \nabla_{\mathbf{x}} u)\Big)
$$

where $\mathsf{f} : \mathbb{R} \to \mathbb{R}^d$, $A : \mathbb{R} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$, and $u : \mathbb{R}^d_+ \to \mathbb{R}$ unknown.

- rough flux $\mathfrak{f}$ ($\mathrm{L}^p$, $p > 1$)
- $A' \geqslant 0$ (degenerate parabolicity)

- Motivation: flow in porous media ($CO_2$ sequestration)
  - heterogeneous layers $\longrightarrow$ discontinuous flux and a lack of diffusion in some directions

# Definition of solutions (kinetic formulation)

(DP) $$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+ \ , \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \ . \end{cases}$$

## Definition

A measurable function $u$ defined on $\mathbb{R}^d_+$ is called a quasi-solution to ($\mathrm{DP}_1$) if $f_k(u), A_{kj}(u) \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d_+)$, $k, j = 1, \dots, d$, and for a.e. $\lambda \in \mathbb{R}$

$$\partial_t |u - \lambda| + \mathsf{div}_{\mathbf{x}} \Big( \mathsf{sgn}(u - \lambda)\, (\mathsf{f}(u) - \mathsf{f}(\lambda)) \Big) - D^2_{\mathbf{x}} \cdot [\mathsf{sgn}(u - \lambda)(A(u) - A(\lambda))] = -\gamma(t, \mathbf{x}, \lambda) \ ,$$

holds in $\mathcal{D}'(\mathbb{R}^d_+)$, where $\gamma \in \mathrm{C}(\mathbb{R}_\lambda; \mathcal{M}_+(\mathbb{R}^d_+))$.

For $A = 0$ coincides with the previous definition of entropy solutions.

# Definition of solutions (kinetic formulation)

(DP) $$\begin{cases} \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+ \ , \\ u|_{t=0} = u_0 \in \mathrm{L}^\infty(\mathbb{R}^d) \ . \end{cases}$$

## Theorem

*If function $u$ is a quasi-solution to (DP$_1$), then the function*

$$h(t, \mathbf{x}, \lambda) := \mathsf{sgn}(u(t, \mathbf{x}) - \lambda) = -\partial_\lambda |u(t, \mathbf{x}) - \lambda|$$

*is a weak solution to the following linear equation (entropy solution):*

$$\partial_t h + \mathsf{div}_{\mathbf{x}} \left( \mathsf{f}' \, h \right) - D^2_{\mathbf{x}} \cdot [A'(\lambda) h] = \partial_\lambda \gamma(t, \mathbf{x}, \lambda) \ .$$

Lions, Perthame, Tadmor (1994)

# Existence of strong traces for $(\mathrm{DP}_1)$

$$(\mathrm{DP_1}) \qquad \partial_t u + \mathrm{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+ \, .$$

$$\operatorname{ess\,lim}_{t\to 0^+} u(t,\cdot) = u_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad ???$$

Kwon (2009): scalar diffusion matrices $A(u) = a(u)I$ without non-degeneracy conditions

Aleksić, Mitrović (2014): traceable fluxes f and ultra-parabolic $A$ (i.e. $A = B \oplus 0$ where $B > 0$) without non-degeneracy conditions

"Fully degenerate" matrices $A$ not covered, e.g.

$$a(\lambda) = \left(\frac{1}{\sqrt{\lambda^2+1}}\begin{bmatrix}\lambda & 1\\ 1 & -\lambda\end{bmatrix}\right)\begin{bmatrix}0 & 0\\ 0 & \lambda^2+1\end{bmatrix}\left(\frac{1}{\sqrt{\lambda^2+1}}\begin{bmatrix}\lambda & 1\\ 1 & -\lambda\end{bmatrix}\right) = \begin{bmatrix}1 & -\lambda\\ -\lambda & \lambda^2\end{bmatrix}$$

# Existence of strong traces for $(\mathrm{DP}_1)$

$$(\mathrm{DP}_1) \qquad \partial_t u + \mathsf{div}_{\mathbf{x}} \mathsf{f}(u) = D^2_{\mathbf{x}} \cdot A(u) \quad \text{in} \quad \mathbb{R}^d_+ \,.$$

$$\operatorname{ess\,lim}_{t\to 0^+} u(t,\cdot) = u_0 \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad ???$$

## Theorem (E., Mitrović)

*Let* $\mathsf{f} \in C^1(\mathbb{R};\mathbb{R}^d)$ *and let* $A \in C^{1,1}(\mathbb{R};\mathbb{R}^{d\times d})$ *be such that*

i) $(\exists \sigma \in C^{0,1}(\mathbb{R};\mathbb{R}^{d\times d}))\ A'(\lambda) = \sigma(\lambda)^T \sigma(\lambda)$*;*

ii) $\sup_{(\tau,\boldsymbol{\xi})\in \mathrm{S}^d} \operatorname{meas}\Big\{\lambda \in \mathbb{R} :\ \tau + \langle \mathsf{f}'(\lambda) \,|\, \boldsymbol{\xi}\rangle = \langle A'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle = 0\Big\} = 0$
*(non-degeneracy condition).*

*Then any bounded quasi-solution* $u$ *to* $(\mathrm{DP}_1)$ *admits the strong trace at* $t = 0$*.*

# Strategy of the proof

1. kinetic formulation
2. existence of a weak trace
3. rescaling (blow-up)
4. equation in new variables
5. localisation principle and non-degeneracy condition

# Strategy of the proof

1. kinetic formulation

   $h = \mathsf{sgn}(u - \lambda) = -\partial_\lambda |u - \lambda|$ satisfies

   $$\partial_t h + \mathsf{div}_{\mathbf{x}} \left( \mathsf{f}' \, h \right) - D^2 \cdot [A'(\lambda) h] = \partial_\lambda \gamma$$

2. existence of a weak trace
3. rescaling (blow-up)
4. equation in new variables
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# Strategy of the proof

1. kinetic formulation
2. existence of a weak trace

   There exists $h_0 \in \mathrm{L}^\infty(\mathbb{R}^{d+1})$, such that

   $$h(t,\cdot,\cdot) \rightharpoonup h_0\,,\ \text{weakly-}\star\text{ in } \mathrm{L}^\infty(\mathbb{R}^{d+1}),\ \text{as } t \to 0,\ t \in E,$$

   where $E$ is a set of full measure
3. rescaling (blow-up)
4. equation in new variables
5. localisation principle and non-degeneracy condition

# Strategy of the proof

1. kinetic formulation
2. existence of a weak trace
3. rescaling (blow-up)

   If $(\exists\alpha > 0)(\forall\rho \in \mathrm{C}^1_c(\mathbb{R}))$

   $$\int_{\mathbb{R}} h\Big(\frac{\hat{t}}{m^\alpha}, \frac{\hat{\mathbf{x}}}{m^\alpha} + \mathbf{y}, \lambda\Big)\rho(\lambda)\, d\lambda \to \int_{\mathbb{R}} h_0(\mathbf{y}, \lambda)\rho(\lambda)\, d\lambda \quad \text{in} \quad \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d_+ \times \mathbb{R}^d)\,,$$

   then $u$ admits the strong trace at $t = 0$ which is equal to $\frac{1}{2}\int_{\mathbb{R}} h_0(\cdot, \lambda)\, d\lambda$
4. equation in new variables
5. localisation principle and non-degeneracy condition

# Strategy of the proof

1. kinetic formulation
2. existence of a weak trace
3. rescaling (blow-up)
4. equation in new variables

   $h^m(\hat{t}, \hat{\mathbf{x}}, \mathbf{y}, \lambda) := h(\frac{\hat{t}}{m}, \frac{\hat{\mathbf{x}}}{m} + \mathbf{y}, \lambda)$ satisfies

   $$\frac{1}{m}\Big(\partial_{\hat{t}} h^m + \mathrm{div}_{\hat{\mathbf{x}}}(\mathsf{f}'\, h^m)\Big) - D^2_{\hat{\mathbf{x}}} \cdot [A'(\lambda) h^m] = \frac{1}{m^2} \partial_\lambda \gamma^m$$

   We test by a suitable function and take $m \to \infty$
5. localisation principle and non-degeneracy condition

# Strategy of the proof

1. kinetic formulation
2. existence of a weak trace
3. rescaling (blow-up)
4. equation in new variables
5. localisation principle and non-degeneracy condition

On the limit we get ($\mu$ is a suitable variant of microlocal defect object):

$$(\forall\phi)\ \langle \mu, F\phi\rangle = 0 \quad \overset{F \text{ non-degenerate}}{\Longrightarrow} \quad \mu \equiv 0 \implies \text{strong convergence of (3)}$$

# Adaptive H-measures

- $(u_n)$ bounded in $\mathrm{L}^2(\mathbb{R}^{2d+2})$ and $(v_n)$ bounded in $\mathrm{L}^2(\mathbb{R}^{2d+1})$
- $\psi$ continuous

$$\langle \mu, \psi \rangle$$
$$= \lim_n \int\limits_{\mathbb{R}^{2d+2}} \psi\Big(\mathbf{y}, \lambda, \frac{(\tau, \boldsymbol{\xi})}{\pi_n(\lambda, \tau, \boldsymbol{\xi})}, \frac{n\langle A'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle}{\pi_n(\lambda, \tau, \boldsymbol{\xi})}\Big) \hat{u}_n(\tau, \boldsymbol{\xi}, \mathbf{y}, \lambda) \overline{\hat{v}_n(\tau, \boldsymbol{\xi}, \mathbf{y})} \, d\tau d\boldsymbol{\xi} d\mathbf{y} d\lambda \,,$$

where

$$\pi_n(\lambda, \tau, \boldsymbol{\xi}) = 1 + |(\tau, \boldsymbol{\xi})| + n\langle A'(\lambda)\boldsymbol{\xi} \,|\, \boldsymbol{\xi}\rangle$$

# And...

# ...thank you for your attention :)

M. E., D. Mitrović (submitted, 2020) arXiv:2008.08307