THE FOURIER TRANSFORM OF THICK DISTRIBUTIONS

RICARDO ESTRADA

1. Plan of the talk

We will talk about 3 topics

Thick distributions and Fourier transforms in one variable

Thick distributions in several variables

Fourier transforms in several variables

The general idea is as follows:

Space of thick test functions,

$$\mathcal{A} \hookrightarrow \mathcal{A}_*$$

corresponding to a space of test functions, $\mathcal{A} = \mathcal{D}$, $\mathcal{S}$, $\mathcal{E}$, etc.

$\mathcal{A}$ is a closed subspace of $\mathcal{A}_*$, so we have a projection of the space of thick distributions onto the space of usual distributions,

$$\Pi : \mathcal{A}'_* \mathcal{A}'$$

To define the Fourier transform we construct a space

$$\mathcal{W}$$

and an isomorphism, the Fourier transform of thick test functions

$$\mathcal{F}_{*,\mathrm{t}} : \mathcal{S}_* \to \mathcal{W}$$

as well as

$$\mathcal{F}^*_{\mathrm{t}} : \mathcal{W} \to \mathcal{S}_*$$

and then construct the Fourier transform of thick distributions by duality

$$\mathcal{F}_* : \mathcal{S}'_* \to \mathcal{W}'$$

$$\mathcal{F}^* : \mathcal{W}' \to \mathcal{S}'_*$$

In one variable, $\mathcal{S}_* \subset \mathcal{S}'$ so we can take

$$\mathcal{F}_{*,\mathrm{t}} = \mathcal{F}^*_{\mathrm{t}} = \mathcal{F},$$

the usual Fourier transform, but this is more complicated in several variables because in that case

$$\mathcal{S}_* \nsubseteq \mathcal{S}'$$

2. Fourier transform in one variable

The Fourier transform of thick distributions in one variable was presented in a 2007 article by Fulling and Estrada that appeared in a volume dedicated to Euler (Estrada, R. and Fulling, S. A., Spaces of test functions and distributions in spaces with thick points, *Int. J. Appl. Math. Stat.* **10** (2007), 25-37).

2.1. **A puzzle.** We begin with a problem that surely would have delighted Euler: Evaluate the integral

$$\int_0^\infty \cos(2kx)\, dx. \tag{2.1}$$

In the classical sense it does not converge, but nevertheless it arises naturally in the spectral theory of simple differential operators and in related applications to, for example, quantum field theory. (It is a simple analogue of integrals that arose in (Bondurant, J.D. and Fulling, S.A., The Dirichlet-to-Robin transform, *J. Phys. A* 8 (2005), 1505–1532.) One expects (2.1) to make sense as a distribution in k, with $k \geq 0$. (It is essentially the orthogonality relation for the Fourier cosine transform, in which k is inherently nonnegative.) We now evaluate the integral in two very plausible ways, getting two different answers.

First, we argue that

$$\begin{aligned} \int_0^\infty \cos(2kx)\, dx &= \frac{1}{2}\int_{-\infty}^\infty \cos(2kx)\, dx \\ &= \frac{1}{2}\int_{-\infty}^\infty e^{2ikx}\, dx \\ &= \pi\delta(2k) \\ &= \frac{\pi}{2}\,\delta(k). \end{aligned} \tag{2.2}$$

On the other hand, we calculate

$$\begin{aligned} \int_0^\infty \cos(2kx)\, dx &= \left.\frac{\sin(2kx)}{2k}\right|_{x=0}^{\infty} \\ &= \lim_{x\to\infty} \frac{\sin(2kx)}{2k}\,. \end{aligned}$$

By definition of a distributional integral, we must evaluate this limit after integrating over a "test function", $f(k)$, with support in $[0,\infty)$:

$$\lim_{x\to\infty}\int_0^\infty \frac{\sin(2kx)}{2k}\, f(k)\, dk = \lim_{x\to\infty}\int_0^\infty \frac{\sin u}{2u}\, f\left(\frac{u}{2x}\right) du \tag{2.3}$$
$$= \frac{1}{2}\, f(0)\int_0^\infty \frac{\sin u}{u}\, du$$
$$= \frac{\pi}{4}\, f(0),$$

where the last step uses a well-known integral.

$$\int_0^\infty \cos(2kx)\, dx = \frac{\pi}{4}\,\delta(k). \tag{2.4}$$

3. Spaces with thick points

Let $a\in\mathbb{R}$. We shall define $\mathcal{D}_{*,a}$, the space of test functions with a *thick point* located at $x=a$, and $\mathcal{D}'_{*,a}$, the corresponding space of distributions. A function ϕ with domain $\mathbb{R}$ belongs to $\mathcal{D}_{*,a}$ if it has compact support, it is smooth in $\mathbb{R}\setminus\{a\}$, and at $x=a$ all its one-sided derivatives,

$$\phi^{(n)}\left(a\pm 0\right) = \lim_{x\to a^{\pm}} \phi^{(n)}\left(x\right) \ \ , \quad n\in\mathbb{N}\ , \tag{3.1}$$

exist. $\mathcal{D}_{*,a}$ has a natural topology, in which $\mathcal{D}\left(\mathbb{R}\right)$ is the closed subspace where $\phi^{(n)}\left(a+0\right)=\phi^{(n)}\left(a-0\right)$, $\forall n\in\mathbb{N}$. The elements of $\mathcal{D}'_{*,a}$ are the distributions defined in the standard way as the linear functionals on this enlarged space of test functions.

One can also define in a similar way the spaces $\mathcal{A}_{*,a}$ and $\mathcal{A}'_{*,a}$ for any of the usual spaces of test functions and distributions. For instance, $\mathcal{E}'_{*,a}$ is the space of compactly supported distributions with a thick point at $x=a$, and $\mathcal{S}'_{*,a}$ the corresponding space of tempered distributions. Without loss of generality we shall take $a=0$ and use the simpler notations $\mathcal{A}_{*}$ and $\mathcal{A}'_{*}$. It is clear that instead of one thick point one could consider a space with a finite number of thick points, or even an infinite (but discrete) set of them.

In fact, the idea of considering functions and generalized functions in spaces with thick points was apparently first proposed by Blanchet and Faye (Blanchet, L. and Faye, G., Hadamard regularization, *J. Math. Phys.* **41** (2000), 7675-7714) in the context of finite parts, pseudo-functions and Hadamard regularization studied by Sellier; their analysis is aimed at the study of the dynamics of point particles in high post-Newtonian approximations of general relativity, and it thus developed in dimension 3.

If X and Y are topological vector spaces with $X \subset Y$, the inclusion, i, being continuous, we shall denote by π the adjoint operator, $\pi = i'$, which is a projection from Y' to X'. In the case of spaces with thick points, one has $\mathcal{A} \subset \mathcal{A}_{*,a}$, and thus we have a projection $\pi : \mathcal{A}'_{*,a} \longrightarrow \mathcal{A}'$, given explicitly as

$$\left\langle \pi\left(f\right), \phi\right\rangle_{\mathcal{A}' \times \mathcal{A}} = \left\langle f, \phi\right\rangle_{\mathcal{A}'_{*,a} \times \mathcal{A}_{*,a}} \ .$$

Every distribution $g \in \mathcal{A}'$ can be extended to $\mathcal{A}'_{*,a}$; that is, there exist distributions $f \in \mathcal{A}'_{*,a}$ such that $\pi\left(f\right) = g$. If f_0 is any extension, then the most general extension is given as

$$f = f_0 + \sum_{j=0}^{n} \alpha_j s_j \ , \tag{3.2}$$

where $s_j = s_{j,a}$ are the distributions that give the *saltus* (jump) of the jth derivative across $x = a$,

$$\left\langle s_j, \phi\right\rangle = \phi^{(j)}\left(a+0\right) - \phi^{(j)}\left(a-0\right) \ , \tag{3.3}$$

where $n \in \mathbb{N}$, and where $\alpha_0, \ldots, \alpha_n$ are arbitrary constants.

We may define the derivatives of the distributions of $\mathcal{A}'_{*,a}$ by the usual duality process,

$$\left\langle f', \phi\right\rangle = -\left\langle f, \phi'\right\rangle .$$

Clearly,

$$\pi\left(f'\right) = \pi\left(f\right)' .$$

Also,

$$s_j = \left(-1\right)^j s_0^{(j)} .$$

We shall consider the one-sided delta functions at the thick point, $\delta_{\pm}\left(x\right) = \delta\left(x - \left(a \pm 0\right)\right)$, defined as

$$\left\langle \delta\left(x - \left(a \pm 0\right)\right), \phi\left(x\right)\right\rangle = \phi\left(a \pm 0\right) \ . \tag{3.4}$$

Observe that $s_0\left(x\right) = \delta\left(x - \left(a+0\right)\right) - \delta\left(x - \left(a-0\right)\right)$, and more generally $\left(-1\right)^j s_j\left(x\right) = \delta^{(j)}\left(x - \left(a+0\right)\right) - \delta^{(j)}\left(x - \left(a-0\right)\right)$.

It is important to observe that the derivative formulas in the space $\mathcal{A}'_{*,a}$ can be somewhat different from the usual derivative formulas. Indeed, suppose that $f \in \mathcal{A}'_{*,a}$ is a regular distribution generated by a function that is of class C^1 in both $(-\infty, a]$ and $[a, \infty)$ but that may have a jump $[f] = f\left(a+0\right) - f\left(a-0\right)$ across the thick point. Then f can also be considered an element of the usual space of distributions $\mathcal{A}'$, and we have the well-known formula

$$\frac{\overline{d}f}{dx} = \frac{df}{dx} + [f]\,\delta\left(x-a\right) \ , \tag{3.5}$$

where the overbar denotes the distributional derivative and df/dx is the ordinary derivative. However, the derivative in the space $\mathcal{A}'_{*,a}$, denoted d^*f/dx, is given by the relation

$$\frac{d^*f}{dx}=\frac{df}{dx}+f\left(a+0\right)\delta_+\left(x\right)-f\left(a-0\right)\delta_-\left(x\right)\ . \tag{3.6}$$

Naturally (3.5) and (3.6) satisfy

$$\pi\left(\frac{d^*f}{dx}\right)=\frac{\overline{d}f}{dx}.$$

Nevertheless, if f is continuous at $x=a$, then the distributional derivative coincides with the ordinary derivative, but in the space $\mathcal{A}'_{*,a}$ we have

$$\frac{d^*f}{dx}=\frac{df}{dx}+f\left(a\right)s_0\left(x\right)\ . \tag{3.7}$$

The general form of the extensions of the Dirac delta function $\delta\left(x-a\right)$ to the thick-point space that are of order 0, that is, that do not contain derivatives of the deltas, is

$$\delta_{*,a,\lambda}\left(x\right)=\lambda\delta\left(x-\left(a+0\right)\right)+\left(1-\lambda\right)\delta\left(x-\left(a-0\right)\right)\ , \tag{3.8}$$

where λ is any constant. The case when $\lambda=1/2$ give us the only such extension,

$$\widetilde{\delta}\left(x-a\right)=\delta_{*,a,1/2}\left(x\right)=\tfrac{1}{2}\left[\delta\left(x-\left(a+0\right)\right)+\delta\left(x-\left(a-0\right)\right)\right], \tag{3.9}$$

that is symmetric with respect to $x=a$.

Let us now consider multiplication in the spaces $\mathcal{A}'_{*,a}$. Any space of distributions $\mathcal{A}'$ has a corresponding Moyal algebra $\mathcal{B}$, the space of multipliers of $\mathcal{A}$ and of $\mathcal{A}'$, i.e., those smooth functions ρ that satisfy $\rho\phi\in\mathcal{A}$, $\forall\phi\in\mathcal{A}$. If $\mathcal{A}=\mathcal{D}$ then $\mathcal{B}=\mathcal{E}$; if $\mathcal{A}=\mathcal{E}$ then $\mathcal{B}=\mathcal{E}$; if $\mathcal{A}=\mathcal{S}$ then $\mathcal{B}=\mathcal{O}_M$. (For more on $\mathcal{O}_M$ and the other spaces see [?] or [?].) In the spaces with thick points, if $\rho\in\mathcal{B}_{*,a}$, then $\rho\phi\in\mathcal{A}_{*,a}$, $\forall\phi\in\mathcal{A}_{*,a}$, and thus we may define the multiplication $\rho f\in\mathcal{A}'_{*,a}$ whenever $f\in\mathcal{A}'_{*,a}$ by the formula

$$\left\langle\rho\left(x\right)f\left(x\right),\phi\left(x\right)\right\rangle=\left\langle f\left(x\right),\rho\left(x\right)\phi\left(x\right)\right\rangle\ . \tag{3.10}$$

On the other hand, if $\rho\in\mathcal{B}_{*,a}$ then the multiplication $\rho\phi$ belongs to $\mathcal{A}_{*,a}$ for any $\phi\in\mathcal{A}$, and thus we can define an operator of multiplication $M_\rho:\mathcal{A}\longrightarrow\mathcal{A}_{*,a}$, and, by duality, a corresponding multiplication operator $M_\rho:\mathcal{A}'_{*,a}\longrightarrow\mathcal{A}'$. Observe that

$$\pi\left(\rho f\right)=M_\rho\left(f\right)\ . \tag{3.11}$$

Notice too that if $\rho_1, \rho_2 \in \mathcal{B}_{*,a}$ then we can perform the operation $\rho_1(\rho_2 f)$, which, naturally, turns out to be $(\rho_1\rho_2)f$. However, the product $M_{\rho_1}M_{\rho_2}$ is not defined.

If $\rho \in \mathcal{E}$, then $\rho(x)\,\delta(x-a) = \rho(a)\,\delta(x-a)$. The corresponding formula when there are thick points is as follows:

$$\rho(x)\,\delta_{*,a,\lambda}(x-a) = \lambda\rho(a+0)\,\delta(x-(a+0)) + (1-\lambda)\,\rho(a-0)\,\delta(x-(a-0)) \ . \tag{3.12}$$

Thus $M_\rho(\delta_{*,a,\lambda}(x-a)) = [\lambda\rho(a+0) + (1-\lambda)\,\rho(a-0)]\,\delta(x)$, and in particular

$$M_\rho\left(\widetilde{\delta}(x-a)\right) = \{\rho\}\,\delta(x-a),$$

where

$$\{\rho\} = (\rho(a+0) + \rho(a-0))/2$$

is the average value at the thick point.

4. The Fourier transform in spaces with thick points

We adopt the simplest definition of the Fourier transform:

$$\widehat{f}(u) = \mathcal{F}\{f(x)\,;u\}$$

is given by the integral

$$\int_{-\infty}^{\infty} f(x)\,e^{ixu}\,dx$$

when the integral exists and defined by duality or other methods when the integral diverges. Naturally, our results will remain valid, modulo trivial modifications, for all the variant conventions, and hence, in particular, for the inverse Fourier transform,

$$\mathcal{F}^{-1}\{f(x)\,;u\} = (2\pi)^{-1}\,\mathcal{F}\{f(x)\,;-u\}\,.$$

If $\phi \in \mathcal{S}_*$ then its Fourier transform $\widehat{\phi}$ is a smooth function, but it will not be of rapid decay at infinity, in general. The behavior of $\widehat{\phi}(u)$ as $|u| \to \infty$ follows from the Erdélyi asymptotic formula (Asymptotic expansions of Fourier integrals involving logarithmic singularities, *SIAM J.* **4** (1956), 38-47),

$$\int_{-\infty}^{\infty} \phi(x)\,e^{ixu}\,dx \sim \frac{c_1}{u} + \frac{c_1}{u^2} + \frac{c_1}{u^3} + \cdots\,, \quad |u| \to \infty\ , \tag{4.1}$$

where

$$c_{n+1} = e^{\pi i(n+1)/2}\left[\phi^{(n)}\right].$$

In fact a smooth function ψ belongs to $\mathcal{F}(\mathcal{S}_*)$ if and only if there exist constants $c_1, c_2, c_3, \ldots$ such that

$$\psi(x) \sim \sum_{n=1}^{\infty} c_n x^{-n} \quad \text{as} \quad |x| \to \infty.$$

Therefore, we introduce the space $\mathcal{W}$ as follows.

Definition. The test-function space $\mathcal{W}$ consists of those functions $\psi \in C^\infty(\mathbb{R})$ that admit an asymptotic expansion of the type

$$\psi(x) \sim \sum_{n=1}^{\infty} c_n x^{-n} \quad \text{as } |x| \to \infty \tag{4.2}$$

for some constants $c_1, c_2, c_3, \ldots$. The space of distributions $\mathcal{W}'$ is the corresponding dual space.

We can now define the Fourier transform of the distributions of the space $\mathcal{S}'_*$.

Definition. If $f \in \mathcal{S}'_*$ then its Fourier transform $\widehat{f} = \mathcal{F}(f)$ is the element of the space $\mathcal{W}'$ defined by

$$\left\langle \widehat{f}(u), \psi(u) \right\rangle = \left\langle f(x), \widehat{\psi}(x) \right\rangle, \quad \psi \in \mathcal{W}. \tag{4.3}$$

Similarly, if $g \in \mathcal{W}'$ then its Fourier transform $\widehat{g} = \mathcal{F}(g)$ is the element of the space $\mathcal{S}'_*$ defined by

$$\langle \widehat{g}(x), \phi(x) \rangle = \left\langle g(u), \widehat{\phi}(u) \right\rangle, \quad \phi \in \mathcal{S}_*. \tag{4.4}$$

The Fourier transform is an isomorphism between the spaces $\mathcal{S}'_*$ and $\mathcal{W}'$, and between the spaces $\mathcal{W}'$ and $\mathcal{S}'_*$.

In order to understand the Fourier transform in these spaces, it is convenient to note several properties of the space $\mathcal{W}'$. This space of generalized functions was introduced in (*Singular Integral Equations,* Birkhäuser, Boston, 2000.) to study the Hilbert transform of distributions. One of the most important characteristics of $\mathcal{W}'$ is that its elements are *not* distributions over $\mathbb{R}$ but rather distributions over the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. We denote by $\delta_{\infty,j}$ the element of $\mathcal{W}'$ given by

$$\langle \delta_{\infty,j}(u), \psi(u) \rangle = c_j \tag{4.5}$$

when $\psi \in \mathcal{W}$ has the development (4.2). Any element $g \in \mathcal{W}'$ admits a "restriction" $\pi g \in \mathcal{S}'$, but that restriction might vanish even if g does not, namely if g is "concentrated at ∞," that is, if it has the form

$$g(u)=\sum_{j=1}^{n} b_j \delta_{\infty,j}(u) \ . \tag{4.6}$$

Each $g\in\mathcal{S}'$ admits "extensions" $\widetilde{g}\in\mathcal{W}'$, but such extensions are not unique, since we could always add a distribution of the form (4.6). Some tempered distributions admit *canonical* extensions to $\mathcal{W}'$, but there is no canonical way to extend *all* elements of $\mathcal{S}'$ to $\mathcal{W}'$.

Observe that when a tempered distribution g admits a canonical extension $\widetilde{g}\in\mathcal{W}'$, then its Fourier transform $\mathcal{F}(g)$, which is an element of $\mathcal{S}'$, admits a canonical extension to the space $\mathcal{S}'_*$ of distributions over the line with a thick point at $x=0$, and this extension is precisely $\mathcal{F}(\widetilde{g})$.

If g is a distribution of compact support, $g\in\mathcal{E}'(\mathbb{R})$, then the equation

$$\langle\widetilde{g},\psi\rangle_{\mathcal{W}'\times\mathcal{W}}=\langle g,\psi\rangle_{\mathcal{E}'\times\mathcal{E}} \tag{4.7}$$

defines a canonical extension. On the other hand, if $g\in\mathcal{S}'$ satisfies the estimate

$$g(u)=O\left(|u|^{\alpha}\right) \quad (\mathrm{C}) \ , \quad \text{as} \quad |u|\to\infty \ , \tag{4.8}$$

in the Cesàro sense, and $\alpha<0$, then g admits a canonical extension given by the Cesàro evaluation

$$\langle\widetilde{g},\psi\rangle_{\mathcal{W}'\times\mathcal{W}}=\langle g,\psi\rangle \quad (\mathrm{C}) \ , \tag{4.9}$$

which exists because $g(u)\psi(u)=O\left(|u|^{\alpha-1}\right)$ (C). Any tempered distribution g satisfies (4.8) for some $\alpha\in\mathbb{R}$, but if $\alpha>0$ the extension to $\mathcal{W}'$ is not canonical but depends on k arbitrary constants if $k-1\le\alpha<k$ for some $k\in\{1,2,3,\ldots\}$, much in the same way that a primitive of order k depends on k arbitrary constants.

Other tempered distributions that admit canonical extensions to $\mathcal{W}'$, obtained by analytic continuation, are the distributions u_+^{α} and u_-^{α} for $\alpha\notin\mathbb{Z}$, the combination $|\widetilde{u}|^{\alpha}=\widetilde{u}_+^{\alpha}+\widetilde{u}_-^{\alpha}$ for $\alpha=0,\pm2,\pm4,\ldots$, and the combination $\operatorname{sgn} u\,|\widetilde{u}|^{\alpha}=\widetilde{u}_+^{\alpha}-\widetilde{u}_-^{\alpha}$ for $\alpha\in\mathbb{C}\setminus2\mathbb{Z}$ Therefore the distribution $\widetilde{u}^{n}$ is defined for *all* integers. In particular, the tempered distribution $1=\left.|u|^{\alpha}\right|_{\alpha=0}$ admits a canonical extension $\widetilde{1}=\left.|\widetilde{u}|^{\alpha}\right|_{\alpha=0}$; this canonical extension is given by the formula

$$\left\langle\widetilde{1},\psi(u)\right\rangle=\text{p.v.}\int_{-\infty}^{\infty}\psi(u)\,du \ , \tag{4.10}$$

the principal value being taken at infinity, i.e., p.v. $\int_{-\infty}^{\infty} = \lim_{A\to\infty} \int_{-A}^{A}$. Alternatively,

$$\left\langle \widetilde{1}, \psi(u) \right\rangle = \int_{-1}^{1} \psi(u)\, du + \int_{|u|>1} \left(\psi(u) - \frac{c_1}{u} \right) du\ . \tag{4.11}$$

5. Some Fourier transforms

We shall now give the Fourier transform of several distributions of the spaces $\mathcal{W}'$ and $\mathcal{S}'_*$. Observe that if a distribution f_0 of $\mathcal{W}'$ is an extension of a tempered distribution f of the space $\mathcal{S}'$, then the Fourier transform $\widehat{f_0}$ is an element of the space $\mathcal{S}'_*$ that extends the tempered distribution $\widehat{f}$. Similar remarks apply to the Fourier transform of the distributions of the space $\mathcal{S}'_*$.

Let us start with the computation of $\mathcal{F}\left\{\widetilde{\delta}(x); u\right\} \in \mathcal{W}'$. Observe that the equation $\left\langle \widetilde{\delta}(x), e^{ixu} \right\rangle = 1$, while correct, just tells us that $\mathcal{F}\left\{\widetilde{\delta}(x); u\right\}$ is a regularization in the space $\mathcal{W}'$ of the tempered distribution 1. Therefore, we proceed as follows:

$$\begin{aligned}
\left\langle \mathcal{F}\left\{\widetilde{\delta}(x); u\right\}, \psi(u) \right\rangle &= \frac{1}{2}\left(\widehat{\psi}(0^+) + \widehat{\psi}(0^-) \right) \\
&= \frac{1}{2} \lim_{x\to 0} \left(\widehat{\psi}(x) + \widehat{\psi}(-x) \right) \\
&= \lim_{x\to 0} \int_{-\infty}^{\infty} \cos xu\, \psi(u)\, du\ .
\end{aligned}$$

We cannot set $x = 0$ in the last integral since that would produce a divergent integral. However, we observe that $\int_{|u|>1} \cos xu\, du/u = 0$ for $x > 0$ and thus obtain

$$\begin{aligned}
\left\langle \mathcal{F}\left\{\widetilde{\delta}(x); u\right\}, \psi(u) \right\rangle &= \lim_{x\to 0} \int_{-1}^{1} \cos xu\, \psi(u)\, du \\
&\qquad + \int_{|u|>1} \cos xu \left(\psi(u) - \frac{c_1}{u} \right) du \\
&= \int_{-1}^{1} \psi(u)\, du + \int_{|u|>1} \left(\psi(u) - \frac{c_1}{u} \right) du \\
&= \left\langle \widetilde{1}, \psi(u) \right\rangle\ ,
\end{aligned}$$

so that

$$\mathcal{F}\left\{\widetilde{\delta}(x); u\right\} = \widetilde{1}\ . \tag{5.1}$$

We can compute $\mathcal{F}\{s_0(x), u\}$ in a similar fashion,

$$\begin{aligned}\langle\mathcal{F}\{s_0(x);u\},\psi(u)\rangle &= \lim_{x\to0^+}\left(\widehat{\psi}(x)-\widehat{\psi}(-x)\right)\\ &= 2i\lim_{x\to0^+}\int_{-\infty}^{\infty}\sin xu\,\psi(u)\,du\\ &= 2i\lim_{x\to0^+}\int_{-\infty}^{\infty}\sin xu\left(\psi(u)-\frac{c_1}{u}\right)du\\ &\qquad+c_1\int_{-\infty}^{\infty}\frac{\sin xu}{u}\,du\\ &= 2\pi ic_1\ ,\end{aligned}$$

so that

$$\mathcal{F}\{s_0(x);u\}=2\pi i\delta_{\infty,1}(u)\ . \tag{5.2}$$

Formulas (5.1) and (5.2) immediately give

$$\mathcal{F}\{\delta_\pm(x);u\}=\widetilde{1}\pm\pi i\delta_{\infty,1}(u)\ , \tag{5.3}$$

where $\delta_\pm(x)=\delta(x-(0\pm0))$. Formulas (5.3), in turn, yield the following limits in the space $\mathcal{W}'$:

$$e^{iu0^\pm}=\lim_{x\to0^\pm}e^{iux}=\widetilde{1}\pm\pi i\delta_{\infty,1}(u)\ . \tag{5.4}$$

If we now use the fact that $\mathcal{F}^{-1}\{f(u);x\}=(2\pi)^{-1}\mathcal{F}\{f(u);-x\}$, we obtain the formulas

$$\mathcal{F}\left\{\widetilde{1};x\right\}=2\pi\widetilde{\delta}(x)\ , \tag{5.5}$$

$$\mathcal{F}\{\delta_{\infty,1}(u);x\}=is_0(x)\ . \tag{5.6}$$

The usual formulas for the computation of the Fourier transforms of derivatives need to be modified in our context, since the product of a function $\psi(u)$ of the space $\mathcal{W}$ by the function u does not belong to $\mathcal{W}$, in general. Therefore, we introduce the modified multiplication operator $M_u:\mathcal{W}\longrightarrow\mathcal{W}$ and its adjoint $M_u':\mathcal{W}'\longrightarrow\mathcal{W}'$ as

$$M_u(\psi)=u\psi(u)-c_1 \tag{5.7}$$

and, of course, $\langle M_u'(g),\psi\rangle=\langle g,M_u(\psi)\rangle$. Then

$$\mathcal{F}\{f'(x);u\}=-iM_u'\mathcal{F}\{f(x);u\}\ . \tag{5.8}$$

Similarly, if $g\in\mathcal{W}'$ then

$$\mathcal{F}\{M_u'g(u);x\}=-i\,\frac{d^*}{dx}\mathcal{F}\{g(u);x\}\ . \tag{5.9}$$

Observe that $M'_u(\delta_{\infty,j}(u)) = \delta_{\infty,j+1}(u)$. Hence

$$\mathcal{F}\{s_j(x);u\} = (-1)^j \mathcal{F}\left\{s_0^{(j)}(x);u\right\} = 2\pi i^{j+1}\delta_{\infty,j+1}(u) \ , \tag{5.10}$$

$$\mathcal{F}\{\delta_{\infty,j}(u);x\} = (-i)^{j-1} s_{j-1}(x) = i^{j-1}s_0^{(j-1)}(x) \ . \tag{5.11}$$

Notice that $M'_u(f)$ is related to the multiplication $uf(u)$, but it is not the same, even if the product is well-defined. For instance, if $f(u) = \delta(u)$ then $u\delta(u)$ vanishes, but $M'_u(\delta(u)) = -\delta_{\infty,1}(u)$ since

$$\langle M'_u\delta(u), \psi(u)\rangle = \langle \delta(u), M_u\psi(u)\rangle = \langle \delta(u), u\psi(u) - c_1\rangle = -c_1 \ .$$

6. An answer

We can now address the puzzle in subsection 2.1.

Now we return to the integral (2.1). Of course it is a Fourier transform, but since it is classically divergent we need to say in which space we are working, or, what is the same, which regularization of the function 1 we are using. If we work in $\mathcal{W}'$ and, consequently, look for a result in $\mathcal{S}'_*$, it is natural because of symmetry arguments to consider the regularization $\widetilde{1}$. Hence,

$$\begin{aligned}
\int_0^\infty \cos(2kx)\,dx &= \frac{1}{2}\int_{-\infty}^\infty \cos(2kx)\,dx \\
&= \frac{1}{2}\int_{-\infty}^\infty e^{2ikx}\,dx \\
&= \frac{1}{2}\mathcal{F}\left\{\widetilde{1};2k\right\} \\
&= \pi\widetilde{\delta}(2k) \\
&= \frac{\pi}{2}\widetilde{\delta}(k) \ .
\end{aligned}$$

The result $(\pi/2)\,\widetilde{\delta}(k)$ holds for k positive or negative. If we want the result for $k > 0$ in the space $\mathcal{S}'$ we need to apply the projection multiplication $M_H : \mathcal{S}'_* \longrightarrow \mathcal{S}'$; that is we need to multiply by the Heaviside function:

$$H(k)\int_0^\infty \cos(2kx)\,dx = \frac{\pi}{2}M_H\left(\widetilde{\delta}(k)\right) = \frac{\pi}{4}\delta(k) \ . \tag{6.1}$$

That is, *both* (2.2) and (2.4) are correct, depending upon context!

7. Thick distributions in several variables

Thick distributions in several variables were first studied by Yunyun Yang and Estrada starting in 2009.

It must be said that the theory of thick distributions in higher dimesions is *very different* from that in one dimension, and thus our methods and results are not just a simple extension of those of [?]. Indeed, if $\mathbf{a} \in \mathbb{R}^{\mathsf{n}}$, the topology of $\mathbb{R}^{\mathsf{n}} \setminus \{\mathbf{a}\}$, $\mathsf{n} \geq 2$, is quite unlike that of $\mathbb{R} \setminus \{a\}$ for $a \in \mathbb{R}$, since the latter space is disconnected, consisting of two unrelated rays, while the former is connected, all directions of approach to the point $\mathbf{a}$ are related, and such behavior imposses strong restrictions on the singularities. In one variable, the derivative of a function with a jump discontinuity at a point may also have a jump discontinuity there, but such situation is not to be expected in higher dimensions, since derivatives of functions with a jump type singularity at a point will have, in general, derivatives that tend to infinity at the point. Therefore, we define test functions as those functions that are smooth away from the thick point but which have *strong* asymptotic expansions of the form

$$\phi(\mathbf{a}+r\mathbf{w}) \sim \sum_{\mathsf{j}=\mathsf{m}}^{\infty} a_{\mathsf{j}}(\mathbf{w})\, r^{\mathsf{j}}, \tag{7.1}$$

as $\mathbf{x} \to \mathbf{a}$ for some $\mathsf{m} \in \mathbb{Z}$. In general if the expansion of ϕ starts at m, then that of $\partial\phi/\partial x_j$ will start at $\mathsf{m}-1$, and more generally, that of $D\phi$, where D is a differential operator of degree p, starts at $\mathsf{m}-\mathsf{p}$; therefore our space of test functions contains functions with developments of the type (7.1) for *any* integer $\mathsf{m} \in \mathbb{Z}$. In one variable [?] it is enough to consider test functions whose expansion starts at $\mathsf{m}=0$, but that approach does not work in dimensions $\mathsf{n} \geq 2$.

8. Space of Test Functions on $\mathbb{R}^{\mathsf{n}}$ with a Thick Point

If $\mathbf{a}$ is a fixed point of $\mathbb{R}^{\mathsf{n}}$, then the space of test functions with a thick point at $\mathbf{x}=\mathbf{a}$ is defined as follows.

Definition 1. *Let $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ denote the vector space of all smooth functions ϕ defined in $\mathbb{R}^{\mathsf{n}} \setminus \{\mathbf{a}\}$, with support of the form $K \setminus \{\mathbf{a}\}$, where K is compact in $\mathbb{R}^{\mathsf{n}}$, that admit a strong asymptotic expansion of the form*

$$\phi(\mathbf{a}+\mathbf{x}) = \phi(\mathbf{a}+r\mathbf{w}) \sim \sum_{\mathsf{j}=\mathsf{m}}^{\infty} a_{\mathsf{j}}(\mathbf{w})\, r^{\mathsf{j}}, \quad \textit{as } \mathbf{x} \to \mathbf{0}\,. \tag{8.1}$$

where m *is an integer (positive or negative), and where the* a_{j} *are smooth functions of* $\mathbf{w}$*, that is,* $a_{\mathsf{j}} \in \mathcal{D}(\mathbb{S})$ *.*

8.1. **The expansion of** $(\partial/\partial\mathbf{x})^{\mathbf{p}}\,\phi$**.** Notice that the definition of the space $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ requires $(\partial/\partial\mathbf{x})^{\mathbf{p}}\,\phi(\mathbf{x})$ to have an asymptotic expansion equal to the term-by-term differentiation of $\sum_{\mathsf{j}=\mathsf{m}}^{\infty} a_{\mathsf{j}}(\mathbf{w})\,r^{\mathsf{j}}$. If

$$\phi(\mathbf{a}+\mathbf{x}) = \phi(\mathbf{a}+r\mathbf{w}) \sim \sum_{\mathsf{j}=\mathsf{m}}^{\infty} a_{\mathsf{j}}(\mathbf{w})\,r^{\mathsf{j}}, \quad \text{as } r \to 0\,. \tag{8.2}$$

we obtain

$$\frac{\partial\phi}{\partial x_i}(\mathbf{a}+r\mathbf{w}) \sim \sum_{\mathsf{j}=\mathsf{m}-1}^{\infty}\left(\frac{\delta a_{\mathsf{j}+1}}{\delta x_i} + (\mathsf{j}+1)\,a_{\mathsf{j}+1}n_i\right) r^{\mathsf{j}}, \quad \text{as } r \to 0\,. \tag{8.3}$$

Iteration of formula (8.3) yields, in turn, the expansion

$$\begin{aligned}&\frac{\partial^2\phi}{\partial x_i\partial x_k}(\mathbf{a}+r\mathbf{w})\\ &\sim \sum_{\mathsf{j}=\mathsf{m}-2}^{\infty}\left(D_{ik}^2 a_{\mathsf{j}+2} + (\mathsf{j}+2)\left(\frac{\delta a_{\mathsf{j}+2}}{\delta x_i}n_k + \right.\right.\\ &\qquad\qquad \left.\left.\frac{\delta a_{\mathsf{j}+2}}{\delta x_k}n_i\right) + (\mathsf{j}+2)\,(\delta_{ik} + \mathsf{j}n_in_k)\,a_{\mathsf{j}+2}\right) r^{\mathsf{j}},\end{aligned} \tag{8.4}$$

as $r \to 0$.

9. Space of Distributions on $\mathbb{R}^n$ with a Thick Point

We can now consider distributions in a space with one thick point.

Definition 2. *The space of distributions on* $\mathbb{R}^n$ *with a thick point at* $\mathbf{x} = \mathbf{a}$ *is the dual space of* $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ *. We denote it* $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ *, or just as* $\mathcal{D}'_{*}(\mathbb{R}^{\mathsf{n}})$ *when* $\mathbf{a} = \mathbf{0}$*.*

We shall call the elements of $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ "thick distributions."

Since $\mathcal{D}(\mathbb{R}^{\mathsf{n}})$ is closed in $\mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$, the Hahn-Banach theorem immediatly yields the following extension result.

Theorem 1. *Let* f *be any distribution in* $\mathcal{D}'(\mathbb{R}^{\mathsf{n}})$ *, then there exist thick distributions* $g \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ *such that* $\pi(g) = f$*.*

Naturally, if $f \in \mathcal{D}'(\mathbb{R}^{\mathsf{n}})$ then there are infinitely many thick distributions g with $\pi(g) = f$. In some cases there is a canonical way to construct such a g, but no general extension procedure exists. We could think of this situation as follows: If $g \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$, then knowing

$\pi(g)$ gives us a lot of information about g, but not enough to know g completely.

It is well known that any locally integrable function f defined in $\mathbb{R}^{\mathsf{n}}$ yields a distribution, usually denoted by the same notation f, by the prescription

$$\langle f, \phi \rangle = \int_{\mathbb{R}^{\mathsf{n}}} f(\mathbf{x})\, \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}\,, \qquad \phi \in \mathcal{D}(\mathbb{R}^{\mathsf{n}})\,. \tag{9.1}$$

If $\mathbf{a} \notin \operatorname{supp} f$, that is, if $f(\mathbf{x}) = 0$ for $|\mathbf{x} - \mathbf{a}| < \varepsilon$ for some $\varepsilon > 0$, then (9.1) will also work in $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$; however, if $\mathbf{a} \in \operatorname{supp} f$ then, in general, the integral $\int_{\mathbb{R}^{\mathsf{n}}} f(\mathbf{x})\, \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}$ would be divergent and thus a thick distribution that one could call "f" cannot be defined in a canonical way. Nevertheless, it is possible in many cases to define a "finite part" distribution $\mathcal{P}f(f(\mathbf{x}))$ which is the canonical thick distribution corresponding to f.

Definition 3. *Let f be a locally integrable function defined in $\mathbb{R}^{\mathsf{n}} \setminus \{\mathbf{a}\}$. The thick distribution $\mathcal{P}f(f(\mathbf{x}))$ is defined as*

$$\langle \mathcal{P}f(f(\mathbf{x})), \phi(\mathbf{x}) \rangle = \mathrm{F.p.} \int_{\mathbb{R}^{\mathsf{n}}} f(\mathbf{x})\, \phi(\mathbf{x})\, \mathrm{d}\mathbf{x}\,, \qquad \phi \in \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})\,, \tag{9.2}$$

provided that the finite part integrals exist for all $\phi \in \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$.

If ψ is somooth in $\mathbb{R}^{\mathsf{n}} \setminus \{\mathbf{a}\}$, and near $\mathbf{x} = \mathbf{a}$ the function ψ has a strong expansion of the form (8.1), then the finite part $\mathcal{P}f(\psi(\mathbf{x}))$ exists as an element of $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$. In particular, when ψ is smooth in all of $\mathbb{R}^{\mathsf{n}}$, $\psi \in \mathcal{E}(\mathbb{R}^{\mathsf{n}})$, then $\mathcal{P}f(\psi(\mathbf{x})) \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$; notice that the *finite part is always needed if there is a thick point,* even if $\pi(\mathcal{P}f(\psi(\mathbf{x}))) = \psi(\mathbf{x})$ in the space $\mathcal{D}'(\mathbb{R}^{\mathsf{n}})$ of standard distributions, so that no finite part is needed there.

Suppose $g(\mathbf{w})$ is a distribution in $\mathbb{S}$. Let us now define the "thick delta function" $g\delta_* \in \mathcal{D}'_*(\mathbb{R}^{\mathsf{n}})$.

$$\langle g\delta_*, \phi \rangle_{\mathcal{D}'_*(\mathbb{R}^{\mathsf{n}}) \times \mathcal{D}_*(\mathbb{R}^{\mathsf{n}})} := \frac{1}{C_{\mathsf{n}-1}} \langle g(\mathbf{w}), a_0(\mathbf{w}) \rangle_{\mathcal{D}'(\mathbb{S}) \times \mathcal{D}(\mathbb{S})}\,,$$

where $C_{\mathsf{n}-1}$ is the surface area of $\mathbb{S}$, the unit sphere in $\mathbb{R}^{\mathsf{n}}$. If g is locally integrable in $\mathbb{S}$, then

$$\langle g\delta_*, \phi \rangle_{\mathcal{D}'_*(\mathbb{R}^{\mathsf{n}}) \times \mathcal{D}_*(\mathbb{R}^{\mathsf{n}})} = \frac{1}{C_{\mathsf{n}-1}} \int_{\mathbb{S}} g(\mathbf{w})\, a_0(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w})\,. \tag{9.3}$$

We sometimes use the notations $g(\mathbf{w})\, \delta_*$ or $g(\mathbf{w})\, \delta_*(\mathbf{x})$ to denote the thick delta $g\delta_*$.

In particular, if $g(\mathbf{x}) \equiv 1$, then we obtain the "plain thick delta function" $\delta_* = g\delta_*$, given as

$$\left\langle \delta_*, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^{\mathsf{n}})\times\mathcal{D}_*(\mathbb{R}^{\mathsf{n}})} = \frac{1}{C_{\mathsf{n}-1}} \int_{\mathbb{S}} a_0(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \ . \tag{9.4}$$

$$\pi\left(g(\mathbf{w})\,\delta_*(\mathbf{x})\right) = I_g \delta(\mathbf{x}) \ , \tag{9.5}$$

where the constant I_g is given by

$$\begin{aligned} I_g &= \frac{1}{C_{\mathsf{n}-1}} \left\langle g(\mathbf{w}), 1 \right\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} \\ &= \frac{1}{C_{\mathsf{n}-1}} \int_{\mathbb{S}} g(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \ , \end{aligned} \tag{9.6}$$

the second expression being valid in case g is locally integrable.

In particular, since $I_1 = 1$, the projection of the plain thick delta function δ_* is no other than the usual delta function in $\mathcal{D}'(\mathbb{R}^{\mathsf{n}})$,

$$\pi(\delta_*) = \delta \,. \tag{9.7}$$

In fact, the notion of thick delta functions can be generalized to a much broader range of the distributions in $\mathcal{D}'_*(\mathbb{R}^{\mathsf{n}})$, the thick delta functions of degree q, so that $g\delta_*$ is the special case when $\mathsf{q} = 0$. We have the following definition.

Definition 4. *(Thick delta functions of degree q) Let $g(\mathbf{w})$ is a distribution in $\mathbb{S}$. The thick delta function of degree q, denoted as $g\delta_*^{[\mathsf{q}]}$, or as $g(\mathbf{w})\,\delta_*^{[\mathsf{q}]}$, acts on a thick test function $\phi(\mathbf{x})$ as*

$$\left\langle g\delta_*^{[\mathsf{q}]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^{\mathsf{n}})\times\mathcal{D}_*(\mathbb{R}^{\mathsf{n}})} = \frac{1}{C_{\mathsf{n}-1}} \left\langle g(\mathbf{w}), a_{\mathsf{q}}(\mathbf{w}) \right\rangle_{\mathcal{D}'(\mathbb{S})\times\mathcal{D}(\mathbb{S})} \ , \tag{9.8}$$

where $\phi(r\mathbf{w}) \sim \sum_{\mathsf{j}=\mathsf{m}}^{\infty} a_{\mathsf{j}}(\mathbf{w})\, r^{\mathsf{j}}$, as $r \to 0^+$.

If g is locally integrable function in $\mathbb{S}$, then

$$\left\langle g\delta_*^{[\mathsf{q}]}, \phi \right\rangle_{\mathcal{D}'_*(\mathbb{R}^{\mathsf{n}})\times\mathcal{D}_*(\mathbb{R}^{\mathsf{n}})} = \frac{1}{C_{\mathsf{n}-1}} \int_{\mathbb{S}} g(\mathbf{w})\, a_{\mathsf{q}}(\mathbf{w})\, \mathrm{d}\sigma(\mathbf{w}) \ . \tag{9.9}$$

Notice, also that

$$\pi\left(g\delta_*^{[\mathsf{q}]}\right) = 0 \,, \qquad \text{whenever } \mathsf{q} < \mathsf{0} \,. \tag{9.10}$$

The projection of the thick deltas for $\mathsf{q} > \mathsf{0}$ is more interesting; observe, in particular, that $\pi\left(\delta_*^{[1]}\right) = 0$, but $\pi\left(C_{\mathsf{n}-1}\delta(\mathbf{w}-\mathbf{e}_k)\,\delta_*^{[1]}\right) = -\partial\delta(\mathbf{x})/\partial x_k$ if $\mathbf{e}_k$ is the k-th unit vector. Furthermore,

$$\pi\left(\delta_*^{[2]}\right) = \frac{1}{2\mathsf{n}} \nabla^2 \delta(\mathbf{x}) \ , \tag{9.11}$$

where $\nabla^2 = \sum_{i=1}^{\mathsf{n}} \partial^2/\partial x_i^2$ is the Laplacian. More generally, we have the following result.

If $g \in \mathcal{D}'(\mathbb{S})$ and $\mathsf{q} \geq 0$ then

$$\pi\left(g\delta_*^{[\mathsf{q}]}\right) = \frac{(-1)^{\mathsf{q}}}{C_{\mathsf{n}-1}} \sum_{\mathsf{j}_1+\cdots+\mathsf{j}_\mathsf{n}=\mathsf{q}} \frac{\left\langle g(\mathbf{w}), \mathbf{w}^{(\mathsf{j}_1,\ldots,\mathsf{j}_\mathsf{n})}\right\rangle}{\mathsf{j}_1!\cdots \mathsf{j}_\mathsf{n}!} \mathbf{D}^{(\mathsf{j}_1,\ldots,\mathsf{j}_\mathsf{n})}\delta(\mathbf{x}) \,. \tag{9.12}$$

There is an important relation between the finite part distributions $\mathcal{P}f\left(r^\lambda\right)$ and the thick delta functions $\delta_*^{[\mathsf{q}]}$.

The thick distributions $\mathcal{P}f\left(r^\lambda\right)$ are analytic functions of λ in the region $\mathbb{C}\setminus\mathbb{Z}$. There are simple poles at all of the integers $\mathsf{k}\in\mathbb{Z}$ with residues

$$\operatorname*{Res}_{\lambda=\mathsf{k}} \mathcal{P}f\left(r^\lambda\right) = C_{\mathsf{n}-1}\delta_*^{[-\mathsf{k}-\mathsf{n}]}. \tag{9.13}$$

The distribution $\mathcal{P}f\left(r^{\mathsf{k}}\right)$ is the finite part of the analytic function[1] at the pole, namely,

$$\mathcal{P}f\left(r^{\mathsf{k}}\right) = \lim_{\lambda\to\mathsf{k}}\left(\mathcal{P}f\left(r^\lambda\right) - \frac{C_{\mathsf{n}-1}\delta_*^{[-\mathsf{k}-\mathsf{n}]}}{\lambda-\mathsf{k}}\right) . \tag{9.14}$$

Formula (9.10) allows us to recover the well known result that $r^\lambda = \pi\left(\mathcal{P}f\left(r^\lambda\right)\right)$, the usual distribution of $\mathcal{D}'(\mathbb{R}^{\mathsf{n}})$ is analytic for $\lambda \neq -\mathsf{n}$, $-\mathsf{n}-1$, $-\mathsf{n}-2, \ldots$ since the residues at the poles $-\mathsf{n}+1$, $-\mathsf{n}+2$, $-\mathsf{n}+3, \ldots$ vanish.

10. Algebraic and Analytic Operations in $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$

Naturally, we define the algebraic and analytic operations in $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ in the same way they are defined for the usual distributions, namely, by duality.

10.1. **Basic Definitions.** Let $f, g \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ and $\phi(\mathbf{x}) \in \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ be a thick test function. Then the sum $f+g$ is given as

$$\langle f+g, \phi\rangle = \langle f, \phi\rangle + \langle g, \phi\rangle \ , \tag{10.1}$$

while if $\lambda\in\mathbb{C}$ then $\lambda f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ is given as

$$\langle \lambda f, \phi\rangle = \lambda\langle f, \phi\rangle \ . \tag{10.2}$$

Translations are handled by the formula

$$\langle f(\mathbf{x}+\mathbf{c}), \phi(\mathbf{x})\rangle = \langle f(\mathbf{x}), \phi(\mathbf{x}-\mathbf{c})\rangle \ , \tag{10.3}$$

[1]If $g(\lambda)$ is analytic for $0 < |\lambda-\lambda_0| < \rho$ and there is a simple pole with residue $a = \operatorname{Res}_{\lambda=\lambda_0} g(\lambda)$ at $\lambda=\lambda_0$, then the finite part of the analytic function g at λ_0 is given by the limit $\lim_{\lambda\to\lambda_0}\left(g(\lambda) - a(\lambda-\lambda_0)^{-1}\right)$.

where $\mathbf{c} \in \mathbb{R}^{\mathsf{n}}$. Here $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ while the translation $f(\mathbf{x}+\mathbf{c})$ belongs to $\mathcal{D}'_{*,\mathbf{a}-\mathbf{c}}(\mathbb{R}^{\mathsf{n}})$; naturally $\phi \in \mathcal{D}_{*,\mathbf{a}-\mathbf{c}}(\mathbb{R}^{\mathsf{n}})$.

Observe that any distribution g of the space $\mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ can be written as $g(\mathbf{x}) = f(\mathbf{x}-\mathbf{a})$ for some $f \in \mathcal{D}'_{*}(\mathbb{R}^{\mathsf{n}})$, and this justifies studying most results in $\mathcal{D}'_{*}(\mathbb{R}^{\mathsf{n}})$ only.

Linear changes of variables are as follows. Let A be a non-singular $\mathsf{n} \times \mathsf{n}$ matrix. If $f \in \mathcal{D}'_{*}(\mathbb{R}^{\mathsf{n}})$ then $f(A\mathbf{x})$ is defined as

$$\langle f(A\mathbf{x}), \phi(\mathbf{x})\rangle = \frac{1}{|\det A|}\left\langle f(\mathbf{x}), \phi\left(A^{-1}\mathbf{x}\right)\right\rangle , \tag{10.4}$$

as in the space $\mathcal{D}'(\mathbb{R}^{\mathsf{n}})$ of usual distributions. In particular, $f(-\mathbf{x})$ is defined as

$$\langle f(-\mathbf{x}), \phi(\mathbf{x})\rangle = \langle f(\mathbf{x}), \phi(-\mathbf{x})\rangle \ . \tag{10.5}$$

10.2. **Multiplication.** The space of multipliers for a space of test functions and for its dual space are the same, their Moyal algebra.

Definition 5. *Let $\rho \in \mathcal{B}$, the space of multipliers of a space of test functions $\mathcal{A}$, that is, $\rho\phi \in \mathcal{A}$, $\forall \phi \in \mathcal{A}$. Then if $f \in \mathcal{A}'$ the multiplication $\rho f \in \mathcal{A}'$ is given by*

$$\langle \rho f, \phi\rangle = \langle f, \rho\phi\rangle \ . \tag{10.6}$$

The space $\mathcal{B}$ is the Moyal algebra of $\mathcal{A}$ and of $\mathcal{A}'$.

Thick distributions can be multiplied by certain multipliers, functions that are smooth away from the thick point, and that behave like test functions near the thick point. Indeed, it is not hard to see that the Moyal algebra of $\mathcal{D}_{*,\mathbf{a}}$, the set of functions ψ, defined in $\mathbb{R}^{\mathsf{n}} \setminus \{\mathbf{a}\}$, such that $\psi\phi \in \mathcal{D}_{*,\mathbf{a}}$, for any $\phi \in \mathcal{D}_{*,\mathbf{a}}$, is precisely $\mathcal{E}_{*,\mathbf{a}}$, the set of all smooth functions in $\mathbb{R}^{\mathsf{n}} \setminus \{\mathbf{a}\}$, that behave like thick test functions at $\mathbf{x} = \mathbf{a}$. On the other hand, the Moyal algebra of the spaces $\mathcal{S}$ and $\mathcal{S}'$ is the space $\mathcal{O}_M$ [**?**] so that the space of multipliers of $\mathcal{S}_{*,\mathbf{a}}$ and $\mathcal{S}'_{*,\mathbf{a}}$ is the space $(\mathcal{O}_M)_{*,\mathbf{a}}$.

Example 1. *The function r^{k} is a multiplier of $\mathcal{D}'_{*}(\mathbb{R}^{\mathsf{n}})$ for any $\mathsf{k} \in \mathbb{Z}$. In particular, the multiplication $r^{\mathsf{k}}\delta_*^{[\mathsf{q}]}$ is defined for any $\mathsf{q} \in \mathbb{Z}$, and a simple computation yields the useful formula*

$$r^{\mathsf{k}}\delta_*^{[\mathsf{q}]} = \delta_*^{[\mathsf{q}-\mathsf{k}]}. \tag{10.7}$$

Observe that also for any $\lambda \in \mathbb{C}$,

$$r^{\mathsf{k}}\mathcal{P}f\left(r^{\lambda}\right) = \mathcal{P}f\left(r^{\lambda+\mathsf{k}}\right) \ . \tag{10.8}$$

If $\rho \in \mathcal{B}_{*,\mathbf{a}}$, then

$$M_\rho : \mathcal{D} \to \mathcal{D}_{*,\mathbf{a}} \\ \phi \mapsto \rho\phi \quad . \tag{10.9}$$

By duality, the corresponding multiplication operator is defined as

$$M'_\rho : \mathcal{D}'_{*,\mathbf{a}} \to \mathcal{D}' \\ f \mapsto \rho f \quad . \tag{10.10}$$

Notice that

$$\pi(\rho f) = M'_\rho(f) \ . \tag{10.11}$$

10.3. **Derivatives of Thick Distributions.** The derivatives of thick distributions are defined in much the same way as the usual distributional derivatives, that is, by duality. If $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$ then its thick distributional derivative $\partial^* f/\partial x_j$ is defined as

$$\left\langle \frac{\partial^* f}{\partial x_j}, \phi \right\rangle = -\left\langle f, \frac{\partial \phi}{\partial x_j} \right\rangle , \qquad \phi \in \mathcal{D}_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}}) \ . \tag{10.12}$$

Let $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$. Then

$$\pi\left(\frac{\partial^* f}{\partial x_j}\right) = \frac{\overline{\partial}\,\pi(f)}{\partial x_j} \ . \tag{10.13}$$

Formula (10.13) has an interesting consequence, as we shall explain next.

Example 2. *Let $f \in \mathcal{D}'_{*,\mathbf{a}}(\mathbb{R}^{\mathsf{n}})$; if $\pi(f) = 0$ (which is perhaps easy to see), then $\pi(\partial^* f/\partial x_j) = 0$ (which is perhaps harder to see). Consider, for instance, $f = g\delta_*^{[-1]}$, a thick delta of order -1; that $\pi\left(g\delta_*^{[-1]}\right) = 0$ is obvious, but the formula $\pi\left((\delta g/\delta x_j - (\mathsf{n}-1)\, n_j g)\, \delta_*^{[0]}\right) = 0$, that follows. In fact, even a particular case, such as $g = n_i$, which gives $\delta g/\delta x_j - (\mathsf{n}-1)\, n_j g = \delta_{ij} - \mathsf{n} n_i n_j$, yields an interesting formula, namely, $\pi\left(n_i n_j \delta_*^{[0]}\right) = \delta_{ij}/\mathsf{n}$.*

In general $\mathcal{P}f(\partial\psi/\partial x_j)$ and $\partial\mathcal{P}f(\psi)/\partial x_j$, even if both exist, *will not be equal.* We shall consider the case when $\psi = r^\lambda$ in detail later on.

11. Derivatives of Thick Deltas

In this section we shall compute the first order derivatives of thick deltas of any order.

Let $g \in \mathcal{D}'(\mathbb{S})$. Then

$$\frac{\partial^*}{\partial x_j}\left(g\delta_*^{[\mathsf{q}]}\right) = \left(\frac{\delta g}{\delta x_j} - (\mathsf{q}+\mathsf{n})\, n_j g\right)\delta_*^{[\mathsf{q}+1]}. \tag{11.1}$$

Observe, in particular, the formula

$$\frac{\partial^*}{\partial x_j}\left(\delta_*^{[\mathsf{q}]}\right) = -(\mathsf{q}+\mathsf{n})\, n_j \delta_*^{[\mathsf{q}+1]}, \tag{11.2}$$

for the derivatives of plain thick deltas.

We can compute the Laplacian of the plain thick deltas as follows,

$$\nabla^2\left(\delta_*^{[\mathsf{q}]}\right) = (\mathsf{q}+\mathsf{n})(\mathsf{q}+2)\,\delta_*^{[\mathsf{q}+2]}. \tag{11.3}$$

In particular, if $\mathsf{m} > 0$,

$$\nabla^{2\mathsf{m}}(\delta_*) = \frac{\Gamma(\mathsf{m}+\mathsf{n}/2)\,\Gamma(1/2)\,(2\mathsf{m})!}{\Gamma(\mathsf{m}+1/2)\,\Gamma(\mathsf{n}/2)}\delta_*^{[2\mathsf{m}]}. \tag{11.4}$$

If we now consider the projection of this identity onto $\mathcal{D}'(\mathbb{R}^{\mathsf{n}})$ and recall that $\pi(\delta_*) = \delta$, we obtain

$$\pi\left(\delta_*^{[2\mathsf{m}]}\right) = \frac{\Gamma(\mathsf{m}+1/2)\,\Gamma(\mathsf{n}/2)}{\Gamma(\mathsf{m}+\mathsf{n}/2)\,\Gamma(1/2)\,(2\mathsf{m})!}\nabla^{2\mathsf{m}}(\delta)\ . \tag{11.5}$$

Formula (11.3) also yields that $\nabla^2\left(\delta_*^{[-2]}\right) = 0$ and $\nabla^2\left(\delta_*^{[-\mathsf{n}]}\right) = 0$.

Notice that since $\delta n_i/\delta x_j = \delta_{ij} - n_i n_j$, we have, more generally than (11.3),

$$\frac{\partial^{*2}}{\partial x_j \partial x_i}\left(\delta_*^{[\mathsf{q}]}\right) = (\mathsf{q}+\mathsf{n})\left((\mathsf{q}+\mathsf{n}+2)\, n_i n_j - \delta_{ij}\right)\delta_*^{[\mathsf{q}+2]}. \tag{11.6}$$

12. Partial Derivatives of $\mathcal{P}f\left(r^\lambda\right)$

Another important set of formulas we want to discuss are the derivatives of $\mathcal{P}f\left(r^\lambda\right)$.

If $\lambda \in \mathbb{C}\setminus\mathbb{Z}$, then

$$\frac{\partial^*}{\partial x_j}\left(\mathcal{P}f\left(r^\lambda\right)\right) = \lambda x_j \mathcal{P}f\left(r^{\lambda-2}\right) = \lambda w_j \mathcal{P}f\left(r^{\lambda-1}\right)\ , \tag{12.1}$$

while if $\mathsf{k} \in \mathbb{Z}$,

$$\frac{\partial^*}{\partial x_j}\left(\mathcal{P}f\left(r^{\mathsf{k}}\right)\right) = \mathsf{k} x_j \mathcal{P}f\left(r^{\mathsf{k}-2}\right) + C_{\mathsf{n}-1} n_j \delta_*^{[-\mathsf{k}-\mathsf{n}+1]}. \tag{12.2}$$

In $\mathbb{R}^3$, $\partial^* \mathcal{P}f\left(r^{-1}\right)/\partial x_j$ is given by

$$\frac{\partial^* \mathcal{P}f\left(r^{-1}\right)}{\partial x_j} = -x_j \mathcal{P}f\left(r^{-3}\right) + 4\pi n_j \delta_*^{[-1]}. \tag{12.3}$$

This is very similar to the usual distributional derivative of $1/r$ except for the extra term $4\pi n_j \delta_*^{[-1]}$. Of course, $\pi\left(4\pi n_j \delta_*^{[-1]}\right) = 0$, so that we recover the well known formula $\overline{\partial}\left(r^{-1}\right)/\partial x_j = -x_j/r^3$.

If we apply the projection operator to (12.1) and (12.2), we obtain the formulas for the partial derivatives of $\mathcal{P}f\left(r^{\lambda}\right)$ in $\mathcal{D}'\left(\mathbb{R}^{\mathsf{n}}\right)$. Since (11.2) yields that $\pi\left(n_j \delta_*^{[\mathsf{q}]}\right) = 0$ unless $\mathsf{q} = 2\mathsf{m}+1$, $\mathsf{m} \geq 0$, in which case

$$\pi\left(n_j \delta_*^{[2\mathsf{m}+1]}\right) = \frac{-\Gamma\left(\mathsf{m}+1/2\right)\Gamma\left(\mathsf{n}/2\right)}{\left(2\mathsf{m}+\mathsf{n}\right)\Gamma\left(\mathsf{m}+\mathsf{n}/2\right)\Gamma\left(1/2\right)\left(2\mathsf{m}\right)!}\frac{\overline{\partial}}{\partial x_j}\nabla^{2\mathsf{m}}\delta\,, \tag{12.4}$$

we obtain $\overline{\partial}/\partial x_j\left(\mathcal{P}f\left(r^{\lambda}\right)\right) = \lambda x_j \mathcal{P}f\left(r^{\lambda-2}\right)$ unless $\lambda = -\mathsf{n}, -\mathsf{n}-2, -\mathsf{n}-4, \ldots$. If $\lambda = -\mathsf{p} = -\mathsf{n}-2\mathsf{m}$,

$$\frac{\overline{\partial}}{\partial x_j}\left(\mathcal{P}f\left(\frac{1}{r^{\mathsf{p}}}\right)\right) = -\mathsf{p}x_j \mathcal{P}f\left(\frac{1}{r^{\mathsf{p}+2}}\right) - \frac{c_{\mathsf{m},\mathsf{n}}}{(2\mathsf{m})!\mathsf{p}}\frac{\overline{\partial}}{\partial x_j}\nabla^{2\mathsf{m}}\delta\,, \tag{12.5}$$

We will write

$$c_{m,n} = \frac{2\Gamma\left(m+1/2\right)\pi^{(n-1)/2}}{\Gamma\left(m+n/2\right)} = \int_{\mathbb{S}} \omega_j^{2m}\, \mathrm{d}\sigma\left(\omega\right)\,, \quad C = c_{0,n}\,. \tag{12.6}$$

Notice that $c_{0,n} = C = 2\pi^{n/2}/\Gamma\left(n/2\right)$, is the surface area of the unit sphere $\mathbb{S}$ of $\mathbb{R}^n$.

Let us now discuss the second-order thick derivatives of r^{λ}. If $\lambda \in \mathbb{C}\setminus\mathbb{Z}$, then

$$\frac{\partial^{*2}\mathcal{P}f\left(r^{\lambda}\right)}{\partial x_i \partial x_j} = \left(\lambda\delta_{ij} + \lambda\left(\lambda-2\right)n_i n_j\right)\mathcal{P}f\left(r^{\lambda-2}\right)\,. \tag{12.7}$$

If $\lambda = \mathsf{k} \in \mathbb{Z}$, then

$$\begin{aligned}\frac{\partial^{*2}\mathcal{P}f\left(r^{\mathsf{k}}\right)}{\partial x_i \partial x_j} &= \left(\mathsf{k}\delta_{ij} + \mathsf{k}\left(\mathsf{k}-2\right)n_i n_j\right)\mathcal{P}f\left(r^{\mathsf{k}-2}\right) \\ &\quad + \left(\delta_{ij} + 2\left(\mathsf{k}-1\right)n_i n_j\right)\delta_*^{[-\mathsf{k}-\mathsf{n}+2]}.\end{aligned} \tag{12.8}$$

When $\mathsf{n} = 3$ and $\mathsf{k} = -1$ we obtain

$$\frac{\partial^{*2}\mathcal{P}f\left(r^{-1}\right)}{\partial x_i \partial x_j} = \left(3x_i x_j - \delta_{ij}r^2\right)\mathcal{P}f\left(r^{-5}\right) + 4\pi\left(\delta_{ij} - 4n_i n_j\right)\delta_*\,. \tag{12.9}$$

Since $\pi\left(n_i n_j \delta_*\right) = (1/3)\,\delta(\mathbf{x})$ in $\mathbb{R}^3$, we obtain the well known formula of Frahm

$$\frac{\overline{\partial}^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{3x_i x_j - r^2\delta_{ij}}{r^5} - \left(\frac{4\pi}{3}\right)\delta_{ij}\delta(\mathbf{x}) \ , \tag{12.10}$$

when we apply the projection operator π to (12.9).

13. Fourier transforms in several variables

The Fourier transform of thick distributions was developed by Yunyun Yang, Jasson Vindas, and Estrada in 2019-2020 (Ricardo Estrada, Jasson Vindas and Yunyun Yang, The Fourier transform of thick distributions, that just appear in *Analysis and Applications*)

We construct isomorphisms

$$\mathcal{F}_* : \mathcal{S}'_*\left(\mathbb{R}^n\right) \longrightarrow \mathcal{W}'\left(\mathbb{R}^n_{\mathrm{c}}\right) \ , \tag{13.1}$$

$$\mathcal{F}^* : \mathcal{W}'\left(\mathbb{R}^n_{\mathrm{c}}\right) \longrightarrow \mathcal{S}'_*\left(\mathbb{R}^n\right) \ , \tag{13.2}$$

that extend the Fourier transform of tempered distributions, namely,

$$\Pi_{\mathcal{W}',\mathcal{S}'}\mathcal{F}_* = \mathcal{F}\Pi_{\mathcal{S}'_*,\mathcal{S}'} \ , \qquad \Pi_{\mathcal{S}'_*,\mathcal{S}'}\mathcal{F}^* = \mathcal{F}\Pi_{\mathcal{W}',\mathcal{S}'} \ ,$$

where $\Pi_{\mathcal{W}',\mathcal{S}'}$ and $\Pi_{\mathcal{S}'_*,\mathcal{S}'}$ are the canonical projections of $\mathcal{S}'_*\left(\mathbb{R}^n\right)$ or $\mathcal{W}'\left(\mathbb{R}^n_{\mathrm{c}}\right)$ onto $\mathcal{S}'\left(\mathbb{R}^n\right)$.

14. Preliminaries

$$F\left\{f(\mathbf{x});\mathbf{u}\right\} = \int_{\mathbb{R}^n} f(\mathbf{x})\, e^{i\mathbf{x}\cdot\mathbf{u}}\,\mathrm{d}\mathbf{x}\,. \tag{14.1}$$

We need the Fourier transform of several distributions in $\mathbb{R}^n$ for later use, especially transforms of the type $\mathcal{F}\left\{\mathcal{P}f\left(r^{-N}\right)a(\mathbf{w});\mathbf{u}\right\}$ where $\mathbf{x} = r\mathbf{w}$ are polar coordinates, and where a is a smooth function defined on the unit sphere $\mathbb{S}$. See (Samko, S. G., On the Fourier transform of the functions $\frac{Y_m\left(\frac{x}{|x|}\right)}{|x|^{n+\alpha}}$, *Soviet Math.* **22** (1978), 60-64; Ricardo Estrada, Jasson Vindas and Yunyun Yang, The Fourier transform of thick distributions, *Analysis and Applications*). We have

$$\mathcal{F}\left\{r^\lambda;\mathbf{u}\right\} = \frac{\pi^{n/2}2^{\lambda+n}\Gamma\left(\frac{\lambda+n}{2}\right)s^{-\lambda-n}}{\Gamma\left(-\frac{\lambda}{2}\right)} \ , \tag{14.2}$$

whenever $\lambda \neq -n, -n-2, -n-4, \ldots$

If $m = 0, 1, 2, \ldots$ then

$$\mathcal{F}\left\{\mathcal{P}f\left(\frac{1}{r^{n+2m}}\right);\mathbf{u}\right\} = \frac{(-1)^m \pi^{n/2}}{m!\Gamma\left(\frac{n}{2}+m\right)}\left(\frac{s}{2}\right)^{2m}\left\{\psi(m+1)+\psi\left(\frac{n}{2}+m\right)-2\ln\left(\frac{s}{2}\right)\right\}. \tag{14.3}$$

Our next task is to find the Fourier transform of distributions of the form $\mathcal{P}f\left(r^{-N}\right)a(\mathbf{w})$ when $a = \mathsf{Y}_k$ is a spherical harmonic of degree k.

If $\mathsf{Y}_k \in \mathcal{H}_k$ and $\lambda \neq -n-k, -n-k-2, -n-k-4, \ldots$

$$\mathcal{F}\left\{r^\lambda \mathsf{Y}_k(\mathbf{w}); s\mathbf{v}\right\} = \frac{i^k \pi^{n/2} 2^{\lambda+n}\Gamma\left(\frac{k+n+\lambda}{2}\right)}{\Gamma\left(\frac{k-\lambda}{2}\right)} s^{-(\lambda+n)}\mathsf{Y}_k(\mathbf{v}). \tag{14.4}$$

If $\mathsf{Y}_k \in \mathcal{H}_k$ and $m = 0, 1, 2, \ldots$ then

$$\mathcal{F}\left\{\mathcal{P}f\left(\frac{1}{r^{n+k+2m}}\right)\mathsf{Y}_k(\mathbf{w}); s\mathbf{v}\right\} = \frac{(-1)^m i^k \pi^{n/2}}{m!\Gamma\left(\frac{n}{2}+k+m\right)}\left(\frac{s}{2}\right)^{2m+k}\left\{\psi(1+m)+\psi\left(\frac{n}{2}+k+m\right)-2\ln\left(\frac{s}{2}\right)\right\}\mathsf{Y}_k(\mathbf{v}). \tag{14.5}$$

Let now a be a smooth function on the sphere, $a \in \mathcal{D}(\mathbb{S})$. Then we can write it in terms of spherical harmonics as

$$a(\mathbf{w}) = \sum_{m=0}^{\infty} \mathsf{Y}_m(\mathbf{w}), \tag{14.6}$$

where $\mathsf{Y}_m = \mathsf{Y}_m\{a\} \in \mathcal{H}_m$ are given as $\mathsf{Y}_m(\mathbf{w}) = \int_{\mathbb{S}} \mathsf{Z}_m(\mathbf{w},\mathbf{v})\, a(\mathbf{v})\, \mathrm{d}\sigma(\mathbf{v})$; here $\mathsf{Z}_m(\mathbf{w},\mathbf{v})$ is the reproducing kernel of $\mathcal{H}_m$, namely [**?**, Thm. 5.38]

$$(n+2m-2)\sum_{q=0}^{[\![m/2]\!]}(-1)^q \frac{n(n+2)\cdots(n+2m-2q-4)}{2^q q!\,(m-2q)!}(\mathbf{w}\cdot\mathbf{v})^{m-2q}. \tag{14.7}$$

We thus obtain the following.

If $\beta \neq 0, 1, 2, \ldots$ then

$$\mathcal{F}\left\{\mathcal{P}f\left(\frac{1}{r^{n+\beta}}\right)a(\mathbf{w});\mathbf{u}\right\} = \mathcal{P}f\left(s^\beta\right)\mathcal{K}_\beta\{a(\mathbf{w});\mathbf{v}\}, \tag{14.8}$$

where $\mathcal{K}_\beta\{a(\mathbf{w});\mathbf{v}\} = \langle K_\beta(\mathbf{w},\mathbf{v}), a(\mathbf{w})\rangle_{\mathbf{w}}$,and

$$K_\beta(\mathbf{w},\mathbf{v}) = \sum_{m=0}^{\infty}\kappa_{\beta,m}\mathsf{Z}_m(\mathbf{w},\mathbf{v}), \quad \kappa_{\beta,m} = \frac{i^m \pi^{n/2} 2^{-\beta}\Gamma\left(\frac{m-\beta}{2}\right)}{\Gamma\left(\frac{m+n+\beta}{2}\right)}. \tag{14.9}$$

The operator $\mathcal{K}_\beta$ is analytic for $\beta \neq 0, 1, 2, \ldots$; for $\beta = q \in \mathbb{N}$ we have the next formula.

If $q=0,1,2,\ldots$ then
(14.10)
$$\mathcal{F}\left\{\mathcal{P}f\left(\frac{1}{r^{n+q}}\right)a\left(\mathbf{w}\right);\mathbf{u}\right\}=s^{q}\left(\mathcal{K}_{q}\left\{a\left(\mathbf{w}\right);\mathbf{v}\right\}+\mathcal{L}_{q}\left\{a\left(\mathbf{w}\right);\mathbf{v}\right\}\ln s\right)\ ,$$
where $\mathcal{K}_{q}\left\{a\left(\mathbf{w}\right);\mathbf{v}\right\}=\left\langle K_{q}\left(\mathbf{w},\mathbf{v}\right),a\left(\mathbf{w}\right)\right\rangle_{\mathbf{w}}$,
$$K_{q}\left(\mathbf{w},\mathbf{v}\right)=\sum_{m=0}^{\infty}\kappa_{q,m}\mathsf{Z}_{m}\left(\mathbf{w},\mathbf{v}\right)\ , \tag{14.11}$$
the constants $\kappa_{q,m}$ being given by (14.9) if $m\neq q,q-2,\ldots$ and as
(14.12)
$$\kappa_{q,q-2m}=\frac{i^{q}\pi^{n/2}2^{-q}}{m!\Gamma\left(\frac{n}{2}+q-m\right)}\left\{\psi\left(1+m\right)+\psi\left(\frac{n}{2}+q-m\right)+2\ln 2\right\}\ ,$$
for $0\leq m\leq[\![q/2]\!]$. On the other hand, $\mathcal{L}_{q}\left\{a\left(\mathbf{w}\right);\mathbf{v}\right\}=\left\langle L_{q}\left(\mathbf{w},\mathbf{v}\right),a\left(\mathbf{w}\right)\right\rangle_{\mathbf{w}}$,
(14.13)
$$L_{q}\left(\mathbf{w},\mathbf{v}\right)=\sum_{m=0}^{[\![q/2]\!]}\lambda_{q,q-2m}\mathsf{Z}_{q-2m}\left(\mathbf{w},\mathbf{v}\right)\ ,\quad \lambda_{q,q-2m}=\frac{-i^{q}2^{-q+1}\pi^{n/2}}{m!\Gamma\left(\frac{n}{2}+q-m\right)}\ .$$

14.1. **The operators $\mathcal{K}_{\beta}$.**
$$K_{\beta}\left(\mathbf{w},\mathbf{v}\right)=\Gamma\left(-\beta\right)e^{-i\pi\beta/2}\left(\mathbf{w}\cdot\mathbf{v}+i0\right)^{\beta}\ , \tag{14.14}$$
a distributional kernel for $\beta\neq0,1,2,\ldots$ that becomes an integral operator if $\Re e\,\beta>0$. Observe that the distribution $\left(t+i0\right)^{\beta}$ is an entire function of β. The singularity of $K_{\beta}\left(\mathbf{w},\mathbf{v}\right)$ at $\beta=q\in\mathbb{N}$ is produced by the term $\Gamma\left(-\beta\right)$,
(14.15)
$$K_{q}\left(\mathbf{w},\mathbf{v}\right)=\frac{\left(-1\right)^{q}e^{-i\pi q/2}}{q!}\left(\psi\left(q+1\right)+\ln\left(\frac{e^{i\pi/2}}{\left(\mathbf{w}\cdot\mathbf{v}+i0\right)}\right)\right)\left(\mathbf{w}\cdot\mathbf{v}\right)^{q},$$
and
$$L_{q}\left(\mathbf{w},\mathbf{v}\right)=\frac{\left(-1\right)^{q}e^{-i\pi q/2}}{q!}\left(\mathbf{w}\cdot\mathbf{v}\right)^{q}. \tag{14.16}$$

If $\beta\neq0,1,2,\ldots$, the coefficients $\kappa_{\beta,m}$ never vanish for $\beta\neq-n-q$, $q=0,1,2,\ldots$, but they could vanish for some m when $\beta=-n-q$, so that the operator $\mathcal{K}_{\beta}$ is an isomorphism of $\mathcal{D}\left(\mathbb{S}\right)$ for $\beta\neq-n-q$, but $\mathcal{K}_{-n-q}\left(\mathcal{D}\left(\mathbb{S}\right)\right)$ is a subspace of finite codimension of $\mathcal{D}\left(\mathbb{S}\right)$. The Fourier inversion formula yields the inverses of the operators $\mathcal{K}_{\beta}$ for $\beta\in\mathbb{C}\setminus\mathbb{Z}$ or for $\beta\in\{1-n,2-n,\ldots,-1\}$ as
$$\mathcal{K}_{\beta}^{-1}\left\{A\left(\mathbf{v}\right);\mathbf{w}\right\}=\frac{1}{\left(2\pi\right)^{n}}\mathcal{K}_{-n-\beta}\left\{A\left(\mathbf{v}\right);-\mathbf{w}\right\}\ ,\quad A\in\mathcal{D}\left(\mathbb{S}\right)\ . \tag{14.17}$$

14.2. **The operators $\mathfrak{K}_q$ and $\mathfrak{L}_q$.** It is convenient to consider a variant of the operators $\mathcal{K}_\beta$ in case $\beta \in \mathbb{Z}$. Let us start with some notation. If $q \in \mathbb{N}$ we denote as $\mathcal{P}_q$ the space of restrictions of homogeneous polynomials of degree q to $\mathbb{S}$, that is $\mathcal{P}_q = \mathcal{H}_q \oplus \mathcal{H}_{q-2} \oplus \mathcal{H}_{q-4} \oplus \cdots$. Let $\mathcal{X}$ be a space of functions or generalized functions over $\mathbb{S}$, as $\mathcal{D}(\mathbb{S})$, $L^2(\mathbb{S})$, or $\mathcal{D}'(\mathbb{S})$, that equals the closure in $\mathcal{X}$ of the sum $\mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$[2]. Then $\mathcal{X}_q$ is the space $\mathcal{X}$ if $1-n \leq q \leq -1$; if $q \geq 0$, $\mathcal{X}_q$ is the sum $\widehat{\bigoplus}_{\mathcal{X}} \mid_{m \neq q, q-2, \ldots} \mathcal{H}_m$, while if $q \leq -n$ then $\mathcal{X}_q = \mathcal{X}_{-n-q}$. Notice that

$$\mathcal{X}_q \oplus \mathcal{P}_{-n-q} = \mathcal{X}\,, \quad q \leq -n\,, \qquad \mathcal{X}_q \oplus \mathcal{P}_q = \mathcal{X}\,, \quad q \geq 0\,. \tag{14.18}$$

We define the operators $\mathfrak{K}_q : \mathcal{D}_q \longrightarrow \mathcal{D}_q$ as $\Pi \mathcal{K}_q \iota$, where ι is the canonical injection of $\mathcal{D}_q$ into $\mathcal{D}(\mathbb{S})$ and Π the canonical projection of $\mathcal{D}(\mathbb{S})$ onto $\mathcal{D}_q$. We can also consider the $\mathfrak{K}_q$ as operators from $\mathcal{D}'_q$ to itself, by duality or employing the expansion (14.9). The Propositions **??** and **??** immediately give the ensuing.

Proposition 1. *The operators $\mathfrak{K}_q$ are isomorphisms of the space $\mathcal{X}_q$ to itself for*[3] $\mathcal{X} = \mathcal{D}(\mathbb{S})$ *or* $\mathcal{D}'(\mathbb{S})$. *Its inverses are given as*

$$\mathfrak{K}_q^{-1}\{A(\mathbf{v});\mathbf{w}\} = \frac{1}{(2\pi)^n}\mathfrak{K}_{-n-q}\{A(\mathbf{v});-\mathbf{w}\}\,, \quad A \in \mathcal{X}_q\,. \tag{14.19}$$

Observe that for $\mathcal{X} = \mathcal{D}(\mathbb{S})$ or $\mathcal{D}'(\mathbb{S})$ we have $\mathcal{X}_q = \mathcal{K}_q(\mathcal{X})$, for $q < 0$. This is not true for $q \geq 0$, but we have $\mathcal{X}_q = \Pi \mathcal{K}_q(\mathcal{X})$ where Π is the canonical projection of $\mathcal{X}$ onto $\mathcal{X}_q$.

The operators $\mathfrak{L}_q : \mathcal{P}_q \longrightarrow \mathcal{P}_q$ are defined as $\Pi \mathcal{L}_q \iota$, where ι is the canonical injection of $\mathcal{P}_q$ into $\mathcal{D}(\mathbb{S})$ and Π the canonical projection of $\mathcal{D}(\mathbb{S})$ onto $\mathcal{P}_q$. They are isomorphisms of the space $\mathcal{P}_q$.

15. The Fourier transform of thick test functions

In this section we will construct a space $\mathcal{W}(\mathbb{R}^n)$ such that it is possible to define an operator

$$\mathcal{F}_{*,\mathrm{t}} : \mathcal{S}_*(\mathbb{R}^n) \longrightarrow \mathcal{W}(\mathbb{R}^n)\,, \tag{15.1}$$

the Fourier transform of *test functions,* which has the expected properties of such a transform.

Let us start by observing that if ϕ is a thick test function in $\mathbb{R}^n$, then in general it is *not* locally integrable at the origin, so that, in general, it does not give a unique distribution. Therefore, we cannot imbed $\mathcal{S}_*(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$ and consequently, if $\phi \in \mathcal{S}_*(\mathbb{R}^n)$ then in general

[2]Such closures will be denoted as $\widehat{\bigoplus}_{\mathcal{X}} \mid_{m=0}^{\infty} \mathcal{H}_m$

[3]The results also holds for $\mathcal{X} = L^2(\mathbb{S})$, but we will not need this case presently.

we cannot define $\mathcal{F}(\phi)$ as a distribution of the space $\mathcal{S}'(\mathbb{R}^n)$[4]. On the other hand, any $\phi \in \mathcal{S}_*(\mathbb{R}^n)$ does have regularizations $f \in \mathcal{S}'(\mathbb{R}^n)$; however f is not unique, since if f_0 is a regularization, then so are all distributions of the form $f_0 + g$, where $\operatorname{supp} g \subset \{\mathbf{0}\}$, that is, where g is a sum of derivatives of the Dirac delta function at the origin. It will be convenient to use the notation

$$\mathcal{S}_{*,\mathrm{reg}}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$$

$\mathcal{S}_{*,\mathrm{reg}}(\mathbb{R}^n)$ for the subspace of $\mathcal{S}'(\mathbb{R}^n)$ whose elements are the regularizations of thick test functions.

The space $\mathcal{W}_{\mathrm{pre}}(\mathbb{R}^n)$ consists of those smooth functions Φ defined in $\mathbb{R}^n$ that admit a strong asymptotic expansion of the form

$$\Phi(s\mathbf{v}) \sim \sum_{q=0}^{Q} \left(A_q(\mathbf{v}) + P_q(\mathbf{v}) \ln s\right) s^q + \sum_{q=1}^{\infty} A_{-q}(\mathbf{v}) s^{-q}, \tag{15.2}$$

where $A_q \in \mathcal{K}_q(\mathcal{D}(\mathbb{S}))$ for $q \le -n$, $A_q \in \mathcal{D}(\mathbb{S})$ for $q > n$, and where the $P_q \in \mathcal{P}_q$ for $q \in \mathbb{N}$.

Our analysis so far yields the ensuing result:

The Fourier transform is an isomorphism of the vector spaces $\mathcal{S}_{*,\mathrm{reg}}(\mathbb{R}^n)$ and $\mathcal{W}_{\mathrm{pre}}(\mathbb{R}^n)$.

15.1. **Delta parts and polynomial parts.** In general it is not possible to separate the contribution to a distribution from a given point; to talk about the "delta part at $\mathbf{x}_0$" of *all* distributions does not make sense. However, *sometimes,* we can actually separate the delta part.

Let $f_0 \in \mathcal{D}'(\mathbb{R}^n \setminus \{\mathbf{0}\})$ be a distribution defined in the complement of the origin. Suppose the pseudofunction $\mathcal{P}f(f_0(\mathbf{x}))$ exists in $\mathcal{D}'(\mathbb{R}^n)$ (respectively in $\mathcal{D}'_*(\mathbb{R}^n)$). Let $f \in \mathcal{D}'(\mathbb{R}^n)$ (respectively in $f \in \mathcal{D}'_*(\mathbb{R}^n)$) be any regularization of f_0. Then the *delta part* at 0 of f is the distribution $f - \mathcal{P}f(f_0(\mathbf{x}))$, whose support is the origin.

In a similar fashion, one may consider the *polynomial part* of distributions. Not all distributions have a well defined polynomial part, but all the elements of $\mathcal{W}_{\mathrm{pre}}(\mathbb{R}^n)$ do. Let us start with the case of a distribution that is homogeneous of degree $q \ge 0$ in $\mathbb{R}^n \setminus \{\mathbf{0}\}$, that is $F_q(\mathbf{u}) = A_q(\mathbf{v}) s^q$, $\mathbf{u} = s\mathbf{v}$ being polar coordinates and $A_q \in \mathcal{D}'(\mathbb{S})$. Then we can write A_q in terms of spherical harmonics as $A_q(\mathbf{v}) = \sum_{m=0}^{\infty} \mathsf{Y}_{m,q}(\mathbf{v})$, where $\mathsf{Y}_{m,q} \in \mathcal{H}_m$. Therefore

$$F_q(\mathbf{u}) = E_q(\mathbf{u}) + \widetilde{F}_q(\mathbf{u}), \tag{15.3}$$

[4]It is possible to consider $\mathcal{F}(\phi)$ as a distribution of the Lizorkin distributional spaces, but for our purposes a different approach is more convenient.

where $E_q = \Pi_{\mathrm{pol}}(F_q)$ is the homogeneous polynomial of degree q given as

$$E_q(\mathbf{u}) = \Pi_{\mathrm{pol}}(F_q) = (\sum_{k\leq q/2} \mathsf{Y}_{q-2k,q}(\mathbf{v}))s^q, \tag{15.4}$$

and $\widetilde{F}_q = F_q - E_q$ is the polynomial free part of F_q.

In the general case when F has the asymptotic expansion of the form

$$F(s\mathbf{v}) \sim \sum_{q=0}^{Q} (A_q(\mathbf{v}) + P_q(\mathbf{v})\ln s)\, s^q + \sum_{q=1}^{\infty} A_{-q}(\mathbf{v})\, s^{-q}, \tag{15.5}$$

then the *polynomial part* of F is the polynomial

$$\Pi_{\mathrm{pol}}(F) = \sum_{q=0}^{Q} \Pi_{\mathrm{pol}}(A_q(\mathbf{v})\, s^q)\ . \tag{15.6}$$

The *polynomial free part* of F is $F - \Pi_{\mathrm{pol}}(F)$.

Let $A \in \mathcal{D}(\mathbb{S})$. If $m \in \mathbb{N}$, then $A \in \mathcal{K}_{-(n+m)}(\mathcal{D}(\mathbb{S}))$ if and only if the function $A(\mathbf{v})\, s^m$ is polynomial free. Similarly, if $A \in \mathcal{D}'(\mathbb{S})$, then $A \in \mathcal{K}_{-(n+m)}(\mathcal{D}'(\mathbb{S}))$ if and only if the distribution $A(\mathbf{v})\, s^m$ is polynomial free.

15.2. **The space $\mathcal{W}(\mathbb{R}^n)$.** The space $\mathcal{S}_{*,\mathrm{reg}}(\mathbb{R}^n)$ admits the representation

$$\mathcal{S}_{*,\mathrm{reg}}(\mathbb{R}^n) = \mathcal{S}_{*,\mathrm{ord}}(\mathbb{R}^n) \oplus \mathcal{D}'_{\{\mathbf{0}\}}(\mathbb{R}^n)\ , \tag{15.7}$$

where $\mathcal{D}'_{\{\mathbf{0}\}}(\mathbb{R}^n)$ is the space of distributions with support at the origin and where $\mathcal{S}_{*,\mathrm{ord}}(\mathbb{R}^n)$ is the space of ordinary parts of regularizations of thick test functions. Clearly the $\mathcal{P}f$ operator is an isomorphism of $\mathcal{S}_{*}(\mathbb{R}^n)$ onto $\mathcal{S}_{*,\mathrm{ord}}(\mathbb{R}^n)$. We define the topology of $\mathcal{S}_{*,\mathrm{ord}}(\mathbb{R}^n)$ by asking $\mathcal{P}f$ to be a topological isomorphism. The space $\mathcal{D}'_{\{\mathbf{0}\}}(\mathbb{R}^n)$ has a topology as a closed subspace of $\mathcal{S}'(\mathbb{R}^n)$. The topology of $\mathcal{S}_{*,\mathrm{reg}}(\mathbb{R}^n)$ is the direct sum topology. Notice that the topology of $\mathcal{S}_{*,\mathrm{reg}}(\mathbb{R}^n)$ is stronger but not equal to the subspace topology inherited from $\mathcal{S}'(\mathbb{R}^n)$. We can now complete the Theorem **??**: *The Fourier transform is a topological isomorphism of the spaces $\mathcal{S}_{*,\mathrm{reg}}(\mathbb{R}^n)$ and $\mathcal{W}_{\mathrm{pre}}(\mathbb{R}^n)$.*

We now define the space $\mathcal{W}(\mathbb{R}^n)$.

The space $\mathcal{W}(\mathbb{R}^n)$ is formed by the polynomial free elements of $\mathcal{W}_{\mathrm{pre}}(\mathbb{R}^n)$, with the subspace topology. Explicitly, $\Phi \in \mathcal{W}$ if it is smooth in $\mathbb{R}^n$ and at infinity it has an asymptotic expansion

$$\Phi(s\mathbf{v}) \sim \sum_{q=0}^{Q} (A_q(\mathbf{v}) + P_q(\mathbf{v})\ln s)\, s^q + \sum_{q=1}^{\infty} A_{-q}(\mathbf{v})\, s^{-q}, \tag{15.8}$$

where $A_q \in \mathcal{D}_q$ for $q \in \mathbb{Z}$ and the $P_q \in \mathcal{P}_q$ are homogeneous polynomials of degree q.

The space $\mathcal{W}(\mathbb{R}^n)$ is exactly the space needed to define the Fourier transform of thick test functions; the condition $A_q \in \mathcal{D}_q$ in the expansion (15.8), which is equivalent to the fact that Φ is polynomial free, will play a very important role in the behavior of the Fourier transform of thick distributions. Notice in fact that

$$\mathcal{W}_{\text{pre}}(\mathbb{R}^n) = \mathcal{W}(\mathbb{R}^n) \oplus \mathcal{P}(\mathbb{R}^n)\ , \tag{15.9}$$

as topological vector spaces. Therefore the space $\mathcal{W}(\mathbb{R}^n)$ can also be constructed as a quotient space. Namely, if we define the equivalence relation $F \sim G$ if $F - G$ is a polynomial, then

$$\mathcal{W}(\mathbb{R}^n) \approx \mathcal{W}_{\text{pre}}(\mathbb{R}^n) / \sim .$$

Similarly, if we consider the equivalence relation $f \sim g$ when $\operatorname{supp}(f-g) \subset \{\mathbf{0}\}$ in $\mathcal{S}_{*,\text{reg}}(\mathbb{R}^n)$, then

$$\mathcal{S}_*(\mathbb{R}^n) \simeq \mathcal{S}_{*,\text{ord}}(\mathbb{R}^n) \simeq \mathcal{S}_{*,\text{reg}}(\mathbb{R}^n) / \sim .$$

When $\phi \in \mathcal{S}_*(\mathbb{R}^n)$ we shall denote by

$$\mathcal{F}_{*,\text{t}}(\phi) = \Pi_{\mathcal{W}_{\text{pre}},\mathcal{W}}(\mathcal{F}(\mathcal{P}f(\phi)))$$

and call it the thick Fourier transform of ϕ. We can also define a Fourier transform in $\mathcal{W}(\mathbb{R}^n)$, $\mathcal{F}_{\text{t}}^* : \mathcal{W}(\mathbb{R}^n) \longrightarrow \mathcal{S}_*(\mathbb{R}^n)$, as

$$\mathcal{F}_{\text{t}}^*\{\Phi(\mathbf{u}); \mathbf{x}\} = (2\pi)^n \mathcal{F}_{*,\text{t}}^{-1}\{\Phi(\mathbf{u}); -\mathbf{x}\}\ . \tag{15.10}$$

We immediately obtain the following important result:

The thick Fourier transform $\mathcal{F}_{*,\text{t}}$ is a topological isomorphism of $\mathcal{S}_*(\mathbb{R}^n)$ onto $\mathcal{W}(\mathbb{R}^n)$. The thick Fourier transform $\mathcal{F}_{\text{t}}^*$ is a topological isomorphism of $\mathcal{W}(\mathbb{R}^n)$ onto $\mathcal{S}_*(\mathbb{R}^n)$.

16. The space $\mathcal{W}'(\mathbb{R}^n_{\text{c}})$

In this section we shall consider the distributions of the space $\mathcal{W}'(\mathbb{R}^n)$. The first thing we would like to point out is that the functions of $\mathcal{W}(\mathbb{R}^n)$ are smooth functions in $\mathbb{R}^n$ with a special type of thick behavior at infinity; therefore the elements of $\mathcal{W}'(\mathbb{R}^n)$ are actually distributions over the space $\mathbb{R}^n_{\text{c}} = \mathbb{R}^n \cup \{\infty\}$, the one point compactification of $\mathbb{R}^n$. From now on we shall also employ the more informative notation $\mathcal{W}'(\mathbb{R}^n_{\text{c}})$ when we want to call attention to the dimension n and the simpler notation $\mathcal{W}'$ when no explicit mention of n is needed. The elements of $\mathcal{W}'$ shall be called *sl−thick distributions,* since the thick test functions of $\mathcal{W}$ have a special type of logarithmic expansion at infinity.

Several distributions defined on $\mathbb{R}^n$ admit canonical extensions to $\mathcal{W}'(\mathbb{R}^n_{\mathrm{c}})$. Indeed, if $\mathcal{W}(\mathbb{R}^n) \subset \mathcal{A}(\mathbb{R}^n)$ continuously and with dense image, where $\mathcal{A}(\mathbb{R}^n)$ is a space of test functions, then $\mathcal{A}'(\mathbb{R}^n)$ is canonically imbedded into $\mathcal{W}'(\mathbb{R}^n_{\mathrm{c}})$. The simplest case is when $\mathcal{A}(\mathbb{R}^n) = \mathcal{E}(\mathbb{R}^n)$, the space of all smooth functions in $\mathbb{R}^n$, which gives that each distribution of compact support, $f \in \mathcal{E}'(\mathbb{R}^n)$ admits a canonical extension to $\mathcal{W}'(\mathbb{R}^n_{\mathrm{c}})$, namely one whose support in $\mathbb{R}^n_{\mathrm{c}}$ is precisely the original support of f,

$$\langle f, \Phi\rangle_{\mathcal{W}'\times\mathcal{W}} = \langle f, \Phi\rangle_{\mathcal{E}'\times\mathcal{E}} \ . \tag{16.1}$$

Actually we can also take $\mathcal{A}(\mathbb{R}^n) = \mathcal{K}(\mathbb{R}^n)$, so that any distribution $f \in \mathcal{K}'(\mathbb{R}^n)$ admits a canonical extension to $\mathcal{W}'(\mathbb{R}^n_{\mathrm{c}})$, given by the Cesàro evaluation,

$$\langle f, \Phi\rangle_{\mathcal{W}'\times\mathcal{W}} = \langle f, \Phi\rangle \quad (\mathrm{C}) \tag{16.2}$$

since $\langle f, \Phi\rangle$ (C) exists whenever $\Phi \in \mathcal{K}(\mathbb{R}^n)$ [**?**] and $\mathcal{W}(\mathbb{R}^n) \subset \mathcal{K}(\mathbb{R}^n)$. We shall employ the same notation for both the distribution of $\mathcal{K}'(\mathbb{R}^n)$ and its canonical extension to $\mathcal{W}'(\mathbb{R}^n_{\mathrm{c}})$. On the other hand, $\mathcal{W}(\mathbb{R}^n)$ is not contained in $\mathcal{S}(\mathbb{R}^n)$, and this means that tempered distributions do not have *canonical* extensions in $\mathcal{W}'(\mathbb{R}^n_{\mathrm{c}})$. In fact, it is not hard to see that actually all elements of $\mathcal{S}'(\mathbb{R}^n)$ have *many* extensions to $\mathcal{W}'(\mathbb{R}^n_{\mathrm{c}})$, but it is not possible to construct a continuous extension procedure.

Another important class of $sl-$thick distributions are the thick deltas at infinity.

If $G \in \mathcal{D}'_q$ then we define $G(\mathbf{v})\,\delta_\infty^{[q]}$, the thick delta at infinity of order q as

$$\left\langle G(\mathbf{v})\,\delta_\infty^{[q]}, \Phi\right\rangle_{\mathcal{W}'\times\mathcal{W}} = \frac{1}{C}\,\langle G, A_q\rangle_{\mathcal{D}'_q\times\mathcal{D}_q} \ , \tag{16.3}$$

if $\Phi \in \mathcal{W}$ has the asymptotic expansion (15.8). Similarly, if $H \in \mathcal{P}'_q = \mathcal{P}_q$ then we define $H(\mathbf{v})\,\delta_{\ln,\infty}^{[q]}$ the thick logarithmic delta of order q at infinity as

$$\left\langle H(\mathbf{v})\,\delta_{\ln,\infty}^{[q]}, \Phi\right\rangle_{\mathcal{W}'\times\mathcal{W}} = \frac{1}{C}\,\langle H, P_q\rangle_{\mathcal{P}'_q\times\mathcal{P}_q} \ . \tag{16.4}$$

Sometimes one may construct extensions of a tempered distribution g by considering a finite part at infinity[5], a construction we shall now denote as $\mathcal{P}f_{\mathcal{W}}(g)$, or later simply as $\mathcal{P}f(g)$ if there is no danger of confusion. Consider for example the distribution $\mathcal{P}f\left(s^\lambda\right)$, $s = |\mathbf{u}|$, of $\mathcal{S}'(\mathbb{R}^n)$: this tempered distribution yields the $sl-$thick distribution $\mathcal{P}f_{\mathcal{W}}\left(s^\lambda\right)$ obtained from the generally divergent integral $\int_{\mathbb{R}^n} s^\lambda \Phi(\mathbf{u})\ d\mathbf{u}$,

[5] Clearly the finite part at infinity does *not exist* for all $g \in \mathcal{S}'(\mathbb{R}^n)$.

$\Phi \in \mathcal{W}$, by taking the radial finite part at $\mathbf{0}$, or at ∞, or at both. Using the ideas of the Example **??** we can see the structure of $\mathcal{P}f_{\mathcal{W}}\left(s^{\lambda}\right)$.

The parametric $sl-$thick distribution $\mathcal{P}f_{\mathcal{W}}\left(s^{\lambda}\right)$ is a meromorphic function of λ, analytic in the region $(\mathbb{C}\setminus\mathbb{Z})\cup\{0,2,4,\ldots\}$, with simple poles at $\lambda=m$, $m\in\{-n-1,-n-3,-n-5,\ldots\}\cup\{-1,-2,\ldots,1-n\}\cup\{1,3,5,\ldots\}$, the residues at these poles being

$$\operatorname{Res}_{\lambda=m}\mathcal{P}f_{\mathcal{W}}\left(s^{\lambda}\right)=-C\delta_{\infty}^{[-n-m]}\left(\mathbf{u}\right)\,, \tag{16.5}$$

and double poles at $\lambda=m$, $m=-n-2q\in\{-n,-n-2,-n-4,\ldots\}$ with singular part

$$\frac{C\delta_{\ln,\infty}^{[2q]}\left(\mathbf{u}\right)}{\left(\lambda-m\right)^{2}}+\frac{c_{q,n}\nabla^{2q}\delta\left(\mathbf{u}\right)}{(2q)!\left(\lambda-m\right)}\,. \tag{16.6}$$

The finite part of $\mathcal{P}f_{\mathcal{W}}\left(s^{\lambda}\right)$ at any pole $\lambda=m$ is precisely $\mathcal{P}f_{\mathcal{W}}\left(s^{m}\right)$.

Many of the constructions that we have discussed can also be done in the space $\mathcal{W}'_{\mathrm{pre}}$. Notice, however, that several distributions of $\mathcal{W}'_{\mathrm{pre}}$ could vanish in $\mathcal{W}$ so that their projection to $\mathcal{W}'$ could be zero. For instance, the plain thick delta $\delta_{\infty}^{[0]}$ is not zero in $\mathcal{W}'_{\mathrm{pre}}$ but it is zero in $\mathcal{W}'$. If one considers the finite part $\mathcal{P}f_{\mathcal{W}_{\mathrm{pre}}}\left(s^{\lambda}\right)$ then it would not be analytic at $\lambda=0,2,4,\ldots$; for instance, it has a simple pole at $\lambda=0$ with residue $-C\delta_{\infty}^{[0]}$.

One of the consequences of the fact that $\mathcal{W}'$ is a space over the compact space $\mathbb{R}_{\mathrm{c}}^{n}$ is that several of the usual operations on $sl-$thick distributions could have additional terms at infinity. This is the case for the linear changes of variables and for the multiplications by polynomials. Curiously, however, derivatives in $\mathcal{W}'$ can be defined in the standard way by duality, since the derivative operators send $\mathcal{W}$ to $\mathcal{W}$,

$$\left\langle\nabla_{j}\left(F\right),\Phi\right\rangle=-\left\langle F,\nabla_{j}\left(\Phi\right)\right\rangle\,,\quad F\in\mathcal{W}',\Phi\in\mathcal{W}\,. \tag{16.7}$$

16.1. **Linear changes of variables in $\mathcal{W}'$.** Let A be a non-singular $n\times n$ matrix. If $\Phi\in\mathcal{W}$ then the function Φ_{A} given by $\Phi_{A}\left(\mathbf{u}\right)=\Phi\left(A\mathbf{u}\right)$ does not belong to $\mathcal{W}$, in general, but it belongs to $\mathcal{W}_{\mathrm{pre}}$. Therefore we define the function of $\mathcal{W}$ obtained by the change of variables, $\tau_{A}^{\mathcal{W}}\left(\Phi\right)$ as

$$\tau_{A}^{\mathcal{W}}\left(\Phi\right)=\Pi_{\mathcal{W}_{\mathrm{pre}},\mathcal{W}}\left(\Phi_{A}\right)\,. \tag{16.8}$$

We can then define the change of variables in $sl-$thick distributions by duality.

Let A be a non-singular $n\times n$ matrix. If $F\in\mathcal{W}'$ then the distribution $\tau_A^{\mathcal{W}'}(F)$, the $sl-$thick version of $F(A\mathbf{u})$, is defined as

$$\left\langle \tau_A^{\mathcal{W}'}(F),\Phi\right\rangle = \frac{1}{\left|\det(A)\right|}\left\langle F,\tau_{A^{-1}}^{\mathcal{W}}(\Phi)\right\rangle . \tag{16.9}$$

A simple example is provided by the delta function at the origin, $\delta(\mathbf{x})$, and the change $A=tI$ for $t\neq 0$. We have $\delta(t\mathbf{x})=|t|^{-n}\delta(\mathbf{x})$, of course, but

$$\tau_{tI}^{\mathcal{W}'}(\delta)(\mathbf{x}) = |t|^{-n}\delta(\mathbf{x}) - |t|^{-n}\ln t\,\delta_{\infty,\ln}^{[0]}(\mathbf{x}) . \tag{16.10}$$

Interestingly, if A is an orthogonal matrix, in particular if it is a rotation, and $F\in\mathcal{K}'(\mathbb{R}^n)$ then the canonical extension of $F(A\mathbf{x})$ is precisely $\tau_A^{\mathcal{W}'}(F)(\mathbf{x})$. Therefore we give the following definitions.

A $sl-$thick distribution $F\in\mathcal{W}'$ is called *radial* if $\tau_A^{\mathcal{W}'}(F)=F$ for all orthogonal matrices A. We say that F is *homogeneous* of order λ if

$$\tau_{tI}^{\mathcal{W}'}(F)(\mathbf{x}) = t^{\lambda}F(\mathbf{x}) , \qquad t>0 . \tag{16.11}$$

Notice that a distribution $F\in\mathcal{K}'(\mathbb{R}^n)$ is radial if and only if its canonical extension is, but (16.10) shows that a corresponding result does not hold for homogeneous distributions. On the other hand, a distribution of the form $G(\mathbf{v})\,\delta_{\infty}^{[q]}$ is radial if and only if G is constant, where we observe that the plain thick delta at infinity $\delta_{\infty}^{[q]}$ is a non zero $sl-$thick distribution for $q\neq 0,2,4,\ldots$ and $q\neq -n,-n-2,-n-4,\ldots$. Furthermore, since the plain thick logarithmic deltas at infinity $\delta_{\ln,\infty}^{[1]},\delta_{\ln,\infty}^{[3]},\delta_{\ln,\infty}^{[5]},\ldots$ vanish, the distributions $c\delta_{\ln,\infty}^{[q]}$ for $q=0,2,4,\ldots$ and c constant are the radial distributions of the form $G(\mathbf{v})\,\delta_{\ln,\infty}^{[q]}$.

It is useful to know the $sl-$thick radial homogeneous distributions.

Let $\lambda\in\mathbb{C}$. Then the set of $sl-$thick radial homogeneous distributions of order λ form a vector space of dimension 1, generated by the distribution

$$\mathcal{P}f_{\mathcal{W}}\left(s^{\lambda}\right) \text{ for } \lambda\in(\mathbb{C}\setminus\mathbb{Z})\cup\{0,2,4,\ldots\} , \tag{16.12}$$

$$\begin{gathered}\delta_{\infty}^{[-n-m]} \quad \text{for } \lambda=m,\\ m\in\{-n-1,-n-3,-n-5,\ldots\}\cup\{1,2,\ldots,n-1\}\cup\{1,3,5,\ldots\} ,\end{gathered} \tag{16.13}$$

$$\delta_{\ln,\infty}^{[-n-m]} \quad \text{for } \lambda=m, \quad m\in\{-n,-n-2,-n-4,\ldots\} . \tag{16.14}$$

16.1.1. *Multiplication by polynomials in $\mathcal{W}'$.* In general if $\Phi \in \mathcal{W}$ then $u_j\Phi(\mathbf{u})$ is in $\mathcal{W}_{\text{pre}}$ but it does not belong to $\mathcal{W}$. Therefore we define the multiplication operator

$$M_{u_j}^{\mathcal{W}} : \mathcal{W} \longrightarrow \mathcal{W}\,, \quad M_{u_j}(\Phi) = \Pi_{\mathcal{W}_{\text{pre}},\mathcal{W}}(u_j\Phi)\ , \tag{16.15}$$

and by duality the operator $M_{u_j}^{\mathcal{W}'} : \mathcal{W}' \longrightarrow \mathcal{W}'$ as

$$\left\langle M_{u_j}^{\mathcal{W}'}(F)\,, \Phi\right\rangle = \left\langle F, M_{u_j}^{\mathcal{W}}(\Phi)\right\rangle\ . \tag{16.16}$$

The multiplication operators $M_p^{\mathcal{W}}$ and $M_p^{\mathcal{W}'}$, where p is a polynomial, can be defined in a similar way.

17. The Fourier transform of thick distributions

The Fourier transform of thick tempered distributions $f \in \mathcal{S}'_*(\mathbb{R}^n)\,,$ $\mathcal{F}_*(f) \in \mathcal{W}'(\mathbb{R}^n_{\text{c}})$ can now be defined in the usual way,

$$\langle \mathcal{F}_*\{f(\mathbf{x})\,;\mathbf{u}\}\,, \Phi(\mathbf{u})\rangle = \langle f(\mathbf{x})\,, \mathcal{F}^*_{\text{t}}\{\Phi(\mathbf{u})\,;\mathbf{x}\}\rangle\ , \quad \Phi \in \mathcal{W}(\mathbb{R}^n_{\text{c}})\ . \tag{17.1}$$

Similarly, the Fourier transform of distributions $G \in \mathcal{W}'(\mathbb{R}^n_{\text{c}})\,, \mathcal{F}^*(G) \in \mathcal{S}'_*(\mathbb{R}^n)$ is defined as

$$\langle \mathcal{F}^*\{G(\mathbf{u})\,;\mathbf{x}\}\,, \phi(\mathbf{x})\rangle = \langle G(\mathbf{u})\,, \mathcal{F}_{*,\text{t}}\{\phi(\mathbf{x})\,;\mathbf{u}\}\rangle\ , \quad \phi \in \mathcal{S}_*(\mathbb{R}^n)\ . \tag{17.2}$$

Theorem 2. *The thick Fourier transform $\mathcal{F}_*$ is a topological isomorphism of $\mathcal{S}'_*(\mathbb{R}^n)$ onto $\mathcal{W}'(\mathbb{R}^n_c)$. The thick Fourier transform $\mathcal{F}^*$ is a topological isomorphism of $\mathcal{W}'(\mathbb{R}^n_c)$ onto $\mathcal{S}'_*(\mathbb{R}^n)$.*

The properties of the Fourier transform of thick distributions are similar to those of the transform in $\mathcal{S}'(\mathbb{R}^n)$ but one must remember that the operations in $\mathcal{W}'(\mathbb{R}^n_{\text{c}})$ may or may not be the standard ones. We have,

$$\mathcal{F}_*\{f(A\mathbf{x})\,;\mathbf{u}\} = \frac{1}{|\det A|}\tau_{A^{-1}}^{\mathcal{W}'}(\mathcal{F}_*\{f(\mathbf{x})\,;\mathbf{u}\})\ , \tag{17.3}$$

for A a non-singular matrix, and in particular, if $t \neq 0$

$$\mathcal{F}_*\{f(t\mathbf{x})\,;\mathbf{u}\} = t^{-n}\tau_{t^{-n}I}^{\mathcal{W}'}(\mathcal{F}_*\{f(\mathbf{x})\,;\mathbf{u}\})\ . \tag{17.4}$$

It follows that $\mathcal{F}_*$ and $\mathcal{F}^*$ send radial thick distributions to radial thick distributions, and homogeneous distributions of degree λ to homogeneous distributions of degree $-n-\lambda$. We also have the usual interchange of multiplication and differentiation,

$$\mathcal{F}_*\{x_jf(\mathbf{x})\,;\mathbf{u}\} = -i\nabla_{u_j}\mathcal{F}_*\{f(\mathbf{x})\,;\mathbf{u}\}\ , \tag{17.5}$$

$$\mathcal{F}_*\left\{\nabla_{x_j}f(\mathbf{x})\,;\mathbf{u}\right\} = -iM_{u_j}^{\mathcal{W}'}\mathcal{F}_*\{f(\mathbf{x})\,;\mathbf{u}\}\ , \tag{17.6}$$

where the modified multiplication operator $M_{u_j}^{\mathcal{W}'}$ is given by (16.16). The formulas for the inverse transforms are a variant of the usual ones,

$$(\mathcal{F}^*)^{-1}\left\{f(\mathbf{x});\mathbf{u}\right\} = \frac{1}{(2\pi)^n}\mathcal{F}_*\left\{f(\mathbf{x});-\mathbf{u}\right\}\ , \tag{17.7}$$

$$(\mathcal{F}_*)^{-1}\left\{F(\mathbf{u});\mathbf{x}\right\} = \frac{1}{(2\pi)^n}\mathcal{F}^*\left\{F(\mathbf{u});-\mathbf{x}\right\}\ . \tag{17.8}$$

Another important property is that the Fourier transforms $\mathcal{F}_*$ or $\mathcal{F}^*$ of extensions of distributions of $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'_*(\mathbb{R}^n)$ or $\mathcal{W}'(\mathbb{R}^n_{\mathrm{c}})$ are extensions of the Fourier transform, that is

$$\Pi_{\mathcal{W}',\mathcal{S}'}\mathcal{F}_*\left\{f(\mathbf{x});\mathbf{u}\right\} = \mathcal{F}\left\{\Pi_{\mathcal{S}'_*,\mathcal{S}'}f(\mathbf{x});\mathbf{u}\right\}\ , \tag{17.9}$$

$$\Pi_{\mathcal{S}'_*,\mathcal{S}'}\mathcal{F}^*\left\{F(\mathbf{u});\mathbf{x}\right\} = \mathcal{F}\left\{\Pi_{\mathcal{W}',\mathcal{S}'}F(\mathbf{u});\mathbf{x}\right\}\ . \tag{17.10}$$

We are now ready to give the Fourier transform of several thick distributions.

Example 3. *Let us compute the Fourier transform $\mathcal{F}_*\left\{\delta_*^{[0]}(\mathbf{x});\mathbf{u}\right\}$ of the plain thick delta function. Since $\delta_*^{[0]}(\mathbf{x})$ is radial and homogenous of degree $-n$, its transform is radial and homogeneous of degree 0. Also, the projection of $\delta_*^{[0]}(\mathbf{x})$ onto $\mathcal{S}'$ is the standard delta function $\delta(\mathbf{x})$, whose transform is the constant function 1. It follows that the only radial, homogeneous of degree 0 $sl-$thick distribution whose projection to $\mathcal{S}'$ is the constant distribution 1 is precisely $\mathcal{P}f_{\mathcal{W}}(1)$. Hence*

$$\mathcal{F}_*\left\{\delta_*^{[0]}(\mathbf{x});\mathbf{u}\right\} = \mathcal{P}f_{\mathcal{W}}(1)\ . \tag{17.11}$$

A similar argument yields

$$\mathcal{F}_*\left\{\delta_*^{[2m]}(\mathbf{x});\mathbf{u}\right\} = \frac{(-1)^m\,\Gamma(m+1/2)\,\Gamma(n/2)}{\Gamma(m+n/2)\,\Gamma(1/2)\,(2m)!}\mathcal{P}f_{\mathcal{W}}\left(s^{2m}\right)\ , \tag{17.12}$$

and by inversion,

$$\mathcal{F}^*\left\{\mathcal{P}f_{\mathcal{W}}\left(s^{2m}\right);\mathbf{x}\right\} = \frac{(-1)^m\,\Gamma(m+n/2)\,\Gamma(1/2)\,(2m)!}{(2\pi)^n\,\Gamma(m+1/2)\,\Gamma(n/2)}\delta_*^{[2m]}(\mathbf{x})\ . \tag{17.13}$$

Example 4. *The ensuing formulas, reminiscent of (14.2), also follow along the same lines,*

$$\mathcal{F}_*\left\{\mathcal{P}f\left(r^\lambda\right);\mathbf{u}\right\} = \frac{\pi^{n/2}2^{\lambda+n}\Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\mathcal{P}f_{\mathcal{W}}\left(s^{-\lambda-n}\right)\ , \tag{17.14}$$

$$\mathcal{F}^*\left\{\mathcal{P}f_{\mathcal{W}}\left(s^\lambda\right);\mathbf{x}\right\} = \frac{\pi^{n/2}2^{\lambda+n}\Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\mathcal{P}f\left(r^{-\lambda-n}\right)\ , \tag{17.15}$$

whenever $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. *Interestingly,* $\mathcal{P}f_{\mathcal{W}}\left(s^{\lambda}\right)$ *is analytic at* $0, 2, 4, \ldots$ *so that (17.13) can be recovered by taking the limit as* $\lambda \to 2m$ *in the right side of (17.15).*

Example 5. *Formulas (17.14) and (17.15) are equalities of meromorphic functions and thus by considering the residues, finite parts, or singular parts at the poles of both sides we obtain the Fourier transform of several thick distributions. Let start with* $m \in \{-n-1, -n-3, -n-5, \ldots\} \cup \{-1, -2, \ldots, 1-n\} \cup \{1, 3, 5, \ldots\}$, *so that* $\lambda = m$ *is a simple pole of the function in (17.15). From the Lemma* **??** *the residue of the left side is* $\mathcal{F}^{*}\left\{-C\delta_{\infty}^{[-n-m]}(\mathbf{u});\mathbf{x}\right\}$, *while if we recall* [**?**, (4.13)] *that* $\mathrm{Res}_{\mu=k}\,\mathcal{P}f\left(r^{\mu}\right) = C\delta_{*}^{[-k-n]}(\mathbf{x})$, *we obtain the residue of the right side as* $Cg(m)\,\delta_{*}^{[m]}(\mathbf{x})$ *where*

$$g(\lambda) = \frac{\pi^{n/2} 2^{\lambda+n} \Gamma\left(\frac{\lambda+n}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}\,. \tag{17.16}$$

Therefore

$$\mathcal{F}^{*}\left\{\delta_{\infty}^{[-n-m]}(\mathbf{u});\mathbf{x}\right\} = -g(m)\,\delta_{*}^{[m]}(\mathbf{x})\,, \tag{17.17}$$

and by inversion,

$$\mathcal{F}_{*}\left\{\delta_{*}^{[m]}(\mathbf{x});\mathbf{u}\right\} = -g(-n-m)\,\delta_{\infty}^{[-n-m]}(\mathbf{u})\,, \tag{17.18}$$

since $g(m)\,g(-n-m) = (2\pi)^{n}$. *Similarly, consideration of the finite parts of both sides of (17.15) yields*

$$\mathcal{F}^{*}\left\{\mathcal{P}f_{\mathcal{W}}(s^{m});\mathbf{x}\right\} = g(m)\left\{\mathcal{P}f\left(r^{-m-n}\right) + \chi_{m}\delta_{*}^{[m]}(\mathbf{x})\right\}\,, \tag{17.19}$$

and

$$\mathcal{F}_{*}\left\{\mathcal{P}f\left(r^{-m-n}\right);\mathbf{u}\right\} = g(-n-m)\left\{\mathcal{P}f_{\mathcal{W}}(s^{m}) + \chi_{-m-n}\delta_{\infty}^{[-n-m]}(\mathbf{u})\right\}\,, \tag{17.20}$$

where

$$\chi_{m} = \chi_{-m-n} = \frac{C}{2}\left(2\ln 2 + \psi\left(\frac{m+n}{2}\right) + \psi\left(\frac{-m}{2}\right)\right)\,. \tag{17.21}$$

Studying the coefficients of order -2 *at the poles of order* 2, $m = -n - 2q$ *for* $q \in \mathbb{N}$ *gives*

$$\mathcal{F}^{*}\left\{\delta_{\ln,\infty}^{[2q]}(\mathbf{u});\mathbf{x}\right\} = \frac{(-1)^{n}\,2^{1-2q}\pi^{n/2}}{q!\Gamma\left(\frac{n+2q}{2}\right)}\delta_{*}^{[-n-2q]}(\mathbf{x})\,, \tag{17.22}$$

and

$$\mathcal{F}_{*}\left\{\delta_{*}^{[-n-2q]}(\mathbf{x});\mathbf{u}\right\} = (-1)^{n}\,2^{n+2q-1}\pi^{n/2}q!\Gamma\left(\frac{n+2q}{2}\right)\delta_{\ln,\infty}^{[2q]}(\mathbf{u})\,. \tag{17.23}$$

We have considered the transform of plain thick deltas so far, now we compute the Fourier transform of general thick deltas.

Example 6. *Let* $\phi \in \mathcal{S}_*$, *with expansion* $\sum_{j=m}^{\infty} a_j r^j$ *at zero and let* $\Phi = \mathcal{F}_{*,t}(\phi) \in \mathcal{W}$, *with expansion* $\sum_{q=0}^{n-m} (A_q(\mathbf{v}) + P_q(\mathbf{v}) \ln s) s^q + \sum_{q=1}^{\infty} A_{-q}(\mathbf{v}) s^{-q}$ *at infinity. Then* $A_q = \mathfrak{K}_q(a_{-n-q})$, *therefore if* $G \in \mathcal{D}_q^{'}$ *then*

$$\left\langle G\delta_{\infty}^{[q]}, \Phi\right\rangle = \frac{1}{C}\left\langle G, A_q\right\rangle = \frac{1}{C}\left\langle G, \mathfrak{K}_q(a_{-n-q})\right\rangle$$
$$= \frac{1}{C}\left\langle \mathfrak{K}_q(G), a_{-n-q}\right\rangle = \left\langle \mathfrak{K}_q(G)\,\delta_*^{[-n-q]}, \phi\right\rangle\,,$$

or

$$\mathcal{F}^*\left\{G(\mathbf{v})\,\delta_{\infty}^{[q]}(\mathbf{u});\mathbf{x}\right\} = \mathfrak{K}_q\left\{G(\mathbf{v});\mathbf{w}\right\}\delta_*^{[-n-q]}(\mathbf{x})\,, \tag{17.24}$$

giving the transform of all *thick deltas at infinity* $G\delta_{\infty}^{[q]}$, *for arbitrary* $q \in \mathbb{Z}$, *since* G *needs to be* $\mathcal{D}_q^{'}$. *Similarly, for* $q \in \mathbb{N}$

$$\mathcal{F}^*\left\{H(\mathbf{v})\,\delta_{\ln,\infty}^{[q]}(\mathbf{u});\mathbf{x}\right\} = \mathfrak{L}_q\left\{H(\mathbf{v});\mathbf{w}\right\}\delta_*^{[-n-q]}(\mathbf{x})\,. \tag{17.25}$$

Example 7. *We now consider the transform of the general thick deltas* $f(\mathbf{w})\,\delta_*^{[m]}(\mathbf{x})$. *Let* $m = -n-q$. *Different formulas arise depending on* m *and* q. *If* $1-n \le m, q \le -1$ *then* $\mathcal{D}_q^{'} = \mathcal{D}_m^{'} = \mathcal{D}^{'}$ *so that inversion of (17.24), remembering (14.19), gives*

$$\mathcal{F}_*\left\{f(\mathbf{w})\,\delta_*^{[m]}(\mathbf{x});\mathbf{u}\right\} = \mathfrak{K}_m\left\{f(\mathbf{w});\mathbf{v}\right\}\delta_{\infty}^{[-n-m]}(\mathbf{u})\,. \tag{17.26}$$

If $m \ge 0$, *that is* $q \le -n$, *we decompose* $f \in \mathcal{D}'(\mathbb{S})$ *as* $f = p_m + f_m$ *where* $f_m \in \mathcal{D}_q^{'} = \mathcal{D}_m^{'}$ *and* $p_m \in \mathcal{P}_m$. *We now notice that* $\mathcal{F}_*\left(p\delta_*^{[m]}\right)$ *is the finite part regularization* $\mathcal{P}f_{\mathcal{W}}(P_m(\mathbf{u}))$ *of a homogeneous polynomial of degree* m, *namely* $P_m = \mathcal{F}\left(\Pi_{\mathcal{S}'_*,\mathcal{S}'}\left(f\delta_*^{[m]}\right)\right)$, *obtaining*

$$\mathcal{F}_*\left\{f(\mathbf{w})\,\delta_*^{[m]}(\mathbf{x});\mathbf{u}\right\} = \mathcal{P}f_{\mathcal{W}}(P_m(\mathbf{u})) + \mathfrak{K}_m\left\{f_m(\mathbf{w});\mathbf{v}\right\}\delta_{\infty}^{[-n-m]}(\mathbf{u})\,. \tag{17.27}$$

In particular, when $m = 0$, *since* $\mathcal{F}_*\left(\delta_*^{[0]}\right) = \mathcal{P}f_{\mathcal{W}}(1)$, *we obtain*

$$\mathcal{F}_*\left\{f(\mathbf{w})\,\delta_*^{[0]}(\mathbf{x});\mathbf{u}\right\} = M\mathcal{P}f_{\mathcal{W}}(1) + \mathfrak{K}_0\left\{f(\mathbf{w}) - M;\mathbf{v}\right\}\delta_{\infty}^{[-n]}(\mathbf{u})\,, \tag{17.28}$$

where M *is the constant* $M = (1/C)\left\langle f(\mathbf{w}), 1\right\rangle$.

Finally, if $m \leq -n$, *i.e.* $q \geq 0$, *the decomposition* $f = p_m + f_m$ *where* $f_m \in \mathcal{D}'_q = \mathcal{D}'_m$ *and* $p_m \in \mathcal{P}_{-n-m} = \mathcal{P}_q$ *yields*

(17.29)

$$\mathcal{F}_* \left\{ f(\mathbf{w})\, \delta_*^{[m]}(\mathbf{x}) ; \mathbf{u} \right\} = (2\pi)^n\, \mathfrak{L}_{-n-m}^{-1} \left\{ p_m(\mathbf{w}) ; -\mathbf{v} \right\} \delta_{\ln,\infty}^{[-n-m]}(\mathbf{u}) + \mathfrak{K}_m \left\{ f_m(\mathbf{w}) ; \mathbf{v} \right\} \delta_{\infty}^{[-n-m]}(\mathbf{u}) \; .$$

Department of Mathematics, Louisana State University, Baton Rouge, LA 70803, USA.

E-mail address: `restrada@math.lsu.edu`