Smoothing and Strichartz estimates for degenerate Schrödinger-type equations

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Introduction

Smoothing and Strichartz estimates are key tools in order to recover existence results for the IVP (initial value problem) associated with (tipically) dispersive equations as well as to determine regularity properties of the solution.

Smoothing estimates describe a gain of smoothness of the solution with respect to the smoothness of the initial data and/or of the inhomogeneous term.

Strichartz estimates describe a gain of integrability of the solution with respect to the integrability of the initial data and/or of the inhomogeneous term.

Introduction

In this talk we shall focus on the validity of such estimates for time-degenerate Schrödinger operators of the form

$$\mathcal{L}_{b,a} := \partial_t - iB(t)a(D), \tag{1}$$

where a is a Fourier multiplier of order m and $B \in C(\mathbb{R})$ is such that $B(0) = 0$. Additionally, B is allowed to vanish either at finitely or at infinitely many points. Since $B(t) = b'(t)$ for $b(t) = \int_0^t B(s)ds$, we shall simply write

$$\mathcal{L}_{b,a} := \partial_t - ib'(t)a(D), \tag{2}$$

so that the solution at time t of the HIVP (homogeneous IVP)

$$\begin{cases} \mathcal{L}_{b,a}u(t,x) = 0, \\ u(s,x) = u_s(x) \end{cases} \tag{3}$$

can be written in a more compact way as

$$u(t,x) = e^{i(b(t)-b(s))a(D)}u_s(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi+i(b(t)-b(s))a(\xi)}\widehat{u_s}(\xi)d\xi.$$

Outline

1. Motivation
2. Comparison principles
3. Global weighted smoothing estimates for $\mathcal{L}_{b,a}$ via comparison principles
4. Weighted Strichartz estimates for $\mathcal{L}_{b,\Delta_x}$
5. Local well-posedness of a semilinear IVP associated with $\mathcal{L}_{b,\Delta_x}$

Motivation

Smoothing and Strichartz estimates in the **constant coefficients** case have been intensively studied by several authors (Kato, Sjölin, Kenig-Ponce-Vega, Caustantin-Saut, Ben-Artzi and Klainerman, Chihara, Kato-Yajima, Linares-Ponce, Sugimoto, Walther, Ginibre-Velo, Keel-Tao and may others).

Smoothing and Strichartz estimates in the **variable coefficients** case have been investigated mainly in the case of **nondegenerate space-variable coefficients** (Doi, Kenig-Ponce-Vega-Rolvung, Staffilani-Tataru, Marzuola, Metcalfe-Tataru).

Global homogeneous smoothing estimates for **nondegenerate time-variable coefficients** Schrödinger equations have been considered by Sugimoto-Ruzhansky, while in the **degenerate** case local smoothing results have been proved by F.-Staffilani (the local posedness of the class considered in the latter case was also studied by Cicognani-Reissig).

Our results complete the picture in the degenerate time-variable coefficients case, namely, we obtain **global homogeneous** smoothing estimates and **Strichartz estimates** in this setting.

Motivation

Operators in our class:

Example 1

$$\mathcal{L}_{b,\Delta} = \mathcal{L}_{\frac{t^{\alpha+1}}{\alpha+1},\Delta} = \partial_t + it^{\alpha}\Delta, \quad \alpha \geq 1;$$

Example 2

$$\mathcal{L}_{b,\Delta} = \mathcal{L}_{e^t - t - 1,\Delta} = \partial_t + i(e^t - 1)\Delta;$$

Example 3

$$\mathcal{L}_{b,\Delta} = \mathcal{L}_{\cos(t),\Delta} := \partial_t u + i\sin(t)\Delta.$$

Comparison principles

Following the work of Sugimoto and Ruzhansky we prove the global smoothing effect for $\mathcal{L}_{b,a}$ by using comparison principles.

Question: Given two operators $P_a(t, x, D_t, D_x)$ and $P_b(t, x, D_t, D_x)$ depending on two functions (symbols) a and b respectively, is it possible to compare (in a suitable sense) the solutions of the HIVP (homogeneous IVP) for P_a and P_b if a and b are comparable (in a suitable sense)?

This is essentially what the comparison principles we refer to do, that is, they translate a relation between a and b in a relation between the solutions of the HIVP for P_a and P_b.

Comparison principles

Example (Sugimoto-Ruzhansky) Let $n = 1$ and $a, \tilde{a} \in C^1(\mathbb{R})$ be real valued and strictly monotone on the support of a measurable function χ, and let $\sigma, \tau \in C^0(\mathbb{R})$. If $\forall \xi \in \mathrm{supp}\chi$ we have

$$\frac{|\sigma(\xi)|}{|a'(\xi)|^{1/2}} \leq C \frac{|\tau(\xi)|}{|\tilde{a}'(\xi)|^{1/2}},$$

then

$$\|\chi(D_x)\sigma(D_x)e^{ita(D)}\varphi\|_{L^2(\mathbb{R}^2_{t,x})} \leq C\|\chi(D_x)\tau(D_x)e^{it\tilde{a}(D)}\varphi\|_{L^2(\mathbb{R}^2_{t,x})}$$

for all $\varphi = \varphi(x)$ smooth enough.

A comparison principle in the radially symmetric case

Even though we proved comparison principles applicable to our setting in the non radially symmetric case, we sate below the one holding in the radially symmetric case since it is the one used to prove global smoothing estimates for $\mathcal{L}_{b,a(|D_x|)}$ by comparison with $\mathcal{L}_{t,\tilde{a}(|D_x|)}$.
Recall that

$$\mathcal{L}_{a,b} = \partial_t - ib'(t)a(D),$$

where $b \in C^1(\mathbb{R})$ is such that $b'(0) = 0$ and b' is possibly vanishing either at finitely or at infinetly many points.
We state below a comparison principle for $\mathcal{L}_{b,a}$ and $\mathcal{L}_{f,\tilde{a}}$, where, in particluar, f will be such that $f(0) = 0$, f is strictly monotone, and $\lim_{t\to\infty} f(t) = \infty$ (for instance $f(t) = t$).

Theorem (F.-Ruzhansky)

Let $a, \tilde{a} \in C^1(\mathbb{R}_+)$ be real-valued strictly monotone on the support of a measurable function χ on $\mathbb{R}_+$, and let $b, f \in C^1(\mathbb{R})$ vanish at 0, with f satisfying the conditions above. Let also $\sigma, \tau \in C^0(\mathbb{R}_+)$ be such that

$$\frac{|\sigma(\rho)|}{|\frac{d}{d\rho}a(\rho)|^{1/2}} \le A\frac{|\tau(\xi)|}{|\frac{d}{d\rho}\tilde{a}(\rho)|^{1/2}}, \tag{4}$$

where $\frac{d}{d\rho}a(\rho) \neq 0$ and $\frac{d}{d\rho}\tilde{a}(\rho) \neq 0$ for all $\rho \in \operatorname{supp}\chi$. Then

(i) If b has a finite number of critical points then

$$\|\chi(|D_x|)|b'(t)|^{1/2}\sigma(|D_x|))e^{ib(t)a(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \tag{5}$$

$$\le CA\|\chi(|D_x|)|f'(t)|^{1/2}\tau(|D_x|))e^{if(t)\tilde{a}(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},$$

where C depends on the nuber of critical points.

Moreover, for any measurable function ω on $\mathbb{R}^n$, we have that (5) holds with $\omega(x)\chi(|D_x|)$ in place of $\chi(|D_x|)$ in both the LHS and the RHS.

Finally, if (4) holds with equality and if b satisfies the same conditions as f, then we get equalities in the previous relations.

(ii) If b is such that the set $\{t \in \mathbb{R}; b'(t) = 0\}$ is countable, then there exists a function $c \in C(\mathbb{R})$ such that

$$\|\chi(|D_x|)|c(t)b'(t)|^{1/2}\sigma(|D_x|))e^{ib(t)a(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)}$$

$$\leq C'A\|\chi(|D_x|)|f'(t)|^{1/2}\tau(|D_x|))e^{if(t)\tilde{a}(|D_x|)}\varphi(x)\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)},$$

where C' depends on b and c.
Moreover, given any measurable function ω on $\mathbb{R}^n$ the previous inequality holds with $\omega(x)\chi(|D_x|)$ in place of $\chi(|D_x|)$ in both the LHS and the RHS.

Conversely, if $\chi \in C^0(\mathbb{R}_+)$, $\omega \neq 0$ on a set of $\mathbb{R}^n$ with positive measure, b satisfies the same conditions as f and (5) (for some $x, \tilde{x}$) or (5) with some measurable function ω is satisfied for any φ, and the norms are finite, then inequality (4) holds.

Weighted global smoothing estimates 1

Standard global homogeneous smoothing estimates:

- Let $n = 1$ and $m > 0$, then

$$\sup_{x\in\mathbb{R}} \||D_x|^{\frac{m-1}{2}} e^{it|D_x|^m}\varphi\|_{L^2(\mathbb{R}_t)} \leq C\|\varphi\|_{L^2(\mathbb{R}_x)}, \quad \text{(Kenig et al.)}$$

- Let $n = 2$ and $m > 0$, then

$$\sup_{x_1\in\mathbb{R}} \||D_{x_2}|^{\frac{m-1}{2}} e^{itD_{x_1}|D_{x_2}|^{m-1}}\varphi\|_{L^2(\mathbb{R}_t\times\mathbb{R}_{x_2})} \leq C\|\varphi\|_{L^2(\mathbb{R}^2)}, \text{(Linares-Ponce)}$$

Weighted global smoothing estimates 1

Generalization of standard global homogeneous smoothing estimates in the time-degenerate case:

- Let $n = 1$ and $m > 0$, then

$$\sup_{x\in\mathbb{R}} \| |c(t)b'(t)|^{1/2} |D_x|^{\frac{m-1}{2}} e^{ib(t)|D_x|^m} \varphi \|_{L^2(\mathbb{R}_t)} \le C \|\varphi\|_{L^2(\mathbb{R}_x)}, \quad \text{(F.-Ruzh.)}$$

- Let $n = 2$ and $m > 0$, then

$$\sup_{x_1\in\mathbb{R}} \| |c(t)b'(t)|^{1/2} |D_{x_2}|^{\frac{m-1}{2}} e^{ib(t)D_{x_1}|D_{x_2}|^{m-1}} \varphi \|_{L^2(\mathbb{R}_t\times\mathbb{R}_{x_2})} \le C \|\varphi\|_{L^2(\mathbb{R}^2)}, \text{(F.-Ruz}$$

Weighted global smoothing estimates 2

Global smoothing estimates in the constant coefficients case:

- Given $n \geq 2$ and $1 - n/2 < \beta < 1/2$, for all $\varphi \in L^2(\mathbb{R}^n)$ we have

$$\||x|^{\beta-1}|D_x|^{\beta}e^{it|D_x|^2}\varphi\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \quad \text{(Sugimoto)}.$$

- For $n, m > 1$, and for all $\varphi \in L^2(\mathbb{R}^n)$ such that

$\mathrm{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$, the following estimate holds

$$\|\langle x\rangle^{-m/2}e^{it|D_x|^m}\varphi\|_{L^2(\mathbb{R}_t\times\mathbb{R}^n_x)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n_x)} \quad \text{(Walther)}.$$

Weighted global smoothing estimates 2

Generalization for time-degenerate Schrödinger operators:

- Given $n \geq 2$ and $1 - n/2 < \beta < 1/2$, for all $\varphi \in L^2(\mathbb{R}^n)$ we have

$$\| |x|^{\beta-1} |c(t)b'(t)|^{1/2} |D_x|^{\beta} e^{ib(t)|D_x|^2} \varphi \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad \text{(F.-Ruzhansky)}$$

- For $n, m > 1$, and for all $\varphi \in L^2(\mathbb{R}^n)$ such that $\mathrm{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$, then, for any $\frac{m-n}{2} < \alpha < \frac{m-1}{2}$, the following estimate holds

$$\| \langle x \rangle^{\alpha - m/2} |c(t)b'(t)|^{1/2} |D_x|^{\alpha} e^{ib(t)|D_x|^m} \varphi \|_{L^2(\mathbb{R}_t \times \mathbb{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbb{R}_x^n)} \quad \text{(F.-Ruzhan}$$

Notations. We shall denote by $L^q_t L^p_x := L^q_t(\mathbb{R}; L^p_x(\mathbb{R}^n))$, and, when not confusing, we shall use the same notation $L^q_t L^p_x := L^q_t([0,T]; L^p_x(\mathbb{R}^n))$ when the time interval is finite.

Definition. (*Admissible pairs*) Given $n \geq 1$ we shall call a pair of exponents (q,p) admissible if $2 \leq q, p \leq \infty$, and

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \quad \text{with} \quad (q,p,n) \neq (2,\infty,2).$$

Theorem. Global weighted Strichartz estimates. (F.-Ruzhansky)
Let $b \in C^1(\mathbb{R})$ be such that it has a either a finite or an infinite (countable) number of critical points. Then, for any (q,p),$(\tilde{q},\tilde{p})$ admissible pairs such that $2 < q, \tilde{q}, p, \tilde{p} < \infty$, the following estimates hold:

the weighted homogeneous Strichartz estimate

$$\| |c(t)b'(t)|^{1/q} e^{ib(t)\Delta}\varphi\|_{L^q_t L^p_x} \leq C(n,q,p)\|\varphi\|_{L^2_x(\mathbb{R}^n)}; \tag{6}$$

the dual weighted homogeneous Strichartz estimate

$$\| \int_{\mathbb{R}} |c(t)b'(s)|^{1/\tilde{q}} e^{-ib(s)\Delta} g(s)ds\|_{L^2_x} \leq C(n,\tilde{q},\tilde{p})\|g\|_{L^{\tilde{q}'}_t L^{\tilde{p}'}_x}; \tag{7}$$

and the weighted inhomogeneous Strichartz estimate

$$\| |c(t)b'(t)|^{1/q} \int_{\mathbb{R}} |c(s)b'(s)|^{1/\tilde{q}} e^{i(b(t)-b(s))\Delta} g(s)ds\|_{L^q_t L^p_x} \leq C(n,q,p,\tilde{q},\tilde{p})\|g\|_{L^{\tilde{q}'}_t L^{\tilde{p}'}_x}. \tag{8}$$

Remark. The previous estimates hold in the endpoint case too.

Theorem. Local weighted Strichartz estimates. (F.-Ruzhansky)
Let $b \in C^1([0,T])$ be vanishing at 0 and such that, for any fixed T, $\sharp\{t \in [0,T]; b'(t) = 0\} = k \geq 1$. Then, for any (q,p) admissible pair such that $2 < q, p < \infty$, the following estimates hold:

the weighted homogeneous Strichartz estimate

$$\| |b'(t)|^{1/q} e^{ib(t)\Delta} \varphi \|_{L^q_t L^p_x} \leq C(n,q,p,k) \|\varphi\|_{L^2_x(\mathbb{R}^n)}, \tag{9}$$

where

$$\|e^{ib(t)\Delta}\varphi\|_{L^\infty_t L^2_x} \leq \|\varphi\|_{L^2_x(\mathbb{R}^n)}; \tag{10}$$

the weighted inhomogeneous Strichartz estimate

$$\| |b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \|_{L^q_t L^p_x} \leq C(n,q,p,k) \| |b'|^{1/q'} g\|_{L^{q'}_t L^{p'}_x}; \tag{11}$$

where

$$\| \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s) ds \|_{L^\infty_t L^2_x} \leq C(n,q,p,k) \| |b'|^{1/q'} g\|_{L^{q'}_t L^{p'}_x}. \tag{12}$$

Local well-posedness of the semilinear problem

Let us consider the semilinear IVP

$$\begin{cases} \partial_t u + ib'(t)\Delta u = \mu|b'(t)||u|^{p-1}u, \quad \mu \in \mathbb{R} \\ u(0,x) = u_0(x), \end{cases} \tag{13}$$

then the following local well-posedness result holds.

Theorem

Let $1 < p < \frac{4}{n} + 1$ and $b \in C^1([0, +\infty))$ be vanishing at 0 and it is either strictly monotone or such that $\sharp\{t \in [0, \tilde{T}]; b'(t) = 0\}$ is finite for any $\tilde{T} < \infty$. Then for all $u_0 \in L^2(\mathbb{R}^n)$ there exists $T = T(\|u_0\|_2, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (13) in the time interval $[0, T]$ with

$$u \in C([0, T]; L^2(\mathbb{R}^n)) \bigcap L^q_t([0, T]; L^{p+1}_x(\mathbb{R}^n))$$

and $q = \frac{4(p+1)}{n(p-1)}$. Moreover the map $u_0 \mapsto u(\cdot, t)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0, T); L^2(\mathbb{R}^n))$, is continuous.

Application

By the previous theorem we have the local well-posedness for the semilinear IVP associated with:

Example 1

$$\mathcal{L}_{b,\Delta} = \mathcal{L}_{\frac{t^{\alpha+1}}{\alpha+1},\Delta} = \partial_t + it^{\alpha}\Delta, \quad \alpha \geq 1$$

$\sharp\{t \in [0, T]; b'(t) = 0\} = 1$ for any $0 < T < \infty$.

Example 2

$$\mathcal{L}_{b,\Delta} = \mathcal{L}_{e^t - t - 1,\Delta} = \partial_t + i(e^t - 1)\Delta,$$

$\sharp\{t \in [0, T]; b'(t) = 0\} = 1$ for any $0 < T < \infty$.

Example 3

$$\mathcal{L}_{b,\Delta} = \mathcal{L}_{\cos(t),\Delta} := \partial_t u + i\sin(t)\Delta$$

$\sharp\{t \in [0, T]; b'(t) = 0\} = k \geq 1$ for any $0 < T < \infty$.

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