Dirac states on the Weyl algebra

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1. Motivation for the abstract Weyl relations

Schrödinger representation (n degrees of freedom): s.a. Q_j and P_j ($j = 1, \ldots, n$) with *Heisenberg (canonical) commutation relations*

$$[Q_j, P_k] = i\delta_{jk} I.$$

If $a, b \in \mathbb{R}^n$, $Q(a) = \sum a_j Q_j$, $P(b) = \sum b_k P_k$, then

$$[Q(a), P(b)] = i(a|b)I.$$

Exponentiation gives $e^{iQ(a)}e^{iP(b)} = e^{-i(a|b)}e^{iP(b)}e^{iQ(a)}$ etc.

Define $z := a + ib \in \mathbb{C}^n$, $W(z) := e^{-\frac{i}{2}(a|b)}e^{iP(b)}e^{iQ(a)} \quad \rightsquigarrow$

$$\boxed{W(z)^* = W(-z), \quad W(y)W(z) = e^{\frac{i}{2}\mathsf{Im}\langle y|z\rangle}W(y+z)}$$

(i) $(y, z) \mapsto \mathsf{Im}\langle y|z\rangle/2$ real symplectic form on $\mathbb{C}^n := \mathbb{R}^n \oplus i\mathbb{R}^n$,
(ii) $t \mapsto W(tz)$ strongly continuous one-parameter unitary group.

2. Weyl algebra [Slawny 1972] and *-representations

Quantum fields (infinitely many degrees of freedom): Replace $\mathbb{C}^n$ by a real symplectic vector space S, e.g. L^2 with $\beta(\varphi, \psi) := \mathrm{Im}\langle\varphi|\psi\rangle/2$.

Thm/Def: The *Weyl algebra* $\mathcal{W}$ over the symplectic space (S, β) is the unique C^*-algebra generated by a set $\{W(z) \mid z \in S\}$ with

$$W(z)^* = W(-z), \quad W(y)W(z) = e^{i\beta(y,z)}W(y+z).$$

Equivalence classes of *-representations: For $\dim S < \infty$ unique (Schrödinger repr.) and in case $\dim S = \infty$ uncountably many.

Let ω be a *state* (i.e., a normalized positive linear functional) on $\mathcal{W}$ and consider the *GNS representation*§ $\pi \colon \mathcal{W} \to B(H)$ associated with ω; let $\Omega \in H$ be its standard *cyclic vector* (*vacuum in physics*).

§(Gelfand-Naimark-Segal) $\mathcal{N} := \{A \in \mathcal{W} \mid \omega(A^*A) = 0\}$ closed left ideal, H is the closure of $\mathcal{W}/\mathcal{N}$ w.r.t. inner product $\langle A + \mathcal{N} | B + \mathcal{N}\rangle := \omega(A^*B)$, $\pi(A)(B + \mathcal{N}) := AB + \mathcal{N}$, $\Omega := 1 + \mathcal{N}$, $\omega(A) = \langle\Omega|\pi(A)\Omega\rangle$.

3. Dirac states [Grundling and Hurst since the 1980's]

A state ω is *regular*, if its GNS representation π is *regular*, i.e., $\forall z \in S$, $t \mapsto \pi(W(tz))$ is a strongly continuous unitary group in H. In this case, the generators $\Phi(z)$ define *fields* ("Wightman setting").

Lemma: ω regular $\iff \lim_{t\to 0} \omega(W(tz)) = 1$ for every $z \in S$.

Def: ω is called a *Dirac state* adapted to the subspace $L \subseteq S$, if

$$\omega(W(y)) = 1 \text{ for all } y \in L.$$

Rem: (i) Dirac states exist for L iff $L \subseteq L^{\perp}$ (β-isotropic)
(ii) in physics L solution set of constraint equations (e.g., $\nabla\Phi = 0$ Coulomb gauge for electromagnetic vector potential in QED).

Prop: (i) ω Dirac state for $L \iff \forall y \in L$: $\pi(W(y))\Omega = \Omega$.
(ii) ω Dirac for $L \neq \{0\} \implies \pi$ not regular.

[Quotient construction $\rightsquigarrow$ C^*-algebra $\mathcal{O}$ of *physical observables*;
Dirac states induce regular states on $\mathcal{O}$.]

4. Normal states on Weyl-von Neumann algebras

$\pi\colon \mathcal{W} \to B(H)$ any *-representation; $\mathcal{W}$ simple $\Rightarrow \pi(\mathcal{W}) \cong \mathcal{W}$.
$\mathcal{W}_\pi :=$ von Neumann algebra generated from $\pi(\mathcal{W})$. ($\pi(\mathcal{W})''$, s/w clos.)

Examples: (i) π irreducible $\Rightarrow \mathcal{W}_\pi = B(H)$. (ii) π *universal repr.* (sum over all GNS repr.), then $\mathcal{W}_\pi$ *enveloping von Neumann algebra*.

Weak* topology on $B(H)$ given by seminorms $A \mapsto |\mathrm{trace}(AT)|$, where T is trace class. A positive linear functional μ on $B(H)$ is weak* continuous iff it is *normal* (respecting incr. s.o.t.-conv. nets).

Physics: Reasonable bounds for states from finite measurements.

Vector functionals $\nu_\xi\colon B(H) \to \mathbb{C}$, $A \mapsto \langle \xi | A\xi \rangle$, are normal.

Prop: Non-existence of normal Dirac states on $\mathcal{W}_\pi$, if π is regular.

Def(Function space V_π on S): For $\xi \in H$ we define $f_\xi\colon S \to \mathbb{C}$ by

$$f_\xi(z) := \nu_\xi(\pi(W(z))) = \langle \xi | \pi(W(z))\xi \rangle$$

and V_π as the *closure of* $\mathrm{span}\{f_\xi \mid \xi \in H\}$ in the Banach space of bounded functions $S \to \mathbb{C}$ with the supremum norm.

5. Description of states on $\mathcal{W}$ by functions $S \to \mathbb{C}$

[Recall: (S, β) symplectic space, $\mathcal{W}$ with generators $W(z)$ ($z \in S$), $\pi\colon \mathcal{W} \to B(H)$ *-representation, $\mathcal{W}_\pi$ vNA generated in $B(H)$, V_π.]

Prop: (i) If μ is a normal state on $\mathcal{W}_\pi$ and $h\colon S \to \mathbb{C}$ defined by $h(z) := \mu(\pi(W(z)))$, then $h \in V_\pi$.

(ii) If S Hilbert sp., ω state on $\mathcal{W}$ s.t. $z \mapsto \omega(W(z))$ Borel meas., π GNS representation associated with ω, then $V_\pi \subseteq B_b(S)$.

(iii) If S Hilbert space and π regular, then $V_\pi \subseteq C_b(S)$.

Examples: (i) $S = \mathbb{R}^{2n}$, π Schrödinger repr., then $V_\pi \subseteq C_0(\mathbb{R}^{2n})$.

(ii) S Hilbert, ω *Fock state*, $\omega(W(z)) = \exp(-\|z\|^2/4)$, then $V_\pi \subseteq C_{b0}(S) := \{f \in C_b(S) \mid \forall \varepsilon > 0 \exists R \geq 0\colon \sup_{\|z\| \geq R} |f(z)| \leq \varepsilon\}$.

(This is larger than $C_0(S) :=$ closure of $C_c(S)$, if $\dim S = \infty$.)

Correspondence: Let $g\colon S \to \mathbb{C}$, $g(0) = 1$. $\exists!$ state ω on $\mathcal{W}$ s.t.

$$\forall z \in S\colon \quad \omega(W(z)) = g(z),$$

iff $G(x, y) := g(x - y) \exp(-i\beta(x, y))$ defines pos. def. kernel.

6. Now specifically $S = L^2(\mathbb{R}^n)$ with $\beta(\varphi, \psi) = \mathrm{Im}\langle\varphi|\psi\rangle/2$

GF aspect: If $g\colon L^2 \to \mathbb{C}$ induces *smooth* function on $\mathscr{D}(\mathbb{R}^n)$ and is *moderate* upon insertion of ε-scaled delta nets $\rightsquigarrow$ $[g] \in \mathcal{G}(\mathbb{R}^n)$.

Dirac state ω_0 (with QED context [Thirring-Narnhofer 1992]) corresponding to *discontinuous* function $g_0(\psi) = 1$, if $\mathrm{Re}\,\psi = 0$, and $g_0(\psi) = 0$ otherwise $\rightsquigarrow$ GNS representation $\pi_0\colon \mathcal{W} \to B(H_0)$ with *non-separable* H_0; vacuum vector state $\nu_0\colon A \mapsto \langle\Omega_0|A\Omega_0\rangle$ is a normal Dirac state on $\mathcal{W}_{\pi_0}$ with $\nu_0 \circ \pi_0 = \omega_0$; π_0 *not regular.*

Lemma: ω_0 is the weak* limit of the regular states ω_k on $\mathcal{W}$ with $\omega_k(\psi) = \exp(-\frac{k^2}{4}\|\mathrm{Re}\,\psi\|^2 - \frac{1}{4k^2}\|\mathrm{Im}\,\psi\|^2) =: g_k(\psi)$.

Observations: (i) Every g_k smooth on $\mathscr{D}(\mathbb{R}^n)$, but 0 in $\mathcal{G}(\mathbb{R}^n)$.
(ii) As function on $\mathscr{S}(\mathbb{R}^n)$, every g_k is the inverse Fourier transform of a Gaussian measure on $\mathscr{S}'(\mathbb{R}^n)$ (background: Bochner-Minlos theorem).

Prop: ω regular state $\leftrightarrow$ g continuous (with G pos. def. kernel)
$\leftrightarrow$ Borel probability measure μ on $\mathscr{S}'$ $\overset{[g \text{ smooth on } \mathscr{D}]}{\rightarrow}$ $[g] \in \mathcal{G}$.

7. $\mathscr{S}(\mathbb{R}^n)$ as Abelian topological group (not loc. compact)

Recall: $\boxed{\widehat{\delta_L} = (2\pi)^{\dim L}\,\delta_{L^\perp}}$, if L subsp. of $\mathbb{R}^d$, δ_L Euclid. surf. measure.

g_0 gives indicator function of $L := \{\psi \in \mathscr{S} \mid \operatorname{Re}\psi = 0\}$. $\boxed{\widehat{g_0} = ?}$

Thm: Dual group $\widehat{\mathscr{S}}$ is isomorphic (as a topological group) to $\mathscr{S}'_{\mathbb{R}}$ and $\mathscr{S}$ has the Pontryagin property. [Smith 1952]

Fourier transform of $\mu \in M(\mathscr{S})$ (complex Borel meas.) $\widehat{\mu}\colon \mathscr{S}'_{\mathbb{R}} \to \mathbb{C}$,

$$\forall u \in \mathscr{S}'_{\mathbb{R}}\colon \quad \widehat{\mu}(u) := \int_{\mathscr{S}} e^{-i\langle u,\varphi\rangle}\,d\mu(\varphi),$$

co-Fourier transform of $\nu \in M(\mathscr{S}'_{\mathbb{R}})$, $\widetilde{\nu}(\varphi) := \int_{\mathscr{S}'_{\mathbb{R}}} e^{i\langle u,\varphi\rangle}\,d\nu(u)$.

$h_0 :=$ indicator function of $L^\perp = \{u \in \mathscr{S}'_{\mathbb{R}} \mid \forall\varphi \in L\colon \langle u,\varphi\rangle = 0\}$,
$K := \{\psi \in \mathscr{S} \mid \operatorname{Im}\psi = 0\}$, then $\mathscr{S} = K \oplus L$ and $\mathscr{S}'_{\mathbb{R}} = L^\perp \oplus K^\perp$.

Prop: $\rho = \rho_1 \otimes \delta_0$, $\mu = \delta_0 \otimes \mu_2$ with *finite positive* measures $\rho_1 \in M(L^\perp)$, $\mu_2 \in M(L)$, then $\boxed{\|\rho\|\,\widehat{(g_0\mu)} = (h_0\rho) * \widehat{\mu}}$.

Intuitively, $\mu_1 \approx 1$ leads to $\|\rho\|\widehat{g_0} \approx \|\rho\|h_0$, thus $\widehat{g_0} \approx h_0$.