A complete characterization of Fujita's blow-up solutions for discrete p-Laplacian paraolic equations

This is a joint work with Prof. Soon-Yeong Chung

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Examples : Blow-up solution and global solution to ODE's

Eq1. $\begin{cases} y'(t) = -2y(t) + 3|y(t)|^2 y(t), \quad t > 0, \\ y(0) = 3 \end{cases}$

Eq2. $\begin{cases} y'(t) = -2y(t) + 3|y(t)|^2 y(t), \quad t > 0, \\ y(0) = 0.5 \end{cases}$

What is blow-up?

Examples : Blow-up solution and global solution to ODE's

Eq1. $$\begin{cases} y'(t) = -2y(t) + 3|y(t)|^2 y(t), & t > 0, \\ y(0) = 3 \end{cases}$$

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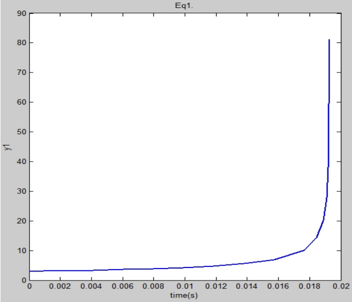
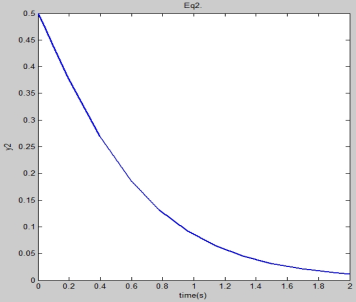




What is Fujita's blow-up?

What is Fujita's blow-up?

The definition of Fujita's blow-up solution

The solution u is Fujita's blows up solution if u blows up in finite time t^* for every nontrivial nonnegative initial data u_0.

What is blow-up?

The definition of the blow-up

We say that the solution u blows up in finite time t^* if u satisfies

$$\lim_{t \to t^*-} \|u(x,t)\|_\infty = \infty.$$

Main equations and goals

Main equations

p-Laplcian parabolic equations under mixed boundary conditions

$$(E)\ :\ \begin{cases} u_t = \Delta_{p,\omega}u + \psi(t)|u|^{q-1}u, & \text{in } S\times(0,t^*),\\ \mu(z)\frac{\partial u}{\partial_p n} + \sigma(z)|u|^{p-2}u = 0, & \text{on } \partial S\times(0,t^*),\\ u(\cdot,0) = u_0 \geq 0\ (u_0 \not\equiv 0), & \text{in } S, \end{cases}$$

where S is a network with a boundary ∂S, t^* is the maximal existence time of the solution u, $p \geq 2$, $q > 0$, and the function ψ is a positive continuous function on $(0,\infty)$.
Here, μ and σ are nonnegative functions with $\mu(z) + \sigma(z) > 0$, $z \in \partial S$ satisfying $\sigma \not\equiv 0$.

Main equations and goals

p-Laplcian parabolic equations under mixed boundary conditions

$$(E) : \begin{cases} u_t = \Delta_{p,\omega} u + \psi(t)|u|^{q-1}u, & \text{in } S \times (0, t^*), \\ \mu(z)\frac{\partial u}{\partial_p n} + \sigma(z)|u|^{p-2}u = 0, & \text{on } \partial S \times (0, t^*), \\ u(\cdot, 0) = u_0 \geq 0 \ (u_0 \not\equiv 0), & \text{in } S. \end{cases}$$

Goals

- When the solutions blow-up or exist globally?
- Can we obtain the blow-up conditions for p, q, and ψ?
- In the case of blow-up phenomena, then the solutions are Fujita's blow-up solutions?
- Can we characterize the parameters p, q, and the function ψ with respect to Fujita's blow-up?

Histories

Fujita (1966)

Fujita firstly began studying Fujita's blow-up solutions to the following Laplacian parabolic equations

$$\begin{cases} u_t = \Delta u + u^q, & x \in \mathbb{R}^N,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \mathbb{R}^N, \end{cases}$$

where $q > 1$. In their results, if $1 < q < q^* = 1 + \frac{2}{N}$, then the solutions blow up in finite time for every nonnegative nontrivial initial data.

(i) If $q > q^*$, then there exist global solutions and blow-up solutions with respect to the initial data u_0.

(ii) Such q^* in the above is called a critical exponent.

H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I 13 (1966) 109–124.

Galaktionov (1994)

Galaktionov obtained Fujita's blow-up solutions for the following p-Laplacian parabolic equations

$$\begin{cases} u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u) + u^q, & x \in \mathbb{R}^N,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \mathbb{R}^N, \end{cases}$$

where $q > p - 1$. In their results, if $p - 1 < q < q^* = p - 1 + \frac{p}{N}$, then the solutions blow up in finite time for every nonnegative nontrivial initial data.

V. A. Galaktionov, *Blow-up for quasilinear heat equations with critical Fujita's exponents*, Proc. Roy. Soc. Edinburgh Sect. A 124 (1994), no. 3, 517–525.

Meier (1990)

Meier obtained Fujita's blow-up solutions for the following parabolic equations

$$\begin{cases} u_t = \Delta u + e^{\beta t}|u|^{q-1}u, & \text{in } \Omega \times (0, t^*), \\ u = 0, & \text{on } \partial\Omega \times (0, t^*), \\ u(\cdot, 0) = u_0 \geq 0 \ (u_0 \not\equiv 0), & \text{in } \Omega, \end{cases}$$

where Ω is a general (bounded or unbounded) domain with smooth boundary $\partial\Omega$. Especially, if Ω is bounded, then the critical exponent is $q^* = 1 + \frac{\beta}{\lambda_0}$, where λ_0 is the first Dirichlet eigenvalue of the Laplace operator Δ.

P. Meier, *On the critical exponent for reaction-diffusion equations*, Arch. Rational Mech. Anal. 109 (1990), no. 1, 63–71.

Zhou, Chen, and Liu (2014)

Zhou et al. obtained Fujita's blow-up solutions for the following discrete parabolic equations

$$\begin{cases} u_t = \Delta_\omega u + e^{\beta t}|u|^{q-1}u, & \text{in } S \times (0, t^*), \\ u = 0, & \text{on } \partial S \times (0, t^*), \\ u(\cdot, 0) = u_0 \geq 0 \ (u_0 \not\equiv 0), & \text{in } S, \end{cases}$$

where S is a network with boundary ∂S. They also obtained the critical exponent q^* as $1 + \frac{\beta}{\lambda_0}$, where λ_0 is the first Dirichlet eigenvalue of the discrete Laplace operator Δ_ω.

W. Zhou, M. Chen, and W. Liu, *Critical exponent and blow-up rate for the ω-diffusion equations on graphs with Dirichlet boundary conditions*, Electron. J. Differential Equations (2014), no. 263, 13 pp.

Histories

Chung, Park, and Choi (2019)

Chung, Park, and Choi obtained Fujita's blow-up solutions for the following discrete parabolic equations

$$\begin{cases} u_t = \Delta_\omega u + \psi(t)|u|^{q-1}u, & \text{in } S \times (0, t^*), \\ u = 0, & \text{on } \partial S \times (0, t^*), \\ u(\cdot, 0) = u_0 \geq 0 \ (u_0 \not\equiv 0), & \text{in } S, \end{cases}$$

where ψ is nonnegative continuous function. They obtained that the solution u is Fujita's blow-up solution iff $q \in \Lambda_\psi$ (*the critical set*), where

$$\Lambda_\psi = \left\{ q > 1 \,|\, \int_0^\infty \psi(t) e^{-(q-1)\lambda_0 t} dt = \infty \right\}.$$

S. -Y. Chung, M. -J. Choi, and J. -H. Park, *Fujita-type blow-up for discrete reaction-diffusion equations on networks*, Publ. Res. Inst. Math. Sci. 55 (2019), 235-258.

The critical set

Remark

(i) In the equation

$$\begin{cases} u_t = \Delta_\omega u + e^{\beta t}|u|^{q-1}u, & \text{in } S \times (0, t^*), \\ u = 0, & \text{on } \partial S \times (0, t^*), \\ u(\cdot, 0) = u_0 \geq 0 \ (u_0 \not\equiv 0), & \text{in } S, \end{cases}$$

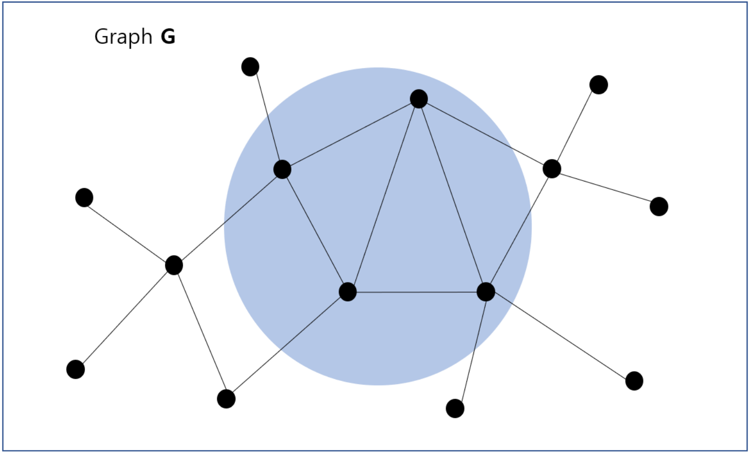
$q \in \Lambda_\psi = \left\{q > 1 \mid \int_0^\infty e^{\beta t} e^{-(q-1)\lambda_0 t} dt = \infty\right\}$ implies that $1 < q \leq 1 + \frac{\beta}{\lambda_0}$. It follows that we can discuss the case of $q = q^*$, by considering the critical set.

(ii) In our results, we define the critical set as

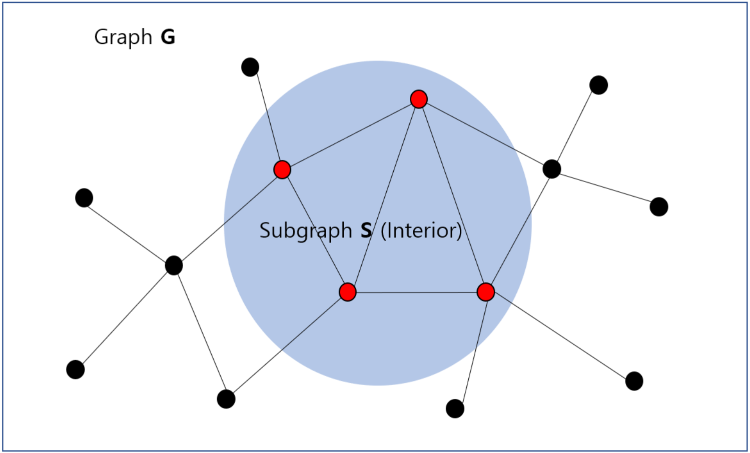
$$\Lambda_{p,\psi} := \left\{q > 1 \,\middle|\, \int_0^\infty \psi(t) e^{-(q-p+1)\lambda_{p,0} t} dt = \infty\right\},$$

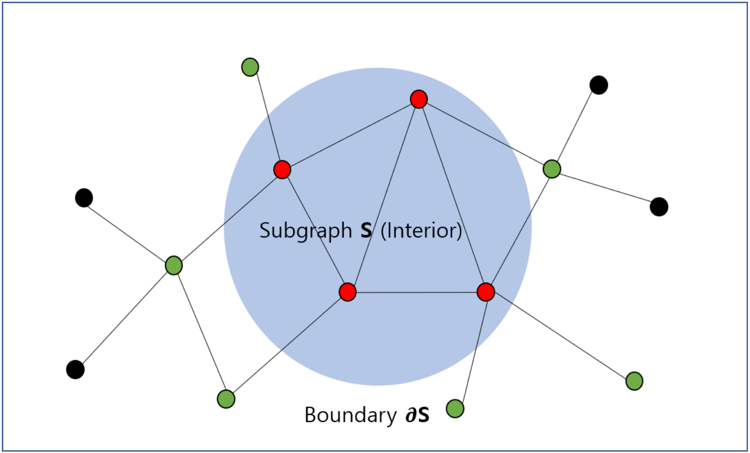
where $\lambda_{p,0}$ is the first eigenvalue of the discrete p-Laplace operator.

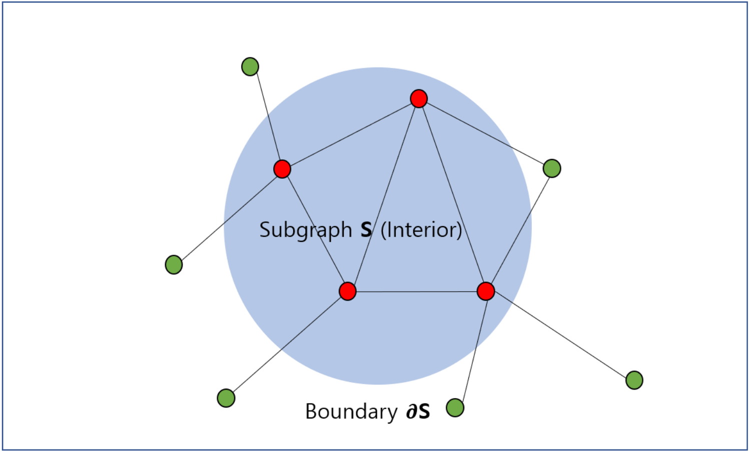
Networks

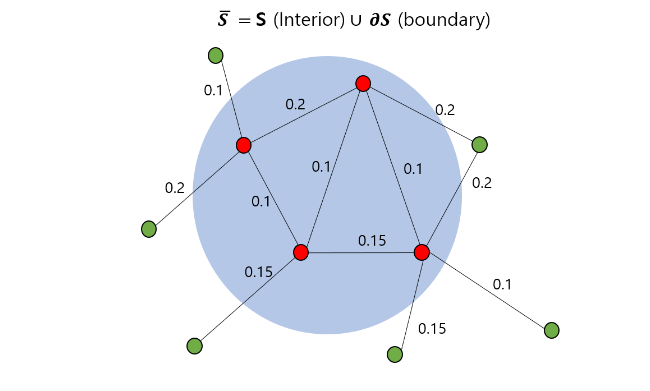


Graph G is simple and connected.


Graph **G**
Subgraph **S** (Interior)


Subgraph **S** (Interior)
Boundary ***∂*S**


Subgraph S (Interior)
Boundary ∂S



A weight ω on a graph $G(\overline{S}, E)$ is a function $\omega : \overline{S} \times \overline{S} \to [0, \infty)$ satisfying

(i) $\omega(x, x) = 0, \quad x \in \overline{S}$,

(ii) $\omega(x, y) = \omega(y, x)$, for all x, y,

(iii) $\omega(x, y) > 0 \Leftrightarrow \{x, y\} \in E$.

- $G(\overline{S}, E : \omega)$ is called a weighted graph or a network.

Discrete p-Laplace operator and p-normal derivative

For a function f (defined on the set $\overline{S}$ of nodes in G), we define

$$\Delta_{p,\omega} f(x) := \sum_{y \in \overline{S}} |f(y) - f(x)|^{p-2}[f(y) - f(x)]\omega(x, y), \quad x \in S,$$

$$\frac{\partial f}{\partial_p n}(x) := \sum_{y \in \overline{S}} |f(x) - f(y)|^{p-2}[f(x) - f(y)]\omega(x, y), \quad x \in \partial S.$$

What is network?

Discrete p-Laplace operator and p-normal derivative

For a function f (defined on the set $\overline{S}$ of nodes in G), we define

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Especially, if $p = 2$, then we have

$$\Delta_{2,\omega} f(x) := \sum_{y \in \overline{S}} [f(y) - f(x)]\omega(x, y), \quad x \in S.$$

Main results

Recall the main equations

Main equations

$$(E)\ :\ \begin{cases} u_t = \Delta_{p,\omega} u + \psi(t)|u|^{q-1}u, & \text{in } S \times (0,t^*),\\ \mu(z)\frac{\partial u}{\partial_p n} + \sigma(z)|u|^{p-2}u = 0, & \text{on } \partial S \times (0,t^*),\\ u(\cdot,0) = u_0 \geq 0\ (u_0 \not\equiv 0), & \text{in } S, \end{cases}$$

where S is a network with a boundary ∂S, t^* is the maximal existence time of the solution u, $p \geq 2$, $q > 0$, and the function ψ is a positive continuous function on $(0,\infty)$.

Here, the boundary condition $B[u] = 0$ on $\partial S \times (0,t^*)$ stands for the boundary condition

$$\mu(z)\frac{\partial u}{\partial_p n}(z,t) + \sigma(z)|u(z,t)|^{p-2} = 0,\ (z,t) \in \partial S \times (0,t^*),$$

where μ and σ are nonnegative functions with $\mu(z) + \sigma(z) > 0$, $z \in \partial S$.

The first eigenvalue of the discrete p-Laplace operator

There exist a nonnegative constant $\lambda_{p,0}$ and nonnegative function ϕ_0 on $\overline{S}$ satisfying

$$\begin{cases} -\Delta_{p,\omega}\phi_0(x) = \lambda_{p,0}|\phi_0(x)|^{p-2}\phi_0(x), & x \in S, \\ \mu(z)\frac{\partial \phi_0}{\partial_p n}(z) + \sigma(z)|\phi_0(z)|^{p-2}\phi_0(z) = 0, & z \in \partial S, \end{cases}$$

Moreover, $\lambda_{p,0}$ can be represented by

$$\lambda_{p,0} = \min_{\substack{u \not\equiv 0 \\ u=0 \text{ on } \partial S \setminus \Gamma}} \frac{\sum_{x,y\in\overline{S}} |u(x)-u(y)|^p \omega(x,y) + \sum_{z\in\Gamma} \frac{\sigma(z)}{\mu(z)}|u(z)|^p}{\sum_{x\in S} |u(x)|^p},$$

where $\Gamma := \{z \in \partial S \,|\, \mu(z) > 0\}$.

Remark

- $\lambda_{p,0} > 0$ if and only if $\sigma \not\equiv 0$.
- $\Lambda_{p,\psi} := \left\{ q > 1 \,|\, \int_0^\infty \psi(t) e^{-(q-p+1)\lambda_{p,0} t} dt = \infty \right\}$.

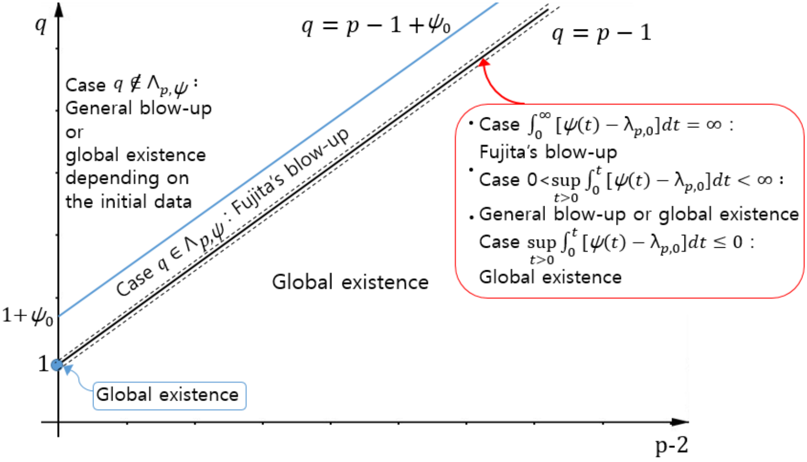
Comparison principle

Comparison principle

Let $T > 0$ (T may be $+\infty$), $p \geq 2$, and $q \geq 1$. Suppose that real-valued functions $u(x, \cdot)$, $v(x, \cdot) \in C[0, T)$ are differentiable in $(0, T)$ for each $x \in \overline{S}$ and satisfy

$$\begin{cases} u_t(x,t) - \Delta_{p,\omega} u(x,t) - \psi(t)|u(x,t)|^{q-1}u(x,t) & \\ \geq v_t(x,t) - \Delta_{p,\omega} v(x,t) - \psi(t)|v(x,t)|^{q-1}v(x,t), & (x,t) \in S \times (0,T)\,, \\ \mu(z)\frac{\partial u}{\partial_p n}(z,t) + \sigma(z)|u(z,t)|^{p-2}u(z,t) & \\ -\mu(z)\frac{\partial v}{\partial_p n}(z,t) + \sigma(z)|v(z,t)|^{p-2}v(z,t) \geq 0, & (z,t) \in \partial S \times (0,T)\,, \\ u\,(x,0) \geq v\,(x,0)\,, & x \in \overline{S}. \end{cases}$$

Then $u\,(x,t) \geq v\,(x,t)$ for all $(x,t) \in \overline{S} \times [0,T)$.



q
$q = p - 1 + \psi_0$
$q = p - 1$
Case $q \notin \Lambda_{p,\psi}$:
General blow-up
or
global existence
depending on
the initial data
Case $q \in \Lambda_{p,\psi}$: Fujita's blow-up
• Case $\int_0^\infty [\psi(t) - \lambda_{p,0}]dt = \infty$:
Fujita's blow-up
• Case $0 < \sup_{t>0} \int_0^t [\psi(t) - \lambda_{p,0}]dt < \infty$:
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Case $\sup_{t>0} \int_0^t [\psi(t) - \lambda_{p,0}]dt \le 0$:
Global existence
Global existence
$1+\psi_0$
1
Global existence
p-2

· $p \geq 2$, $q > 0$.

Main results : The case $p-1 \neq q$ or $q \leq 1$

The case $q < p-1$ or $q \leq 1$

Let $q < p-1$ or $q \leq 1$. Then every solution u to the equation (E) exists globally.

The case $q > p-1$

Let u be the solution to the equation (E).

(i) Suppose that $q \in \Lambda_{p,\psi}$. Then u blows up at finite time t^* for every nontrivial and nonnegative initial data u_0.

(ii) Suppose that $q \notin \Lambda_{p,\psi}$. Then u blows up at finite time t^* whenever the initial data u_0 is sufficiently large.

(iii) Suppose that $q \notin \Lambda_{p,\psi}$. Then u exists globally whenever the initial data u_0 is sufficiently small.

Sketch of proof the case $q > p - 1$

Let us consider two ODE problems

$$\begin{cases} z'(t) = -\lambda_{p,0} z^{p-1}(t) + k\psi(t) z^q(t), \ t > t_0, \\ z(t_0) = z_0 > 0, \end{cases} \tag{1}$$

and

$$\begin{cases} w'(t) = -\lambda_{p,0} w(t) + k\psi(t) w^{q-p+2}(t), \ t > t_0, \\ w(t_0) = w_0 > 0. \end{cases} \tag{2}$$

We will use the Comparision Principle. By taking k, z_0, w_0 suitably small or large, we can obtain the following relations

$$w(t)\phi_0(x) \le z(t)\phi_0(x) \le u(x,t) \ \text{ or } \ w(t)\phi_0(x) \ge z(t)\phi_0(x) \ge u(x,t).$$

Also, the solution w can be expressed as

$$w(t) = \left[\frac{1}{e^{(q-p+1)\lambda_{p,0}(t-t_0)} \left[w_0^{-(q-p+1)} - k(q-p+1) \int_{t_0}^{t} \psi(\tau) e^{-(q-p+1)\lambda_{p,0}(\tau - t_0)} d\tau \right]} \right]^{\frac{1}{q-p+1}},$$

we can obtain the desired results.

The case $p-1=q>1$

Let u be the solution to the equation (E).

(i) Suppose that $\int_0^\infty [\psi(\tau)-\lambda_{p,0}]d\tau=\infty$. Then u blows up at finite time t^* for every nontrivial and nonnegative initial data u_0.

(ii) Suppose that $0<\sup_{t>0}\int_0^t [\psi(\tau)-\lambda_{p,0}]d\tau<\infty$. Then u blows up at finite time t^* whenever the initial data u_0 is sufficiently large.

(iii) Suppose that $0<\sup_{t>0}\int_0^t [\psi(\tau)-\lambda_{p,0}]d\tau<\infty$. Then u exists globally whenever the initial data u_0 is sufficiently small.

(iv) Suppose that $\sup_{t>0}\int_0^t [\psi(\tau)-\lambda_{p,0}]d\tau\le 0$. Then u exists globally for every initial data u_0.

Sketch of proof the case $p-1=q>1$

Consider the following ODE problem:

$$\begin{cases} z'(t) = k(t)(\psi(t) - \lambda_{p,0}) z^{p-1}(t), \ t > t_0, \\ z(t_0) = z_0 > 0, \end{cases}$$

where k is a positive real-valued function and z_0 is a positive constant which are determined later. Then we have

$$z(t) = \left[\frac{1}{z_0^{2-p} - (p-2)k(t)\int_{t_0}^{t} [\psi(\tau) - \lambda_{p,0}] d\tau} \right]^{\frac{1}{p-2}},$$

for all $t \geq t_0$. Now, we define $v(x,t) := \phi_0(x) z(t)$ on $\overline{S} \times [t_0, t^*)$. Then we obtain

$$\begin{aligned} & v_t(x,t) - \Delta_{p,\omega} v(x,t) - \psi(t) v^{p-1}(x,t) \\ =& \phi_0(x) k(t) [\psi(t) - \lambda_{p,0}] z^{p-1}(t) + \lambda_{p,0} \phi_0^{p-1}(x) z^{p-1}(t) - \psi(t) \phi_0^{p-1}(x) z^{p-1}(t) \\ =& z^{p-1}(t) \phi_0(x) \left(\psi(t) - \lambda_{p,0}\right) \left(k(t) - \phi_0^{p-2}(x) \right). \end{aligned}$$

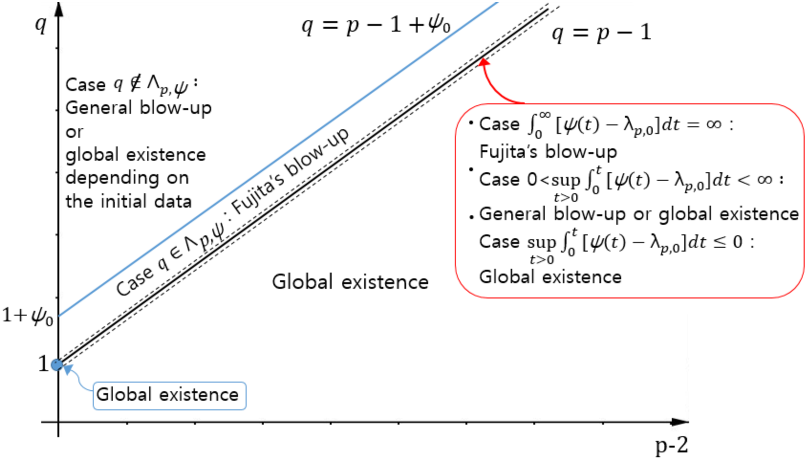
By choosing $k(t)$ and z_0 suitably, we can obtain from comparison principle that

$$\left[\frac{[\phi_0(x)]^{p-2}}{\left[\min_{x\in S\cup\Gamma}\frac{u(x,t_0)}{\phi_0(x)}\right]^{2-p}-(p-2)\min_{x\in S}\phi_0^{p-2}(x)\int_{t_0}^{t}[\psi(\tau)-\lambda_{p,0}]d\tau}\right]^{\frac{1}{p-2}}$$

$$\leq u(x,t)\leq$$

$$\left[\frac{[\phi_0(x)]^{p-2}}{\left[\max_{x\in S\cup\Gamma}\frac{u(x,t_0)}{\phi_0(x)}\right]^{2-p}-(p-2)\max_{x\in S}\phi_0^{p-2}(x)\int_{t_0}^{t}[\psi(\tau)-\lambda_{p,0}]d\tau}\right]^{\frac{1}{p-2}}.$$

Conclusion



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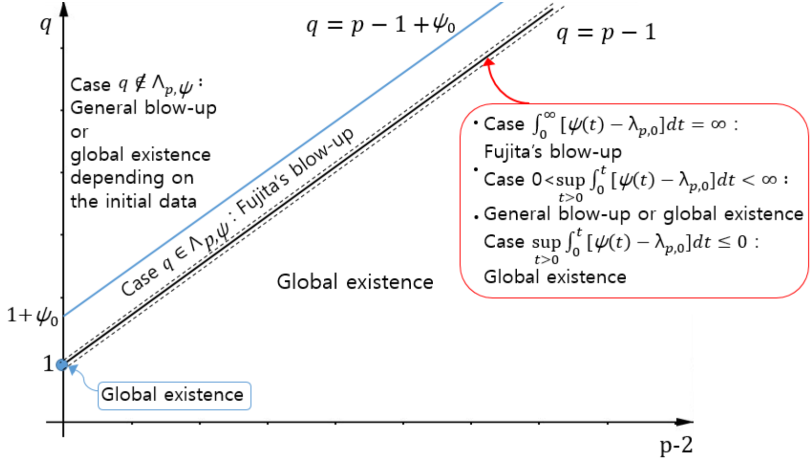
Conclusion



For instance, if $\psi(t) = e^{\beta t}$, then the blue line is $q = p - 1 + \frac{\beta}{\lambda_{p,0}}$, since

$$\Lambda_{p,\psi} = \left\{ q > 1 \mid q \leq p - 1 + \frac{\beta}{\lambda_{p,0}} \right\}.$$

thank you