

Convolution on Weighted Spaces of Functions and Distributions

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Outline

We discuss two classes of locally convex spaces that possess **extremal** domains X , Y and codomains Z for which convolution

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$$\mathcal{C}_0(w) = \mathcal{C}_0(G; w), \quad \mathcal{M}(w) = \mathcal{M}(G; w).$$

- ② C-closed distribution spaces on Euclidean space

$$F \subseteq \mathcal{D}' \text{ such that } F^{**} = F, \quad \mathcal{D}' = \mathcal{D}'(\mathbb{R}^d)$$

endowed with a weighted L^1 -type topology.

The following presents a particular aspect of

-  T. K., R. H., *Convolution operators on weighted spaces of continuous functions and supremal convolution*,
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Notations:

G = a fixed locally compact group

$$\mathcal{U}^+ = \mathcal{U}^+(G) = \{G \rightarrow \overline{\mathbb{R}}_+ \text{ upper semicontinuous}\}$$

$$\overline{\mathbb{R}}_+ = [0, \infty] \text{ with the convention } 0 \cdot \infty = 0$$

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where $\|f\|_{\infty, w} := \sup_{x \in G} |f(x)|w(x)$.

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Weighted Banach Spaces

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Definition

Define $\mathcal{M}(1/w)$ as the space

$$\mathcal{M}(1/w) := \left\{ \mu \in \mathcal{M}_{\{w < \infty\}}; \|\mu\|_{1,1/w} < \infty \right\} \quad (\cong \mathcal{C}_0^\sim(w)')$$

endowed with the norm $\|\mu\|_{1,1/w} := \int (1/w(x)) d|\mu|(x)$.

Convolution and Supremal Convolution

Theorem (cf. Edwards 1965, Gaudry 1967, T. K. and R. H. 2019)

Let $w, v, u \in \mathcal{U}^+$. The following are equivalent:

$$*: \mathcal{M}(1/w) \times \mathcal{M}(1/v) \longrightarrow \mathcal{M}(1/u) \text{ bounded with } C \leq 1. \quad (1)$$

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Definition (cf. Moreau 1969)

Multiplicative **supremal convolution** is defined point-wise as

$$(w \blacksquare v)(z) := \sup_{\substack{x \cdot y = z \\ x, y \in G}} w(x)v(y) \quad \text{for } z \in G, w, v: G \rightarrow \overline{\mathbb{R}}_+.$$

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Note: For $w = v = u$ Equation (3) means $1/w$ is submultiplicative.

Definition (cf. Quantales and their Applications, Rosenthal 1990)

A **Quantale** is a triple (Q, \leq, \bullet) such that

(Q, \leq) is a complete order, (Q₁)

(Q, \bullet) is a semigroup, (Q₂)

$\sup(A \bullet b) = (\sup A) \bullet b$ for all $A \subseteq Q, b \in Q$, (Q_{3a})

$\sup(a \bullet B) = a \bullet (\sup B)$ for all $a \in Q, B \subseteq Q$. (Q_{3b})

Quantales and Residual Division

Definition (cf. Quantales and their Applications, Rosenthal 1990)

A **Quantale** is a triple (Q, \leq, \bullet) such that

$$(Q, \leq) \text{ is a complete order,} \quad (Q_1)$$

$$(Q, \bullet) \text{ is a semigroup,} \quad (Q_2)$$

$$\sup(A \bullet b) = (\sup A) \bullet b \quad \text{for all } A \subseteq Q, b \in Q, \quad (Q_{3a})$$

$$\sup(a \bullet B) = a \bullet (\sup B) \quad \text{for all } a \in Q, B \subseteq Q. \quad (Q_{3b})$$

Proposition + Definition (cf. Rosenthal)

Quantales (Q, \leq, \bullet) have right (left) **residual divisions** $\not\bullet$ ($\not\bullet$):

$$c \not\bullet b := \max \{a \in Q; a \bullet b \leq c\} \quad \text{for } b, c \in Q, \quad (Q_{4a})$$

$$a \not\bullet c := \max \{b \in Q; a \bullet b \leq c\} \quad \text{for } a, c \in Q. \quad (Q_{4b})$$

Extremal Domains for Convolution

Proposition (T. K. and R. H., FCAA 2019, Props. 5.2(c) and 5.3(a)
+ Rosenthal, Prop. 3.1.2)

The triple $(\mathcal{U}^+, \leq, \lceil \blacksquare \rceil)$ is a quantale with the composition

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Conclusions

Let $w, u \in \mathcal{U}^+$. Consider the condition

$$*: \mathcal{M}(1/w) \times Y \longrightarrow \mathcal{M}(1/u) \text{ bounded with } C \leq 1.$$

The largest solution Y of type $\mathcal{M}(1/v)$ is given by $v = w \blacktriangleright u$.

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The following presents some methods used in

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Notations:

$$\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$$

$$\mathcal{D}' = \mathcal{D}'(\mathbb{R}^d)$$

$$\mathfrak{B}(\mathcal{D}) = \{\Phi \subseteq \mathcal{D} \text{ bounded}\}$$

Convolution of Distributions and the C-Dual

Definition (cf. Schwartz 1954, Shiraishi 1959, etc.)

Distributions $f, g \in \mathcal{D}'$ are called **convolvable** iff

$$\phi(\hat{x} + \hat{y}) \cdot (f \otimes g) \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d}) \quad \text{for all } \phi \in \mathcal{D}.$$

The *convolute* $f * g$ is defined as

$$\langle f * g, \phi \rangle := \langle \phi(\hat{x} + \hat{y}) \cdot (f \otimes g), 1 \rangle \quad \text{for } \phi \in \mathcal{D}.$$

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Definition (cf. Yoshinaga and Ogata 1958)

Convolvability induces a **Galois connection**. For $F \subseteq \mathcal{D}'$ define

$$\text{c-dual:} \quad F^* := \{g \in \mathcal{D}' ; \forall f \in F : (f, g) \text{ convolvable}\},$$

$$\text{c-closure:} \quad F^{**} := (F^*)^*,$$

$$\text{c-closed:} \quad :\iff: F = F^{**}.$$



Proposition (T.K. and R.H.)

Convolution induces a composition of c-closed spaces via:

$$F \tilde{*} G := (F * G)^{**} \quad \text{for } F, G \subseteq \mathcal{D}' \text{ c-closed, convolvable.}$$

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This defines a “**quantale with partially defined composition**”.

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- ② F^{**} = smallest hull for F (in the sense of c-closedness).
- ③ $F^{**} \tilde{*} F^*$ = smallest codomain for F (i.t.s.o. c-closedness).

Extending Schwartz' description of Fractional Calculus

Schwartz described fractional integrals as convolution operators

$$I_+^\alpha : \mathcal{D}'_+ \rightarrow \mathcal{D}'_+, \quad f \mapsto Y_\alpha * f$$

with the convolution kernels

$$Y_\alpha(t) := t_+^{\alpha-1}/\Gamma(\alpha) \quad \text{for } \alpha > 0,$$

$$Y_\alpha := D^n Y_{\alpha+n} \quad \text{for } n \in \mathbb{N}, \alpha > -n,$$

and with (co-)domain $\mathcal{D}'_+ := \{f \in \mathcal{D}'(\mathbb{R}) ; -\infty < \inf \text{supp } f\}$.

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For $\alpha \in \mathbb{R} \setminus -\mathbb{N}$ the c-dual of Y_α is

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The largest joint domain for I_+^α , $\alpha \in \mathbb{R}$ is (cf. R. H. and T. K. 2020)

$$\{f \in \mathcal{D}'(\mathbb{R}) ; \forall \phi \in \mathcal{D} : \phi * f(t) \rightarrow 0 \text{ rapidly as } t \rightarrow -\infty\}.$$

Outlook to “Glassy fractional relaxation...”

Consider the linear combination of Y_α kernels

$$L := \sum_{k=1}^n \mu_k Y_{\beta_k} \neq 0 \quad \beta_1, \dots, \beta_n > 0, \quad \mu_1, \dots, \mu_n \in \mathbb{R}.$$

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Using properties of $(\mathcal{D}'_+, *)$ and Y_α one infers well-definedness of

$$K := \left(\sum_{k=1}^n \lambda_k Y_{\alpha_k} \right) * \left(\delta - \sum_{l=1}^m \mu_l Y_{\beta_l} \right)^{*,-1} \quad \lambda_k, \mu_l \in \mathbb{R}.$$

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Problem

- Calculate an asymptotic expansion of the kernels K .

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Problem

- ① Calculate an asymptotic expansion of the kernels K .
- ② Describe convolution modules on which kernels K operate.

Problem

Let $F \subseteq \mathcal{D}'$. Endow the spaces appearing in

$$*: F^{**} \times F^* \longrightarrow F^{**} \tilde{*} F^* \quad (\text{B})$$

with topologies and bornologies systematically such that (B) becomes hypocontinuous.

Adding Topologies

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Ansatz (cf. §30, Topological Vector Spaces, Köthe 1969)

Inspired by Köthes' "normal topology" on perfect spaces $\lambda = \lambda^{\times \times}$

$$p_y: x \mapsto p_y(x) := \sum_{n=1}^{\infty} |x_n| \cdot |y_n| \quad y \in \lambda^{\times},$$

we let the elements of F^* generate seminorms on F .

An Analogue of Köthes' "Normal Topology"

Definition (function-valued seminorms)

Let $\Phi \in \mathfrak{B}(\mathcal{D})$. Define the Φ -modulus as

$$|f|_{\Phi} := \sup_{\phi \in \Phi} |\phi * f| \quad \text{for } f \in \mathcal{D}'.$$

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Definition (a topology for c-closed spaces)

Let $F \subseteq \mathcal{D}'$ be c-closed. The topology $\mathfrak{T}^*(F)$ on F is generated by

$$p_{\Phi,g}: f \mapsto \||f|_{\Phi} \cdot |\check{g}|_{\Phi}\|_1 \quad \text{where } \Phi \in \mathfrak{B}(\mathcal{D}), g \in F^*.$$

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Proposition (T. K. and R. H.)

Let $F \subseteq \mathcal{D}'$ be c-closed.

- ① The space $(F, \mathfrak{T}^*(F))$ is complete.

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Proposition (T. K. and R. H.)

Let $F \subseteq \mathcal{D}'$ be c-closed.

- ① The space $(F, \mathfrak{T}^*(F))$ is complete.
- ② The set \mathcal{D} is dense in $(F, \mathfrak{T}^*(F))$.

Definition (a bornology for c-closed spaces)

Let $F \subseteq \mathcal{D}'$ be c-closed. The bornology $\mathfrak{B}^*(F)$ on F consists of the subsets $B \subseteq F$ that satisfy

$$\sup_{f \in B} |f|_\Phi \in F \quad \text{for all } \Phi \in \mathfrak{B}(\mathcal{D}).$$

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Theorem (T. K. and R. H.)

Let $F, G, H \subseteq \mathcal{D}'$ be c-closed such that

$$(F, G) \text{ convolvable and } F * G \subseteq H.$$

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Definition (a bornology for c-closed spaces)

Let $F \subseteq \mathcal{D}'$ be c-closed. The bornology $\mathfrak{B}^*(F)$ on F consists of the subsets $B \subseteq F$ that satisfy

$$\sup_{f \in B} |f|_\Phi \in F \quad \text{for all } \Phi \in \mathfrak{B}(\mathcal{D}).$$

Theorem (T. K. and R. H.)

Let $F, G, H \subseteq \mathcal{D}'$ be c-closed such that

$$(F, G) \text{ convolvable and } F * G \subseteq H.$$

Convolution is $(\mathfrak{B}^*(F), \mathfrak{B}^*(G))$ -hypocontinuous and bilinear for

$$*: (F, \mathfrak{T}^*(F)) \times (G, \mathfrak{T}^*(G)) \longrightarrow (H, \mathfrak{T}^*(H)).$$

A Relation to Convoluter Spaces

Consider the convoluter space (cf. Debrouwere and Vindas 2020)

$$\mathcal{O}'_C(\mathcal{D}, L^1_W) := \{f \in \mathcal{D}'; (\phi \mapsto \phi * f) \in \mathcal{L}(\mathcal{D}, L^1_W)\}$$

with the topology induced by $\mathcal{L}_b(\mathcal{D}, L^1_W)$, where

$$L^1_W := \left\{ g \in L^1_{\text{loc}}; \forall w \in W : \|g\|_{1,w} = \|g \cdot w\|_1 < \infty \right\}.$$

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Theorem (T. K. and R. H.)

Let $F \subseteq \mathcal{D}'$ be c-closed. It holds $(F, \mathfrak{T}^*(F)) = \mathcal{O}'_C(\mathcal{D}, L^1_W)$ with

$$W := \{|\check{f}|_\Phi ; f \in F^*, \Phi \in \mathfrak{B}(\mathcal{D})\}.$$

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The space $\mathcal{L}_s(\mathcal{D}, L^1_W)$ induces the same topology on $\mathcal{O}'_C(\mathcal{D}, L^1_W)$.

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The latter relates to topological characterizations of subsets and filters via regularization. (cf. Props. 17, 19 in C. Bargetz et al 2017)

Sketch of the Proof (non-trivial direction)

Theorem (cf. Malliavin and Dixmier 1978, Voigt 1984)

Let $\Phi \in \mathfrak{B}(\mathcal{D})$. There exist $\Psi \in \mathfrak{B}(\mathcal{D})$, $\theta_1, \dots, \theta_{2^d} \in \mathcal{D}$ such that

$$\Phi \subseteq \Psi * \theta_1 + \dots + \Psi * \theta_{2^d}.$$

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Let $\Phi \in \mathfrak{B}(\mathcal{D})$. There exist $\theta_1, \dots, \theta_{2^d} \in \mathcal{D}$ and $K \subseteq \mathbb{R}^d$ compact such that

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Lemma

It holds $\|1_K * v\|_{1,w} = \|v\|_{1,1-K*w}$ for $w, v: \mathbb{R}^d \rightarrow \mathbb{R}_+$ measurable and $K \subseteq \mathbb{R}^d$ compact.



Important References

-  T. K., R. H., *Convolution operators on weighted spaces of continuous functions and supremal convolution*, Ann. Math. Pura Appl. (2019)
-  T. K., R. H., *Weyl integrals on weighted spaces*, FCAA (2019)
-  R. H., T. K., *Maximal Domains for Fractional Derivatives and Integrals*, Mathematics, MDPI (2020)
-  T. K., R. H., *Glassy fractional relaxation and convolution modules of distributions* (submitted)

-  K. I. Rosenthal, *Quantales and Their Application*, (1990)
-  A. Debrouwere and J. Vindas, *Topological properties of convolutor space via the short-time Fourier transform*, arXiv preprint arXiv:1801.09246
-  C. Bargetz, E. A. Nigsch and N. Ortner, *Convolvability and regularization of distributions*, Ann. Mat. Pura Appl. (2017)
-  J. Voigt, *Factorization in Fréchet Algebras*, Jour. London Math. Soc. (1984)
-  P. Malliavin and J. Dixmier, *Factorisations de fonctions et de vecteurs indefiniment differentiables*, Bull. Sci. Math. (1978)