Selected Topics in Almost Periodicity

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PREFACE

The theory of almost periodic functions is still very popular and unavoidable in the world of mathematics. The main purpose of this monograph, entitled "Selected Topics in Almost Periodicity", is to present the recent research results of author in adequate detail.

In the existing literature, there are numerous research articles dealing with the almost periodic (automorphic) properties and asymptotically almost periodic (automorphic) properties of abstract Volterra integro-differential equations in Banach spaces, degenerate or non-degenerate in time variable. Special attention has been paid to fractional integro-differential equations and inclusions, primarily from their invaluable importance in modeling of real world phenomena appearing in physics, chemistry, biology, economy, aerodynamics etc. This is probably the first research monograph considering uniformly recurrent solutions and c-almost periodic solutions of abstract Volterra integro-differential equations as well as Stepanov, Weyl and Doss generalizations of almost periodic functions in Lebesgue spaces with variable coefficients. We have tried to aggregate many complicated and miscellaneous parts into a stable, compact unity.

This monograph is consisting of two chapters, which are further divided into sections and subsections. As in my previously published monographs [**232**]-[**236**], the numbering of theorems, propositions, lemmas, corollaries, definitions, etc., is done by chapter and section; we sort the reference list in alphabetical order. The readers should be familiar with the fundamentals of functional analysis and integration theory, the basic theory of abstract differential equations in Banach spaces, the basic theory of vector-valued almost periodic functions and vector-valued almost automorphic functions.

Conventional wisdom says you should know your target audience. Concerning the groups of people the book would interest, we would like to mention experts in the fields of almost periodicity and almost automorphicity, researchers in abstract partial differential equations, experts from all areas of functional analysis and PhD students in mathematics. We have tried the reference list to be avoided from any form of plagiarism. The book is not intended to be a thorough and exhaustive study.

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NOTATION

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: the natural numbers, integers, rationals, reals, complexes.
For any $s \in \mathbb{R}$, we denote $\lfloor s \rfloor = \sup\{l \in \mathbb{Z} : s \geqslant l\}$ and $\lceil s \rceil = \inf\{l \in \mathbb{Z} : s \leqslant l\}$.
$\operatorname{Re} z$, $\operatorname{Im} z$: the real and imaginary part of a complex number $z \in \mathbb{C}$; $|z|$: the modul of z, $\arg(z)$: the argument of a complex number $z \in \mathbb{C} \smallsetminus \{0\}$.
$\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.
$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leqslant r\}$ $(z_0 \in \mathbb{C},\ r > 0)$.
$\Sigma_\alpha = \{z \in \mathbb{C} \smallsetminus \{0\} : |\arg(z)| < \alpha\}$, $\alpha \in (0, \pi]$.
$\operatorname{card}(G)$: the cardinality of G.
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
$\mathbb{N}_n = \{1, \cdots, n\}$.
$\mathbb{N}_n^0 = \{0, 1, \cdots, n\}$.
$\mathbb{R}^n$: the real Euclidean space, $n \geqslant 2$.
If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index, then we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$.
$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$.
$f^{(\alpha)} := \partial^{|\alpha|} f / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$; $D^\alpha f := (-i)^{|\alpha|} f^{(\alpha)}$.
If (X, τ) is a topological space and $F \subseteq X$, then the interior, the closure, the boundary, and the complement of F with respect to X are denoted by $\operatorname{int}(F)$ (or F°), $\overline{F}$, ∂F and F^c, respectively.
If Z is a vector space over the field $\mathbb{F} \in \{\mathbb{R},\ \mathbb{C}\}$, then for each non-empty subset F of Z by $span(F)$ we denote the smallest linear subspace of Z which contains F.
X : a complex Banach space.
$L(X, Y)$: the space of all continuous linear mappings between complex Banach spaces X and Y, $L(X) = L(X, X)$.
X^* : the dual space of X.
A : a linear operator on X.
$\mathcal{A}$: a multivalued linear operator on X (MLO).
If F is a subspace of X, then we denote by $\mathcal{A}_{|F}$ the part of $\mathcal{A}$ in F.
$\chi_\Omega(\cdot)$: the characteristic function, defined to be identically one on Ω and zero elsewhere.
$\Gamma(\cdot)$: the Gamma function.
If $\alpha > 0$, then $g_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$; $g_0(t) \equiv$ the Dirac delta distribution.

If $1 \leqslant p < \infty$, $(X, \| \cdot \|)$ is a complex Banach space, and $(\Omega, \mathcal{R}, \mu)$ is a measure space, then $L^p(\Omega, X, \mu)$ denotes the space which consists of those strongly μ-measurable functions $f : \Omega \to X$ such that $\|f\|_p := (\int_\Omega \|f(\cdot)\|^p d\mu)^{1/p}$ is finite; $L^p(\Omega, \mu) \equiv L^p(\Omega, \mathbb{C}, \mu)$.

$L^\infty(\Omega, X, \mu)$: the space which consists of all strongly μ-measurable, essentially bounded functions.

$\|f\|_\infty = ess \sup_{t\in\Omega} \|f(t)\|$, the norm of a function $f \in L^\infty(\Omega, X, \mu)$.

$L^p(\Omega : X) \equiv L^p(\Omega, X) \equiv L^p(\Omega, X, \mu)$, if $p \in [1, \infty]$ and $\mu = m$ is the Lebesgue measure; $L^p(\Omega) \equiv L^p(\Omega : \mathbb{C})$.

$L^p_{loc}(\Omega : X)$: the space consisting of those Lebesgue measurable functions $u(\cdot)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'} \in L^p(\Omega' : X)$; $L^p_{loc}(\Omega) \equiv L^p_{loc}(\Omega : \mathbb{C})$ $(1 \leqslant p \leqslant \infty)$.

Assume that $I = \mathbb{R}$ or $I = [0, \infty)$. By $C_b(I : X)$ we denote the space consisting of bounded continuous functions from I into X; $C_0(I : X)$ denotes the closed subspace of $C_b(I : X)$ consisting of functions vanishing as the absolute value of the argument tends to plus infinity. By $BUC(I : X)$ we denote the space consisting of all bounded uniformly continuous functions from I to X. The sup-norm turns these spaces into Banach's.

$C^k(\Omega : X)$: the space of k-times continuously differentiable functions $(k \in \mathbb{N}_0)$ from a non-empty subset $\Omega \subseteq \mathbb{C}$ into X; $C(\Omega : X) \equiv C^0(\Omega : X)$.

$\mathcal{D} = C_0^\infty(\mathbb{R})$, $\mathcal{E} = C^\infty(\mathbb{R})$ and $\mathcal{S} = \mathcal{S}(\mathbb{R})$: the Schwartz spaces of test functions.

If $\emptyset \neq \Omega \subseteq \mathbb{R}$, then by $\mathcal{D}_\Omega$ we denote the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\mathrm{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$.

$\mathcal{D}' := L(\mathcal{D}, \mathbb{C})$: the space consisting of all scalar-valued distributions.

If $k \in \mathbb{N}$, $p \in [1, \infty]$ and Ω is an open non-empty subset of $\mathbb{R}^n$, then we denote by $W^{k,p}(\Omega : X)$ the Sobolev space which consists of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every $i \in \mathbb{N}_k^0$ and for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant k$, one has $D^\alpha u \in L^p(\Omega : X)$.

$W^{k,p}_{loc}(\Omega : X)$: the space of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'} \in W^{k,p}(\Omega' : X)$.

$\mathcal{F}$, $\mathcal{F}^{-1}$: the Fourier transform and its inverse transform.

$L^1_{loc}([0, \infty))$, resp. $L^1_{loc}([0, \tau))$: the space of scalar-valued locally integrable functions on $[0, \infty)$, resp. $[0, \tau)$.

J_t^α : the Riemann-Liouville fractional integral of order $\alpha > 0$.

D_t^α : the Riemann-Liouville fractional derivative of order $\alpha > 0$.

$\mathbf{D}_t^\alpha$: the Caputo fractional derivative of order $\alpha > 0$.

$D_{t,+}^\gamma$: the Weyl-Liouville fractional derivative.

$E_{\alpha,\beta}(z)$: the Mittag-Leffler function $(\alpha > 0, \beta \in \mathbb{R})$; $E_\alpha(z) \equiv E_{\alpha,1}(z)$.

$\Psi_\gamma(t)$: the Wright function $(0 < \gamma < 1)$.

$\mathrm{supp}(f)$: the support of function $f(t)$.

$L^{p(x)}(\Omega : X)$: the Lebesgue space with variable exponent $p(x)$.

Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $1 \leqslant p < \infty$.

$P_c(I : X)$: the space of all continuous c-periodic functions $f : I \to X$ $(c > 0)$.

$AP(I : X)$: the Banach space consisting of all almost periodic functions from the interval I into X, equipped with the sup-norm.

$UR(I : X)$: the collection of all uniformly recurrent functions from the interval I into X, equipped with the sup-norm.

$AAP(I : X)$: the Banach space consisting of all asymptotically almost periodic functions from the interval I into X, equipped with the sup-norm.

$AUR(I : X)$: the collection of all asymptotically uniformly recurrent functions from the interval I into X, equipped with the sup-norm.

$AP(I \times Y : X)$: the set consisting of all almost periodic functions $f : I \times Y \to X$.

$UR(I \times Y : X)$: the set consisting of all uniformly recurrent functions $f : I \times Y \to X$.

$AAP(I \times Y : X)$: the set consisting of all asymptotically almost periodic functions $f : I \times Y \to X$.

$AUR(I \times Y : X)$: the set consisting of all asymptotically uniformly recurrent functions $f : I \times Y \to X$.

$AP_{\odot_g}(I \times Y : X)$: the collection of all two-parameter $\odot_g$-almost periodic functions.

$AP_{\odot_g,b}(I \times Y : X)$: the collection of all two-parameter $\odot_g$-almost periodic functions on bounded sets.

$UR_b(I \times Y : X)$: the collection of all two-parameter uniformly recurrent functions on bounded sets.

$e - W^p_{ap,\mathbf{K}}(I \times Y : X)$: the collection of all equi-Weyl p-almost periodic functions.

$W^p_{ap,\mathbf{K}}(I \times Y : X)$: the collection of all Weyl p-almost periodic functions.

$W^p_{0,\mathbf{K}}(I \times Y : X)$: the collection of all Weyl p-vanishing function.

$e - W^p_{0,\mathbf{K}}(I \times Y : X)$: the collection of all equi-Weyl p-vanishing function.

$L^p_S(I : X)$: the space of all Stepanov p-bounded functions.

$L^{p(x)}_S(I : X)$: the space of all Stepanov $p(x)$-bounded functions.

$APS^p(I : X)$: the Banach space of all Stepanov p-almost periodic functions $I \to X$, equipped with the Stepanov norm.

$AAPS^p(I : X)$: the Banach space of all asymptotically Stepanov p-almost periodic functions $f : I \to X$, equipped with the Stepanov norm.

$AURS^p(I : X)$: the collection of all asymptotically Stepanov p-uniformly recurrent functions $f : I \to X$.

$AAPS^p(I \times Y : X)$: the vector space consisting of all Stepanov p-almost periodic functions $f : I \times Y \to X$.

$APS^{p(x)}(I : X)$: the space of all Stepanov $p(x)$-almost periodic functions $f : I \to X$.

$AAPS^{p(x)}(I : X)$: the space of all asymptotically Stepanov $p(x)$-almost periodic functions $f : I \to X$.

$AURS^{p(x)}(I : X)$: the collection of all asymptotically Stepanov $p(x)$-uniformly recurrent functions $f : I \to X$.

$AAPS^{p(x)}(I\times Y:X)$: the vector space consisting of all Stepanov p-almost periodic functions $f:I\times Y\rightarrow X$.

$e-W_{ap}^{p}(I:X)$: the collection of all equi-Weyl-p-almost periodic functions $f:I\rightarrow X$.

$W_{ap}^{p}(I:X)$: the collection of all Weyl-p-almost periodic functions $f:I\rightarrow X$.

$W_{0}^{p}([0,\infty):X)$ and $e-W_{0}^{p}([0,\infty):X)$: the collections consisting of all Weyl-p-vanishing functions and equi-Weyl-p-vanishing functions, respectively.

$\mathrm{B}^{p}(I:X)$ and $B^{p}(I:X)$: the sets consisting of all Besicovitch-Doss-p-almost periodic functions $f:I\rightarrow X$ and all Besicovitch-p-almost periodic functions $f:I\rightarrow X$, respectively.

$\mathrm{D}^{p}(I:X)$: the class consisting of all Doss-p-almost periodic functions $f:I\rightarrow X$.

$B_{0}^{p}([0,\infty):X)$: the vector space consisting of all Besicovitch-p-vanishing functions.

$ANP_{0}(I:X)$: the linear span of almost anti-periodic functions $f:I\rightarrow X$; $ANP(I:X)$: the linear closure of $ANP_{0}(I:X)$ in $AP(I:X)$.

$AA(\mathbb{R}:X)$ and $AA_{c}(\mathbb{R}:X)$: the Banach spaces consisting of all almost automorphic functions and compactly almost automorphic functions, respectively, equipped with the sup-norm.

$W^{p}AA(\mathbb{R}:X)$: the vector space consisting of all Weyl-p-almost automorphic functions.

$B^{p}AA(\mathbb{R}:X)$: the vector space consisting of all Besicovitch p-almost automorphic functions.

$\mathcal{P}_{p,k}(I:X)$: the vector space consisting of all Bloch (p,k)-periodic functions.

$Q-AAP(I:X)$: the set consisting of all quasi-asymptotically almost periodic functions from I into X.

$Q-AUR(I:X)$: the set consisting of all quasi-asymptotically uniformly recurrent functions from I into X.

$S^{p}Q-AAP(I:X)$: the set consisting of all Stepanov p-quasi-asymptotically almost periodic functions from I into X.

$S^{p(x)}Q-AAP(I:X)$, $S^{p(x)}Q-AUR(I:X)$ and $S^{p(x)}SAP_{\omega}(I:X)$: the set consisting of all Stepanov $p(x)$-quasi-asymptotically almost periodic functions from I into X, the set consisting of all Stepanov $p(x)$-quasi-asymptotically uniformly recurrent functions from I into X and the set consisting of all Stepanov $p(x)$-asymptotically ω-periodic functions, respectively.

$\mathcal{S}B_{k}(I:X)$: the space of all semi-Bloch k-periodic functions from I into X.

$\mathcal{SANP}(I:X)$: the space consisting of all semi-anti-periodic functions from I into X.

$APS^{p(x)}(I:X)$: the space of all Stepanov $p(x)$-almost periodic functions.

$AAPS^{p(x)}(I:X)$: the space of all asymptotically Stepanov $p(x)$-almost periodic functions.

$APS^{p(x)}(I\times Y:X)$: the space of all Stepanov $p(x)$-almost periodic functions $f:I\times Y\rightarrow X$.

$AAPS^{p(x)}(I \times Y : X)$: the space of all asymptotically Stepanov $p(x)$-almost periodic functions $f : I \times Y \to X$.

$(e-)W_{ap}^{(p,\phi,F)}(I : X)$: the collection of all (equi)-Weyl-(p, ϕ, F)-almost periodic functions $f : I \to X$.

$(e-)W_{ap}^{(p,\phi,F)_i}(I : X)$: the collection of all (equi)-Weyl-$(p, \phi, F)_i$-almost periodic functions $f : I \to X$ $(i = 1, 2)$.

$(e-)W_{ap}^{[p,\phi,F]}(I : X)$: the collection of all (equi)-Weyl-$[p, \phi, F]$-almost periodic functions $f : I \to X$.

$(e-)W_{ap}^{[p,\phi,F]_i}(I : X)$: the collection of all (equi)-Weyl-$[p, \phi, F]_i$-almost periodic functions $f : I \to X$ $(i = 1, 2)$.

$W_{\phi,F,0}^{p(x)}([0, \infty) : X)$ and $e - W_{\phi,F,0}^{p(x)}([0, \infty) : X)$ $[W_{\phi,F,0}^{p(x);1}([0, \infty) : X)$ and $e - W_{\phi,F,0}^{p(x);1}([0, \infty) : X)/W_{\phi,F,0}^{p(x);2}([0, \infty) : X)$ and $e - W_{\phi,F,0}^{p(x);2}([0, \infty) : X)]$: the sets consisting of all Weyl-(p, ϕ, F)-vanishing functions and equi-Weyl-(p, ϕ, F)-vanishing functions [Weyl-$(p, \phi, F)_1$-vanishing functions and equi-Weyl-$(p, \phi, F)_1$-vanishing functions/Weyl-$(p, \phi, F)_2$-vanishing functions and equi-Weyl-$(p, \phi, F)_2$-vanishing functions].

$UR_{\omega,c}(I : X)$, $AP_{\omega,c}(I : X)$, $AA_{\omega,c}(I : X)$ and $AA_{\omega,c;\mathbf{c}}(I : X)$: the space of all (ω, c)-uniformly recurrent functions, the space of all (ω, c)-almost periodic functions, the space of all (ω, c)-almost automorphic functions and the space of all compactly (ω, c)-almost automorphic, respectively.

$S^pUR_{\omega,c}(I : X)$, $S^pAP_{\omega,c}(I : X)$ and $S^pAA_{\omega,c}(I : X)$: the space of all Stepanov (p, ω, c)-uniformly recurrent functions, the space of all Stepanov (p, ω, c)-almost periodic functions and the space of all Stepanov (p, ω, c)-almost automorphic functions, respectively.

$AS^pUR_{\omega,c}(I : X)$, $AS^pAP_{\omega,c}(I : X)$ and $AS^pAA_{\omega,c}(I : X)$: the space of all asymptotically Stepanov (p, ω, c)-uniformly recurrent functions, the space of all asymptotically Stepanov (p, ω, c)-almost periodic functions and the space of all asymptotically Stepanov (p, ω, c)-almost automorphic functions, respectively.

$UR_{\omega,c,i}(I : X)$ and $AP_{\omega,c,i}(I : X)$: the space of all (ω, c)-uniformly recurrent functions of type i and the space of all (ω, c)-almost periodic functions of type i, respectively $(i = 1, 2)$.

$UR_c(I : X)$: the set consisting of all c-uniformly recurrent functions from the interval I into X.

$AP_c(I : X)$: the set consisting of all c-almost periodic functions from the interval I into X.

$\mathcal{SP}_{c,i}(I : X)$: the set of all semi-c-periodic functions of type i, where $i = 1, 2$.

$\mathcal{SP}_{c,i,+}(I : X)$: the set of all semi-c-periodic functions of type i_+, where $i = 1, 2$.

$e - W_{ur}^{(p(x),\phi,F)}(I : X)$: the set of all equi-Weyl-$(p(x), \phi, F)$-uniformly recurrent functions.

$e - W_{ur}^{(p(x),\phi,F)_1}(I : X)$: the set of all equi-Weyl-$(p(x), \phi, F)_1$-uniformly recurrent functions.

$e - W_{ur}^{(p(x),\phi,F)_2}(I : X)$: the set of all equi-Weyl-$(p(x),\phi,F)_2$-uniformly recurrent functions.
$e - W_{ur}^{[p(x),\phi,])}(I : X)$: the set of all equi-Weyl-$[p(x),\phi,F]$-uniformly recurrent functions.
$e - W_{ur}^{[p(x),\phi,F]_1}(I : X)$: the set of all equi-Weyl-$[p(x),\phi,F]_1$-uniformly recurrent functions.
$e - W_{ur}^{[p(x),\phi,F]_2}(I : X)$: the set of all equi-Weyl-$[p(x),\phi,F]_2$-uniformly recurrent functions.
$Q - AUR_{\mathbf{B}}(I \times Y : X)$: the set consisting of all quasi-asymptotically uniformly recurrent, uniformly on $\mathbf{B}$ functions from $I \times Y$ into X.
$UR_{\omega,c}(I : X)$, $AP_{\omega,c}(I : X)$, $AA_{\omega,c}(I : X)$ and $AA_{\omega,c;\mathbf{c}}(I : X)$: the space of all (ω,c)-uniformly recurrent functions, the space of all (ω,c)-almost periodic functions, the space of all (ω,c)-almost automorphic functions and the space of all compactly (ω,c)-almost automorphic functions, respectively.
$S^pUR_{\omega,c}(I : X)$, $S^pAP_{\omega,c}(I : X)$ and $S^pAA_{\omega,c}(I : X)$: the space of all Stepanov (p,ω,c)-uniformly recurrent functions, the space of all Stepanov (p,ω,c)-almost periodic functions and the space of all Stepanov (p,ω,c)-almost automorphic functions, respectively.
$UR_{\omega,c,i}(I : X)$ and $AP_{\omega,c,i}(I : X)$: the space of all (ω,c)-uniformly recurrent functions of type i and the space of all (ω,c)-almost periodic functions of type i, respectively $(i = 1, 2)$.
$S^pUR_{\omega,c,2}([0,\infty) : X)$ and $S^pAP_{\omega,c,2}([0,\infty) : X)$: the collection of all Stepanov (p,ω,c)-uniformly recurrent functions of type 2 and the collection of all Stepanov (p,ω,c)-almost periodic functions of type 2, respectively.
$S^{p(x)}UR_{\omega,c,2}([0,\infty) : X)$ and $S^{p(x)}AP_{\omega,c,2}([0,\infty) : X)$: the collection of all Stepanov $(p(x),\omega,c)$-uniformly recurrent functions of type 2 and the collection of all Stepanov $(p9x),\omega,c)$-almost periodic functions of type 2, respectively.
$PAP_{0;\omega,c,i}(\mathbb{R} \times Y : X)$: the space of $(\omega,c,1)$-pseudo ergodic vanishing functions $(i = 1, 2)$.
$AP_{\omega,c,i}(\mathbb{R} \times Y : X)$, resp. $AA_{\omega,c,i}(\mathbb{R} \times Y : X)$: the space of all (ω,c,i)-almost periodic, resp. (ω,c,i)-almost automorphic, functions.$(i = 1, 2)$.
$PAP_{\omega,c}(\mathbb{R} : X)$, resp. $PAA_{\omega,c}(\mathbb{R} : X)$: the space of all (ω,c)-pseudo almost periodic, resp. (ω,c)-pseudo almost automorphic, functions.
$PAP_{\omega,c,i}(\mathbb{R} \times Y : X)$, resp. $PAA_{\omega,c,i}(\mathbb{R} \times Y : X)$: the space of all (ω,c,i)-pseudo almost periodic, resp. (ω,c,i)-pseudo almost automorphic, functions.
$\mathcal{B}_{AP_{w,c}}$: the space of smooth (w,c) $-$almost periodic functions defined on $\mathbb{R}$.
$\mathcal{B}'_{AP_{w,c}}$: the set of (w,c) $-$almost periodic distributions.
$\mathcal{B}'_{0+}$: the space of bounded distributions vanishing at infinity.
$\mathcal{B}'_{aap}(\mathbb{R}_+)$: the space of asymptotically almost periodic Schwartz distributions.
$(e-)W^p_{ap;c}(I : X)$: the collection of all (equi-)Weyl-(p,c)-almost periodic functions.
$SAP_{\omega;c}(I : X)$ and $SAP_c(I : X)$: the sets of all S-asymptotically (ω,c)-periodic functions and S_c-asymptotically periodic functions $(\omega \in I,\ c \in \mathbb{C} \smallsetminus \{0\})$.

$S^{p(x)}Q - AAP_c(I : X)$: the set consisting of all Stepanov $p(x)$-quasi-asymptotically c-almost periodic functions from I into X.

$S^pQ - AAP_c(I : X)$: the set consisting of all Stepanov p-quasi-asymptotically c-almost periodic functions from I into X.

$Q - AAP_c(I : X)$: the collection of all quasi-asymptotically c-almost periodic functions from I into X, respectively.

$Q - AAP_{c;\mathcal{F}}(I \times Y : X)$: the collection consisting of all quasi-asymptotically c-almost periodic functions $F : I \times Y \to X$ on $\mathcal{F}$.

INTRODUCTION

The class of almost periodic functions was introduced by a Danish mathematician H. Bohr [**75**] (1925), the younger brother of the Nobel Prize-winning physicist N. Bohr, and later generalized by many others. The class of almost automorphic functions was introduced by an American mathematician S. Bochner [**73**] (1962). The theories of almost periodic functions and almost automorphic functions are still very active fields of investigations of numerous authors, full of open problems, conjectures, hypotheses and possibilities for further expansions.

There is an enormous literature devoted to the study of almost periodic and almost automorphic solutions of the abstract differential equations of the first order. PEROV [**309**]-[**310**] [**49**]-[**50**] opisati rezultate rada The study of almost periodic solutions of the abstract Volterra integro-differential equations was initiated by J. Prüss in [**319**, Section 11.4], where the author has analyzed the almost periodic solutions, Stepanov almost periodic solutions and asymptotically almost periodic solutions of the following abstract integro-differential equation

$$u'(t)=\int_0^{\infty}A_0(s)u'(t-s)\,ds+\int_0^{\infty}dA_1(s)u(t-s)+f(t),\quad t\in\mathbb{R};$$

here $A_0\in L^1([0,\infty):L(Y,X))$, $t\mapsto A_1(t)\in L(Y,X)$, $t\geqslant 0$ is locally of bounded variation, X and Y are Banach spaces such that Y is densely and continuously embedded into X. Almost immediately after that, Q.-P. Vu [**345**] has investigated the almost periodicity of the abstract Cauchy problem

$$u'(t)=Au(t)+\int_0^{\infty}dBu(\tau)u(t-\tau)+f(t),\quad t\in\mathbb{R},$$

where A is a closed linear operator acting on a Banach space X, $(B(t))_{t\geqslant 0}$ is a family of closed linear operators on X and $f:\mathbb{R}\to X$ is continuous.

It is very difficult and unpleasant to say precisely who was the first to study the almost periodic solutions of the abstract fractional differential equations (for almost periodic type solutions of abstract diffrerential equations with integer order derivatives, we refer the reader to [**31, 32, 38, 39, 59, 88, 89, 90, 163, 215, 260, 303, 344, 369**]). J. Mu, Y. Zhoa and L. Peng [**297**] have recently investigated the periodic solutions and S-asymptotically periodic solutions to fractional evolution equation

$$D_{t,+}^{\gamma}u(t)=-Au(t)+g(t),\ t\in\mathbb{R}$$

and its semilinear analogue

$$D_{t,+}^{\gamma}u(t) = -Au(t) + g(t, u(t)),\ t \in \mathbb{R},$$

where $D_{t,+}^{\gamma}$ denotes the Weyl-Liouville fractional derivative of order $\gamma \in (0,1)$, A is the infinitesimal generator of an exponentially decaying strongly continuous semigroup of operators and $g : \mathbb{R}\times X \to X$ satisfies certain assumptions (see also the article [**8**] by R. Agarwal, B. de Andrade and C. Cuevas). Later, the author of this monograph extended their results to the abstract fractional differential inclusion

$$D_{t,+}^{\gamma}u(t) \in -\mathcal{A}u(t) + g(t),\ t \in \mathbb{R}$$

and its semilinear analogue

$$D_{t,+}^{\gamma}u(t) \in -\mathcal{A}u(t) + g(t, u(t)),\ t \in \mathbb{R},$$

where $\mathcal{A}$ is a closed multivalued linear operator satisfying condition (P) below. The obtained results enable one to consider the almost periodic type solutions of the following fractional Poisson heat equations

$$\begin{cases} \frac{\partial}{\partial t}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t, m(x)v(t,x)), \quad t \in \mathbb{R},\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

$$\begin{cases} \mathbf{D}_t^{\gamma}[m(x)v(t,x)] = \Delta v(t,x) + bv(t,x), \quad t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega; \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

and the following fractional semilinear equation with higher order differential operators in the Hölder space $X = C^{\alpha}(\overline{\Omega})$:

$$\begin{cases} \mathbf{D}_t^{\gamma}u(t,x) = -\sum_{|\beta|\leqslant 2m} a_{\beta}(t,x)D^{\beta}u(t,x) - \sigma u(t,x) + f(t, u(t,x)),\ t \geqslant 0,\ x \in \Omega; \\ u(0,x) = u_0(x), \quad x \in \Omega; \end{cases}$$

see [**234**] for more details. Let us also recall that R. Ponce [**318**] has investigated the bounded mild solutions of the following non-degenerate fractional integro-differential equation

$$(1) \qquad D_{t,+}^{\gamma}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t, u(t)), \quad t \in \mathbb{R},$$

where A is a closed linear operator, $a \in L^1([0,\infty))$ is a scalar-valued kernel and $f(\cdot,\cdot)$ satisfies some Lipschitz type conditions. In particular, almost periodic solutions of (1) have been analyzed.

In the non-degenerate case, many results concerning the existence and uniqueness of almost periodic type solutions and almost automorphic type solutions to the abstract (semilinear) fractional differential equations have recently been given by numerous authors. In almost all these results (in the linear setting, the quite exceptional are some examples and results presented by S. Zaidman [**359**, Examples 4, 5, 7, 8; pp. 32-34], which have been employed by numerous authors so far, for various purposes; we will also use these examples to illustrate our results about the existence and uniqueness of almost periodic type solutions of the abstract integro-differential equations), the basic key is to investigate the invariance of certain kinds of generalized almost periodicity and generalized almost automorphicity under the actions of the infinite convolution product

$$t \mapsto \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R}$$

and the finite convolution product

$$t \mapsto \int_{0}^{\infty} R(t-s)f(s)\,ds, \quad t \geqslant 0.$$

Here, it is commonly assumed that $(R(t))_{t\geqslant 0} \subseteq L(X,Y)$ is a non-degenerate strongly continuous operator family between the Banach spaces X and Y which exponentially or, at least, polynomially decays as $t \to +\infty$. In [**234**], we have investigated the case $(R(t))_{t>0} \subseteq L(X,Y)$ is a degenerate strongly continuous operator family which decays similarly as $t \to +\infty$, but we have allowed $(R(t))_{t>0}$ to have a removable singularity at zero; by that we basically mean that there exists a number $\zeta \in (0,1)$ such that the operator family $(t^{\zeta}R(t))_{t\geqslant 0}$ is well defined and strongly continuous at the point $t=0$. The integral generator of $(R(t))_{t\geqslant 0}$ is not single-valued any longer and this is the main reason why we have employed the multivalued linear approach to the abstract integro-differential equations in [**234**], which is also obeyed in this monograph. For the theory of abstract degenerate differential equations of first order, mention should be made of the research monographs [**87**] by R. W. Caroll and R. W. Showalter, [**167**] by A. Favini, A. Yagi, [**317**] by M. V. Plekhanova, V. E. Fedorov and [**337**] by G. A. Sviridyuk, V. E. Fedorov. The well-posedness of the abstract degenerate Cauchy problem

$$Bu(t) = f(t) + \int_{0}^{t} a(t-s)Au(s)\,ds,\ t \in [0,\tau),$$

where $0 < \tau \leqslant \infty$, $t \mapsto f(t)$, $t \in [0,\tau)$ is a continuous mapping, $a \in L^1_{loc}([0,\tau))$ and A, B are closed linear operators, has been thoroughly analyzed in the monograph [**236**], which provides the reader a valuable information about the abstract degenerate Volterra integro-differential equations (for scalar-valued Volterra integro-differential equations, we refer the reader to the monograph [**191**] by G. Gripenberg, S. O. Londen, O. J. Staffans).

We will say just a few words about periodic solutions of the abstract degenerate Volterra integro-differential equations. In [**37**], V. Barbu and A. Favini have considered 1-periodic solutions of abstract degenerate differential equation $(d/dt)(Bu(t)) = Au(t), t \geqslant 0$, accompanied with inital condition $(Bu)(0) = (Bu)(1)$, by using P. Grisvard's sum of operators method and some results from investigation of J. Prüss [**320**] in the non-degenerate case. The authors reduced the above problem to $v'(t) \in \mathcal{A}v(t)$, $t \geqslant 0$, $v(0) = v(1)$, where the multivalued linear operator $\mathcal{A}$ is given by $\mathcal{A} = AB^{-1}$. The main problem is whether the inclusion $1 \in \rho(\mathcal{A})$ holds or not; recall that J. Prüss [**320**] have proved that $1 \in \rho(A)$ if and only if $2\pi i\mathbb{Z} \subseteq \rho(A)$ and $\sup(\{\|(2\pi i n - A)^{-1}\| : n \in \mathbb{Z}\}) < \infty$, provided that A generates a non-degenerate strongly continuous semigroup. Applications are given to the Poisson heat equation in $H^{-1}(\Omega)$ and $L^2(\Omega)$, as well as to some systems of ordinary differential equations. On the other hand, C. Lizama and R. Ponce [**275**] have analyzed the existence of 2π-periodic solutions to the following abstract inhomogeneous linear equation

$$\frac{d}{dt}(Bu(t)) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t), \quad t \geqslant 0, \tag{2}$$

subjected with the initial condition $(Bu)(0) = (Bu)(2\pi)$. The authors also considered the maximal regularity of (2) in periodic Besov, Triebel-Lizorkin and Lebesgue vector-valued function spaces.

It is also worth noting that S. Abbas, V. Kavitha and R. Murugesu have recently analyzed Stepanov-like (weighted) pseudo almost automorphic solutions to the following fractional order abstract integro-differential equation:

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t), Ku(t)), \quad t \in \mathbb{R},$$

where

$$Ku(t) = \int_{-\infty}^{t} k(t-s)h(s, u(s))\,ds, \quad t \in \mathbb{R},$$

$1 < \alpha < 2$, A is a sectorial operator with domain and range in X, of negative sectorial type $\omega < 0$, the function $k(t)$ is exponentially decaying, the functions $f : \mathbb{R} \times X \times X \to X$ and $h : \mathbb{R} \times X \to X$ are Stepanov-like weighted pseudo almost automorphic in time for each fixed elements of $X \times X$ and X, respectively, satisfying some extra conditions ([**3**]).

The study of differential equations with discontinuous arguments was initiated by A. D. Myshkis [**299**] in 1977. The analysis of asymptotically anti-periodic solutions for nonlinear differential first-order equations with piecewise constant argument carried out by W. Dimbour and V. Valmorin [**150**] has been recently reconsidered and extended for asymptotically Bloch periodic solutions for nonlinear fractional differential inclusions with piecewise constant argument by M. Kostić and D. Velinov in [**253**]. We have considered the following fractional differential Cauchy inclusion with piecewise constant argument:

$$\mathbf{D}_t^{\gamma}u(t) \in \mathcal{A}u(t) + A_0u\big(\lfloor t\rfloor\big) + g\big(t, u\big(\lfloor t\rfloor\big)\big),\ t > 0; \quad u(0) = u_0,$$

where $A_0 \in L(X)$, $g : [0,\infty) \times X \to X$ is a given function, and $\mathbf{D}_t^{\gamma}u(t)$ denotes the Caputo fractional derivative or order γ, taken in a weakaned sense (cf. the paragraph preceding Definition 2.5.21). It is also worth noting that A. Chávez, S. Castillo and M. Pinto [**95**] have analyzed the the existence of a unique almost automorphic solution on R for the following differential equations with a piecewise constant argument

$$y'(t) = A(t)y(t) + B(t)y(\lfloor t\rfloor) + f(t, y(t), y(\lfloor t\rfloor)), \quad t \in \mathbb{R}, \tag{3}$$

where $A(t)$ and $B(t)$ are almost automorphic $p \times p$ complex matrices and $f : \mathbb{R} \times \mathbb{C}^p \times \mathbb{C}^p \to \mathbb{C}^p$ is an almost automorphic function satsifying a condition of Lipschitz type. The study carried out in [**95**] leans heavily on the use of results on discontinuous almost automorphic functions, exponential dichotomies and the Banach fixed point theorem. The almost periodic solutions of (3) were considered for the first time by R. Yuan and J. Hong in [**358**] (1997); for more details about differential equations with a piecewise constant argument (DEPCA), the reader may consult the articles [**114**] by K. L. Cooke and J. Wiener, [**331**] by S. M. Shah and J. Wiener, as well as the articles [**14, 105, 106, 298, 305, 315, 357**] and the list of references cited therein.

There is a vast amount of articles in the existing literature which consider almost automorphic type solutions for various classes of integro-differential equations. Let us only mention our analysis (the joint work with Prof. G. M. N'Guérékata [**195**]) of the following abstract multi-term fractional differential inclusion:

$$\mathbf{D}_t^{\alpha_n} u(t) + \sum_{i=1}^{n-1} A_i \mathbf{D}_t^{\alpha_i} u(t) \in \mathcal{A}\mathbf{D}_t^{\alpha} u(t) + f(t), \quad t \geqslant 0,$$
$$u^{(k)}(0) = u_k, \quad k = 0, \cdots, \lceil \alpha_n \rceil - 1,$$

where $n \in \mathbb{N} \setminus \{1\}$, $A_1, \cdots, A_{n-1}$ are bounded linear operators on a Banach space X, $\mathcal{A}$ is a closed multivalued linear operator on X, $0 \leqslant \alpha_1 < \cdots < \alpha_n$, $0 \leqslant \alpha < \alpha_n$, $f(\cdot)$ is an X-valued function, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α ([**52**], [**233**]). Many excellent examples have been presented in the monograph [**135**] by T. Diagana; see also the monograph [**21**] by M. Amerio and G. Prouse for almost periodic solutions of functional equations, [**66**] by P. H. Bezandry and T. Diagana for almost periodic solutions of stochastic differential equations, [**96**] by D. N. Cheban for asymptotically almost periodic solutions of linear and nonlinear equations, [**212**] by Y. Hino, T. Naito, N. V. Minh and J. S. Shin and [**194**] by G. M. N'Guérékata for spectral analysis of almost periodic functions and Massera type theorems ([**284**]), [**214**] by R. Hsu for weakly almost periodic functions, and [**334**] by G. Tr. Stamov for almost periodic solutions of impulsive differential equations (see also the recent article [**354**] by P. Yang, Y.-R. Wang and M. Fečkan). Concerning almost periodic and almost almost automorphic solutions of the abstract functional integro-differential equations, we refer the reader to [**2**], [**12**], [**9**], [**92**]-[**93**], [**162**], [**161**], [**208**], [**356**]; for almost periodic and almost automorphic solutions of abstract nonlinear integro-differential equations, see the reference lists in the article [**108**] and the monograph [**234**]. Concerning semilinear Cauchy inclusions, we can also recommend the monograph [**221**] by M. Kamenskii, V. Obukhovskii and P. Zecca for another approach obeyed.

Concerning the existence and uniqueness of almost periodic type solutions of inhomogeneous evolution equations of first order, the notions of hyperbolic evolution systems and Green's functions are incredible important; for more details on the subject, we refer the reader to P. Acquistapace [**4**], P. Acquistapace, B. Terreni [**5**], Y.-H. Chang, J.-S. Chen [**91**], T. Diagana [**135**], R. Schnaubelt [**328**] and the list of references in [**234**]. Let us recall that a family $\{U(t,s) : t \geqslant s,\ t,\ s \in \mathbb{R}\}$ of bounded linear operators on X is said to be an evolution system if and only if the following holds:

(a) $U(s,s) = I$, $U(t,s) = U(t,r)U(r,s)$ for $t \geqslant r \geqslant s$ and $t,\ r,\ s \in \mathbb{R}$,
(b) $\{(\tau, s) \in \mathbb{R}^2 : \tau > s\} \ni (t,s) \mapsto U(t,s)x$ is continuous for any fixed element $x \in X$.

If the family $A(\cdot)$ satisfies the following condition introduced by P. Acquistapace and B. Terreni in [**5**] (with $\omega = 0$):

(H1): There is a real number $\omega \geqslant 0$ such that the family of closed linear operators $A(t)$, $t \in \mathbb{R}$ acting on X satisfies $\overline{\Sigma_\phi} \subseteq \rho(A(t) - \omega)$,

$$\big\|R(\lambda : A(t) - \omega)\big\| = O\Big(\big(1 + |\lambda|\big)^{-1}\Big), \quad t \in \mathbb{R},\ \lambda \in \overline{\Sigma_\phi},\ \text{and}$$

$$\Big\|(A(t) - \omega)R(\lambda : A(t) - \omega)\big[R(\omega : A(t)) - R(\omega : A(s))\big]\Big\| = O\Big(|t - s|^{\mu}|\lambda|^{-\nu}\Big),$$

for any $t,\ s \in \mathbb{R}$, $\lambda \in \overline{\Sigma_\phi}$, where $\phi \in (\pi/2, \pi)$, $0 < \mu,\ \nu \leqslant 1$ and $\mu + \nu > 1$,

then we have the existence of an evolution system $U(\cdot,\cdot)$ generated by $A(\cdot)$, satisfying the following properties:

1. $U(\cdot, s) \in C^1((s, \infty) : L(X))$ for all $s \in \mathbb{R}$,
2. $\partial_t U(t, s) = A(t)U(t, s)$, $s \in \mathbb{R}$, $t > s$,
3. $\|A(t)^k U(t, s)\| \leqslant Const. \cdot (t - s)^{-k}$, $t > s$, $k \in \mathbb{N}_0$,
4. $\|A(t)U(t, s)R(\omega : A(s))\| \leqslant Const.$, $t > s$,
5. $\|U(t, s)(\omega - A(s))^{\alpha} x\| \leqslant Const. \cdot (\mu - \alpha)^{-1}(t - s)^{-\alpha}\|x\|$, for $0 < t - s \leqslant 1$, $k = 0, 1$, $0 \leqslant \alpha < \nu$, $x \in D((\omega - A(s))^{\alpha})$,
6. $\partial_s^+ U(t, s)x = -U(t, s)A(s)x$, for $s \in \mathbb{R}$, $t > s$, $x \in D(A(s))$ and $A(s)x \in \overline{D(A(s))}$.

In many concrete situations, it is very difficult to verify the validity of the following non-trivial condition

(H2): The evolution system $U(\cdot,\cdot)$ generated by $A(\cdot)$ is hyperbolic (or, equivalently, has exponential dichotomy), i.e., there exist a family of projections $(P(t))_{t\in\mathbb{R}} \subseteq L(X)$, being uniformly bounded and strongly continuous in t, and constants $M',\ \omega > 0$ such that the following holds, with $Q := I - P$ and $Q(\cdot) := I - P(\cdot)$:
 (a) $U(t, s)P(s) = P(t)U(t, s)$ for all $t \geqslant s$,
 (b) the restriction $U_Q(t, s) : Q(s)X \to Q(t)X$ is invertible for all $t \geqslant s$ (here we set $U_Q(s, t) = U_Q(t, s)^{-1}$),
 (c) $\|U(t, s)P(s)\| \leqslant M' e^{-\omega(t-s)}$ and $\|U_Q(s, t)Q(t)\| \leqslant M' e^{-\omega(t-s)}$ for all $t \geqslant s$.

If the choice $P(t) = I$ for all $t \in \mathbb{R}$ is possible, then $U(\cdot,\cdot)$ is called exponentially stable. Further on, we say that $U(\cdot,\cdot)$ is (bounded) exponentially bounded if and only if there exist real constants $M > 0$ and $(\omega = 0)$ $\omega \in \mathbb{R}$ such that $\|U(t, s)P(s)\| \leqslant M e^{-\omega(t-s)}$ for all $t \geqslant s$.

The associated Green's function $\Gamma(\cdot,\cdot)$, defined by

$$\Gamma(t, s) := \begin{cases} U(t, s)P(s),\ t \geqslant s,\ t,\ s \in \mathbb{R}, \\ -U_Q(t, s)Q(s),\ t < s,\ t,\ s \in \mathbb{R}, \end{cases}$$

satisfies

$$\|\Gamma(t, s)\| \leqslant M' e^{-\omega|t-s|}, \quad t,\ s \in \mathbb{R},$$

where M' is the constant appearing in the formulation of (H2). If the function $f : \mathbb{R} \to X$ is continuous, then the function

$$u(t) := \int_{-\infty}^{+\infty} \Gamma(t,s)f(s)\, ds, \quad t \in \mathbb{R}$$

is a unique mild solution of the abstract Cauchy problem

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{4}$$

i.e., $u(\cdot)$ is a unique bounded continuous function on $\mathbb{R}$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\, d\tau, \quad t \geqslant s;$$

see e.g. [**328**] and [**135**, Lemma 9.11, p. 234]. Furthermore, if the function $f : [0,\infty) \to X$ is continuous, then we say that the function

$$u(t) := U(t,0)x + \int_0^t U(t,s)f(s)\, ds, \quad t \geqslant 0.$$

is a mild solution of the abstract Cauchy problem

$$u'(t) = A(t)u(t) + f(t), \quad t > 0;\ u(0) = x. \tag{5}$$

The almost periodic and almost automorphic solutions of the abstract Cauchy problems (4)-(5) and their semilinear analogues have been investigated in a great number of research papers. Without going into full details, we will only refer the readers to the research monographs [**135**] by T. Diagana, [**234**] by M. Kostić, the articles [**40**] by M. Baroun, L. Maniar, R. Schnaubelt, [**41**] by M. Baroun, K. Ezzinbi, K. Khalil, L. Maniar and the list of references therein. Concerning the applications of evolution systems in the theory of the second-order nonautonomous differential equations, mention should be made of the paper [**361**] by D. A. Zakora (almost periodic solutions of such equations have been investigated in [**340**]).

In this research monograph, we present several recent results concerning various generalizations of almost periodic functions. The organization and main ideas of monograph, which consists of two chapters, can be described as follows. The first chapter is devoted to the recapitulation of basic concepts we will need later on. We reconsider linear and multivalued linear operators in Banach spaces, integration and strongly continuous semigroups in Banach spaces as well as the basic definitions and results from fractional calculus and the theory of abstract degenerate Volterra integro-differential equations. After introducing these basic concepts, in Subsection 1.1.1 we recall the main definitions and results about Lebesgue spaces with variable exponents $L^{p(x)}$.

Chapter 2 consists of ten sections. Section 2.1, Section 2.2 and Section 2.3 are of introductory charachter and there we recollect the basic definitions and results about almost periodic functions and almost automorphic functions. Composition principles for Weyl almost periodic functions analyzed in Subsection 2.2.1, Proposition 2.3.1 and the conclusion clarified in Example 2.3.5 are the only new contributions of ours given in these sections. The main aims of Section 2.4, which considers uniformly recurrent functions and $\odot_g$-almost periodic functions, will be explained within itself. Various classes of generalized almost periodic functions in the Lebesgue spaces with variable exponents have been analyzed in Section 2.5, Section 2.6 and Section 2.7. Section 2.5 consists of six subsections. In our joint papers with T. Diagana [**142**]-[**143**], we have recently introduced and analyzed several important classes of (asymptotically) Stepanov almost periodic functions and (asymptotically) Stepanov almost automorphic functions in the Lebesgue spaces with variable exponents (see also the earlier papers [**145**]-[**146**] by T. Diagana and M. Zitane). The material of Subsection 2.5.1, Subsection 2.5.2 and Subsection 2.5.3 is taken from [**142**].

The classes introduced by H. Weyl [**350**] and A. S. Kovanko [**258**] are enormously larger compared with the class of Stepanov almost periodic functions; the main purpose of papers [**241**]-[**247**] has been to initiate the study of generalized (asymptotical) almost periodicity that intermediates Stepanov and Weyl concept. In these papers, we have introduced the class of Stepanov p-quasi-asymptotically almost periodic functions and proved that this class contains all asymptotically Stepanov p-almost periodic functions and makes a subclass of the class consisting of all Weyl p-almost periodic functions ($p \in [1, \infty)$), taken in the sense of Kovanko's approach [**258**]. The main aim of Subsection 2.5.4-Subsection 2.5.7 is to continue the research studies raised in [**168**] and [**246**]-[**247**] by investigating several various classes of asymptotically Weyl almost periodic functions in Lebesgue spaces with variable exponents $L^{p(x)}$. The material of these subsections are taken from the paper [**249**], whose main ideas can be briefly described as follows. In Definition 2.5.22-Definition 2.5.24, we introduce the classes of (equi-)Weyl-(p, ϕ, F)-almost periodic functions and (equi-)Weyl-$(p, \phi, F)_i$-almost periodic functions, where $i = 1, 2$. The main aim of Proposition 2.5.26 is to clarify some inclusions between these spaces provided that the function $\phi(\cdot)$ is convex and satisfies certain extra conditions. In order to ensure the translation invariance of generalized Weyl almost periodic functions with variable exponent, in Definition 2.5.28-Definition 2.5.30 we introduce the classes of (equi-)Weyl-$[p, \phi, F]$-almost periodic functions and (equi-)Weyl-$[p, \phi, F]_i$-almost periodic functions, where $i = 1, 2$. Several useful comments about these spaces have been provided in Remark 2.5.31. In Example 2.5.33-Example 2.5.34, we focus our attention on the following special case: $p(x) \equiv p \in [1, \infty)$, $\phi(x) = x$ and $F(l, t) = l^{(-1)/p\sigma}$, $\sigma \in \mathbb{R}$, which is the most important for the investigations of generalized almost periodicity which stands between the Stepanov and Weyl concepts. In Subsection 2.5.5, we introduce and analyze various types of Weyl ergodic components with variable exponent and asymptotically Weyl almost periodic functions with variable exponent. The introduced classes of generalized (asymptotically) Weyl almost periodic functions are new even in the case that the function $p(x)$ has a constant value $p \geqslant 1$ and $\phi(x) \neq x$ or $F(l, t) \neq l^{(-1)/p(t)}$. From the application point of view, Subsection 2.5.6 is very important because there we examine the invariance of generalized Weyl almost periodicity with variable exponent under the action of convolution products and the convolution invariance of Weyl almost periodic functions with variable exponent. In order to do that, we shall basically follow the method proposed in the proof of Theorem 2.5.45. In Subsection 2.5.7, we consider the case in which the exponent $p(x) \equiv p \in [1, \infty)$ is constant and solution operator family $(R(t))_{t>0} \subseteq L(X,Y)$ has a certain growth order around the points zero, plus infinity, providing also some illustrative applications in the qualitative analysis of solutions to the abstract degenerate fractional differential equations with Weyl-Liouville or Caputo derivatives. Two-parameter asymptotically Weyl almost periodic functions with variable exponents and related composition principles will be considered somewhere else.

Section 2.6 is broken down into three subsections. In Subsection 2.6.1, we analyze Stepanov uniformly recurrent functions in the Lebesgue spaces with variable exponents. Doss almost periodic functions and Doss uniformly recurrent functions in Lebesgue spaces with variable exponents are investigated in Subsection 2.6.2, while the invariance of generalized Doss almost periodicity with variable exponent under the actions of convolution products is investigated in Subsection 2.6.3.

Section 2.7 is broken down into six subsections. Subsection 2.7.1 introduces the notion of several different types of generalized (equi-)Weyl almost periodicity in Lebesgue spaces with variable exponents. The spaces introduced in Definition 2.7.1-Definition 2.7.3 may not be translation invariant, in general, which is not the case with the spaces introduced in Definition 2.7.5-Definition 2.7.7. The main aim of Subsection 2.7.1 is to explain without proofs how the structural results and characterizations established for generalized (equi-)Weyl almost periodic functions in [**249**] can be straightforwardly extended for the corresponding classes of generalized (equi-)Weyl uniformly recurrent functions. In Definition 2.7.8, we introduce the class of quasi-asymptotically uniformly recurrent functions (it is worth noting that some classes of generalized Stepanov and Weyl $p(x)$-almost periodic type functions and $p(x)$-uniformly recurrent type functions have not been considered elsewhere even for the constant coefficients $p(x) \equiv p \in [1, \infty)$). Proposition 2.7.9 shows that any asymptotically uniformly recurrent function is quasi-asymptotically uniformly recurrent; the converse statement is generally false, as a class of very simple counterexamples shows. In Proposition 2.7.10, we prove that the sum of a quasi-asymptotically uniformly recurrent function and a continuous function vanishing at infinity is again quasi-asymptotically uniformly recurrent. In Theorem 2.7.14, we revisit [**247**, Theorem 2.5] once more and examine some extra conditions under which a quasi-asymptotically uniformly recurrent function is (asymptotically) uniformly recurrent. Subsection 2.7.3 introduces and investigates several different classes of Stepanov quasi-asymptotically uniformly recurrent type functions in the Lebesgue spaces with variable exponents. The notion introduced in this subsection, in which we reconsider and slightly improve several known results from [**247**] in our new framework, is new even for the constant coefficients $p(x) \equiv p \in [1, \infty)$, and can be used to intermediate the concepts of the quasi-asymptotical almost periodicity (quasi-asymptotical uniform recurrence, S-asymptotical ω-periodicity) and its Stepanov generalizations with constant exponents. In Proposition 2.7.23, we reconsider the assertion of [**142**, Proposition 4.5] for the Stepanov quasi-asymptotically uniformly recurrent functions (see also Corollary 2.7.24 and Proposition 2.7.25). Any Stepanov p-quasi-asymptotically almost periodic function is Weyl p-almost periodic, and clearly, any (quasi-)asymptotical almost periodic function is Stepanov p-quasi-asymptotically almost periodic for any finite exponent $p \geqslant 1$ (see [**247**, Proposition 2.12]); as observed here, the same holds for the related concepts of quasi-asymptotical uniform recurrence. The main objective in Proposition 2.7.26 is to state and prove a general result in this direction. In Subsection 2.7.4, we clarify the main composition principles for the class of quasi-asymptotically uniformly recurrent functions. Our main contributions are given in Subsection 2.7.5, where we examine the invariance of generalized quasi-asymptotical uniform recurrence with variable exponents under the actions of convolution products. Some applications to the abstract Volterra integro-differential equations are presented in Subsection 2.7.6. The material of Section 2.6 and Section 2.7 is taken from our recent papers obtained in a coauthorship with Prof. W.-S. Du [**251**]-[**252**].

The definitions and basic properties of (ω, c)-periodic and (ω, c)-pseudo periodic functions were introduced and analyzed by E. Alvarez, A. Gómez and M. Pinto in [**17**]-[**16**], motivated by some known results regarding the qualitative properties of solution to Mathieu's linear differential equation

$$y''(t) + [a - 2q\cos 2t]y(t) = 0,$$

arising in modeling of railroad rails and seasonally forced population dynamics ($\omega > 0$, $c \in \mathbb{C} \smallsetminus \{0\}$). The linear delayed equations can have (ω, c)-periodic solutions, as well (see e.g., [**17**, Example 2.5]). The notions of anti-periodicity and Bloch periodicity are special cases of the notion of an (ω, c)-periodicity.

The authors of [**17**] have analyzed the existence and uniqueness of mild (ω, c)-periodic solutions to the abstract semilinear integro-differential equation (1). Further on, E. Alvarez, S. Castillo and M. Pinto have analyzed in [**16**] the existence and uniqueness of mild (ω, c)-pseudo periodic solutions to the abstract semilinear differential equation of the first order:

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R},$$

where A generates a strongly continuous semigroup. The authors have proved the existence of positive (ω, c)-pseudo periodic solutions to the Lasota-Wazewska equation with (ω, c)-pseudo periodic coefficients

$$y'(t) = -\delta y(t) + h(t)e^{-a(t)y(t-\tau)}, \quad t \geqslant 0.$$

This equation describes the survival of red blood cells in the blood of an animal (see e.g., M. Wazewska-Czyzewska and A. Lasota [**349**]). Concerning the applications to time varying impulsive differential equations, mention should be made of the article [**346**] by J. R. Wang, L. Ren and Y. Zhou; cf. also the article [**7**] by M. Agaoglou, M. Fečkan, A. P. Panagiotidou, the article [**290**] by G. Mophou, G. M. N'Guérékata and the article [**266**] by M. Li, J. R. Wang and M. Fečkan.

In Section 2.8, we analyze various types of (ω, c)-almost periodic functions, (ω, c)-uniformly recurrent functions and (compactly) (ω, c)-almost automorphic functions. The classes of (ω, c)-uniformly recurrent functions of type i and (ω, c)-almost periodic functions of type i ($i = 1, 2$) are introduced and analyzed in Subsection 2.8.1. Composition principles for (ω, c)-almost periodic type functions are analyzed in Subsection 2.8.2. The classes of (ω, c)-pseudo almost periodic functions, (ω, c)-pseudo almost automorphic functions and related applications are studied in

Subsection 2.8.4. Subsection 2.8.5 introduces and investigates $(\omega, c)-$almost periodicity (resp. asymptotic $(w, c)-$almost periodicity) in the setting of Schwartz-Sobolev distributions (for simplicity, we will consider only scalar-valued distributions because the extensions to the vector-valued case are straightforward); in the next subsection, we apply our abstract theoretical results in the study of the existence of distributional $(w, c)-$almost periodic solutions of linear differential systems. In [**319**, Chapter II], J. Prüss has analyzed abstract non-scalar Volterra equations. Applications have been given in the analysis of viscoelastic Timoshenko beam model, Midlin-Timoshenko plate model and viscoelastic Kirchhoff plate model, with the corresponding materials being non-synchronous, as well as in the analysis of some problems of linear thermoviscoelasticity and electrodynamics. In Subsection 2.8.7, we initiate the study of asymptotically (ω, c)-almost periodic type solutions of abstract degenerate non-scalar Volterra equations.

The organization and main ideas of Section 2.9, which consists of seven subsections, is given as follows. The notion of c-almost periodicity and the notion of c-uniform recurrence, where $c \in \mathbb{C} \setminus \{0\}$, are introduced in Definition 2.9.2 and Definition 2.9.4, respectively (in case $c = 1$, we recover the usual notions of almost periodicity and uniform recurrence, while in case $c = -1$, we recover the usual notions of almost anti-periodicity and uniform anti-recurrence); the main idea is the use of difference $f(\cdot + \tau) - cf(\cdot)$ in place of the usually considered difference $f(\cdot + \tau) - f(\cdot)$. After that, in Definition 2.9.5 and Proposition 2.9.6, we introduce the notion of semi-c-periodicity and prove some necessary and sufficient conditions for a continuous function $f : I \to X$ to be semi-c-periodic. Proposition 2.9.11 is crucially important in our analysis because it states that there does not exist a c-uniformly recurrent function $f : I \to X$ if $|c| \neq 1$. The invariance of c-almost type periodicity under the actions of convolution products is also analyzed here. The composition theorems for c-almost periodic type functions are analyzed in Subsection 2.9.1 (the structural results in this subsection are given without proofs, which can be deduced similarly as in our previous research studies; it is also worth noting that we present numerous illustrative examples and comments about the problems considered). In Subsection 2.9.2, we will present some illustrative applications of our abstract results in the analysis of the existence and uniqueness of c-almost periodic type solutions to the abstract (semilinear) Volterra integro-differential inclusions. The class of semi-c-periodic functions with general parameter $c \in \mathbb{C} \setminus \{0\}$ is introduced and analyzed in Subsection 2.9.3; the main result of this subsection is Theorem 2.9.45 which states that the notion of c-periodicity and semi-c-periodicity are equivalent for $|c| \neq 1$. The material of Section 2.8 and Section 2.9 is obtained in a couathorship with Prof. M. Pinto, M. T. Khalladi, A. Rahmani and D. Velinov ([**223**]-[**227**]).

Let $p > 0$ and $k \in \mathbb{R}$. Recall that a bounded continuous function $f : I \to X$ is said to be Bloch (p, k)-periodic, or Bloch periodic with period p and Bloch wave vector or Floquet exponent k if and only if $f(x + p) = e^{ikp} f(x)$, $x \in I$, with $p > 0$ and $k \in \mathbb{R}$. The study of Bloch (p, k)-periodic functions is an important subject of applied functional analysis. The Bloch periodic functions and almost Bloch periodic functions are widely used in biology, physics, probability, modeling, solid mechanics and many other areas (see the papers [**203**] by M. F. Hasler, [**204**] by M. F. Hasler, G. M. N'Guérékata, [**253**] by M. Kostić, D. Velinov and references cited therein). As is well known, the notion of an anti-periodic function is a special case of the notion of a Bloch (p, k)-periodic function (a bounded continuous function $f : I \to X$ is said to $f(\cdot)$ is anti-periodic if and only if there exists $p > 0$ such that $f(x + p) = -f(x)$, $x \in I$; any such function needs to be periodic of period $2p$). For more details about anti-periodic type functions and their applications, we refer the reader to [**101, 150, 197, 253, 271, 272**] and references cited therein. Semi-Bloch k-periodic functions are investigated in Subsection 2.9.4 (the results are obtained in a coauthorship with Prof. B. Chaouchi, S. Pilipović and D. Velinov [**94**]). The genesis of paper [**94**] is motivated by reading the research article [**25**] by J. Andres and D. Pennequin, where the authors have introduced and analyzed the class of semi-periodic functions (sequences) and related applications to differential (difference) equations; see also [**26**]. Of course, a semi-periodic function is nothing else but a semi-c-periodic function with $c = 1$.

The class of S-asymptotically ω-periodic functions, introduced by H. Henríquez et al. [**209**] for case $I = \mathbb{R}$ and M. Kostić [**247**] for case $I = [0, \infty)$, are reconsidered in Subsection 2.9.6, where we introduce the class of S-asymptotically (ω, c)-periodic functions. Quasi-asymptotically c-almost periodic functions and related composition principles are investigated in Subsection 2.9.7.

Several notes and appendicies are provided in the final section of monograph, where we particularly analyze recurrent strongly continuous semigroups of operators.

CHAPTER 1

PRELIMINARIES

1.1. Linear operators and integration in Banach spaces, strongly continuous semigroups and fixed point theorems

In this section, we recollect the indispensable things about vector-valued functions, closed operators, integration and strongly continuous semigroups in Banach spaces. We also recall the basic fixed point theorems we will employ later on. In Subsection 1.1.1, we explore the basic definitions and results about the Lebesgue spaces with variable exponents $L^{p(x)}$.

Vector-valued functions, closed operators. Generally, by $(X, \|\cdot\|)$ we denote a Banach space over the field of complex numbers. If $(Y, \|\cdot\|_Y)$ is another Banach space over the field of complex numbers, then by $L(X,Y)$ we denote the space consisting of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. The topologies on $L(X,Y)$ and X^*, the dual space of X, are introduced in the usual way. If not stated otherwise, by I we denote the identity operator on X. If X and Y are two Banach spaces such that Y is continuously embedded in X, then we write $Y \hookrightarrow X$.

We say that a linear operator $A : D(A) \to X$ is closed if and only if the graph of the operator A, defined by $G_A := \{(x, Ax) : x \in D(A)\}$, is a closed subset of $X \times X$. The null space and range of A are denoted by $N(A)$ and $R(A)$, respectively. Let us recall that a linear operator $A : D(A) \to X$ is closed if and only if, for every sequence (x_n) in $D(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Ax_n = y$, the following holds: $x \in D(A)$ and $Ax = y$; a linear operator A is called closable if and only if there exists a closed linear operator B such that $A \subseteq B$. Assuming that F is a linear submanifold of X, then we define the part of A in F by $D(A_{|F}) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_{|F}x := Ax$, $x \in D(A_{|F})$.

The power A^n of A is defined inductively ($n \in \mathbb{N}_0$); set $D_\infty(A) := \bigcap_{n\geqslant 1} D(A^n)$. For a closed linear operator A acting on X, we introduce the adjoint A^* of $X^* \times X^*$ by

$$A^* := \Big\{ \big(x^*, y^*\big) \in X^* \times X^* : x^*(Ax) = y^*(x) \text{ for all } x \in D(A) \Big\}.$$

In the case that A is densely defined, then A^* is single-valued, closed and also known as the adjoint operator of A. If $\alpha \in \mathbb{C} \smallsetminus \{0\}$, A and B are linear operators, we define the operators αA, $A+B$ and AB in the usual way. The Gamma function will be denoted by $\Gamma(\cdot)$ and the principal branch will be always used to take the powers. Set, for every $\alpha > 0$,

$$g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha), \quad t > 0,$$

$g_0(t) \equiv$ the Dirac delta distribution and $0^\zeta := 0$. Set $\Sigma_\alpha := \{z \in \mathbb{C} \smallsetminus \{0\} : |\arg(z)| < \alpha\}$, $\alpha \in (0, \pi]$.

By $C(\Omega : X)$ we denote the space consisting of all continuous functions $f : \Omega \to X$, where $\emptyset \neq \Omega \subseteq \mathbb{C}^n$ ($n \in \mathbb{N}$); $C(\Omega) \equiv C(\Omega : \mathbb{C})$. Let $0 < \tau \leqslant \infty$ and $a \in L^1_{loc}([0, \tau))$. Then we say that the function $a(t)$ is a kernel on $[0, \tau)$ if and only if for each $f \in C([0, \tau))$ the assumption $\int_0^t a(t-s)f(s)\, ds = 0$, $t \in [0, \tau)$ implies $f(t) = 0$, $t \in [0, \tau)$. If $s \in \mathbb{R}$ and $n \in \mathbb{N}$, we define $\lfloor s \rfloor := \sup\{l \in \mathbb{Z} : s \geqslant l\}$, $\lceil s \rceil := \inf\{l \in \mathbb{Z} : s \leqslant l\}$, $\mathbb{N}_n := \{1, \cdots, n\}$ and $\mathbb{N}_n^0 := \{0, 1, \cdots, n\}$. If X, $\mathrm{Y} \neq \emptyset$, put $\mathrm{Y}^{\mathrm{X}} := \{f \mid f : \mathrm{X} \to \mathrm{Y}\}$.

Let $I = \mathbb{R}$ or $I = [0, \infty)$. By $C_b(I : X)$ we denote the space consisting of all bounded continuous functions from I into X; the symbol $C_0(I : X)$ denotes the closed subspace of $C_b(I : X)$ consisting of those functions $f : I \to X$ such that $\lim_{|t| \to \infty} \|f(t)\| = 0$. By $BUC(I : X)$ we denote the space consisting of all bounded uniformly continuous functions from I to X; $C_b(I) \equiv C_b(I : \mathbb{C})$, $C_0(I) \equiv C_0(I : \mathbb{C})$ and $BUC(I) \equiv BUC(I : \mathbb{C})$. Equipped with the sup-norms, these vector spaces are the Banach spaces.

Regarding analytical functions with values in Banach spaces and locally convex spaces, we refer the reader to [**30**] and [**236**] (for almost periodic and almost automorphic functions in locally convex spaces and general vector topological spaces, we refer the reader to [**234**, Section 3.11] and references cited in the first part of this section).

Integration in Banach spaces. The following definition is elementary.

Definition 1.1.1. (i) A function $f : I \to X$ is said to be simple if and only if there exist $k \in \mathbb{N}$, elements $z_i \in X$, $1 \leqslant i \leqslant k$ and Lebesgue measurable subsets Ω_k, $1 \leqslant i \leqslant k$ of I, such that $m(\Omega_i) < \infty$, $1 \leqslant i \leqslant k$ and

$$f(t) = \sum_{i=1}^{k} z_i \chi_{\Omega_i}(t), \quad t \in I. \tag{6}$$

(ii) A function $f : I \to X$ is said to be measurable if and only if there exists a sequence (f_n) in X^I such that, for every $n \in \mathbb{N}$, $f_n(\cdot)$ is a simple function and $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$.

(iii) Let $-\infty < a < b < \infty$ and $a < \tau \leqslant \infty$. A function $f : [a, b] \to X$ is said to be absolutely continuous if and only if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that for any finite collection of open subintervals (a_i, b_i), $1 \leqslant i \leqslant k$ of $[a, b]$ with $\sum_{i=1}^k (b_i - a_i) < \delta$, we have $\sum_{i=1}^k \|f(b_i) - f(a_i)\| < \varepsilon$; a function $f : [a, \tau) \to X$ is said to be absolutely continuous if and only if for every $\tau_0 \in (a, \tau)$, the function $f_{|[a,\tau_0]} : [a, \tau_0] \to X$ is absolutely continuous.

If $f : I \to X$ and (f_n) is a sequence of measurable functions satisfying that $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$, then the function $f(\cdot)$ is measurable, as well.

The Bochner integral of a simple function $f : I \to X$, $f(t) = \sum_{i=1}^{k} z_i \chi_{\Omega_i}(t)$, $t \in I$ is defined by

$$\int_I f(t)\,dt := \sum_{i=1}^{k} z_i m(\Omega_i).$$

The definition of Bochner integral does not depend on the representation (6), as easily approved.

We say that a measurable function $f : I \to X$ is Bochner integrable if and only if there exists a sequence of simple functions (f_n) in X^I such that $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$ and

$$\lim_{n\to\infty} \int_I \big\|f_n(t) - f(t)\big\|\,dt = 0; \tag{7}$$

if this is the case, the Bochner integral of $f(\cdot)$ is defined by

$$\int_I f(t)\,dt := \lim_{n\to\infty} \int_I f_n(t)\,dt.$$

This definition does not depend on the choice of a sequence of simple functions (f_n) in X^I satisfying $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$ and (7). It is well known that $f : I \to X$ is Bochner integrable if and only if $f(\cdot)$ is measurable and the function $t \mapsto \|f(t)\|$, $t \in I$ is integrable. For any Bochner integrable function $f : [0,\infty) \to X$, we have $\int_0^\infty f(t)\,dt = \lim_{\tau\to+\infty} \int_0^\tau f_{|[0,\tau]}(t)\,dt$.

The space of all Bochner integrable functions from I into X is denoted by $L^1(I : X)$; endowed with the norm $\|f\|_1 := \int_I \|f(t)\|\,dt$, $L^1(I : X)$ is a Banach space. It is said that a function $f : [0,\infty) \to X$ is locally (Bochner) integrable if and only if $f(\cdot)_{|[0,\tau]}$ is Bochner integrable for every $\tau > 0$. The space of all locally integrable functions from $[0,\infty)$ into X is denoted by $L^1_{loc}([0,\infty) : X)$. If $f : [a,b] \to X$ is Bochner integrable, where $-\infty < a < b < +\infty$, then the function $F(t) := \int_a^t f(s)\,ds$, $t \in [a,b]$ is absolutely continuous and $F'(t) = f(t)$ for a.e. $t \geqslant 0$. Basically, we will not distinguish a function and its restriction to any subinterval of its domain.

Theorem 1.1.2. (i) *(The dominated convergence theorem) Suppose that (f_n) is a sequence of Bochner integrable functions from X^I and that there exists an integrable function $g : I \to \mathbb{R}$ such that $\|f_n(t)\| \leqslant g(t)$ for a.e. $t \in I$ and $n \in \mathbb{N}$. If $f : I \to X$ and $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in I$, then $f(\cdot)$ is Bochner integrable, $\int_I f(t)\,dt = \lim_{n\to\infty} \int_I f_n(t)\,dt$ and $\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|\,dt = 0$.*

(ii) *(The Fubini theorem) Let I_1 and I_2 be segments in $\mathbb{R}$ and let $I = I_1 \times I_2$. Suppose that $F : I \to X$ is measurable and $\int_{I_1}\int_{I_2} \|f(s,t)\|\,dt\,ds < \infty$. Then $f(\cdot,\cdot)$ is Bochner integrable, the repeated integrals $\int_{I_1}\int_{I_2} f(s,t)\,dt\,ds$ and $\int_{I_2}\int_{I_1} f(s,t)\,ds\,dt$ exist and equal to the integral $\int_I f(s,t)\,ds\,dt$.*

Suppose now that $1 \leqslant p < \infty$ and $(\Omega, \mathcal{R}, \mu)$ is a measure space. By $L^p(\Omega : X)$ we denote the space of all strongly μ-measurable functions $f : \Omega \to X$ such that $\|f\|_p := (\int_\Omega \|f(\cdot)\|^p d\mu)^{1/p}$ is finite. The space $L^\infty(\Omega : X)$ consists of all strongly

μ-measurable, essentially bounded functions; this space is a Banach space equipped with the norm $\|f\|_\infty := ess\sup_{t\in\Omega}\|f(t)\|$, $f \in L^\infty(\Omega : X)$. Let us recall that we identify functions that are equal μ-almost everywhere on Ω. The famous Riesz–Fischer theorem states that $(L^p(\Omega : X), \|\cdot\|_p)$ is a Banach space for all $p \in [1,\infty]$; furthermore, $(L^2(\Omega : X), \|\cdot\|_2)$ is a Hilbert space. If $\lim_{n\to\infty} f_n = f$ in $L^p(\Omega : X)$, then there exists a subsequence (f_{n_k}) of (f_n) such that $\lim_{k\to\infty} f_{n_k}(t) = f(t)$ μ-almost everywhere. If the Banach space X is reflexive, then $L^p(\Omega : X)$ is reflexive for all $p \in (1,\infty)$ and its dual is isometrically isomorphic to $L^{\frac{p}{p-1}}(\Omega : X)$. We refer the reader to [**30**] and [**234**] for more details about the absolutely continuous functions. The space consisting of all X-valued functions that are absolutely continuous on any closed subinterval of $[0,\infty)$ will be denoted by $AC_{loc}([0,\infty) : X)$.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ $(n \in \mathbb{N})$. By $C^k(\Omega : X)$ we denote the space of k-times continuously differentiable functions $(k \in \mathbb{N})$ $f : \Omega \to X$. The space $L^p_{loc}(\Omega : X)$ for $1 \leqslant p \leqslant \infty$ is defined in the usual way $(T,\ \tau > 0)$; $L^p_{loc}(\Omega) \equiv L^p_{loc}(\Omega : \mathbb{C})$.

Assume now that $k \in \mathbb{N}$ and $p \in [1,\infty]$. Then the Sobolev space $W^{k,p}(\Omega : X)$ consists of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every $i \in \mathbb{N}^0_k$ and for every multi-index $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leqslant k$, we have $D^\alpha u \in L^p(\Omega, X)$. At this place, the derivative D^α is taken in the sense of distributions. By $W^{k,p}_{loc}(\Omega : X)$ we denote the space of those X-valued distributions $u \in \mathcal{D}'(\Omega : X)$ such that, for every bounded open subset Ω' of Ω, one has $u_{|\Omega'} \in W^{k,p}(\Omega' : X)$.

We will use the following simple lemma:

LEMMA 1.1.3. *Let $-\infty < a < b < \infty$, let $1 \leqslant p' < p'' < \infty$, and let $f \in L^{p''}([a,b] : X)$. Then $f \in L^{p'}([a,b] : X)$ and*

$$\left[\frac{1}{b-a}\int_a^b \|f(s)\|^{p'}\,ds\right]^{1/p'} \leqslant \left[\frac{1}{b-a}\int_a^b \|f(s)\|^{p''}\,ds\right]^{1/p''}.$$

Strongly continuous semigroups in Banach spaces. An operator family $(T(t))_{t\geqslant 0} \subseteq L(X)$ is said to be a strongly continuous semigroup if and only if the following holds:

(i) $T(0) = I$,
(ii) $T(t+s) = T(t)T(s),\ t,\ s \geqslant 0$ and
(iii) the mapping $t \mapsto T(t)x,\ t \geqslant 0$ is continuous for every fixed $x \in X$.

The linear operator

$$A := \left\{(x,y) \in X \times X : \lim_{t\to 0+}\frac{T(t)x - x}{t} = y\right\} \tag{8}$$

is said to be the infinitesimal generator of $(T(t))_{t\geqslant 0}$. A strongly continuous semigroup (group) $(T(t))_{t\geqslant 0}$ is also said to be C_0-semigroup; if the condition (i) is neglected, then the operator $T(0)$ is a projection and then we say that $(T(t))_{t\geqslant 0}$ is a degenerate C_0-semigroup.

In both cases, degenerate and non-degenerate, we know that there exist finite constants $M \geqslant 1$ and $\omega \geqslant 0$ such that $\|T(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$. The famous Hille-Yosida theorem states that a linear operator A generates a non-degenerate strongly continuous semigroup $(T(t))_{t\geqslant 0}$ satisfying the estimate $\|T(t)\| \leqslant Me^{\omega t}$, $t \geqslant 0$ for some finite constants $M \geqslant 1$ and $\omega \geqslant 0$ if and only if A is closed, densely defined, $(\omega, \infty) \subseteq \rho(A)$ and

$$\left\|(\lambda - A)^{-n}\right\| \leqslant \frac{M}{(\lambda - \omega)^n}, \quad \lambda > \omega, \ n \in \mathbb{N}.$$

If not stated otherwise, then we will always assume that a C_0-semigroup $(T(t))_{t\geqslant 0}$ is non-degenerate.

If $(T(t))_{t\in\mathbb{R}} \subseteq L(X)$ satisifes (i), (ii) for all $t,\ s \in \mathbb{R}$ and (iii) for $t \in \mathbb{R}$, then we say that $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group, C_0-group for short. Similarly as above, if condition (i) is neglected, then we say that $(T(t))_{t\in\mathbb{R}}$ is a degenerate strongly continuous group, degenerate C_0-group for short. The infinitesimal generator of $(T(t))_{t\in\mathbb{R}}$ is defined through (8); in the degenerate case, the infinitesimal generator is a closed multivalued linear operator on X; see Section 1.2 below.

For more details about the theory of strongly continuous semigroups, the reader may consult the monographs [**160, 233, 236, 306**] and references quoted therein; for the theory of integrated semigroups and C-regularized semigroups, we refer the reader to [**30, 127, 232, 233, 368**] and references quoted therein.

Fixed point theorems. In this part, we remind the readers of the Banach contraction principle and its well known generalization, the Bryant fixed point theorem; for further information about the fixed point theory, the reader may consult the monographs [**10**] and [**184**].

Let (E, d) be a metric space. Then $T : E \to E$ is called a contraction mapping on E if and only if there exists a constant $q \in [0, 1)$ such that $d(T(x), T(y)) \leqslant qd(x, y)$ for all $x,\ y \in E$.

THEOREM 1.1.4. *(The Banach contraction principle, 1922) Let (E, d) be a complete metric space, and let $T : E \to E$ be a contraction mapping. Then T admits a unique fixed point x in X (i.e. $T(x) = x$).*

THEOREM 1.1.5. *(The Bryant fixed point theorem, 1968) Let (E, d) be a complete metric space, and let $T : E \to E$ be such that there is an integer $n \in \mathbb{N}$ such that $T^n : E \to E$ is a contraction mapping. Then T has a unique fixed point x in E.*

1.1.1. Lebesgue spaces with variable exponents $L^{p(x)}$. The monograph [**147**] by L. Diening, P. Harjulehto, P. Hästüso and M. Ruzicka is of invaluable importance in the study of Lebesgue spaces with variable exponents.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}$ be a nonempty subset and let $M(\Omega : X)$ stand for the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Furthermore, $\mathcal{P}(\Omega)$ denotes the vector space of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, set

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geqslant 0, \quad 1 \leqslant p(x) < \infty, \\ 0, & 0 \leqslant t \leqslant 1, \quad p(x) = \infty, \\ \infty, & t > 1, \quad p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_\Omega \varphi_{p(x)}(\|f(x)\|)\,dx. \tag{9}$$

We define the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent by

$$L^{p(x)}(\Omega : X) := \Big\{ f \in M(\Omega : X) : \lim_{\lambda \to 0+} \rho(\lambda f) = 0 \Big\}.$$

Equivalently,

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \Big\};$$

see e.g., [**147**, p. 73]. For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ in the following manner (see the doctoral dissertation of W. A. J. Luxemburg [**279**] for further information):

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\Big\{ \lambda > 0 : \rho(f/\lambda) \leqslant 1 \Big\}.$$

Equipped with the above norm, the space $L^{p(x)}(\Omega : X)$ becomes a Banach space (see e.g., [**147**, Theorem 3.2.7] for the scalar-valued case), coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) = p \geqslant 1$ is a constant function. For any $p \in M(\Omega)$, we set

$$p^- := \text{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \text{esssup}_{x\in\Omega} p(x).$$

Define

$$C_+(\Omega) := \big\{ p \in M(\Omega) : 1 < p^- \leqslant p(x) \leqslant p^+ < \infty \text{ for a.e. } x \in \Omega \big\}$$

and

$$D_+(\Omega) := \big\{ p \in M(\Omega) : 1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty \text{ for a.e. } x \in \Omega \big\}.$$

For $p \in D_+(\Omega)$, the space $L^{p(x)}(\Omega : X)$ behaves nicely, with almost all fundamental properties of the Lesbesgue space with constant exponent $L^p(\Omega : X)$ being retained; in this case, we know that the function $\rho(\cdot)$ given by (9) is modular in the sense of [**147**, Definition 2.1.1], as well as that

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) : \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty \Big\}.$$

Furthermore, if $p \in C_+(\Omega)$, then $L^{p(x)}(\Omega : X)$ is uniformly convex and thus reflexive ([**164**]).

We will use the following lemma (see e.g., [**147**, Lemma 3.2.20, (3.2.22); Corollary 3.3.4; p. 77] for the scalar-valued case):

LEMMA 1.1.6. (i) *(The Hölder inequality) Let $p,\ q,\ r \in \mathcal{P}(\Omega)$ such that*
$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$
Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and
$$\|uv\|_{q(x)} \leqslant 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) *Let Ω be of a finite Lebesgue's measure and let $p,\ q \in \mathcal{P}(\Omega)$ such $q \leqslant p$ a.e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$.*

(iii) *Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leqslant \|g\| \leqslant \|f\|$ a.e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leqslant \|f\|_{p(x)}$.*

For additional details upon Lebesgue spaces with variable exponents $L^{p(x)}$, we refer the reader to the following sources: [**145**], [**146**], [**164**] and [**302**].

1.2. Multivalued linear operators

This section aims to present a brief synopsis of definitions and results from the theory of multivalued linear operators that we will use later on. For more details, we refer to the monograph [**120**] by R. Cross.

Suppose that X and Y are two Banach spaces. A multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a multivalued linear operator (MLO) if and only if the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;
(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then we say that $\mathcal{A}$ is an MLO in X. Let us recall that, for every $x,\ y \in D(\mathcal{A})$ and $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda|+|\eta| \neq 0$, we have $\lambda\mathcal{A}x+\eta\mathcal{A}y = \mathcal{A}(\lambda x+\eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear submanifold of Y and $\mathcal{A}x = f+\mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$ and $N(\mathcal{A}) := \mathcal{A}^{-1}0 := \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ (we call that the range and kernel space of $\mathcal{A}$, respectively). The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It follows that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$. If $N(\mathcal{A}) = \{0\}$, i.e., if $\mathcal{A}^{-1}$ is single-valued, then $\mathcal{A}$ is said to be injective.

Assuming that $\mathcal{A},\ \mathcal{B} : X \to P(Y)$ are two MLOs, we define its sum $\mathcal{A}+\mathcal{B}$ by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x+\mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. Clearly, $\mathcal{A}+\mathcal{B}$ is likewise an MLO.

Suppose now that $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ be two MLOs, where Z is a complex Banach space. The product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{BA}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{BA}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. We have $\mathcal{BA} : X \to P(Z)$ is an MLO and $(\mathcal{BA})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$. The scalar multiplication of an MLO $\mathcal{A} : X \to P(Y)$ with the number $z \in \mathbb{C}$, $z\mathcal{A}$ for short, is defined by $D(z\mathcal{A}) := D(\mathcal{A})$ and $(z\mathcal{A})(x) := z\mathcal{A}x$, $x \in D(\mathcal{A})$.

The integer powers of an MLO $\mathcal{A} : X \to P(X)$ are defined inductively as follows: $\mathcal{A}^0 =: I$; if $\mathcal{A}^{n-1}$ is defined, set
$$D(\mathcal{A}^n) := \big\{x \in D(\mathcal{A}^{n-1}) : D(\mathcal{A}) \cap \mathcal{A}^{n-1}x \neq \emptyset\big\},$$

and

$$\mathcal{A}^n x := \big(\mathcal{A}\mathcal{A}^{n-1}\big)x = \bigcup_{y \in D(\mathcal{A}) \cap \mathcal{A}^{n-1}x} \mathcal{A}y, \quad x \in D(\mathcal{A}^n).$$

Assume that $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : X \to P(Y)$ are two MLOs. Then the inclusion $\mathcal{A} \subseteq \mathcal{B}$ is equivalent to saying that $D(\mathcal{A}) \subseteq D(\mathcal{B})$ and $\mathcal{A}x \subseteq \mathcal{B}x$ for all $x \in D(\mathcal{A})$.

It is said that an MLO operator $\mathcal{A} : X \to P(Y)$ is closed if and only if for any sequences (x_n) in $D(\mathcal{A})$ and (y_n) in Y such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$ we have that the suppositions $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

Assume that $\mathcal{A} : X \to P(Y)$ is an MLO. Then $\overline{\mathcal{A}} : X \to P(Y)$ is an MLO, as well, so that any MLO has a closed linear extension, in contrast to the usually considered single-valued linear operators.

Let $\mathcal{A}$ be an MLO in X and $C \in L(X)$. The C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the C-resolvent of $\mathcal{A}$. If $C = I$, then we say that $\rho(\mathcal{A}) \equiv \rho_C(\mathcal{A})$ is the resolvent set of $\mathcal{A}$ and the mapping $\lambda \mapsto (\lambda - \mathcal{A})^{-1}$ is called the resolvent of $\mathcal{A}$ ($\lambda \in \rho(\mathcal{A})$). For the generalized resolvent equations and the analytical properties of C-resolvents of multivalued linear operators, we refer the reader to [**236**].

Suppose now that $(-\infty, 0] \subseteq \rho(\mathcal{A})$ as well as that there exist finite numbers $M \geqslant 1$ and $\beta \in (0, 1]$ such that

$$\|R(\lambda : \mathcal{A})\| \leqslant M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \leqslant 0.$$

Then there are two positive numbers $c > 0$ and $M_1 > 0$ such that the resolvent set of $\mathcal{A}$ contains an open region $\Omega = \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leqslant (2M_1)^{-1}(c - \operatorname{Re}\lambda)^{\beta},\ \operatorname{Re}\lambda \leqslant c\}$ of complex plane around the half-line $(-\infty, 0]$, where we have the estimate $\|R(\lambda : \mathcal{A})\| = O((1+|\lambda|)^{-\beta})$, $\lambda \in \Omega$. Let Γ' be the upwards oriented curve $\{\xi \pm i(2M_1)^{-1}(c-\xi)^{\beta} : -\infty < \xi \leqslant c\}$. Following A. Favini and A. Yagi [**167**], we define the fractional power

$$\mathcal{A}^{-\theta} := \frac{1}{2\pi i} \int_{\Gamma'} \lambda^{-\theta} \big(\lambda - \mathcal{A}\big)^{-1} \, d\lambda \in L(X),$$

for $\theta > 1 - \beta$. Set $\mathcal{A}^{\theta} := (\mathcal{A}^{-\theta})^{-1}$ ($\theta > 1 - \beta$). Then the semigroup properties $\mathcal{A}^{-\theta_1}\mathcal{A}^{-\theta_2} = \mathcal{A}^{-(\theta_1+\theta_2)}$ and $\mathcal{A}^{\theta_1}\mathcal{A}^{\theta_2} = \mathcal{A}^{\theta_1+\theta_2}$ hold for $\theta_1,\ \theta_2 > 1 - \beta$ (it is worth recalling that the fractional power $\mathcal{A}^{\theta}$ need not be injective and that the meaning of $\mathcal{A}^{\theta}$ is understood in the MLO sense for $\theta > 1 - \beta$).

For any $\theta \in (0, 1)$, the vector space

$$X_{\mathcal{A}}^{\theta} := \bigg\{ x \in X : \sup_{\xi > 0} \xi^{\theta} \Big\| \xi \big(\xi + \mathcal{A}\big)^{-1} x - x \Big\| < \infty \bigg\},$$

endowed with the norm

$$\|\cdot\|_{X^{\theta}_{\mathcal{A}}} := \|\cdot\| + \sup_{\xi>0} \xi^{\theta} \Big\| \xi\big(\xi+\mathcal{A}\big)^{-1}\cdot - \cdot \Big\|,$$

becomes the Banach space.

We will use conditions (P) and (QP) henceforth:

(P) There exist finite constants $c,\ M>0$ and $\beta\in(0,1]$ such that

$$\Psi := \Psi_c := \Big\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \geqslant -c\big(|\operatorname{Im}\lambda|+1\big)\Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda:\mathcal{A})\| \leqslant M\big(1+|\lambda|\big)^{-\beta}, \quad \lambda\in\Psi.$$

(QP): There exist finite numbers $0<\beta\leqslant 1$, $0<d\leqslant 1$, $M>0$ and $0<\eta'<\eta''\leqslant 1$ such that

$$\Psi_{d,\pi\eta''/2} := \Big\{\lambda\in\mathbb{C} : |\lambda|\leqslant d \text{ or } \lambda\in\overline{\Sigma_{\pi\eta''/2}}\Big\} \subseteq \rho(\mathcal{A})$$

and

$$\|R(\lambda:\mathcal{A})\| \leqslant M\big(1+|\lambda|\big)^{-\beta}, \quad \lambda\in\Psi_{d,\pi\eta''/2}.$$

hence, the resolvent set of a multivalued linear operator $\mathcal{A}$ satisfying (QP) can be strictly contained in an acute angle. In the single-valued linear case, the class of almost sectorial operators $A=\mathcal{A}$ satisfying condition (P) is crucially important; for more details about almost sectorial operators and their applications, we refer the reader to the papers [**307**] by F. Periago, [**308**] by F. Periago and B. Straub, the monographs [**233**]-[**234**] and references cited therein.

1.3. Fractional calculus and solution operator families

Fractional calculus and fractional differential equations play an important role in various fields of theoretical and applied science, such as engineering, physics, chemistry, mechanics, electricity, economics, control theory and image processing. For further information about fractional calculus and fractional differential equations, we refer the reader to the monographs by K. Diethelm [**148**], C. Goodrich, A. C. Peterson [**?**], A. A. Kilbas, H. M. Srivastava, J. J. Trujillo [**229**], V. Kiryakova [**230**], F. Mainardi [**280**], S. G. Samko, A. A. Kilbas, O. I. Marichev [**326**] and M. Kostić [**232**]-[**236**], as well as to the doctoral dissertation of E. Bazhlekova [**52**].

Suppose that $\alpha>0$, $m=\lceil\alpha\rceil$ and $I=(0,T)$ for some $T\in(0,\infty]$. Then the Riemann-Liouville fractional integral J_t^{α} of order α is defined by

$$J_t^{\alpha} f(t) := \big(g_{\alpha} * f\big)(t), \quad f\in L^1(I:X),\ t\in I.$$

The Caputo fractional derivative $\mathbf{D}_t^{\alpha}u(t)$ is defined for those functions $u\in C^{m-1}([0,\infty):X)$ for which $g_{m-\alpha}*(u-\sum_{k=0}^{m-1}u_k g_{k+1})\in C^m([0,\infty):X)$, by

$$\mathbf{D}_t^{\alpha}u(t) = \frac{d^m}{dt^m}\left[g_{m-\alpha} * \left(u - \sum_{k=0}^{m-1} u_k g_{k+1}\right)\right].$$

It is worth noticing that the existence of Caputo fractional derivative $\mathbf{D}_t^{\alpha}u$ for $t\geqslant 0$ implies the existence of Caputo fractional derivative $\mathbf{D}_t^{\zeta}u$ for $t\geqslant 0$ and any $\zeta \in (0, \alpha)$. At some places, we will use a slightly weakened notion of Caputo fractional derivatives, as explicitly emphasized.

The Riemann-Liouville fractional derivative D_t^α of order α is defined for those functions $f \in L^1(I : X)$ satisfying $g_{m-\alpha} * f \in W^{m,1}((0,\infty) : X)$, by

$$D_t^\alpha f(t) := \frac{d^m}{dt^m} J_t^{m-\alpha} f(t), \quad t \in I.$$

The Riemann-Liouville fractional integrals and derivatives satisfy the following equalities:

$$J_t^\alpha J_t^\beta f(t) = J_t^{\alpha+\beta} f(t), \quad D_t^\alpha J_t^\alpha f(t) = f(t),$$

for $f \in L^1(I : X)$ and

$$J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} (g_{m-\alpha} * f)^{(k)}(0) g_{\alpha+k+1-m}(t)$$

for any $f \in L^1(I : X)$ with $g_{m-\alpha} * f \in W^{m,1}(I : X)$.

The Weyl-Liouville fractional derivative $D_{t,+}^\gamma u(t)$ of order $\gamma \in (0,1)$ is defined for those continuous functions $u : \mathbb{R} \to X$ such that $t \mapsto \int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\,ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^\gamma u(t) := \frac{d}{dt} \int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\,ds, \quad t \in \mathbb{R}.$$

Set $D_{t,+}^1 u(t) := -(d/dt)u(t)$. For more details about the subject, the reader may consult the article [**297**].

The Mittag-Leffler functions and the Wright functions play an incredible role in fractional calculus. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C};$$

set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$.

The asymptotic behaviour of entire function $E_{\alpha,\beta}(z)$ is given by the following important result (see e.g., [**351**, Theorem 1.1]):

THEOREM 1.3.1. *Let $0 < \sigma < \frac{1}{2}\pi$. Then, for every $z \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{N} \setminus \{1\}$,*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum_s Z_s^{1-\beta} e^{Z_s} - \sum_{j=1}^{m-1} \frac{z^{-j}}{\Gamma(\beta - \alpha j)} + O\big(|z|^{-m}\big),$$

where Z_s is defined by $Z_s := z^{1/\alpha} e^{2\pi i s/\alpha}$ and the first summation is taken over all those integers s satisfying $|\arg(z) + 2\pi s| < \alpha(\frac{\pi}{2} + \sigma)$.

Let $\gamma \in (0,1)$. Then the Wright function $\Phi_\gamma(\cdot)$ is defined by

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\gamma-\gamma n)}, \quad z \in \mathbb{C}.$$

Let us recall that $\Phi_\gamma(\cdot)$ is an entire function as well as that:

(i) $\Phi_\gamma(t) \geqslant 0$, $t \geqslant 0$,
(ii) $\int_0^\infty e^{-\lambda t}\gamma s t^{-1-\gamma}\Phi_\gamma(t^{-\gamma}s)\,dt = e^{-\lambda^\gamma s}$, $\lambda \in \mathbb{C}_+$, $s > 0$, and
(iii) $\int_0^\infty t^r \Phi_\gamma(t)\,dt = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$.

The asymptotic expansion of the Wright function $\Phi_\gamma(\cdot)$, as $|z| \to \infty$ in the sector $|\arg(z)| \leqslant \min((1-\gamma)3\pi/2, \pi) - \varepsilon$, is given by

$$\Phi_\gamma(z) = Y^{\gamma-1/2}e^{-Y}\left(\sum_{m=0}^{M-1} A_m Y^{-M} + O\Big(|Y|^{-M}\Big)\right),$$

where $Y = (1-\gamma)(\gamma^\gamma z)^{1/(1-\gamma)}$, $M \in \mathbb{N}$ and A_m are certain real numbers (see e.g., [**52**]).

Solution operator families. Suppose now that $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, $C_2 \in L(X)$ is injective, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$.

We will use the following general definition:

DEFINITION 1.3.2. ([**236**]) Suppose $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C_1 \in L(Y,X)$, and $C_2 \in L(X)$ is injective.

(i) Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)} \subseteq L(Y,X) \times L(X)$ if and only if the mappings $t \mapsto R_1(t)y$, $t \geqslant 0$ and $t \mapsto R_2(t)x$, $t \in [0,\tau)$ are continuous for every fixed $x \in X$ and $y \in Y$, as well as the following conditions hold:

$$\left(\int_0^t a(t-s)R_1(s)y\,ds, R_1(t)y - k(t)C_1 y\right) \in \mathcal{A},\ t \in [0,\tau),\ y \in Y \text{ and} \tag{10}$$

$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2 x,\ \text{whenever } t \in [0,\tau) \text{ and } (x,y) \in \mathcal{A}. \tag{11}$$

(ii) Let $(R_1(t))_{t\in[0,\tau)} \subseteq L(Y,X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_1-existence family $(R_1(t))_{t\in[0,\tau)}$ if and only if (10) holds.
(iii) Let $(R_2(t))_{t\in[0,\tau)} \subseteq L(X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ if and only if (11) holds.

Let us recall that $R(R_1(0) - k(0)C_1) \subseteq \mathcal{A}0$ and, if $a(t)$ is a kernel on $[0,\tau)$, then $R_2(t)\mathcal{A}$ is single-valued for any $t \in [0,\tau)$ and $R_2(t)y = 0$ for any $y \in \mathcal{A}0$ and $t \in [0,\tau)$.

DEFINITION 1.3.3. ([**236**]) Suppose that $0 < \tau \leqslant \infty$, $k \in C([0,\tau))$, $k \neq 0$, $a \in L^1_{loc}([0,\tau))$, $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, $C \in L(X)$ is injective and $C\mathcal{A} \subseteq$

$\mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (a,k)-regularized C-resolvent family with a subgenerator $\mathcal{A}$ if and only if $(R(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized C-uniqueness family having $\mathcal{A}$ as subgenerator, $R(t)C = CR(t)$ and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in [0,\tau))$.

If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geqslant 0$, then we also say that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; 0-times integrated (a,C)-resolvent family is further abbreviated to (a,C)-resolvent family. We will accept a similar terminology for mild (a,k)-regularized C_1-existence families and mild (a,k)-regularized C_2-uniqueness families.

Suppose that $(R_1(t),R_2(t))_{t\in[0,\tau)}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Then we have

$$\begin{aligned}\big(a*R_2\big)(s)R_1(t)y - R_2(s)\big(a*R_1\big)(t)y\\ = k(t)\big(a*R_2\big)(s)C_1y - k(s)C_2\big(a*R_1\big)(t)y, \quad t\in[0,\tau),\ y\in Y.\end{aligned}$$

The integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ (mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family $(R_1(t),R_2(t))_{t\in[0,\tau)}$) is defined by

$$\mathcal{A}_{int} := \left\{(x,y)\in X\times X : R_2(t)x - k(t)C_2x = \int_0^t a(t-s)R_2(s)y\,ds,\ t\in[0,\tau)\right\};$$

we define the integral generator of an (a,k)-regularized C-regularized family $(R(t))_{t\in[0,\tau)}$ in the same way as above. The integral generator $\mathcal{A}_{int}$ is an MLO in X which extends any subgenerator of $(R_2(t))_{t\in[0,\tau)}$ $((R(t))_{t\in[0,\tau)})$ in the set theoretical sense; furthermore, the assumption $R_2(t)C_2 = C_2R_2(t)$, $t\in[0,\tau)$ implies that $C_2^{-1}\mathcal{A}_{int}C_2 = \mathcal{A}_{int}$ so that $C^{-1}\mathcal{A}_{int}C = \mathcal{A}_{int}$ for resolvent families.

Concerning the vector-valued Laplace transform, we can recommend for the reader the monographs [**30, 236, 352**]. The following condition on a scalar-valued function $k(t)$ will be used:

(P1): $k(t)$ is Laplace transformable, i.e., it is locally integrable on $[0,\infty)$ and there exists $\beta\in\mathbb{R}$ such that $\tilde{k}(\lambda) := \mathcal{L}(k)(\lambda) := \lim_{b\to\infty}\int_0^b e^{-\lambda t}k(t)\,dt := \int_0^\infty e^{-\lambda t}k(t)\,dt$ exists for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda > \beta$. Put $\operatorname{abs}(k)$:=inf$\{\operatorname{Re}\lambda : \tilde{k}(\lambda) \text{ exists}\}$, and denote by $\mathcal{L}^{-1}$ the inverse Laplace transform.

We have the following ([**236**]):

THEOREM 1.3.4. *Suppose $\mathcal{A}$ is a closed* MLO *in X, $C_1\in L(Y,X)$, $C_2\in L(X)$, C_2 is injective, $\omega_0\geqslant 0$ and $\omega\geqslant \max(\omega_0,\operatorname{abs}(|a|),\operatorname{abs}(k))$.*

(i) *Let $(R_1(t),R_2(t))_{t\geqslant 0}\subseteq L(Y,X)\times L(X)$ be strongly continuous, and let the family $\{e^{-\omega t}R_i(t) : t\geqslant 0\}$ be equicontinuous for $i=1,2$.*
   (a) *Suppose $(R_1(t),R_2(t))_{t\geqslant 0}$ is a mild (a,k)-regularized (C_1,C_2)-existence and uniqueness family with a subgenerator $\mathcal{A}$. Then, for every $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>\omega$ and $\tilde{a}(\lambda)\tilde{k}(\lambda)\neq 0$, the operator*

$I-\tilde{a}(\lambda)\mathcal{A}$ *is injective,* $R(C_1) \subseteq R(I-\tilde{a}(\lambda)\mathcal{A})$,

$$\tilde{k}(\lambda)\big(I-\tilde{a}(\lambda)\mathcal{A}\big)^{-1}C_1 y = \int_0^\infty e^{-\lambda t}R_1(t)y\,dt,\ y \in Y, \tag{12}$$

$$\left\{\frac{1}{\tilde{a}(z)} : \operatorname{Re} z > \omega,\ \tilde{k}(z)\tilde{a}(z) \neq 0\right\} \subseteq \rho_{C_1}(\mathcal{A}) \tag{13}$$

and

$$\tilde{k}(\lambda)C_2 x = \int_0^\infty e^{-\lambda t}\big[R_2(t)x - \big(a * R_2\big)(t)y\big]\,dt, \quad \text{whenever } (x,y) \in \mathcal{A}. \tag{14}$$

(b) *Let* (13) *hold, and let* (12) *and* (14) *hold for any* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$. *Then* $(R_1(t), R_2(t))_{t \geqslant 0}$ *is a mild* (a,k)*-regularized* (C_1, C_2)*-existence and uniqueness family with a subgenerator* $\mathcal{A}$.

(ii) *Let* $(R_1(t))_{t\geqslant 0}$ *be strongly continuous, and let the family* $\{e^{-\omega t}R_1(t) : t \geqslant 0\}$ *be equicontinuous. Then* $(R_1(t))_{t\geqslant 0}$ *is a mild* (a,k)*-regularized* C_1*-existence family with a subgenerator* $\mathcal{A}$ *if and only if for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, *one has* $R(C_1) \subseteq R(I-\tilde{a}(\lambda)\mathcal{A})$ *and*

$$\tilde{k}(\lambda)C_1 y \in \big(I-\tilde{a}(\lambda)\mathcal{A}\big)\int_0^\infty e^{-\lambda t}R_1(t)y\,dt, \quad y \in Y.$$

(iii) *Let* $(R_2(t))_{t\geqslant 0}$ *be strongly continuous, and let the family* $\{e^{-\omega t}R_2(t) : t \geqslant 0\}$ *be equicontinuous. Then* $(R_2(t))_{t\geqslant 0}$ *is a mild* (a,k)*-regularized* C_2*-uniqueness family with a subgenerator* $\mathcal{A}$ *if and only if for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, *the operator* $I-\tilde{a}(\lambda)\mathcal{A}$ *is injective and* (14) *holds.*

THEOREM 1.3.5. *Let* $(R(t))_{t\geqslant 0} \subseteq L(X)$ *be a strongly continuous operator family such that there exists* $\omega \geqslant 0$ *satisfying that the family* $\{e^{-\omega t}R(t) : t \geqslant 0\}$ *is equicontinuous, and let* $\omega_0 > \max(\omega, \operatorname{abs}(|a|), \operatorname{abs}(k))$. *Suppose that* $\mathcal{A}$ *is a closed* MLO *in* X *and* $C\mathcal{A} \subseteq \mathcal{A}C$.

(i) *Assume that* $\mathcal{A}$ *is a subgenerator of the global* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying* (10) *for all* $x = y \in X$. *Then, for every* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > \omega_0$ *and* $\tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0$, *the operator* $I-\tilde{a}(\lambda)\mathcal{A}$ *is injective,* $R(C) \subseteq R(I-\tilde{a}(\lambda)\mathcal{A})$, *as well as*

$$\tilde{k}(\lambda)\big(I-\tilde{a}(\lambda)\mathcal{A}\big)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt,\ x \in X,\ \operatorname{Re}\lambda > \omega_0,\ \tilde{a}(\lambda)\tilde{k}(\lambda) \neq 0, \tag{15}$$

$$\left\{\frac{1}{\tilde{a}(\lambda)} : \operatorname{Re}\lambda > \omega_0,\ \tilde{k}(\lambda)\tilde{a}(\lambda) \neq 0\right\} \subseteq \rho_C(\mathcal{A}) \tag{16}$$

and $R(s)R(t) = R(t)R(s),\ t,\ s \geqslant 0$.

(ii) *Assume* (15)-(16). *Then* $\mathcal{A}$ *is a subgenerator of the global* (a, k)*-regularized* C*-resolvent family* $(R(t))_{t\geqslant 0}$ *satisfying* (10) *for all* $x = y \in X$ *and* $R(s)R(t) = R(t)R(s),\ t,\ s \geqslant 0.$

CHAPTER 2

ALMOST PERIODIC TYPE FUNCTIONS AND SOLUTIONS TO INTEGRO-DIFFERENTIAL EQUATIONS

In this chapter, we investigate vector-valued almost periodic type functions and almost periodic type solutions of the abstract Volterra integro-differential equations in Banach spaces, which could be degenerate or non-degenerate in time variable. Special attention is paid to the analysis of various classes of abstract semilinear fractional Cauchy inclusions.

We start by recalling the basic features of almost periodic functions and asymptotically almost periodic functions in Banach spaces.

2.1. Almost periodic functions and asymptotically almost periodic functions

The notion of almost periodicity was introduced by the famous Danish mathematician H. Bohr around 1924-1926 ([**75**]) and later generalized by many others (cf. [**18**], [**115**], [**117**], [**135**], [**170**], [**192**]-[**193**], [**212**], [**265**], [**333**] and [**359**] for more details on the subject). Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $f : I \to X$ be continuous. Given $\varepsilon > 0$, we call $\tau > 0$ an ε-period for $f(\cdot)$ if and only if

$$\|f(t + \tau) - f(t)\| \leqslant \varepsilon, \quad t \in I. \tag{17}$$

By $\vartheta(f, \varepsilon)$ we denote the set of all ε-periods for $f(\cdot)$. We say that $f(\cdot)$ is almost periodic if and only if for each $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta(f, \varepsilon)$. It is said that $f(\cdot)$ is weakly almost periodic if and only if for each $x^* \in X^*$ the function $x^*(f(\cdot))$ is almost periodic. Any weakly almost periodic function $f \in BUC(I : X)$ with relatively compact range in X is almost periodic; see e.g., [**30**, Proposition 4.5.12].

By $AP(I : X)$ we denote the space consisting of all almost periodic functions from the interval I into X; equipped with the sup-norm, $AP(I : X)$ is a Banach space. This space contains the space $P_c(I : X)$ consisting of all continuous functions $f : I \to X$ of period $c > 0$; by $P(I : X)$ we denote the space consisting of all continuous functions $f : I \to X$ for which there exists $c > 0$ such that $f(\cdot)$ is of period $c > 0$.

The notion of an almost periodic strongly continuous semigroup was introduced by H. Bart and S. Goldberg in [**43**]. The translation semigroup $(W(t))_{t \geqslant 0}$ on

$AP([0,\infty):X)$, defined by $[W(t)f](s) := f(t+s)$, $t \geqslant 0$, $s \geqslant 0$, $f \in AP([0,\infty):X)$ is consisting solely of surjective isometries $W(t)$ $(t \geqslant 0)$ and can be extended to a C_0-group $(W(t))_{t\in\mathbb{R}}$ of isometries on $AP([0,\infty):X)$, where $W(-t) := W(t)^{-1}$ for $t > 0$. Moreover, the mapping $\mathbb{E} : AP([0,\infty):X) \to AP(\mathbb{R}:X)$, defined by

$$[\mathbb{E}f](t) := [W(t)f](0), \quad t \in \mathbb{R},\ f \in AP([0,\infty):X),$$

is a linear surjective isometry and $\mathbb{E}f(\cdot)$ is the unique almost periodic extension of a function $f(\cdot)$ from $AP([0,\infty):X)$ to the whole real line. Let us recall that $[\mathbb{E}(Bf)] = B(\mathbb{E}f)$ for all $B \in L(X)$ and $f \in AP([0,\infty):X)$.

In the following theorem, we collect the fundamental properties of almost periodic vector-valued functions; by c_0 we denote the Banach space of all numerical sequences tending to zero, equipped with the sup-norm.

THEOREM 2.1.1. *Let $f \in AP(I:X)$. Then the following holds:*

(i) $f \in BUC(I:X)$;

(ii) *if $g \in AP(I:X)$, $h \in AP(I:\mathbb{C})$, $\alpha,\ \beta \in \mathbb{C}$, then $\alpha f + \beta g$ and $hf \in AP(I:X)$;*

(iii) *Bohr's transform of $f(\cdot)$,*

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} f(s)\,ds,$$

exists for all $r \in \mathbb{R}$ and

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t} \int_\alpha^{t+\alpha} e^{-irs} f(s)\,ds$$

for all $\alpha \in I,\ r \in \mathbb{R}$. The element $P_r(f)$ is called the Bohr coefficient or the Bohr-Fourier coefficient of $f(\cdot)$;

(iv) *if $P_r(f) = 0$ for all $r \in \mathbb{R}$, then $f(t) = 0$ for all $t \in I$;*

(v) *Bohr's spectrum $\sigma(f) := \{r \in \mathbb{R} : P_r(f) \neq 0\}$ is at most countable;*

(vi) *if X does not contain an isomorphic copy of c_0, $I = \mathbb{R}$ and $g(t) = \int_0^t f(s)\,ds$ $(t \in \mathbb{R})$ is bounded, then $g \in AP(\mathbb{R}:X)$;*

(vii) *if $(g_n)_{n\in\mathbb{N}}$ is a sequence in $AP(I:X)$ and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to g, then $g \in AP(I:X)$;*

(viii) *if $f' \in BUC(I:X)$, then $f' \in AP(I:X)$;*

(ix) *(Spectral synthesis) $f \in \overline{span\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\}}$;*

(x) *$R(f)$ is relatively compact in X;*

(xi) *(Supremum formula) we have*

$$\|f\|_\infty = \sup_{t\geqslant t_0} \|f(t)\|, \quad t_0 \in I;$$

(xii) *(Convolution invariance) if $I = \mathbb{R}$ and $g \in L^1(\mathbb{R})$, then $g*f \in AP(\mathbb{R}:X)$, where*

$$(g*f)(t) = \int_{-\infty}^{\infty} g(t-s)f(s)\,ds, \quad t \in \mathbb{R};$$

(xiii) *if* $n \in \mathbb{N}$ *and* $f_1 \in AP(I : X_1), \cdots, f_n \in AP(I : X_n)$, *then* $(f_1, \cdots, f_n) \in AP(I : X_1 \times \cdots \times X_n)$. *Here,* X_i *is a complex Banach space for all* $i = 1, \cdots, n$;

(xiv) *if* $f_1 \in AP(I : X_1), \cdots, f_n \in AP(I : X_n)$, *then for each* $\varepsilon > 0$ *there exists a common relatively dense set* $\vartheta(f_1, \cdots, f_n, \varepsilon)$ *of* ε*-periods for any of these functions. Here,* X_i *is a complex Banach space for all* $i = 1, \cdots, n$;

(xv) *(Bochner's criterion) Let* $I = \mathbb{R}$. *Then* $f(\cdot)$ *is almost periodic if and only if for any real sequence* (b_n) *there exists a subsequence* (a_n) *of* (b_n) *such that* $(f(\cdot + a_n))$ *converges in* $BUC(\mathbb{R} : X)$.

Before proceeding any further, we would like to mention that the necessary and sufficient condition for X to contain c_0 is given in [**30**, Theorem 4.6.14]. The importance of such condition has been recognized already by H. Bohr and later employed frequently (see e.g., the fomulation of Kadet's theorem [**30**, Theorem 4.6.11]).

By $AP_\Lambda(I : X)$, where Λ is a non-empty subset of I, we denote the vector subspace of $AP(I : X)$ consisting of all functions $f \in AP(I : X)$ satisfying that $\sigma(f) \subseteq \Lambda$; $AP_\Lambda(I : X)$ is a closed subspace of $AP(I : X)$ and therefore a Banach space. For numerous equivalent criteria stating the necessary and sufficient conditions for the almost periodicity of a given function, we refer the reader to [**234**] and references quoted therein.

In the case that $I = [0, \infty)$, the notion of asymptotical almost periodicity was introduced by A. S. Kovanko [**257**] in 1929 and later rediscovered, in a slightly different form, by M. Fréchet [**174**] in 1941 (for comprehensive information about the subject, we refer to [**96**], [**135**], [**192**]-[**193**], [**324**]-[**325**], [**353**] and [**366**]). A function $f \in C_b(I : X)$ is said to be asymptotically almost periodic if and only if for every $\varepsilon > 0$ we can find numbers $l > 0$ and $M > 0$ such that every subinterval of I of length l contains, at least, one number τ such that $\|f(t + \tau) - f(t)\| \leqslant \varepsilon$ provided $|t|, |t+\tau| \geqslant M$. The space consisting of all asymptotically almost periodic functions from I into X is denoted by $AAP(I : X)$. It is well known that (see W. M. Ruess, W. H. Summers [**323**]-[**325**] for the case that $I = [0, \infty)$ and C. Zhang [**365**]-[**366**] for the case that $I = \mathbb{R}$) the following statements are equivalent:

(i) $f \in AAP(I : X)$.

(ii) There exist uniquely determined functions $g \in AP(\mathbb{R} : X)$ and $\phi \in C_0(I : X)$ such that $f = g + \phi$.

The functions g and ϕ from (ii) are called the principal and corrective terms of the function f, respectively. If there exist functions $g \in P(\mathbb{R} : X)$ (of period $c > 0$) and $\phi \in C_0(I : X)$ such that $f = g + \phi$, then we say that $f(\cdot)$ is asymptotically periodic (asymptotically c-periodic).

By $C_0(I \times Y : X)$ we denote the space of all continuous functions $h : I \times Y \to X$ such that $\lim_{|t| \to \infty} h(t, y) = 0$ uniformly for y in any compact subset of Y. A continuous function $f : I \times Y \to X$ is called uniformly continuous on bounded sets, uniformly for $t \in I$ if and only if for every $\varepsilon > 0$ and every bounded subset K of Y there exists a number $\delta_{\varepsilon,K} > 0$ such that $\|f(t, x) - f(t, y)\| \leqslant \varepsilon$ for all $t \in I$

and all $x,\ y \in K$ satisfying that $\|x-y\| \leqslant \delta_{\varepsilon,K}$. If $f : I \times Y \to X$, then we define $\hat{f} : I \times Y \to L^p([0,1] : X)$ by $\hat{f}(t,y) := f(t+\cdot, y)$, $t \geqslant 0$, $y \in Y$.

The following definition and related composition principle can be found, e.g., in [**234**]:

Definition 2.1.2. Let $1 \leqslant p < \infty$.

(i) A function $f : I \times Y \to X$ is called almost periodic if and only if $f(\cdot,\cdot)$ is bounded, continuous as well as for every $\varepsilon > 0$ and every compact $K \subseteq Y$ there exists $l(\varepsilon,K) > 0$ such that every subinterval $J \subseteq I$ of length $l(\varepsilon,K)$ contains a number τ with the property that $\|f(t+\tau,y) - f(t,y)\| \leqslant \varepsilon$ for all $t \in I$, $y \in K$. The collection of such functions will be denoted by $AP(I \times Y : X)$.

(ii) A function $f : I \times Y \to X$ is said to be asymptotically almost periodic if and only if it is bounded continuous and admits a decomposition $f(t) = g(t)+q(t)$, $t \in I$, where $g \in AP(\mathbb{R}\times Y : X)$ and $q \in C_0(I\times Y : X)$. Denote by $AAP(I \times Y : X)$ the vector space consisting of all such functions.

Theorem 2.1.3. (i) *Let $f \in AP(I \times Y : X)$ and $h \in AP(I : Y)$. Then the mapping $t \mapsto f(t,h(t))$, $t \in I$ belongs to the space $AP(I : X)$.*

(ii) *Let $f \in AAP(I \times Y : X)$ and $h \in AAP(I : Y)$. Then the mapping $t \mapsto f(t,h(t))$, $t \geqslant 0$ belongs to the space $AAP(I : X)$.*

Let us recall that $f(\cdot)$ is anti-periodic if and only if there exists $p > 0$ such that $f(x+p) = -f(x)$, $x \in I$. Any such function needs to be periodic, as it can be easily proved. Given $\varepsilon > 0$, we call $\tau > 0$ an ε-antiperiod for $f(\cdot)$ if and only if

$$\|f(t+\tau) + f(t)\| \leqslant \varepsilon, \quad t \in I.$$

By $\vartheta_{ap}(f,\varepsilon)$ we denote the set of all ε-antiperiods for $f(\cdot)$. The notion of an almost anti-periodic function has recently been introduced in [**254**, Definition 2.1] as follows:

Definition 2.1.4. It is said that $f(\cdot)$ is almost anti-periodic if and only if for each $\varepsilon > 0$ the set $\vartheta_{ap}(f,\varepsilon)$ is relatively dense in $[0,\infty)$.

We know that any almost anti-periodic function needs to be almost periodic. Denote by $ANP_0(I : E)$ the linear span of almost anti-periodic functions from I into X. Then [**254**, Theorem 2.3] implies that $ANP_0(I : E)$ is a linear subspace of $AP(I : E)$ as well as that the linear closure of $ANP_0(I : E)$ in $AP(I : E)$, denoted by $ANP(I : E)$, satisfies

$$ANP(I : E) = AP_{\mathbb{R}\smallsetminus\{0\}}(I : E). \tag{18}$$

Later, we will generalize the notion of almost anti-periodicity by introducing the notion of almost c-periodicity (see Section 2.9).

Within the theory of topological dynamical systems, the notion of recurrence plays an important role; for more details, the reader may consult the research monographs [**129**] by J. de Vries and [**159**] by T. Eisner et al. Following A. Haraux and P. Souplet [**202**], we introduce the following notion:

DEFINITION 2.1.5. It is said that a continuous function $f : I \to X$ is uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to\infty} \sup_{t\in I} \bigl\| f(t+\alpha_n) - f(t) \bigr\| = 0. \tag{19}$$

It is well known that any almost periodic function is uniformly recurrent, while the converse statement is not true in general. It is worth noting that the convergence of the above limit is uniform in the variable $t \in \mathbb{R}$, so that the notion of a uniformly recurrent function should not be mistakenly identified with the notion of a reccurent function in the continuous Bebutov system [**53**], where the author has analyzed the usual Fréchet space $C(\mathbb{R})$ and the topology of uniform convergence on compact sets (cf. also Subection 2.3.9 in the monograph [**61**] by G. Bertotti and I. D. Mayergoyz, the paper [**124**] by L. I. Danilov and references cited therein for further information in this direction).

Let us recall that the notion of a pseudo almost periodic function was introduced by C. Zhang in his doctoral dissertation [**362**] (cf. also [**363**]-[**364**]). Henceforth, $PAP_0(\mathbb{R} : X)$ stands for the space consisting of all pseudo-ergodic components, i.e., the bounded continuous functions $\Phi : \mathbb{R} \to X$ such that

$$\lim_{l\to\infty} \frac{1}{2l} \int_{-l}^{l} \|\Phi(s)\|\, ds = 0.$$

Concerning the space $PAP_0(\mathbb{R} : \mathbb{C})$, it should be recalled that $f \in PAP_0(\mathbb{R} : \mathbb{C})$ if and only if $f \cdot f \in PAP_0(\mathbb{R} : \mathbb{C})$.

We say that a continuous function $f : \mathbb{R} \to X$ is pseudo almost periodic if and only if it admits a decomposition $f = g + q$, where $g \in AP(\mathbb{R} : X)$ and $q \in PAP_0(\mathbb{R} : X)$. It is well known that, if such a decomposition exists, then it must be unique. The space consisting of all pseudo almost periodic functions will be denoted by $PAP(\mathbb{R} : X)$ henceforth.

EXAMPLE 2.1.6. Define

$$f(t) := \frac{1}{2t} \int_{-t}^{t} s|\sin s|^{s^N}\, ds, \quad t \in \mathbb{R},$$

where $N > 6$. From [**13**, Example p. 1143] we know that $\lim_{t\to+\infty} f(t) = 0$ and therefore $\cdot|\sin \cdot|^{\cdot^N} \in PAP_0(\mathbb{R} : \mathbb{C})$ for $N > 6$.

For more details about pseudo almost periodic functions, see the book [**136**] by T. Diagana and the doctoral dissertation of C. Zhang [**362**]. Mention should be made of the monograph [**304**] by A. A. Pankov.

The almost periodic and almost automorphic functions on time scales and their applications to the abstract Volterra integro-differential equations have been recently considered by numerous mathematicians (for time scale calculus, we warmly recommend the monograph [**72**] by M. Bochner and A. Peterson). For more details about this problematic, we refer the reader to [**132, 133, 134, 219, 267, 268, 269, 273, 274, 291**] and references cited therein.

Concerning Hartman almost periodic functions, we recommend for the reader the article [**113**] by G. Cohen and V. Losert.

2.2. Stepanov, Weyl and Besicovitch classes

Suppose that $1 \leqslant p < \infty$, $l > 0$ and $f,\ g \in L^p_{loc}(I : X)$, where $I = \mathbb{R}$ or $I = [0, \infty)$. We define the Stepanov 'metric' by

$$D^p_{S_l}\big[f(\cdot), g(\cdot)\big] := \sup_{x \in I}\left[\frac{1}{l}\int_x^{x+l} \|f(t) - g(t)\|^p\, dt\right]^{1/p}.$$

Then, for every two numbers $l_1,\ l_2 > 0$, there exist two positive real constants $k_1,\ k_2 > 0$ independent of $f,\ g$, such that

$$k_1 D^p_{S_{l_1}}\big[f(\cdot), g(\cdot)\big] \leqslant D^p_{S_{l_2}}\big[f(\cdot), g(\cdot)\big] \leqslant k_2 D^p_{S_{l_1}}\big[f(\cdot), g(\cdot)\big],$$

as well as that there exists

$$D^p_W\big[f(\cdot), g(\cdot)\big] := \lim_{l \to \infty} D^p_{S_l}\big[f(\cdot), g(\cdot)\big] \tag{20}$$

in $[0, \infty]$. The distance appearing above is called the Weyl distance of $f(\cdot)$ and $g(\cdot)$. The Stepanov and Weyl 'norm' of $f(\cdot)$ are defined by

$$\|f\|_{S^p_l} := D^p_{S_l}\big[f(\cdot), 0\big] \quad \text{and} \quad \|f\|_{W^p} := D^p_W\big[f(\cdot), 0\big],$$

respectively.

Before proceeding further, we would like to note that it is not clear how we can define the Stepanov distance by considering a general variable exponent $p \in \mathcal{P}(I)$ in place of the constant coefficient $p \geqslant 1$ above; morover, it is not clear whether the formula (20) can be generalized in this context.

Henceforth we assume that $l_1 = l_2 = 1$. It is said that a function $f \in L^p_{loc}(I : X)$ is Stepanov p-bounded, S^p-bounded for short, if and only if

$$\|f\|_{S^p} := \sup_{t \in I}\left(\int_t^{t+1} \|f(s)\|^p\, ds\right)^{1/p} < \infty.$$

Equipped with the above norm, the space $L^p_S(I : X)$ consisting of all S^p-bounded functions is a Banach space. A function $f \in L^p_S(I : X)$ is said to be Stepanov p-almost periodic, S^p-almost periodic shortly, if and only if the function $\hat{f} : I \to L^p([0, 1] : X)$, defined by

$$\hat{f}(t)(s) := f(t + s), \quad t \in I,\ s \in [0, 1], \tag{21}$$

is almost periodic. We say that the function $f \in L^p_S(I : X)$ is asymptotically Stepanov p-almost periodic if and only if there exist two locally p-integrable functions $g : \mathbb{R} \to X$ and $q : I \to X$ satisfying the following conditions:

(i) g is S^p-almost periodic,
(ii) $\hat{q}$ belongs to the class $C_0(I : L^p([0, 1] : X))$,
(iii) $f(t) = g(t) + q(t)$ for all $t \in I$.

Recall, if $f(\cdot)$ is an (asymptotically) almost periodic function, then $f(\cdot)$ is also (asymptotically) Stepanov p-almost periodic for $1 \leqslant p < \infty$. The converse statement is false, however ([**264**]):

EXAMPLE 2.2.1. Assume that $\alpha,\ \beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well-defined irrational number. Then the functions

$$f(t) := \sin\left(\frac{1}{2+\cos\alpha t+\cos\beta t}\right), \quad t \in \mathbb{R} \tag{22}$$

and

$$g(t) := \cos\left(\frac{1}{2+\cos\alpha t+\cos\beta t}\right), \quad t \in \mathbb{R} \tag{23}$$

are Stepanov p-almost periodic but not almost periodic ($1 \leqslant p < \infty$). The case $\alpha = 1$ and $\beta = \sqrt{2}$ has been further analyzed by A. Nawrocki in [**301**], who proved with the help of Liouville's theorem and some results from the theory of continuous fractions [**301**, Theorem 1, Theorem 2] that

$$\lim_{t\to+\infty} \frac{t^{-2-\varepsilon}}{2+\cos t+\cos\sqrt{2}t} = 0, \quad \varepsilon > 0$$

and

$$\lim_{t\to+\infty} \frac{t^{-2}}{2+\cos t+\cos\sqrt{2}t}$$

does not exist. Recall, the function $t \mapsto 1/(2+\cos t+\cos\sqrt{2}t)$, $t \in \mathbb{R}$ is well defined, continuous and unbounded.

Denote by $APS^p(I : X)$ and $AAPS^p(I : X)$ the space consisting of all S^p-almost periodic functions $f : I \to X$ and the space consisting of all asymptotically S^p-almost periodic functions $f : I \to X$, respectively. The Bochner theorem asserts that any uniformly continuous function which is also Stepanov p-almost periodic needs to be almost periodic ($1 \leqslant p < \infty$); the Bochner theorem for Stepanov p-almost periodic functions has been established by Z. Hu and A. B. Mingarelli in [**213**, Theorem 1].

DEFINITION 2.2.2. Let $1 \leqslant p < \infty$. A function $f : I\times Y \to X$ is called Stepanov p-almost periodic if and only if $\hat{f} : I \times Y \to L^p([0,1] : X)$ is almost periodic.

Recall that a bounded continuous function $f : I \times Y \to X$ is asymptotically almost periodic if and only if for every $\varepsilon > 0$ and every compact $K \subseteq Y$ there exist $l(\varepsilon, K) > 0$ and $M(\varepsilon, K) > 0$ such that every subinterval $J \subseteq I$ of length $l(\varepsilon, K)$ contains a number τ with the property that $\|f(t+\tau, y) - f(t, y)\| \leqslant \varepsilon$ provided $|t|,\ |t+\tau| \geqslant M(\varepsilon, K)$, $y \in K$. The notion of an asymptotically Stepanov p-almost periodic function $f(\cdot,\cdot)$ is introduced in [**234**] for case $I = [0,\infty)$ as follows:

DEFINITION 2.2.3. Let $1 \leqslant p < \infty$. A function $f : I \times Y \to X$ is said to be asymptotically S^p-almost periodic if and only if $\hat{f} : I \times Y \to L^p([0,1] : X)$ is asymptotically almost periodic. The collection of such functions will be denoted by $AAPS^p(I \times Y : X)$.

Let $\omega \in I$. Then we say that a bounded continuous function $f : I \to X$ is S-asymptotically ω-periodic if and only if $\lim_{|t|\to\infty} \|f(t+\omega) - f(t)\| = 0$. Denote by $SAP_\omega(I : X)$ the space consisting of all such functions. A Stepanov p-bounded function $f(\cdot)$ is said to be Stepanov p-asymptotically ω-periodic if and only if

$$\lim_{|t|\to\infty} \int_t^{t+1} \left\|f(s+\omega) - f(s)\right\|^p ds = 0.$$

If we denote by $S^pSAP_\omega(I : X)$ the space consisting of all such functions, then we have that $SAP_\omega(I : X) \subseteq S^pSAP_\omega(I : X)$ and the inclusion is strict (for more details, see H. R. Henríquez [**206**] and H. R. Henríquez, M. Pierri, P. Táboas [**209**]).

The (Stepanov) quasi-asymptotically almost periodic functions have been analyzed in [**247**]. For our further work, it will be necessary to recall the following definition:

Definition 2.2.4. Suppose that $I = [0, \infty)$ or $I = \mathbb{R}$.

(i) A bounded continuous function $f : I \to X$ is said to be quasi-asymptotically almost periodic if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\|f(t+\tau) - f(t)\| \leqslant \varepsilon, \text{ provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

Denote by $Q - AAP(I : X)$ the set consisting of all quasi-asymptotically almost periodic functions from I into X.

(ii) Let $f \in L^p_S(I : X)$. Then it is said $f(\cdot)$ is Stepanov p-quasi-asymptotically almost periodic if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\int_t^{t+1} \|f(s+\tau) - f(s)\|^p ds \leqslant \varepsilon^p, \text{ provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

Denote by $S^pQ - AAP(I : X)$ the set consisting of all Stepanov p-quasi-asymptotically almost periodic functions from I into X.

Let us recall that for each number $p \in [1, \infty)$ we have that $Q - AAP(I : X) \subseteq S^pQ - AAP(I : X)$ as well as that any asymptotically Stepanov p-almost periodic function is Stepanov p-quasi-asymptotically almost periodic. Furthermore, if $1 \leqslant p \leqslant q < \infty$, then $S^qQ - AAP(I : X) \subseteq S^pQ - AAP(I : X)$ and for any function $f \in L^p_S(I : X)$, we have that $f(\cdot)$ is Stepanov p-quasi-asymptotically almost periodic if and only if the function $\hat{f} : I \to L^p([0, 1] : X)$, defined by (21), is quasi-asymptotically almost periodic. It is said that $f(\cdot)$ is Stepanov quasi-asymptotically almost periodic if and only if $f(\cdot)$ is Stepanov 1-quasi-asymptotically almost periodic. Any asymptotically almost periodic function $f : I \to X$ is quasi-asymptotically almost periodic. Furthermore, we have $SAP_\omega(I : X) \subseteq Q-AAP(I : X)$ and $S^pSAP_\omega(I : X) \subseteq S^pQ - AAP(I : X)$.

Let $1 \leqslant p < \infty$. In order to introduce the Besicovitch-p-almost periodic functions and the Besicovitch-Doss-p-almost periodic functions, suppose that X and Y are two complex Banach spaces (see also the article [**125**] by L. I. Danilov for the corresponding notion in complete metric spaces). If $f \in L^p_{loc}(\mathbb{R} : X)$, then we define

$$\|f\|_{\mathcal{M}^p} := \limsup_{t \to +\infty} \left[\frac{1}{2t} \int_{-t}^{t} \|f(s)\|^p \, ds \right]^{1/p};$$

if $f \in L^p_{loc}([0, \infty) : X)$, then

$$\|f\|_{\mathcal{M}^p} := \limsup_{t \to +\infty} \left[\frac{1}{t} \int_{0}^{t} \|f(s)\|^p \, ds \right]^{1/p};$$

see also J. Marcinkiewicz's article [**282**] and M. A. Picardello's article [**313**].

In any case, $\| \cdot \|_{\mathcal{M}^p}$ is a seminorm on the space $\mathcal{M}^p(I : X)$ consisting of those $L^p_{loc}(I : X)$-functions $f(\cdot)$ for which $\|f\|_{\mathcal{M}^p} < \infty$. Denote $K_p(I : X) := \{f \in \mathcal{M}^p(I : X) : \|f\|_{\mathcal{M}^p} = 0\}$ and

$$M_p(I : X) := \mathcal{M}^p(I : X)/K_p(I : X).$$

The seminorm $\| \cdot \|_{\mathcal{M}^p}$ on $\mathcal{M}^p(I : X)$ induces the norm $\| \cdot \|_{M^p}$ on $M^p(I : X)$ under which $M^p(I : X)$ is complete so that $(M^p(I : X), \| \cdot \|_{M^p})$ is a Banach space.

Now we are able to introduce the following notion:

Definition 2.2.5. Let $1 \leqslant p < \infty$. We say that a function $f \in L^p_{loc}(I : X)$ is Besicovitch-p-almost periodic if and only if there exists a sequence of X-valued trigonometric polynomials converging to $f(\cdot)$ in $(M^p(I : X), \| \cdot \|_{M^p})$.

The vector space consisting of all Besicovitch-p-almost periodic functions is denoted by $B^p(I : X)$. It is well known that $B^p(I : X)$ is a closed subspace of $M^p(I : X)$ and therefore a Banach space equipped with the norm $\| \cdot \|_{M^p}$.

The Besicovitch class can be equivalently introduced in a Bohr-like manner, by using the notion of satisfactorily uniform sets (see e.g. [**62**] and [**23**, Definition 5.10, Definition 5.11]). We will not use this approach henceforth.

We define the Besicovitch 'distance' of functions $f,\ g \in L^p_{loc}(I : X)$ by

$$D_{B^p}[f(\cdot), g(\cdot)] := \|f - g\|_{\mathcal{M}^p};$$

the Besicovitch 'norm' of $f \in L^p_{loc}(I : X)$ is defined by

$$\|f\|_{B^p} := D_{B^p}[f(\cdot), 0] := \|f\|_{\mathcal{M}^p}.$$

We say that $f(\cdot)$ is Besicovitch p-bounded if and only if $\|f\|_{\mathcal{M}^p} < \infty$. Recall that

$$\|f - g\|_\infty \geqslant D_{S^p_l}[f(\cdot), g(\cdot)] \geqslant D_{W^p}[f(\cdot), g(\cdot)] \geqslant D_{B^p}[f(\cdot), g(\cdot)],$$

for $1 \leqslant p < \infty$, $l > 0$ and $f,\ g \in L^p_{loc}(I : X)$, as well as that the assumption $\|f\|_{\mathcal{M}^p} = 0$ does not imply $f = 0$ a.e. on I.

The notion of a Besicovitch-Doss-p-almost periodic function is introduced in [**234**] following the fundamental analyses of R. Doss [**155**]-[**156**]:

DEFINITION 2.2.6. Let $1 \leqslant p < \infty$. It is said that $f \in L^p_{loc}(I : X)$ is Besicovitch-Doss-p-almost periodic if and only if the following conditions hold:

(i) (B^p-boundedness) We have $\|f\|_{\mathcal{M}^p} < \infty$.

(ii) (B^p-continuity) We have

$$\lim_{\tau\to 0} \limsup_{t\to+\infty} \left[\frac{1}{2t} \int_{-t}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{\tau\to 0+} \limsup_{t\to+\infty} \left[\frac{1}{t} \int_{0}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} = 0,$$

in the case that $I = [0, \infty)$.

(iii) (Doss-p-almost periodicity) For every $\varepsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t\to+\infty} \left[\frac{1}{2t} \int_{-t}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} < \varepsilon, \tag{24}$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t\to+\infty} \left[\frac{1}{t} \int_{0}^{t} \|f(s+\tau) - f(s)\|^p \, ds \right]^{1/p} < \varepsilon,$$

in the case that $I = [0, \infty)$, is relatively dense in I.

(iv) For every $\lambda \in \mathbb{R}$, we have that

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \frac{1}{l} \left[\frac{1}{2t} \int_{-t}^{t} \left\| \left(\int_{x}^{x+l} - \int_{0}^{l} \right) e^{i\lambda s} f(s) \, ds \right\|^p dx \right]^{1/p} = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \frac{1}{l} \left[\frac{1}{t} \int_{0}^{t} \left\| \left(\int_{x}^{x+l} - \int_{0}^{l} \right) e^{i\lambda s} f(s) \, ds \right\|^p dx \right]^{1/p} = 0,$$

in the case that $I = [0, \infty)$.

The vector space consisting of all Besicovitch-Doss-p-almost periodic functions $f : I \to X$ in the sense of Definition 2.2.6 will be denoted by $\mathrm{B}^p(I : X)$. In the case that $X = \mathbb{C}$, an intriguing result of R. Doss says that $\mathrm{B}^p(I : X) = B^p(I : X)$. It is still an unsolved problem whether the equality $\mathrm{B}^p(I : X) = B^p(I : X)$ holds in vector-valued case.

2.2.1. Composition principles for Weyl almost periodic functions. The notion of an (equi-)Weyl-p-almost periodic function plays an important role in our investigations (cf. [**234**, Section 2.3] for more details):

DEFINITION 2.2.7. Let $1 \leqslant p < \infty$ and $f \in L^p_{loc}(I : X)$.

(i) We say that the function $f(\cdot)$ is equi-Weyl-p-almost periodic, $f \in e-W_{ap}^p(I:X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|f(t+\tau)-f(t)\right\|^p\,dt\right]^{1/p} \leqslant \varepsilon.$$

(ii) We say that the function $f(\cdot)$ is Weyl-p-almost periodic, $f \in W_{ap}^p(I:X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l\to\infty}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|f(t+\tau)-f(t)\right\|^p\,dt\right]^{1/p} \leqslant \varepsilon.$$

It is well known that $APS^p(I:X) \subseteq e-W_{ap}^p(I:X) \subseteq W_{ap}^p(I:X)$ and $e-W_{ap}^p(I:X) \subseteq B^p(I:X)$. In the remainder of this subsection, we will present some the research results obtained recently in [**242**], which have not been presented in any other research monograph by now.

The following definition is slightly different from the corresponding definitions introduced recently in [**56**] and [**243**] for the class of equi-Weyl-p-almost periodic functions, with only one pivot space $X = Y$:

Definition 2.2.8. (i) A function $F : I\times Y \to X$ is said to be equi-Weyl p-almost periodic in $t \in I$ uniformly with respect to compact subsets of Y iff $f(\cdot,u) \in L_{loc}^p(I:X)$ for each fixed element $u \in Y$ and if for each $\varepsilon > 0$ and each compact K of Y there exist two numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{u\in K}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|F(t+\tau,u)-F(t,u)\right\|^p\,dt\right]^{1/p} < \varepsilon.$$

We denote by $e-W_{ap,\mathbf{K}}^p(I\times Y:X)$ the vector space consisting of all such functions.

(ii) A function $F : I\times Y \to X$ is said to be Weyl p-almost periodic in $t \in I$ uniformly with respect to compact subsets of Y if $f(\cdot,u) \in L_{loc}^p(I:X)$ for each fixed element $u \in Y$ and if for each $\varepsilon > 0$ and each compact K of Y we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ satisfying that there exists a finite number $l(\varepsilon,\tau) > 0$ such that

$$\sup_{u\in K}\sup_{x\in I}\left[\frac{1}{l}\int_x^{x+l}\left\|F(t+\tau,u)-F(t,u)\right\|^p\,dt\right]^{1/p} < \varepsilon, \quad l \geqslant l(\varepsilon,\tau).$$

We denote by $W_{ap,\mathbf{K}}^p(I\times Y:X)$ the vector space consisting of all such functions.

The following definition is known in the case that $X = Y$ (cf. [**243**]):

DEFINITION 2.2.9. Let $q : [0,\infty) \times Y \to X$ be such that $q(\cdot, u) \in L^p_{loc}([0,\infty) : X)$ for each fixed element $u \in Y$.

(i) It is said that $q(\cdot,\cdot)$ is Weyl p-vanishing uniformly with respect to compact subsets of Y if and only if for each compact set K of Y we have:

$$\lim_{t\to\infty} \lim_{l\to\infty} \sup_{\xi \geqslant 0, u\in K} \left[\frac{1}{l} \int_{\xi}^{\xi+l} \bigl\| q(t+s,u) \bigr\|^p \, ds \right]^{1/p} = 0.$$

(ii) It is said that $q(\cdot,\cdot)$ is equi-Weyl p-vanishing uniformly with respect to compact subsets of Y if and only if for each compact set K of Y we have:

$$\lim_{l\to\infty} \lim_{t\to\infty} \sup_{\xi \geqslant 0, u\in K} \left[\frac{1}{l} \int_{\xi}^{\xi+l} \bigl\| q(t+s,u) \bigr\|^p \, ds \right]^{1/p} = 0.$$

We denote by $W^p_{0,\mathbf{K}}(I \times Y : X)$ and $e - W^p_{0,\mathbf{K}}(I \times Y : X)$ the classes consisting of all Weyl p-vanishing functions, uniformly with respect to compact subsets of Y and all equi-Weyl p-vanishing functions, uniformly with respect to compact subsets of Y, respectively.

Similarly, for the class of (equi-)Weyl p-almost periodic functions, we have the following result which is not comparable with [**56**, Theorem 3] in the case of consideration of equi-Weyl p-almost periodic functions, with $I = \mathbb{R}$ and $X = Y$:

THEOREM 2.2.10. *Suppose that the following conditions hold:*

(i) $F \in (e-)W^p_{ap,\mathbf{K}}(I \times Y : X)$ *with* $p > 1$, *and there exist a number* $r \geqslant \max(p, p/(p-1))$ *and a function* $L_F \in L^r_S(I)$ *such that*

$$\|F(t,x) - F(t,y)\| \leqslant L_F(t)\|x-y\|_Y, \quad t \in I,\ x,\ y \in Y. \tag{25}$$

(ii) $x \in (e-)W^p_{ap}(I : Y)$, *and there exists a set* $E \subseteq I$ *with* $m(E) = 0$ *such that* $K := \{x(t) : t \in I \setminus E\}$ *is relatively compact in* Y.

(iii) *For every* $\varepsilon > 0$, *there exist two numbers* $l > 0$ *and* $L > 0$ *such that any interval* $I' \subseteq I$ *of length* L *contains a number* $\tau \in I'$ *such that*

$$\sup_{t\in I, u\in K} \left[\frac{1}{l} \int_t^{t+l} \bigl\| F(s+\tau,u) - F(s,u) \bigr\|^p \, ds \right]^{1/p} \leqslant \varepsilon \tag{26}$$

and

$$\sup_{t\in I} \left[\frac{1}{l} \int_t^{t+l} \bigl\| x(s+\tau) - x(s) \bigr\|^p_Y \, ds \right]^{1/p} \leqslant \varepsilon \tag{27}$$

in the case of consideration of equi-Weyl p*-almost periodic functions, resp., there exists a finite number* $L > 0$ *such that any interval* $I' \subseteq I$ *of length* L *contains a number* $\tau \in I'$ *satisfying that there exists a number* $l(\varepsilon,\tau) > 0$ *so that* (26)-(27) *hold for all numbers* $l \geqslant l(\varepsilon,\tau)$, *in the case of consideration of Weyl* p*-almost periodic functions.*

Then $q := pr/(p+r) \in [1,p)$ *and* $F(\cdot, x(\cdot)) \in (e-)W^q_{ap}(I : X)$.

PROOF. Without loss of generality, we may assume that $X = Y$. Since the function $L_F(\cdot)$ is Stepanov r-bounded, equivalently, Weyl r-bounded, the measurability and S^p-boundedness of function $F(\cdot, x(\cdot))$ follow similarly as in the proof of [**276**, Theorem 2.2]. Applying the Hölder inequality and an elementary calculation involving the estimate (25) and condition (ii), we get that, for every $t,\ \tau \in I$ and $l > 0$,

$$
\begin{aligned}
&\frac{1}{l}\int_t^{t+l} \big\|F(s+\tau, x(s+\tau)) - F(s, x(s))\big\|^q\, ds \\
\leqslant &\frac{1}{l}\Bigg[\Bigg(\int_t^{t+l} L_F^r(s+\tau)\, ds\Bigg)^{1/r}\Bigg(\int_t^{t+l} \big\|x(s+\tau)-x(s)\big\|^p\, dt\Bigg)^{1/p} \\
&+\Bigg(\int_t^{t+l} \big\|F(s+\tau, x(s)) - F(s, x(s))\big\|^q\, ds\Bigg)^{1/q}\Bigg] \\
\leqslant &\frac{1}{l}\Bigg[\Bigg(\int_t^{t+l} L_F^r(s+\tau)\, ds\Bigg)^{1/r}\Bigg(\int_t^{t+l} \big\|x(s+\tau)-x(s)\big\|^p\, dt\Bigg)^{1/p} \\
&+\Bigg(\int_t^{t+l} \Big(\sup_{u\in K}\big\|F(s+\tau, u) - F(s, u)\big\|\Big)^q\, ds\Bigg)^{1/q}\Bigg].
\end{aligned}
$$

The remaining part of proof is almost the same for both classes of functions, equi-Weyl p-almost periodic functions and Weyl p-almost periodic functions; because of that, we will consider only the first class up to the end of proof. Let $\varepsilon > 0$ be given. By (iii), there exist two numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a number $\tau \in I'$ such that (26)-(27) hold. Since the validity of (26)-(27) with given numbers $l > 0$ and $\tau \in I$ implies the validity of (26)-(27) with numbers nl and $\tau \in I$ $(n \in \mathbb{N})$, we may assume that the number $l > 0$ is as large as we want to be. Then, due to Lemma 1.1.3, we obtain the existence of a finite number $M > 0$ such that:

$$
\frac{1}{l}\Bigg(\int_t^{t+l} L_F^r(s+\tau)\, ds\Bigg)^{1/r} \leqslant M l^{(1/r)-1}\|L_F\|_{W^r}, \quad t \in I
$$

and

$$
\begin{aligned}
&\frac{1}{l}\Bigg(\int_t^{t+l} L_F^r(s+\tau)\, ds\Bigg)^{1/r}\Bigg(\int_t^{t+l} \big\|x(s+\tau)-x(s)\big\|^p\, dt\Bigg)^{1/p} \\
\leqslant &M l^{(1/p)+(1/r)-1}\|L_F\|_{W^r} = l^{(1/q)-1}\|L_F\|_{W^r} \leqslant \|L_F\|_{W^r}, \quad t \in I.
\end{aligned}
$$

For the estimation of term

$$
\frac{1}{l}\Bigg(\int_t^{t+l} \Big(\sup_{u\in K}\big\|F(s+\tau, u) - F(s, u)\big\|\Big)^q\, ds\Bigg)^{1/q}, \quad t \in I
$$

we can use the trick employed for proving [**276**, Lemma 2.1]. Since K is totally bounded, there exist an integer $k \in \mathbb{N}$ and a finite subset $\{x_1, \cdots, x_k\}$ of K such that

$K \subseteq \bigcup_{i=1}^{k} B(x_i, \varepsilon)$, where $B(x, \varepsilon) := \{y \in X : \|x - y\| \leqslant \varepsilon\}$. Applying Minkowski's inequality and a simple argumentation similar to that used in the proof of above-mentioned lemma, we get the existence of a finite positive real number $c_q > 0$ such that

$$\begin{aligned}
&\frac{1}{l}\left(\int_t^{t+l}\left(\sup_{u\in K}\big\|F(s+\tau,u)-F(s,u)\big\|\right)^q ds\right)^{1/q}\\
&\leqslant \frac{c_q}{l}\left[\varepsilon\left(\int_t^{t+l}\left[L_F^q(s+\tau)+L_F^q(s)\right] ds\right)^{1/q}\right.\\
&\left.+\sum_{i=1}^{k}\left(\int_t^{t+l}\big\|F(s+\tau,x_i)-F(s,x_i)\big\|^q ds\right)^{1/q}\right].
\end{aligned}$$

The term $\frac{1}{l}(\int_t^{t+l}[L_F^q(s+\tau)+L_F^q(s)]\,ds)^{1/q}$ can be estimated by using Lemma 1.1.3 in the following way:

$$\begin{aligned}
&\leqslant \frac{1}{l}\left(\int_{t+\tau}^{t+l+\tau} L_F^q(s)\,ds\right)^{1/q} + \frac{1}{l}\left(\int_t^{t+l} L_F^q(s)\,ds\right)^{1/q}\\
&\leqslant M l^{(-1/r)+(1/q)-1}\big\|L_F\big\|_{W^r} l^{1/r} \leqslant M\big\|L_F\big\|_{W^r}, \quad t \in I.
\end{aligned}$$

Similarly, using Lemma 1.1.3 and (iii), we get

$$\begin{aligned}
&\frac{1}{l}\sum_{i=1}^{k}\left(\int_t^{t+l}\big\|F(s+\tau,x_i)-F(s,x_i)\big\|^q ds\right)^{1/q}\\
&\leqslant \frac{1}{l} l^{(1/q)-(1/p)}\sum_{i=1}^{k}\left(\int_t^{t+l}\big\|F(s+\tau,x_i)-F(s,x_i)\big\|^p ds\right)^{1/p} \leqslant \varepsilon l^{(1/q)-1}, \quad t \in I.
\end{aligned}$$

This completes the proof of theorem. □

REMARK 2.2.11. To the best knowledge of the author, it is not known whether the assumptions $F \in (e-)W_{ap}^p(I \times Y : X)$ and $x \in (e-)W_{ap}^p(I : Y)$ imply the validity of condition (iii), as for the class of Stepanov p-almost periodic functions.

The following result for the class of Weyl p-almost periodic functions can be also deduced with the help of argumentation contained in [**276**] (compare with Theorem 2.4.49, where we have analyzed the Stepanov class):

THEOREM 2.2.12. *Suppose that $p,\ q \in [1, \infty)$, $r \in [1, \infty]$, $1/p = 1/q + 1/r$ and the following conditions hold:*

- (i) $F \in W_{ap,\mathbf{K}}^p AP(I \times Y : X)$ *and there exists a function* $L_F \in L_S^r(I)$ *such that* (25) *holds.*
- (ii) $x \in W_{ap}^q AP(I : Y)$, *and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{x(t) : t \in I \smallsetminus \mathrm{E}\}$ *is relatively compact in* Y.

(iii) *For every $\varepsilon > 0$, there exists a finite number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a number $\tau \in I'$ satisfying that there exists a number $l(\varepsilon, \tau) > 0$ so that* (26) *holds for all numbers $l \geqslant l(\varepsilon, \tau)$ and* (27) *holds for all numbers $l \geqslant l(\varepsilon, \tau)$, with the number p replaced by q therein.*

Then $F(\cdot, x(\cdot)) \in W^p_{ap}AP(I : X)$.

After proving Theorem 2.2.10, the subsequent composition principle for asymptotically (equi-)Weyl p-almost periodic functions follows almost immediately; cf. also [**243**, Theorem 3.4] for a similar result in this direction.

THEOREM 2.2.13. *Suppose that $p > 1$, $r \geqslant \max(p, p/(p-1))$, $q = pr/(p + r)$, and the conditions* (i)-(iii) *of* Theorem 2.2.10 *hold with the interval $I = [0, \infty)$ and the functions $F(\cdot, \cdot)$, $x(\cdot)$ replaced therein with the functions $G(\cdot, \cdot)$, $y(\cdot)$. Suppose, further, that the following holds:*

(i) *The function $Q := F - G : [0, \infty) \times Y \to X$ is in class $(e-)W^{q'}_{0,\mathbf{K}}([0, \infty) \times Y : X)$ for some number $q' \in [1, \infty)$.*

(ii) *The function $z : [0, \infty) \to Y$ is in class $(e-)W^{q''}_0([0, \infty) : Y)$ for some number $q'' \in [1, \infty)$.*

(iii) *$x(t) = y(t) + z(t)$ for a.e. $t \geqslant 0$, and there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{x(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact in Y.*

Then the mapping $t \mapsto F(t, x(t))$, $t \geqslant 0$ is in class $(e-)W^q_{ap}([0, \infty) : X) + (e-)W^{q'}_0([0, \infty) : X) + (e-)W^{q'''}_0([0, \infty) : X)$, provided $q''' \in [1, \infty)$ and $1/r + 1/q'' = 1/q'''$.

PROOF. It is clear that $F(t, x(t)) = \big[G(t, x(t)) - G(t, y(t))\big] + G(t, y(t)) + Q(t, x(t))$, $t \geqslant 0$. By Theorem 2.2.10, we know that $G(\cdot, y(\cdot)) \in (e-)W^q_{ap}([0, \infty) : X)$. Keeping in mind (i) and (iii), the function $t \mapsto Q(t, x(t))$, $t \geqslant 0$ is in class $(e-)W^{q'}_0([0, \infty) : X)$ by definition (see the notions of classes $W^p_{0,\mathbf{K}}(I \times Y : X)$ and $e - W^p_{0,\mathbf{K}}(I \times Y : X)$ introduced in Definition 2.2.9). Therefore, it suffices to show that the mapping $t \mapsto G(t, x(t)) - G(t, y(t))$, $t \geqslant 0$ is in class $(e-)W^{q'''}_0([0, \infty) : X)$. But, this follows similarly as in the proof of [**243**, Theorem 3.4], with the exponents p, q, r replaced therein with the exponents q''', q'', r, respectively. $\square$

An analogue of [**243**, Theorem 3.4] for the class of asymptotically Weyl p-almost periodic functions can be also deduced by means of Theorem 2.2.12.

2.3. Almost automorphic type functions

Suppose that $f : \mathbb{R} \to X$ is continuous. Then we say that $f(\cdot)$ is almost automorphic if and only if for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to X$ such that

$$\lim_{n\to\infty} f\big(t + a_n\big) = g(t) \quad \text{and} \quad \lim_{n\to\infty} g\big(t - a_n\big) = f(t), \tag{28}$$

pointwise for $t \in \mathbb{R}$ (see the foundational paper [**73**] by S. Bochner for the scalar-valued case). If this is the case, we have $f \in C_b(\mathbb{R} : X)$ and that the limit function $g(\cdot)$ is bounded on $\mathbb{R}$ but not necessarily continuous on $\mathbb{R}$. If the convergence of limits appearing in (28) is uniform on compact subsets of $\mathbb{R}$, then we say that $f(\cdot)$ is compactly almost automorphic. The vector space consisting of all almost automorphic, resp., compactly almost automorphic functions, is denoted by $AA(\mathbb{R} : X)$, resp., $AA_{\mathbf{c}}(\mathbb{R} : X)$. By Bochner's criterion [**135**], any almost periodic function is compactly almost automorphic. The converse statement is not true, however [**171**]. Recall that P. R. Bender proved in his doctoral dissertation [**58**] that an almost automorphic function $f(\cdot)$ is compactly almost automorphic if and only if it is uniformly continuous (1966, Iowa State University).

The almost automorphy of a function $f : \mathbb{R} \to X$ can be also introduced in the following equivalent way: A function $f : \mathbb{R} \to X$ is said to be almost automorphic if and only if for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) such that

$$\lim_{m\to\infty} \lim_{n\to\infty} f\big(t + a_n - a_m\big) = f(t), \quad t \in \mathbb{R}.$$

An interesting example of an almost automorphic function that is not compactly almost automorphic has been constructed by W. A. Veech ([**341**]-[**342**])

$$f(t) := \frac{2 + e^{it} + e^{it\sqrt{2}}}{\big|2 + e^{it} + e^{it\sqrt{2}}\big|}, \quad t \in \mathbb{R}. \tag{29}$$

Let $I = \mathbb{R}$ or $I = [0, \infty)$. A continuous function $f : I \to X$ is said to be asymptotically (compactly) almost automorphic, if and only if there exist a function $q \in C_0(I : X)$ and a (compactly) almost automorphic function $h : \mathbb{R} \to X$ such that $f(t) = h(t) + q(t)$, $t \in I$. Any asymptotically almost periodic function $f : I \to X$ is asymptotically (compactly) almost automorphic. Asymptotically almost periodic functions and asymptotically (compactly) almost automorphic functions form closed subspaces of $C_b(\mathbb{R} : X)$ equipped with the sup-norm.

For the sake of completeness, we will include the proof of following simple proposition:

PROPOSITION 2.3.1. (i) *Suppose that* $f \in AA(\mathbb{R} : \mathbb{C})$ *and* $g \in AA(\mathbb{R} : X)$. *Then* $fg \in AA(\mathbb{R} : X)$.

(ii) *Suppose that* $f \in AA_{\mathbf{c}}(\mathbb{C} : \mathbb{R})$ *and* $g \in AA_{\mathbf{c}}(\mathbb{R} : X)$. *Then* $fg \in AA_{\mathbf{c}}(\mathbb{R} : X)$.

PROOF. Suppose that (b_n) is a given real sequence. Then there exist a subsequence (a_n) of (b_n) and a map $g : \mathbb{R} \to X$ such that (28) holds pointwise for $t \in \mathbb{R}$, with the function $g(\cdot)$ replaced therein with the function $h_1(\cdot)$. Further on, there exist a subsequence (a_{n_k}) of (a_n) and a map $h_2 : \mathbb{R} \to \mathbb{C}$ such that $\lim_{k\to\infty} f(t + a_{n_k}) = h_2(t)$ and $\lim_{k\to\infty} h_2(t - a_{n_k}) = f(t)$, pointwise for $t \in \mathbb{R}$. This simply implies that $\lim_{k\to\infty} f(t + a_{n_k})g(t + a_{n_k}) = h_1(t)h_2(t)$ and $\lim_{k\to\infty} h_1(t - a_{n_k})h_2(t - a_{n_k}) = f(t)g(t)$, pointwise for $t \in \mathbb{R}$, finishing the proof of (i). The proof of (ii) follows from (i) and the fact that the pointwise product of two bounded uniformly continuous functions is a uniformly continuous function. $\square$

Let $p \in [1, \infty)$. Then a function $f \in L^p_{loc}(\mathbb{R} : X)$ is said to be Stepanov p-almost automorphic (see e.g., G. M. N'Guérékata and A. Pankov [**196**], and V. Casarino

[**88**]-[**90**] for a slightly different approach) if and only if for every real sequence (a_n), there exist a subsequence (a_{n_k}) and a function $g \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \int_t^{t+1} \Bigl\| f\bigl(a_{n_k} + s\bigr) - g(s)\Bigr\|^p \, ds = 0 \text{ and } \lim_{k\to\infty} \int_t^{t+1} \Bigl\| g\bigl(s - a_{n_k}\bigr) - f(s)\Bigr\|^p \, ds = 0$$

for each $t \in \mathbb{R}$; a function $f \in L^p_{loc}(I : X)$ is called asymptotically Stepanov p-almost automorphic if and only if there exist an S^p-almost automorphic function $g : \mathbb{R} \to X$ and a function $q \in L^p_S(I : X)$ such that $f(t) = g(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : X))$. Any Stepanov p-almost automorphic function $f(\cdot)$ has to be Stepanov p-bounded. Furthermore, if $1 \leqslant p \leqslant q < \infty$ and a function $f(\cdot)$ is (asymptotically) Stepanov q-almost automorphic, then $f(\cdot)$ is (asymptotically) Stepanov p-almost automorphic. We say that a function $f(\cdot)$ is (asymptotically) Stepanov almost automorphic if and only if $f(\cdot)$ is (asymptotically) Stepanov 1-almost automorphic. Let us recall that any uniformly continuous Stepanov almost periodic (automorphic) function $f(\cdot)$ is almost periodic (automorphic). The vector space consisting of all S^p-almost automorphic functions, resp., asymptotically S^p-almost automorphic functions, will be denoted by $AAS^p(\mathbb{R} : X)$, resp., $AAAS^p([0,\infty) : X)$. By the (asymptotical) Stepanov almost automorphy we mean (asymptotical) Stepanov 1-almost automorphy. Recall, the (asymptotical) Stepanov p-almost periodicity of $f(\cdot)$ for some $p \in [1,\infty)$ implies the (asymptotical) Stepanov p-almost automorphy of $f(\cdot)$.

Example 2.3.2. ([**152**]) Let $\varepsilon \in (0, 1/2)$, and let $f(t) := \sin(1/(2 + \cos n + \cos \sqrt{2}n))$, provided that $n \in \mathbb{Z}$ and $t \in (n-\varepsilon, n+\varepsilon)$. Otherwise, we define $f(t) := 0$. Then for each $p \in [1,\infty)$ we have that $f(\cdot)$ is S^p-almost automorphic.

Let us recall that any uniformly continuous Stepanov almost periodic (automorphic) function $f(\cdot)$ is almost periodic (automorphic); see [**151**, Theorem 3.3]. The following lemma can be deduced by using an elementary argumentation involving [**218**, Proposition 3.1], the above-mentioned theorem and a simple observation that any uniformly continuous function $q \in C_0(I : L^p([0,1] : X))$ belongs to the space $C_0(I : X)$:

Lemma 2.3.3. *Let $f : I \to X$ be uniformly continuous and $p \in [1,\infty)$.*

(i) *If $f(\cdot)$ is asymptotically Stepanov p-almost periodic, then $f(\cdot)$ is asymptotically almost periodic.*

(ii) *If $f(\cdot)$ is asymptotically Stepanov p-almost automorphic, then $f(\cdot)$ is asymptotically almost automorphic.*

The concepts of Weyl almost automorphy and Weyl pseudo almost automorphy were introduced by S. Abbas [**1**] in 2012:

Definition 2.3.4. Let $p \geqslant 1$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is Weyl p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to+\infty} \lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \Bigl\| f\bigl(t + s_{n_k} + x\bigr) - f^*(t + x)\Bigr\|^p \, dx = 0 \tag{30}$$

and

$$\lim_{k\to+\infty}\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Bigl\|f^{*}\bigl(t-s_{n_k}+x\bigr)-f(t+x)\Bigr\|^{p}\,dx=0 \tag{31}$$

for each $t\in\mathbb{R}$. The set of all such functions are denoted by $W^{p}AA(\mathbb{R}:X)$.

The set $W^{p}AA(\mathbb{R}:X)$, equipped with the usual operations of pointwise addition of functions and multiplication of functions with scalars, has a linear vector structure. As the next illustrative example shows, the Weyl p-almost automorphicity does not imply the Besicovitch p-unboundedness:

EXAMPLE 2.3.5. Let $p=1$ and let $h(x):=\sqrt{|x|}$, $x\in\mathbb{R}$. Then $h(\cdot)$ is Weyl (1-)almost automorphic with the limit function $h^{*}\equiv h$. This simply follows from the fact that for each numbers $t,\ \omega\in\mathbb{R}$ we have

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}|h(t+x+\omega)-h(t+x)|\,dx=0.$$

The class of Besicovitch p-almost automorphic functions has been analyzed by F. Bedouhene, N. Challali, O. Mellah, P. Raynaud de Fitte and M. Smaali in [**54**]. This class extends the class of Weyl p-almost automorphic functions and its full importance lies in the fact that we do allow now the possible non-existence of limit

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Bigl\|f\bigl(t+s_{n_k}+x\bigr)-f^{*}(t+x)\Bigr\|^{p}\,dx,$$

resp.,

$$\lim_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Bigl\|f^{*}\bigl(t-s_{n_k}+x\bigr)-f(t+x)\Bigr\|^{p}\,dx$$

in (30), resp., (31).

DEFINITION 2.3.6. Let $p\geqslant 1$. Then we say that a function $f\in L^{p}_{loc}(\mathbb{R}:X)$ is Besicovitch p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^{*}\in L^{p}_{loc}(\mathbb{R}:X)$ such that

$$\lim_{k\to\infty}\limsup_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Bigl\|f\bigl(t+s_{n_k}+x\bigr)-f^{*}(t+x)\Bigr\|^{p}\,dx=0$$

and

$$\lim_{k\to\infty}\limsup_{l\to+\infty}\frac{1}{2l}\int_{-l}^{l}\Bigl\|f^{*}\bigl(t-s_{n_k}+x\bigr)-f(t+x)\Bigr\|^{p}\,dx=0$$

for each $t\in\mathbb{R}$. The set of all such functions are denoted by $B^{p}AA(\mathbb{R}:X)$.

As in the case of Weyl almost automorphic functions, we can prove that the set $B^{p}AA(\mathbb{R}:X)$, equipped with the usual operations, has a linear vector structure. Let us stress once more that it is not clear how we can prove that a Besicovitch p-almost periodic function is Besicovitch p-almost automorphic ([**234**]).

We refer the reader to S. Abbas [**1**] for the notion of Weyl p-pseudo almost automorphicity. For more details about the class of Besicovitch p-pseudo almost automorphic functions, we refer the reader to [**234**].

For more details about about almost periodic functions (sequences), almost automorphic functions (sequences) and their applications, we refer the reader to [**3, 11, 68, 74, 81, 82, 102, 103, 104, 141, 200, 231, 332**] and [**110, 111, 112, 138, 139, 179, 180, 240, 273, 274, 285, 316, 360**].

2.4. Almost periodic type functions and densities

We will first describe the main ideas and aims of this section, which consists of three subsections. Albeit the definitions of an almost periodic function and a uniformly recurrent function are quite easy and understandable, the class consisting of all almost periodic functions and the class consisting of all uniformly recurrent functions are sometimes very unpleasant and difficult to deal with. For example, already H. Bohr has marked in his pioneering papers that it is not so satisfactory to introduce the concept of almost periodicity by requiring that for each number $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is unbounded (see e.g., [**76**]). A bounded uniformly continuous function $f : I \to \mathbb{R}$ satisfying this property need not be almost periodic, its Bohr-Fourier coefficients cannot be defined in general, and moreover, if two bounded uniformly continuous functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ satisfy this property, then its sum $f + g : I \to \mathbb{R}$ need not satisfy this property (see [**75**, part I, pp. 117-118]). Furthermore, saying that for each number $\varepsilon > 0$ the set $\vartheta(f, \varepsilon)$ is unbounded is equivalent to saying that $f(\cdot)$ is uniformly recurrent; hence, the sum of two bounded uniformly continuous uniformly recurrent functions is not uniformly recurrent, in general. Taking into account Proposition 2.4.31 below, we get that the sum of two bounded uniformly continuous $\odot_g$-almost periodic functions is not $\odot_g$-almost periodic, in general. This example can be also used for proving the fact that the pointwise product of two bounded uniformly continuous, uniformly recurrent ($\odot_g$-almost periodic) functions is not uniformly recurrent ($\odot_g$-periodic), in general.

The above observation of H. Bohr has motivated us to further analyze some very specific examples of generalized almost periodic functions in more detail (see [**44**] for a non-updated list of unsolved problems in the theory). First of all, we recall that B. Basit and H. Güenzler have constructed, in [**47**, Example 3.2], a bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ such that its first antiderivative $t \mapsto \int_0^t f(s)\, ds$, $t \in \mathbb{R}$ is almost periodic, while the function $f(\cdot)$ itself is not uniformly continuous, not Stepanov almost periodic, not almost automorphic as well as

$$\sup_{t \in [-2,0]} |f(t+\tau) - f(t)| \geqslant 1 \text{ for all } \tau \geqslant 2. \tag{32}$$

The construction concretely goes as follows. Define a continuous 2^{n+1}-periodic function $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(t) := \sin(2^n \pi t)$ for $t \in [2^n - 1, 2^n]$, $f_n(t) := 0$ for $t \in [-2^n, 2^n - 1)$, and extend it 2^{n+1}-periodically to the whole real axis. Then $\operatorname{supp}(f_n) = [2^n - 1, 2^n] + 2^{n+1}\mathbb{Z}$, which simply implies that $\operatorname{supp}(f_n)$ and $\operatorname{supp}(f_m)$ are disjunct sets for each integers $n,\ m \in \mathbb{N}$ with $n \neq m$. Therefore, the function $f(x) := \sum_{n=1}^{\infty} f_n(x)$, $x \in \mathbb{R}$ is well-defined. This function satisfies all above properties, and we will provide a small contribution here by proving that the set $\vartheta(f, \varepsilon)$ is empty for each number $\varepsilon \in (0, 1)$:

$\triangle$. Suppose that $\tau \in \vartheta(f,\varepsilon)$. Due to (32), we have $\tau \in (0,2)$ so that there exist two possibilities: $\tau \in (0,1)$ or $\tau \in [1,2)$. In the first case, there exists a sufficiently large number $n \in \mathbb{N}$ such that $(2^n+1)-(2^n-1+2^{-n-1}) > \tau$. Let $t = 2^n - 1 + 2^{-n-1}$; then $t+\tau \in (2^n, 2^n+1)$ and therefore $f(t) = 1$ while $f(t+\tau) = 0$ so that $|f(t+\tau) - f(t)| = 1 > \varepsilon$. In the second case, there exists a sufficiently large number $n \in \mathbb{N}$ such that $\tau > 2^{-n-1}$. In this case, take $t = 2^n - 2^{-n-1}$; then $t+\tau \in (2^n, 2^n+1)$ and therefore $f(t) = -1$ while $f(t+\tau) = 0$ so that $|f(t+\tau) - f(t)| = 1 > \varepsilon$.

Essentially, the functions $f(\cdot)$ satisfying that there exists a number $\varepsilon \in (0,1)$ such that the set $\vartheta(f,\varepsilon)$ is bounded will not occupy our attention henceforth. In connection with the above example, we would like to propose the following question:

QUESTION 2.4.1. Suppose that $f : I \to X$ is a bounded, continuous and Stepanov almost periodic. Is it true that $\vartheta(f,\varepsilon) \neq \emptyset$ ($\vartheta(f,\varepsilon)$ is unbounded) for all $\varepsilon > 0$?

More concretely, assume that $\alpha,\ \beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well defined irrational number. Let the function $f(\cdot)$ and $g(\cdot)$ be given through (22) and (23), respectively. Is it true that $\vartheta(f,\varepsilon) \neq \emptyset$ ($\vartheta(f,\varepsilon)$ is unbounded) [$\vartheta(g,\varepsilon) \neq \emptyset$ ($\vartheta(g,\varepsilon)$ is unbounded)] for all $\varepsilon > 0$?

We continue by observing that A. Haraux and P. Souplet have proved, in [**202**, Theorem 1.1], that the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\Big(\frac{t}{2^n}\Big)\, dt, \quad t \in \mathbb{R}, \tag{33}$$

is uniformly continuous, uniformly recurrent and unbounded. From the argumentation given in the proof of the above-mentioned theorem, it immediately follows that the function $f(\cdot)$ given by (33) is neither Besicovitch almost periodic [**234**] nor asymptotically Stepanov almost automorphic. The reason for that is quite simple, this function is even and enjoys the property that

$$\limsup_{t\to+\infty} \frac{1}{2t}\int_{-t}^{t} f(s)\, ds = +\infty.$$

Since the concepts of H. Weyl and A. S. Besicovitch suggest very general ways of approaching almost automorphicity ([**234**]), it is logical to ask whether the function $f(\cdot)$ is Weyl almost automorphic. We will prove the following result:

THEOREM 2.4.2. *The function $f(\cdot)$, given by* (33)*, is Weyl p-almost automorphic for any finite exponent $p \geqslant 1$ and satisfies that for each number $\tau \in \mathbb{R}$ the function $f(\cdot+\tau) - f(\cdot)$ belongs to the space $ANP(\mathbb{R} : \mathbb{C})$.*

Concerning this contribution, it is worth noting that the unbounded functions $f : \mathbb{R} \to \mathbb{R}$ such that for each number $\tau \in \mathbb{R}$ the function $f(\cdot+\tau)-f(\cdot)$ belongs to the space $AP(\mathbb{R} : \mathbb{C})$ have been analyzed by A. M. Samoilenko and S. I. Trofimchuk in [**327**] (let us recall that the bounded functions satisfying this condition are always almost periodic due to the famous Loomis theorem [**277**]; see also the results obtained in the articles [**60**] by I. Berg and [**338**] by R. Terras). Let us also note that the function $f(\cdot)$, given by (33), has been employed by H. Y. Zhao and M. Fečkan in [**367**], for proving the fact that for each finite real numbers $M,\ L>0$ the set consisting of all almost periodic functions $h:\mathbb{R}\to\mathbb{R}$ such that $|h(t)|\leqslant M$, $t\in\mathbb{R}$ and $|h(t_1)-h(t_2)|\leqslant L|t_1-t_2|$, $t_1,\ t_2\in\mathbb{R}$ is not precompact in $C(\mathbb{R})$.

Further on, in [**202**, Theorem 1.2], A. Haraux and P. Souplet have proved that for each real number $c>0$ the function $h(\cdot)=\min(c,f(\cdot))$, where $f(\cdot)$ is given by (33), is bounded uniformly continuous, uniformly recurrent and not asymptotically almost periodic. Since the function $h(\cdot)$ is uniformly continuous, Lemma 2.3.3(ii) below implies that $h(\cdot)$ is asymptotically Stepanov p-almost automorphic ($p\geqslant 1$) if and only if $h(\cdot)$ is asymptotically almost automorphic. But, this is actually not the case because [**202**, Lemma 2.1] can be improved in the following manner:

LEMMA 2.4.3. *Let $\omega:\mathbb{R}\to[0,\infty)$ be Lipschitz continuous and such that the set $\omega([0,+\infty))$ is unbounded. Define, for each finite number $c>\liminf_{t\to+\infty}\omega(t)$, the function $\omega_1:\mathbb{R}\to[0,\infty)$ by $\omega_1(t):=\min(c,\omega(t))$, $t\in\mathbb{R}$. Then the restriction of function $\omega_1(\cdot)$ to the non-negative real axis is not asymptotically almost automorphic.*

The proof of Lemma 2.4.3 is almost the same as that of [**202**, Lemma 2.1]. The only thing worth noticing is that the existence of an almost automorphic function $\omega_1^*(\cdot)$ such that $\lim_{t\to+\infty}|\omega_1(t)-\omega_1^*(t)|=0$ implies, as in the proof of the above-mentioned lemma, that $\omega_1^*\equiv c$; this follows by using the same arguments, almost directly from definition of almost automorphicity (we do not need the fact that the limits in the second part of proof are uniform on $\mathbb{R}$).

We will extend [**202**, Theorem 1.2] in the following way:

THEOREM 2.4.4. *Let the function $f(\cdot)$ be given by* (33)*, and let $c>0$. Then the function $h(t):=\min(c,f(t))$, $t\in\mathbb{R}$ is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

Concerning this contribution, we have made a decision to further analyze the function constructed by H. Bohr on pp. 113–115 of the first part of his landmark trilogy [**75**]. In actual fact, the results obtained by A. M. Fink in his doctoral dissertation [**172**] tell us that this function is uniformly continuous (nonexpansive, in fact), uniformly recurrent and not almost periodic. The construction of this function goes as follows. Let $\tau_1:=1,\ \tau_2>2$ and let the sequence $(\tau_n)_{n\in\mathbb{N}}$ of positive real numbers satisfy $\tau_n>2\sum_{i=1}^{n-1}i\tau_i$ for all $n\in\mathbb{N}$. Let the sequence $(f_n:\mathbb{R}\to\mathbb{R})_{n\in\mathbb{N}}$ be defined as follows. Set $f_1(x):=1-|x|$ for $|x|\leqslant 1$ and $f_1(x):=0$, otherwise. If the functions $f_1(\cdot),\cdot\cdot\cdot,f_{n-1}(\cdot)$ are already defined, set

$$f_n(x):=f_{n-1}(x)+\sum_{m=1}^{n-1}\frac{n-m}{n}\Big[f_{n-1}\big(x-m\tau_n\big)+f_{n-1}\big(x+m\tau_n\big)\Big],\quad x\in\mathbb{R}.$$

Then

$$\big|f_n(x+\tau_n)-f_n(x)\big|\leqslant\frac{1}{n},\quad n\in\mathbb{N},\ x\in\mathbb{R},$$

and the function

$$f(x) := \lim_{n\to+\infty} f_n(x), \quad x \in \mathbb{R} \tag{34}$$

is well defined, even and satisfies that $0 \leqslant f(x) \leqslant 1$ for all $x \in \mathbb{R}$. It is worth observing that this function also satisfies all clarified properties of function $h(\cdot)$ from Theorem 2.4.4:

THEOREM 2.4.5. *The function $f : \mathbb{R} \to \mathbb{R}$, given by* (34)*, is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

In Example 2.4.37, we will show that, for some concrete choices of sequences $(\tau_n)_{n\in\mathbb{N}}$, the function $f : \mathbb{R} \to \mathbb{R}$, given by (34), is Weyl p-almost automorphic for each finite exponent $p \geqslant 1$. Since any Stepanov p-quasi-asymptotically almost periodic function is Weyl-p-almost periodic ($p \geqslant 1$) in the sense of A. S. Kovanko's approach (see [**247**, Proposition 2.11]), it is quite reasonable to ask the following:

QUESTION 2.4.6. Is it true that the function $f(\cdot)$, given by (34), is (equi-)Weyl-p-almost periodic for some (each) finite exponent $p \geqslant 1$?

We would like to note that the function used by J. de Vries in [**129**, point 6., p. 208] can serve as a much simpler example of a bounded uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying all clarified properties of functions examined in Theorem 2.4.4 and Theorem 2.4.5: Let $(p_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers such that $p_i | p_{i+1}$, $i \in \mathbb{N}$ and $\lim_{i\to\infty} p_i/p_{i+1} = 0$. Define the function $f_i : [-p_i, p_i] \to [0, 1]$ by $f_i(t) := |t|/p_i$, $t \in [-p_i, p_i]$ and extend the function $f_i(\cdot)$ periodically to the whole real axis; the obtained function, denoted by the same symbol $f_i(\cdot)$, is of period $2p_i$ ($i \in \mathbb{N}$). Set

$$f(t) := \sup\{f_i(t) : i \in \mathbb{N}\}, \quad t \in \mathbb{R}. \tag{35}$$

We will prove the following:

THEOREM 2.4.7. *The function $f : \mathbb{R} \to \mathbb{R}$, given by* (35)*, is bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic.*

We proceed with much elementary things, by considering a general continuous function $f : I \to X$. Suppose first that there exists a number $\varepsilon > 0$ such that $\vartheta(f, \varepsilon) \neq \emptyset$, say $\tau \in \vartheta(f, \varepsilon)$. Setting $M := \sup_{t\in I, |t|\leqslant\tau} \|f(t)\|$, it can be simply proved by induction that we have $\|f(t)\| \leqslant M + n\varepsilon$ for all $t \in I$ with $|t| \in [n\tau, (n+1)\tau]$ ($n \in \mathbb{N}$). Hence, $\|f(t)\| \leqslant M + |t|\varepsilon/\tau$ for all $t \in I$ with $|t| \in [n\tau, (n + 1)\tau]$ ($n \in \mathbb{N}$), so that

$$\|f(t)\| \leqslant M + |t|\varepsilon/\tau, \quad t \in \mathbb{R} \tag{36}$$

and the function $f(\cdot)$ is linearly bounded as $|t| \to +\infty$. Further on, it is clear that the assumption $\vartheta(f, \varepsilon) \neq \emptyset$ for each $\varepsilon > 0$ implies that $\vartheta(f, \varepsilon)$ is infinite for each $\varepsilon > 0$ as well as that there does not exist a finite constant M such that the interval $[0, M]$ contains the union of sets $\vartheta(f, \varepsilon)$ when $\varepsilon > 0$; this is a simple consequence of the fact that for each $\varepsilon > 0$ we have $j\vartheta(f, \varepsilon/n) \subseteq \vartheta(f, \varepsilon)$ for all $j = 1, \cdots, n$. Let us observe that a linear function $f : I \to \mathbb{C}$ can serve as an example of a function for which the growth order in (36) cannot be improved and for which the assumption $\vartheta(f, \varepsilon) \neq \emptyset$ for each $\varepsilon > 0$ does not imply the existence of a number $\varepsilon_0 > 0$ such that the set $\vartheta(f, \varepsilon_0)$ is unbounded.

To the best of our knowledge, this is the first systematic study of vector-valued uniformly recurrent functions. In this section, we attempt to further profile the sets of ε-periods of uniformly recurrent functions by introducing the class of $\odot_g$-almost periodic functions, which is simply defined by using the notions of lower and upper (Banach) densities for the subsets of the non-negative real axis (we feel it is our duty to say that we have only partially succeeded in our mission because it is very difficult to practically control and give intrinsic characterizations of ε-periods). The lower and upper (Banach) m_n-densities for the subsets of $\mathbb{N}$, considered recently in [**244**], are discrete analogues of the lower and upper (Banach) g-densities considered in this paper. In the discrete setting, these densities play an important role in the field of linear chaos, for example, in definitions of frequent hypercyclicity and reiterative m_n-distributional chaos of linear continuous operators on Fréchet spaces. In the continuous setting, these densities play an important role in the qualitative analysis of solutions to the abstract (fractional) integro-differential equations in Fréchet spaces; see e.g., the recent research monograph [**235**] by the author and references cited therein for a brief introduction to the theory of linear chaos. We generalize the notion of almost periodicity by analyzing several different types of (Stepanov) $\odot_g$-almost periodicity for functions with values in complex Banach spaces. Speaking-matter-of-factly, we analyze the lower and upper (Banach) g-densities of sets $\vartheta(f, \varepsilon)$, where $\varepsilon > 0$ and $g : [0, \infty) \to [1, \infty)$ is an increasing mapping satisfying condition (38) below.

The organization of section can be briefly described as follows. Subsection 2.4.1 investigates the lower and upper (Banach) g-densities for the subsets of the non-negative real line; in this subsection, we present our first significant contributions, Theorem 2.4.10 and Theorem 2.4.11, in which we transfer the main result of paper [**186**] by G. Grekos, V. Toma and J. Tomanová to the continuous setting and reconsider the notion and several recent results from [**244**].

In Subsection 2.9.64, we analyze $\odot_g$-almost periodic functions, uniformly recurrent functions and their Stepanov generalizations. With the notation explained below, we say that a continuous function $f : I \to X$ is $\odot_g$-almost periodic if and only if for each $\varepsilon > 0$ we have $\odot_g(\vartheta(f, \varepsilon)) > 0$; see Definition 2.4.12, in which the symbol $\odot_g$ denotes exactly one of the densities $\underline{d}_{gc}$, $\overline{d}_{gc}$, $\underline{Bd}_{l;gc}$, $\underline{Bd}_{u;gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$. In the paragraph following Definition 2.4.12, we collect the basic properties of $\odot_g$-almost periodic functions and uniformly recurrent functions. The main purpose of Proposition 2.4.13 is to clarify the supremum formula for uniformly recurrent functions; in Proposition 2.4.14, we prove that any almost periodic function $f : I \to X$ is $\odot_g$-almost periodic. All introduced concepts are equivalent in case $g(x) \equiv x$, and reduced then to the concept of almost periodicity (Proposition 2.4.15). After that, in Proposition 2.4.16, we prove that the almost periodicity is equivalent with the $\underline{Bd}_{l;gc}$-almost periodicity and $\underline{Bd}_{u;gc}$-almost periodicity for every increasing mapping $g(\cdot)$ satisfying the condition (38).

Definition 2.4.20 introduces the notions of asymptotical uniform recurrence and asymptotical $\odot_g$-almost periodicity, while Proposition 2.4.21 restates all results from Subsection 2.9.64 proved by then in this context. We introduce the notion of (asymptotical) Stepanov p-uniform recurrence and (asymptotical) Stepanov $(p, \odot_g)$-almost periodicity in Definition 2.4.22. The main purpose of Theorem 2.4.24 is to show that any asymptotically uniformly recurrent, quasi-asymptotically almost periodic function is asymptotically almost periodic; the Stepanov analogue of this statement is also considered here. Proposition 2.4.26 shows that the uniform recurrence and asymptotical almost automorphicity (asymptotical almost periodicity) implies almost automorphicity (almost periodicity), for the usually considered classes and Stepanov classes. Further on, in Theorem 2.4.28 and Proposition 2.4.29, we prove that any uniformly continuous (asymptotically) Stepanov p-uniformly recurrent [(asymptotically) Stepanov $(p, \odot_g)$-almost periodic/Stepanov p-quasi-asymptotically almost periodic] function $f : I \to X$ is asymptotically uniformly recurrent [asymptotically $\odot_g$-almost periodic, quasi-asymptotically almost periodic].

Proposition 2.4.31 clarifies an interesting result which shows that for any (asymptotically) uniformly continuous, uniformly recurrent function we can find an increasing mapping $g : [0, \infty) \to [1, \infty)$ such that (38) holds and $f(\cdot)$ is (asymptotically) $\cdot_g$-almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$ (see also Remark 2.4.32, where we use the densities $\overline{Bd}_{l:gc}(\cdot)$ and $\overline{Bd}_{u:gc}(\cdot)$). In Example 2.4.35, we prove that the compactly almost automorphic function constructed by A. M. Fink in [**171**] is not asymptotically uniformly recurrent; the proofs of Theorem 2.4.2, Theorem 2.4.4, Therorem 2.4.5 and Theorem 2.4.7 are provided after that.

We investigate the existence and uniqueness of uniformly recurrent and $\odot_g$-almost periodic type solutions of abstract integro-differential equations in Banach spaces in a concise, semi-heuristical manner, paying special attention to the invariance of (asymptotical) uniform recurrence and (asymptotical) $\odot_g$-almost periodicity under the actions of convolution products.

The function $sign : \mathbb{R} \to \{-1, 0, 1\}$ is defined by $sign(t) := -1$ (0, 1) if and only if $t < 0$ ($t = 0$, $t > 0$); if $c \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, then we define $cA := \{ca : a \in A\}$. Let us recall that a function $f : (0, \infty) \to \mathbb{R}$ is called subadditive if and only if $f(x + y) \leqslant f(x) + f(y)$, $x,\ y > 0$. A continuous version of Fekete's lemma states that, for every measurable subadditive function $f : (0, \infty) \to \mathbb{R}$, the limit $\lim_{t\to+\infty} \frac{f(t)}{t}$ exists in $[-\infty, \infty)$ and $\lim_{t\to+\infty} \frac{f(t)}{t} = \inf_{t>0} \frac{f(t)}{t}$ (see e.g., [**210**, Theorem 6.6.1]). We will use the following simple lemma:

LEMMA 2.4.8. *There do not exist $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that*

$$sign\Big(\cos\big((n + k)\pi\sqrt{2}\big)\Big) = sign\Big(\cos\big(n\pi\sqrt{2}\big)\Big), \quad n \in \mathbb{Z},\ |n| \geqslant n_0. \tag{37}$$

PROOF. Since $\cos(n\pi\sqrt{2}) \neq 0$ for all $n \in \mathbb{Z}$, it is clear that (37) is equivalent to saying that $\cos((n+k)\pi\sqrt{2}) \cdot \cos(n\pi\sqrt{2}) > 0$, $n \in \mathbb{Z}$, $|n| \geqslant n_0$. If $k \in \mathbb{N}$ satisfies the above condition and $k\pi\sqrt{2} = 2k_0\pi + a$ for some numbers $k_0 \in \mathbb{Z}$ and $a \in (0, 2\pi)$, then we get from the above: $\cos(n\pi\sqrt{2}+a)\cdot\cos(n\pi\sqrt{2})>0$, $n\in\mathbb{Z}$, $|n|\geqslant n_0$. This cannot be satisfied because the set $\{e^{in\pi\sqrt{2}}:n\in\mathbb{Z},\ |n|\geqslant n_0\}$ is dense in the unit sphere and $\cos x=\mathrm{Re}(e^{ix})$, $x\in\mathbb{R}$. □

2.4.1. Lower and upper (Banach) g-densities. Unless stated otherwise, in this subsection we will always assume that $g:[0,\infty)\to[1,\infty)$ is an increasing mapping satisfying that there exists a finite number $L\geqslant 1$ such that

$$x\leqslant Lg(x),\quad x\geqslant 0, \tag{38}$$

which clearly implies $\liminf_{x\to+\infty} g(x)/x>0$. If $A\subseteq[0,\infty)$ and $a,\ b\geqslant 0$, then we define $A(a,b):=\{x\in A\ ;\ x\in[a,b]\}$.

For simplicity and better exposition, in this subsection we will use the Lebesgue measure $m(\cdot)$ on the non-negative real line, only, which will be sufficiently enough for our analyses of uniformly continuous $\odot_g$-almost periodic functions; we feel it is our duty to say that the general case is much more complicated and almost not considered below.

Let us define (cf. [**235**]-[**244**] for more details):

(i) The lower g-density of A, denoted in short by $\underline{d}_{gc}(A)$, as follows

$$\underline{d}_{gc}(A):=\liminf_{x\to+\infty}\frac{m(A(0,g(x)))}{x};$$

(ii) the upper g-density of A, denoted in short by $\overline{d}_{gc}(A)$, as follows

$$\overline{d}_{gc}(A):=\limsup_{x\to+\infty}\frac{m(A(0,g(x)))}{x},$$

as well as:

(i) the lower $l;gc$-Banach density of A, denoted in short by $\underline{Bd}_{l;gc}(A)$, as follows

$$\underline{Bd}_{l;gc}(A):=\liminf_{x\to+\infty}\liminf_{y\to+\infty}\frac{m(A(y,y+g(x)))}{x};$$

(ii) the lower $u;gc$-Banach density of A, denoted in short by $\underline{Bd}_{u;gc}(A)$, as follows

$$\underline{Bd}_{u;gc}(A):=\limsup_{x\to+\infty}\liminf_{y\to+\infty}\frac{m(A(y,y+g(x)))}{x};$$

(iii) the (upper) $l;gc$-Banach density of A, denoted in short by $\overline{Bd}_{l;gc}(A)$, as follows

$$\overline{Bd}_{l;gc}(A):=\liminf_{x\to+\infty}\limsup_{y\to+\infty}\frac{m(A(y,y+g(x)))}{x};$$

(iv) the (upper) $u;fc$-Banach density of A, denoted in short by $\overline{Bd}_{u;gc}(A)$, as follows

$$\overline{Bd}_{u;gc}(A):=\limsup_{x\to+\infty}\limsup_{y\to+\infty}\frac{m(A(y,y+g(x)))}{x}.$$

REMARK 2.4.9. It is worth noting that, for every set $A \subseteq [0,\infty)$, we have

$$\begin{aligned}
&\liminf_{x\to+\infty}\limsup_{y\to+\infty}\frac{m([I\smallsetminus A](y,y+g(x)))}{x}\\
&\quad=\liminf_{x\to+\infty}\limsup_{y\to+\infty}\left[\frac{g(x)-m(A(y,y+g(x)))}{x}\right]\\
&\quad=\liminf_{x\to+\infty}\left[\frac{g(x)}{x}-\liminf_{y\to+\infty}\frac{m(A(y,y+g(x)))}{x}\right]. \qquad (39)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\limsup_{x\to+\infty}\limsup_{y\to+\infty}\frac{m([I\smallsetminus A](y,y+g(x)))}{x}\\
&\quad=\limsup_{x\to+\infty}\left[\frac{g(x)}{x}-\liminf_{y\to+\infty}\frac{m(A(y,y+g(x)))}{x}\right], \qquad (40)
\end{aligned}$$

$$\liminf_{x\to+\infty}\frac{m([I\smallsetminus A](0,g(x)))}{x}=\liminf_{x\to+\infty}\left[\frac{g(x)}{x}-\limsup_{x\to+\infty}\frac{m(A(0,g(x)))}{x}\right] \qquad (41)$$

and

$$\limsup_{x\to+\infty}\frac{m([I\smallsetminus A](0,g(x)))}{x}=\limsup_{x\to+\infty}\left[\frac{g(x)}{x}-\liminf_{x\to+\infty}\frac{m(A(0,g(x)))}{x}\right]. \qquad (42)$$

Case $g(x) := (1+|x|)^q$, $x \geqslant 0$ is the most important ($q \geqslant 1$), when we denote the corresponding densities by $\underline{d}_{qc}(A)$, $\overline{d}_{qc}(A)$, $\underline{Bd}_{l;qc}(A)$, $\underline{Bd}_{u;qc}(A)$, $\underline{Bd}_{l;qc}(A)$ and $\underline{Bd}_{l;qc}(A)$. Arguing similarly as in [**244**, Example 2.1(i)], for each number $q > 1$ we can simply construct a set $A \subseteq [0,\infty)$ such that $\overline{Bd}_{l;qc}(A) = 0$ and $\overline{Bd}_{u;qc}(A) = +\infty$; using the construction given in [**244**, Example 2.1(ii)], for each number $q > 1$ we can simply construct a set $A \subseteq [0,\infty)$ such that $\overline{d}_{qc}(A) = +\infty$ and $\overline{Bd}_{u;qc}(A) = 0$ so that the case $q > 1$ is not standard. Further on, if $q = 1$, then we get the usual concepts of lower and upper Banach densities: in this case, we have the following

THEOREM 2.4.10. *Let $A \subseteq [0,\infty)$. Then we have*

$$\begin{aligned}
&\underline{Bd}_{l;1c}(A)=\underline{Bd}_{u;1c}(A)\\
&\quad=\sup_{x>0}\liminf_{y\to+\infty}\frac{m(A(y,y+x))}{x}=\sup_{x>0}\inf_{y\geqslant 0}\frac{m(A(y,y+x))}{x}:=\underline{Bd}_c(A) \qquad (43)
\end{aligned}$$

and

$$\begin{aligned}
&\overline{Bd}_{l;1c}(A)=\overline{Bd}_{u;1c}(A)\\
&\quad=\inf_{x>0}\limsup_{y\to+\infty}\frac{m(A(y,y+x))}{x}=\inf_{x>0}\sup_{y\geqslant 0}\frac{m(A(y,y+x))}{x}:=\overline{Bd}_c(A). \qquad (44)
\end{aligned}$$

PROOF. Using the continuous version of Fekete's lemma, for the proof of first equality in (44) it suffices to show that the function

$$F(x) := \limsup_{y \to +\infty} m(A(y, y+x)), \quad x > 0$$

is subadditive, i.e., that for each fixed real numbers $x_1,\ x_2 > 0$ we have

$$\lim_{t \to +\infty} \sup_{t \geqslant y} m\big(A(t, t+x_1+x_2)\big) \leqslant \lim_{t \to +\infty} \sup_{t \geqslant y} m\big(A(t, t+x_1)\big) + \lim_{t \to +\infty} \sup_{t \geqslant y} m\big(A(t, t+x_2)\big).$$

This follows immediately if we prove that for each real number $y \geqslant 0$ we have

$$m\big(A(t, t+x_1+x_2)\big) \leqslant \sup_{t \geqslant y} m\big(A(t, t+x_1)\big) + \sup_{t \geqslant y} m\big(A(t, t+x_2)\big).$$

But, this is a simple consequence of the fact that for each real number $y \geqslant 0$ we have $t + x_1 \geqslant y$ and

$$m\big(A(t, t+x_1+x_2)\big) \leqslant m\big(A(t, t+x_1)\big) + m\big(A(t+x_1, t+x_1+x_2)\big);$$

see also P. Ribenboim's paper [**321**]. Since

$$\limsup_{y \to +\infty} \frac{m(A(y, y+x))}{x} \leqslant \sup_{y \geqslant 0} \frac{m(A(y, y+x))}{x} \leqslant \liminf_{x \to +\infty} \sup_{y \geqslant 0} \frac{m(A(y, y+x))}{x},$$

for the proof of (44) it remains to be shown that

$$\liminf_{x \to +\infty} \sup_{y \geqslant 0} \frac{m(A(y, y+x))}{x} \leqslant \overline{Bd}_{u;1c}(A). \tag{45}$$

For this, we will slightly adapt the arguments proposed in the proof of discrete version of this statement, given in [**186**]. Define

$$D = \big\{x \in [0,1] : \forall L > 0\ \exists \text{interval } I' \subseteq [0, \infty) \text{ s.t. } m(I') \geqslant L \text{ and } m(A \cap I')/m(I') \geqslant x\big\}.$$

Repeating literally the arguments given in [**186**, Subsection 2.1], we obtain that $\liminf_{x \to +\infty} \sup_{y \geqslant 0} \frac{m(A(y,y+x))}{x} \leqslant b := \sup D$. The proof of (44) will be completed if one shows that $b \leqslant \inf_{x>0}(\limsup_{y \to +\infty} m(A(y, y+x))/x)$. Suppose the contrary. Then there are a positive real number $x_0 > 0$ and two real numbers $x_1,\ x_2 \in [0,1]$ such that $x_1 < x_2 < b$ and

$$\limsup_{y \to +\infty} m(A(y, y+x_0)) < x_0 x_1.$$

By definition of $\limsup_{y \to +\infty} \cdot$, this implies that there exists a positive real number $y_0 > 0$ such that $m(A(y, y+x_0)) < x_0 x_1$ for all $y \geqslant y_0$. We will prove that there exists a sufficiently large number $L > 0$ such that every subinterval $I' \subseteq I$ with $m(I') \geqslant L$ satisfies $m(A \cap I') < x_2 m(I')$, showing that $x_2 \notin D$ and implying the contradiction. To see this, suppose that $I' = [y, y+h]$ for some $h > 0$. Then there exists $q \in \mathbb{N}_0$ such that $q x_0 \leqslant h < (q+1)x_0$ and therefore

$$\begin{aligned} m(A(y, y+h)) &\leqslant y_0 + m\big(A(y_0, y+h)\big) \leqslant y_0 + \sum_{j=0}^{q} m\big(A(y_0 + j x_0, y_0 + (j+1)x_0)\big) \\ &\leqslant y_0 + (q+1)x_0 x_1 \leqslant y_0 + x_0 x_1 + q x_0 x_1 < y_0 + x_0 x_1 + h x_1 < h x_2, \end{aligned}$$

for any $h > 0$ sufficiently large. The proof of (45) follows from (39)-(40) and (44). $\square$

By the proof of Theorem 2.4.10, it follows that for each subset $A \subseteq [0, \infty)$ we have

$$\overline{Bd}_c(I \setminus A) + \underline{Bd}_c(A) = 1. \tag{46}$$

Since the case $g(x) \equiv x$ is very special in our analysis, we will also prove the following result which is well known in the discrete case (we then write $\underline{d}_c(A) \equiv \underline{d}_{gc}(A)$ and $\overline{d}_c(A) \equiv \overline{d}_{gc}(A)$):

THEOREM 2.4.11. *Let $A \subseteq [0, \infty)$. Then we have*

$$0 \leqslant \underline{Bd}_c(A) \leqslant \underline{d}_c(A) \leqslant \overline{d}_c(A) \leqslant \overline{Bd}_c(A) \leqslant 1.$$

PROOF. The only non-trivial parts are $\underline{Bd}_c(A) \leqslant \underline{d}_c(A)$ and $\overline{d}_c(A) \leqslant \overline{Bd}_c(A)$; due to (46), it suffices to show that $\overline{d}_c(A) \leqslant \overline{Bd}_c(A)$. Suppose the contrary. Due to (44) and definition of $\limsup_{x \to +\infty} \cdot$, it follows that

$$\lim_{t \to +\infty} \sup_{t \geqslant x} \frac{m(A(0,t))}{t} > \inf_{x>0} \sup_{y \geqslant 0} \frac{m(A(y, y+x))}{x}.$$

Since the mapping in the above limit is monotonically decreasing in variable t, we get the existence of positive real numbers $\delta > 0$, $x_0 > 0$ and $y_0 > 0$ such that

$$\frac{m(A(0,y))}{y} \geqslant \frac{m\big(A(z, z+x_0)\big)}{x_0} + \delta, \quad y \geqslant y_0, \ z \geqslant 0. \tag{47}$$

Due to (47), we get

$$m(A(0,y)) \leqslant \sum_{j=0}^{\lfloor y/x_0 \rfloor} m\big(A(jx_0, (j+1)x_0\big) \leqslant \Big(\lfloor y/x_0 \rfloor + 1\Big)\left(\frac{m(A(0,y))}{y} - \delta\right)x_0,$$

i.e.,

$$\left(1 - \frac{x_0}{y}\Big(\lfloor y/x_0 \rfloor + 1\Big)\right)\frac{m(A(0,y))}{y} \leqslant -\delta x_0 \Big(\lfloor y/x_0 \rfloor + 1\Big)/y, \quad y \geqslant y_0.$$

After taking the limits as $y \to +\infty$, we obtain $0 \leqslant -\delta$, which is a contradiction. $\square$

For more details about densities, see also Section 2.10. Let us finally note that, in the combinatorial and additive number theory, the sets with positive upper Banach density play a major role; see e.g., [**181**, Section 5.7, Section 5.8]. A great number of results about the lower and upper (Banach) densities, known for subsets of integers, cannot be so easily reformulated and reconsidered for the subsets of the non-negative real axis. This is not the case with the statements of [**244**, Proposition 2.5-Proposition 2.7, Corollary 2.2], which can be simply reformulated for (Banach) g-densities; details can be left to the interested reader.

2.4.2. $\odot_g$-Almost periodic functions, uniformly recurrent functions and their Stepanov generalizations. We will always assume henceforth that $g : [0, \infty) \to [1, \infty)$ is an increasing mapping satisfying that there exists a finite number $L \geqslant 1$ such that (38) holds. Let $\odot_g$ denote exactly one of the symbols $\underline{d}_{gc}$, $\overline{d}_{gc}$, $\underline{Bd}_{l;gc}$, $\underline{Bd}_{u;gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$.

We start by introducing the following notion:

DEFINITION 2.4.12. Let $f : I \to X$ be continuous. Then it is said that $f(\cdot)$ is $\odot_g$-almost periodic if and only if for each $\varepsilon > 0$ we have $\odot_g(\vartheta(f, \varepsilon)) > 0$.

We will use hereafter the following fundamental properties of $\odot_g$-almost periodic functions and uniformly recurrent functions, collected as follows (for parts (iv)-(vi), see [**62**, pp. 3-4]; for parts (vii)-(viii), see [**265**, p. 3]):

(i) Any constant function is $\odot_g$-almost periodic, and for any $\odot_g$-almost periodic (uniformly recurrent) function $f(\cdot)$ we have that the function $\|f(\cdot)\|$ is $\odot_g$-almost periodic (uniformly recurrent). Any $\odot_g$-almost periodic function is uniformly recurrent.

(ii) Since for each $\varepsilon > 0$ and $c \in \mathbb{C} \setminus \{0\}$ we have $\vartheta(cf, \varepsilon) = \vartheta(f, \varepsilon/|c|)$, the $\odot_g$-almost periodicity of function $f(\cdot)$ implies the $\odot_g$-almost periodicity of function $cf(\cdot)$. Similarly, the uniform recurrence of function $f(\cdot)$ implies the uniform recurrence of function $cf(\cdot)$.

(iii) The set consisting of all $\odot_g$-almost periodic (uniformly recurrent) functions is translation invariant in the sense that for each $\tau \in I$ and any $\odot_g$-almost periodic (uniformly recurrent) function $f(\cdot)$, the function $f(\cdot + \tau)$ is also $\odot_g$-almost periodic (uniformly recurrent).

(iv) If $(f_n(\cdot))$ is a sequence of $\odot_g$-almost periodic (uniformly recurrent) functions and $(f_n(\cdot))$ converges uniformly to a function $f : I \to X$, then the function $f(\cdot)$ is $\odot_g$-almost periodic (uniformly recurrent).

(v) If $X = \mathbb{C}$, $\inf_{x \in I} |f(x)| > m > 0$ and $f(\cdot)$ is a bounded $\odot_g$-almost periodic (uniformly recurrent) function, then the function $1/f(\cdot)$ is likewise a bounded $\odot_g$-almost periodic (uniformly recurrent).

(vi) If $f(\cdot)$ is a bounded $\odot_g$-almost periodic (uniformly recurrent) function and $g : [0, \infty) \to X$ is continuous, then the mapping $g(\|f(\cdot)\|)$ is bounded and $\odot_g$-almost periodic (uniformly recurrent).

(vii) If $f(\cdot)$ is a bounded $\odot_g$-almost periodic (uniformly recurrent) function and $r > 0$, then the function $\|f(\cdot)\|^r$ is bounded and $\odot_g$-almost periodic (uniformly recurrent).

Furthermore, it can be simply shown that:

(viii) If $f : \mathbb{R} \to X$ is a bounded $\odot_g$-almost periodic (uniformly recurrent) function and $\psi \in L^1(\mathbb{R})$, then the function $(\psi * f)(\cdot)$ is bounded, uniformly continuous and $\odot_g$-almost periodic (uniformly recurrent).

(ix) If $f : [0, \infty) \to X$ is uniformly recurrent and belongs to the space $C_0([0, \infty) : X)$, then $f \equiv 0$.

(x) If $f : \mathbb{R} \to X$ is $\odot_g$-almost periodic (uniformly recurrent), then the function $\check{f} : \mathbb{R} \to X$, defined by $\check{f}(\cdot) := f(-\cdot)$, is $\odot_g$-almost periodic (uniformly recurrent). If, additionally, $f_{|[0,\infty)}(\cdot) \in C_0([0,\infty) : X)$ or $\check{f}_{|[0,\infty)}(\cdot) \in C_0([0,\infty) : X)$, then $f \equiv 0$.

(xi) If $a \in I$ and the function $f(\cdot)$ is $\odot_g$-almost periodic (uniformly recurrent), then the function $f(\cdot + a) - f(\cdot)$ is $\odot_g$-almost periodic (uniformly recurrent).

For the sake of completeness, we will include short proofs of the following two propositions (the first proposition improves the corresponding result for almost periodic functions; for almost automorphic functions, see [**234**, Lemma 3.9.9]):

Proposition 2.4.13. *(Supremum formula) Suppose that $f : I \to X$ is uniformly recurrent. Then we have*

$$\sup_{t\in I} \|f(t)\| = \sup_{t\geqslant a} \|f(t)\| \in [0,\infty], \quad a \in I.$$

Proof. Let $a \in I$, $t \in I$ and $\varepsilon > 0$ be fixed. It suffices to show that

$$\|f(t)\| \leqslant \varepsilon + \sup_{s\geqslant a} \|f(s)\|.$$

In order to do that, take any strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and (19) holds. Let $n \in \mathbb{N}$ be such that $t + \alpha_n \geqslant a$. Then $\|f(t+\alpha_n) - f(t)\| \leqslant \varepsilon$ and therefore

$$\|f(t)\| \leqslant \varepsilon + \|f(t+\alpha_n)\| \leqslant \varepsilon + \sup_{s\geqslant a} \|f(s)\|,$$

as claimed. $\square$

Proposition 2.4.14. *Any almost periodic function $f : I \to X$ is $\odot_g$-almost periodic.*

Proof. Let us recall that any almost periodic function is uniformly continuous. Using this fact, it can be easily seen that for each $\varepsilon > 0$ there exist two finite constants $\delta > 0$ and $l > 0$ such that any segment $[y, y + g(x)]$ for $x \geqslant L(1 + l)$ and $y \geqslant 0$ contains the segment $[y, y + x/L]$ (cf. (38)) and therefore at least $\lfloor x/Ll \rfloor \geqslant 1$ disjunct intervals of length δ whose elements are ε-periods for $f(\cdot)$; see also [**62**, Corollary, p. 2]. This clearly implies $\odot_g(\vartheta(f,\varepsilon)) > \delta/Ll > 0$. $\square$

Now we will prove the following

Proposition 2.4.15. *Let $f : I \to X$ be continuous and $g(x) \equiv x$. Then $f(\cdot)$ is almost periodic if and only if $f(\cdot)$ is $\odot_g$-almost periodic.*

Proof. Having in mind Proposition 2.4.14 and Theorem 2.4.11, it suffices to show that any $\underline{Bd}_c$-almost periodic function $f : I \to X$ is almost periodic. Towards this end, it suffices to show that any set $A \subseteq [0,\infty)$ satisfying $\underline{Bd}_c(A) > 0$ is relatively dense. Otherwise, for every real number $L > 0$, we have that there exists an interval I_L of length L which does not contain any ε-period of $f(\cdot)$. Thus, an unbounded set $\bigcup_{n\in\mathbb{N}} I_{2^n}$ does not contain any ε-period of $f(\cdot)$, which immediately implies that $\underline{Bd}_c(A) = 0$ by definition. $\square$

Concerning the notions of $\underline{Bd}_{l;gc}$-almost periodicity and $\underline{Bd}_{u;gc}$-almost periodicity, the things are pretty clear. In the following proposition, whose discrete analogue has been considered in [**244**, Proposition 2.4], we will prove that these notions are equivalent with the almost periodicity:

PROPOSITION 2.4.16. *Let $f : I \to X$ be continuous and let $g : [0,\infty) \to [1,\infty)$ be an increasing mapping satisfying that there exists a finite number $L \geqslant 1$ such that* (38) *holds. Then $f(\cdot)$ is almost periodic if and only if $f(\cdot)$ is $\underline{Bd}_{l;gc}$-almost periodic if and only if $f(\cdot)$ is $\underline{Bd}_{u;gc}$-almost periodic.*

PROOF. Due to Proposition 2.4.14 and the fact that any $\underline{Bd}_{l;gc}$-almost periodic function is $\underline{Bd}_{u;gc}$-almost periodic, it suffices to show that any $\underline{Bd}_{u;gc}$-almost periodic function is almost periodic. Suppose the contrary and fix a number $x > 0$. Then there exists a number $\varepsilon > 0$ such that, for every $n \in \mathbb{N}$, there exists an interval $I_n = [y_n, y_n + 2n + 2g(x)] \subseteq [0,\infty)$ of length $2n + 2g(x)$ such that the set $\vartheta(f,\varepsilon)$ does not meet I_n. Then, for every $n \in \mathbb{N}$, the interval $I_n' = [y_n + n + g(x), y_n + 2n + 2g(x)]$ does not meet $\vartheta(f,\varepsilon)$ and has the length $n+g(x) \geqslant g(x)$. This implies $m(([\vartheta(f,\varepsilon)](y_n+n+g(x), y_n+2n+2g(x))) = 0$. Hence, $\liminf_{y\to+\infty} m([\vartheta(f,\varepsilon)](y, y+x)) = 0$, which contradicts condition $\underline{Bd}_{u;gc}(\vartheta(f,\varepsilon)) > 0$. $\square$

REMARK 2.4.17. Let $f : I \to X$ be continuous and let $c \in I \setminus \{0\}$. Define the function $f_c : I \to X$ by $f_c(t) := f(ct)$, $t \in I$. Then we have $|c|\vartheta(f,\varepsilon) \subseteq \vartheta(f_c,\varepsilon)$ for all $\varepsilon > 0$, which simply implies that for any uniformly recurrent function $f(\cdot)$ we have that the function $f_c(\cdot)$ is uniformly recurrent. Due to Proposition 2.4.16 and the corresponding statement for almost periodic functions, the same holds for $\odot_g$-almost periodicity with $\odot_g \in \{\underline{Bd}_{l;gc}, \underline{Bd}_{u;gc}\}$. If $\odot_g$ is one of the densities $\underline{d}_{gc}$, $\overline{d}_{gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$, then directly from their definitions and the definition of $\odot_g$-almost periodicity we may conclude, keeping in mind the fact that for any Lebesgue measurable subset $A \subseteq [0,\infty)$ the set cA is also Lebesgue measurable with $m(cA) = cm(A)$, that the $\odot_g$-almost periodicity of function $f(\cdot)$ implies the $\odot_g$-almost periodicity of function $f_c(\cdot)$ for any $c \in I \setminus \{0\}$ with $|c| \leqslant 1$. Assume now that $\odot_g$ is one of the above four densities and $|c| > 1$. In this case, it is almost inevitable to impose some additional conditions on the function $g(\cdot)$ under which the $\odot_g$-almost periodicity of function $f(\cdot)$ implies the $\odot_g$-almost periodicity of function $f_c(\cdot)$. For example, it is very natural to assume additionally that $g(\cdot)$ is continuous, strictly increasing as well as that there exist two numbers $t_0 > 0$ and $\delta > 0$ such that $|c|g(t) \leqslant g(t/\delta)$ for all $t \geqslant t_0$. For the Banach density $\overline{Bd}_{u;gc}$, the claimed statement then follows from the computation ($x > 0$ satisfies that $t = g^{-1}(g(x)/c) \geqslant t_0$):

$$\begin{aligned}\limsup_{y\to+\infty} \frac{m(cA(y, y+g(x)))}{x} &= \limsup_{y\to+\infty} \frac{cm(A(y/c, y/c + (g(x)/c)))}{x} \\ = \limsup_{y\to+\infty} \frac{m(A(y, y + (g(x)/c)))}{x} &= \limsup_{y\to+\infty} \frac{m(A(y, y+g(t)))}{g^{-1}(cg(t))}\end{aligned}$$

$$= \limsup_{y\to+\infty} \frac{m(A(y, y+g(t)))}{t} \frac{t}{g^{-1}(cg(t))} \geqslant \delta \limsup_{y\to+\infty} \frac{m(A(y, y+g(t)))}{t}.$$

For the Banach density $\overline{Bd}_{l;gc}$ and for the densities $\underline{d}_{gc}$, $\overline{d}_{gc}$, the claimed statement follows similarly.

REMARK 2.4.18. (see also [**202**, Lemma 2.1]) If $f : \mathbb{R} \to \mathbb{R}$ is a (uniformly) continuous, $\odot_g$-almost periodic (uniformly recurrent) function, $\varepsilon > 0$, $c \in \mathbb{R}$ and $\tau \in \vartheta(f, \varepsilon)$, then $\tau \in \vartheta(\min(c, f), \varepsilon)$ and the function $\min(c, f(\cdot))$ is (uniformly) continuous and $\odot_g$-almost periodic (uniformly recurrent).

REMARK 2.4.19. Let $f : \mathbb{R} \to \mathbb{R}$ be an almost periodic function such that there exist two real numbers a and b such that $a < 0 < b$ and an analytic function $F : \{z \in \mathbb{C} : a < \operatorname{Re} z < b\} \to \mathbb{C}$ such that $F(ix) = f(x)$ for all $x \in \mathbb{R}$. Then the function $h : \mathbb{R} \to \mathbb{R}$, defined by $h(x) := sign(f(x))$, $x \in \mathbb{R}$ is Stepanov p-almost periodic for any finite exponent $p \geqslant 1$. For $p = 1$, this has been proved in [**264**, Theorem 5.3.1, p. 210], while the general case follows from the consideration given in [**234**, Example 2.2.3(i)] (we feel duty bound to say that we have made small mistakes in the formulations of conditions in [**234**, Example 2.2.2, Example 2.2.3(ii)] by neglecting the necessary condition on the analytical extensibility of function $f((-i)\cdot)$ to the strip $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$). The Bochner criterion is essentially employed in the proof of the above-mentioned theorem and we would like to observe here that the above condition on the analytical extensibility of function $f((-i)\cdot)$ can be neglected in some situations, even for the uniform recurrence and $\odot_g$-almost periodicity. More precisely, let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly recurrent function (an $\odot_g$-almost periodic function) satisfying that

$$(\exists L \geqslant 1)\ (\forall \varepsilon > 0)\ (\forall y \in \mathbb{R})\ m\big(\{x \in [y, y+1] : |f(x)| \leqslant \varepsilon\}\big) \leqslant L\varepsilon.$$

Then the function $h(\cdot)$, defined above, is uniformly recurrent ($\odot_g$-almost periodic), which follows from the foregoing arguments.

Now we will introduce the following definition:

DEFINITION 2.4.20. (i) Suppose that $f \in C(I : X)$. Then we say that the function $f(\cdot)$ is asymptotically uniformly recurrent if and only if there exist a uniformly recurrent function $h : \mathbb{R} \to X$ and a function $\phi \in C_0(I : X)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$.

(ii) Suppose that $f \in C(I : X)$. Then we say that the function $f(\cdot)$ is asymptotically $\odot_g$-almost periodic if and only if there exist an $\odot_g$-almost periodic function $h : \mathbb{R} \to X$ and a function $\phi \in C_0(I : X)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$.

Assume that the function $f : [0, \infty) \to X$ is continuous and the function $h : [0, \infty) \to X$ is continuous. For each $\varepsilon > 0$ and $M > 0$, we define

$$\vartheta_M(f, \varepsilon) := \{\tau > 0 : \|f(t+\tau) - f(t)\| \leqslant \varepsilon,\ t \geqslant M\}.$$

Then it is clear that the assumption $M_1 \leqslant M_2$ implies $\vartheta_{M_1}(f, \varepsilon) \subseteq \vartheta_{M_2}(f, \varepsilon)$. Furthermore, if $\phi \in C_0([0, \infty) : X)$ and $\varepsilon > 0$, then we have the existence of a number $M > 0$ such that

$$\begin{aligned}\|[h+\phi](t+\tau)-[h+\phi](t)\| &\leqslant \|h(t+\tau)-h(t)\|+\|\phi(t+\tau)-\phi(t)\| \\ &\leqslant \|h(t+\tau)-h(t)\|+\frac{\varepsilon}{2},\ t \geqslant M,\end{aligned}$$

so that $\vartheta(h,\varepsilon/2) \subseteq \vartheta_M(h+\phi,\varepsilon)$. Therefore, for any asymptotically $\odot_g$-almost periodic function $f : [0,\infty) \rightarrow X$ we have that for each $\varepsilon > 0$ there exists $M > 0$ such that $\odot_g(\vartheta_M(f,\varepsilon)) > 0$ (a similar statement holds for the Stepanov classes). In the case that $g(x) \equiv x$, then we also have the converse: if for each $\varepsilon > 0$ there exists $M > 0$ such that $\odot_g(\vartheta_M(f,\varepsilon)) > 0$, then the function $f(\cdot)$ is asymptotically almost periodic; if $\odot_g$ is $\underline{Bd}_{l;gc}$ or $\underline{Bd}_{u;gc}$, then the converse also holds in general case. For the remaining four densities, it seems very conceivable that the converse does not hold in general case.

From this definition and previously proved results in this section, it is clear that we have the following:

Proposition 2.4.21. (i) *Any asymptotically almost periodic function is asymptotically $\odot_g$-almost periodic, and any asymptotically $\odot_g$-almost periodic function is asymptotically uniformly recurrent.*

(ii) *Let $f : I \rightarrow X$ be continuous and $g(x) \equiv x$. Then $f(\cdot)$ is asymptotically almost periodic if and only if $f(\cdot)$ is asymptotically $\odot_g$-almost periodic.*

(iii) *Let $f : I \rightarrow X$ be continuous and let $g : [0,\infty) \rightarrow [1,\infty)$ be an increasing mapping satisfying that there exists a finite number $L \geqslant 1$ such that* (38) *holds. Then $f(\cdot)$ is asymptotically almost periodic if and only if $f(\cdot)$ is asymptotically $\underline{Bd}_{l;gc}$-almost periodic if and only if $f(\cdot)$ is asymptotically $\underline{Bd}_{u;gc}$-almost periodic.*

Now we have an open door to introduce the concepts of (asymptotical) Stepanov $p(x)$-uniform recurrence and (asymptotical) Stepanov $(p(x),\odot_g)$-almost periodicity:

Definition 2.4.22. (i) Let $p \in \mathcal{P}([0,1])$. A function $f \in L^{p(x)}_{loc}(I : X)$ is said to be Stepanov $p(x)$-uniformly recurrent if and only if the function $\hat{f} : I \rightarrow L^{p(x)}([0,1] : X)$, defined by (21), is uniformly recurrent.

(ii) Let $p \in \mathcal{P}([0,1])$. A function $f \in L^{p(x)}_{loc}(I : X)$ is said to be Stepanov $(p(x),\odot_g)$-almost periodic if and only if the function $\hat{f} : I \rightarrow L^{p(x)}([0,1] : X)$, defined by (21), is $\odot_g$-almost periodic.

If $p(x) \equiv p \in [1,\infty)$, then we alo say that the function $f(\cdot)$ is Stepanov p-uniformly recurrent (Stepanov $(p,\odot_g)$-almost periodic).

Definition 2.4.23. (i) Let $p \in \mathcal{P}([0,1])$. A function $f \in L^{p(x)}_{loc}(I : X)$ is said to be asymptotically Stepanov $p(x)$-uniformly recurrent if and only if there exist a Stepanov $p(x)$-uniformly recurrent function $h : \mathbb{R} \rightarrow X$ and a function $q \in L^{p(x)}_{S}(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^{p(x)}([0,1] : X))$.

(ii) Let $p \in \mathcal{P}([0,1])$. A function $f \in L^{p(x)}_{loc}(I : X)$ is said to be asymptotically Stepanov $(p(x),\odot_g)$-almost periodic if and only if there exist a Stepanov

$(p(x), \odot_g)$-almost periodic function $h : \mathbb{R} \to X$ and a function $q \in L_S^{p(x)}(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^{p(x)}([0,1] : X))$.

If $p(x) \equiv p \in [1, \infty)$, then we alo say that the function $f(\cdot)$ is asymptotically Stepanov p-uniformly recurrent (asymptotically Stepanov $(p, \odot_g)$-almost periodic).

We can simply state the analogues of Proposition 2.4.14-2.4.16 and Proposition 2.4.21 for the Stepanov classes. Taking into account Proposition 2.4.16 and Proposition 2.4.21(iii), in the remainder of section we will always assume, if not explicitly stated otherwise, that $\odot_g$ denotes exactly one of the densities $\underline{d}_{gc}$, $\overline{d}_{gc}$, $\overline{Bd}_{l;gc}$ or $\overline{Bd}_{u;gc}$. Before proceeding any further, we would like to note that we can similarly introduce and analyze the concepts of $\odot_g$-almost anti-periodicity and Stepanov $(p, \odot_g)$-almost anti-periodicity ([**234**]).

The following result, which is closely related with [**247**, Theorem 2.5, Theorem 2.10], plays a significant role in the proof of Theorem 2.4.4:

THEOREM 2.4.24. (i) *Suppose that the function $f : I \to X$ is asymptotically uniformly recurrent and quasi-asymptotically almost periodic. Then the function $f(\cdot)$ is asymptotically almost periodic.*

(ii) *Suppose that $p \in \mathcal{P}([0,1])$, the function $f \in L_S^{p(x)}(I : X)$ is asymptotically Stepanov $p(x)$-uniform recurrent and Stepanov $p(x)$-quasi-asymptotically almost periodic. Then the function $f(\cdot)$ is asymptotically Stepanov $p(x)$-almost periodic.*

PROOF. The proof of theorem essentially follows from the argumentation contained in the proof of [**234**, Theorem 2.5]; for the sake of completeness, we will include all details of proof. Suppose that the function $f : I \to X$ satisfies the assumptions in (i). Then there exist a uniformly recurrent function $h(\cdot)$ and a function $q \in C_0(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\|[h(t+\tau) - h(t)] + [q(t+\tau) - q(t)]\| \leqslant \varepsilon, \text{ provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau). \tag{48}$$

Since $f(\cdot)$ is bounded and $q \in C_0(I : X)$, we have that $h(\cdot)$ is bounded. The above implies the existence of a finite number $M_1(\varepsilon, \tau) \geqslant M(\varepsilon, \tau)$ such that

$$\|h(t+\tau) - h(t)\| \leqslant 2\varepsilon, \text{ provided } t \in I \text{ and } |t| \geqslant M_1(\varepsilon, \tau). \tag{49}$$

Define the function $H : I \to X$ by $H(t) := h(t+\tau) - h(t)$, $t \in I$. Then the function $H(\cdot)$ is bounded and, due to the property (xi), we have that the function $H(\cdot)$ is uniformly recurrent. Applying supremum formula clarified in Proposition 2.4.13 and (49), we get

$$\sup_{t \in I} \|H(t)\| = \sup_{t \geqslant M_1(\varepsilon,\tau)} \|H(t)\| = \sup_{t \geqslant M_1(\varepsilon,\tau)} \|h(t+\tau) - h(t)\| \leqslant 2\varepsilon.$$

Hence, $\|h(t+\tau) - h(t)\| \leqslant 2\varepsilon$ for all $t \in I$ and $h(\cdot)$ is almost periodic by definition, which completes the proof of part (i). For part (ii), observe first that there exist an Stepanov p-uniformly recurrent function $h(\cdot)$ and a function $q \in L_S^p(I : X)$ such that $f(t)=h(t)+q(t)$, $t\in I$ and $\hat{q}\in C_0(I:L^p([0,1]:X))$. Repeating verbatim the arguments given in the proof of part (i), with the function $f(\cdot)$ replaced therein with the function $\hat{f}(\cdot)$, we get that the function $\hat{h}:I\rightarrow L^p([0,1]:X)$ is asymptotically almost periodic. This simply completes the proof of (ii). □

EXAMPLE 2.4.25. Define

$$f(t):=\Biggl(\frac{4n^2t^2}{(t^2+n^2)^2}\Biggr)_{n\in\mathbb{N}},\ t\geqslant 0.$$

Then $f\in Q-AAA([0,\infty):c_0)\cap BUC([0,\infty):c_0)$ and $f(\cdot)$ is not asymptotically almost automorphic (see [**247**, Example 2.6, Theorem 2.5]). Due to Theorem 2.4.24(ii) and Lemma 2.3.3(i), we have that the function $f(\cdot)$ is not asymptotically Stepanov (1-)uniformly recurrent.

The results presented in the subsequent proposition are expected to a certain extent:

PROPOSITION 2.4.26. *Let $p\in\mathcal{P}([0,1])$.*

(i) *If $f:\mathbb{R}\rightarrow X$ is uniformly recurrent and asymptotically almost automorphic, then $f(\cdot)$ is almost automorphic.*
(ii) *If $f:I\rightarrow X$ is uniformly recurrent and asymptotically almost periodic, then $f(\cdot)$ is almost periodic.*
(iii) *If $f:\mathbb{R}\rightarrow X$ is Stepanov $p(x)$-uniformly recurrent and asymptotically Stepanov $p(x)$-almost automorphic, then $f(\cdot)$ is Stepanov $p(x)$-almost automorphic.*
(iv) *If $f:I\rightarrow X$ is Stepanov $p(x)$-uniformly recurrent and asymptotically Stepanov $p(x)$-almost periodic, then $f(\cdot)$ is Stepanov $p(x)$-almost periodic.*

PROOF. We will prove only (i). Suppose that $f:\mathbb{R}\rightarrow X$ is uniformly recurrent and asymptotically almost automorphic. Then there exist a function $h\in AA(\mathbb{R}:X)$, a function $q\in C_0(\mathbb{R}:X)$ and a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that (19) holds and $f(t)=h(t)+q(t)$ for all $t\in\mathbb{R}$. Fix a number $t\in\mathbb{R}$. Then $\lim_{n\rightarrow+\infty}q(t+\alpha_n)=0$ and, in combination with (19), we get

$$\lim_{n\rightarrow+\infty}h\bigl(t+\alpha_n\bigr)=f(t)\quad\text{and}\quad\lim_{n\rightarrow+\infty}f\bigl(t-\alpha_n\bigr)=f(t). \tag{50}$$

Since $h(\cdot)$ is almost automorphic, we can extract a subsequence (β_n) of (α_n) such that there exists a mapping $f_1:\mathbb{R}\rightarrow X$ satisfying

$$\lim_{n\rightarrow+\infty}h\bigl(t+\beta_n\bigr)=f_1(t)\quad\text{and}\quad\lim_{n\rightarrow+\infty}f_1\bigl(t-\beta_n\bigr)=h(t)\quad\text{for all}\quad t\in\mathbb{R}. \tag{51}$$

The uniqueness of the first limits in (50) and (51) yields $f_1(t)=f(t)$. Using the uniqueness of the second limits in (50) and (51), we get $f(t)=h(t)$, which completes the proof of (i). □

Combining Theorem 2.4.24 and Proposition 2.4.26, we may deduce the following:

COROLLARY 2.4.27. *Let $p \in \mathcal{P}([0,1])$.*

(i) *If $f : I \to X$ is uniformly recurrent and asymptotically almost periodic, then $f(\cdot)$ is almost periodic.*

(ii) *If $f \in L_S^{p(x)}(I : X)$ is Stepanov $p(x)$-uniform recurrent and Stepanov $p(x)$-quasi-asymptotically almost periodic, then $f(\cdot)$ is Stepanov $p(x)$-almost periodic.*

In the following theorem, we reconsider the statements given in Lemma 2.3.3 for the (asymptotical) Stepanov $p(x)$-uniform recurrence and (asymptotical) Stepanov $(p(x), \odot_g)$-almost periodicity:

THEOREM 2.4.28. *Let $p \in \mathcal{P}([0,1])$.*

(i) *If the function $h : I \to X$ is uniformly recurrent, $\phi \in C_0(I : X)$ and $f(t) = h(t) + \phi(t)$ for all $t \in I$, then*

$$\{h(t) : t \in I\} \subseteq \overline{\{f(t) : t \in I\}}. \tag{52}$$

(ii) *If $h : I \to X$ is uniformly continuous and Stepanov $p(x)$-uniformly recurrent (Stepanov $(p(x), \odot_g)$-almost periodic), then the function $h(\cdot)$ is uniformly recurrent ($\odot_g$-almost periodic).*

(iii) *If $f : I \to X$ is uniformly continuous and asymptotically Stepanov $p(x)$-uniformly recurrent (asymptotically Stepanov $(p(x), \odot_g)$-almost periodic), then the function $f(\cdot)$ is asymptotically uniformly recurrent (asymptotically $\odot_g$-almost periodic).*

PROOF. Part (i) can be simply deduced as follows. Let the numbers $t \in \mathbb{R}$ and $\varepsilon > 0$ be fixed. It is clear that there exists a strictly increasing sequence (α_n) of positive real numbers such that $\|h(t) - h(t + \alpha_n)\| < \varepsilon/2$, $n \in \mathbb{N}$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\big\|h(t) - f(t + \alpha_n)\big\| \leqslant \big\|h(t) - h(t + \alpha_n)\big\| + \big\|q(t + \alpha_n)\big\| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This, in turn, implies (52). For the proofs of (ii) and (iii), it suffices to consider case $p(x) \equiv 1$. If the function $h : I \to X$ satisfies the requirements of (ii), then for each $\sigma \in (0,1)$ the function $h_\sigma : I \to X$, given by

$$h_\sigma(t) := \frac{1}{\sigma} \int_t^{t+\sigma} h(s)\, ds, \quad t \in I, \tag{53}$$

is continuous and, due to the uniform continuity of $h(\cdot)$, we have the existence of a number $\delta \in (0,1)$ such that $\|h(t') - h(t'')\| < \varepsilon$, provided $t',\ t'' \in I$ and $|t' - t''| < \delta$. Therefore, if $\sigma \in (0, \delta)$, then we have

$$\big\|h_\sigma(t) - h(t)\big\| \leqslant \frac{1}{\sigma} \int_t^{t+\sigma} \|h(s) - h(t)\|\, ds < \varepsilon, \quad t \in \mathbb{R}, \tag{54}$$

and $\lim_{\sigma \to 0+} h_\sigma(t) = h(t)$ uniformly in $t \in I$. By property (iv) from the beginning of section, it suffices to show that for each fixed number $\sigma \in (0,1)$ the function $h_\delta(\cdot)$ is uniformly recurrent ($\odot_g$-almost periodic). But, this follows from the argumentation given on [**62**, p. 80], where it has been proved that for each number $\varepsilon > 0$ we have $\vartheta(\hat{h}, \sigma\varepsilon) \subseteq \vartheta(h_\sigma, \varepsilon)$. This completes the proof of (ii). To deduce (iii), observe that there exist a Stepanov 1-uniformly recurrent (Stepanov $(1, \odot_g)$-almost periodic) function $h(\cdot)$ and a function $q \in L^1_S(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^1([0,1] : X))$. Using (i) and the arguments contained in the proof of [**218**, Proposition 3.1], we get that the both functions $h(\cdot)$ and $q(\cdot)$ are uniformly continuous. This yields that $q \in C_0(I : X)$ and, due to part (ii), $h(\cdot)$ is uniformly recurrent ($\odot_g$-almost periodic). The proof of the theorem is thereby completed. $\square$

In [**353**, Proposition 12], R. Xie and C. Zhang have proved that any uniformly continuous function $f \in S^pSAP_\omega(I : X)$ belongs to the space $AP_\omega(I : X)$; see [**353**] for the notion. As already mentioned, we have $S^pSAP_\omega(I : X) \subseteq S^pQ - AAP(I : X)$ and it is reasonable to ask whether we can extend the above result by showing that any uniformly continuous function $f \in S^pQ - AAP(I : X)$ belongs to the space $Q - AAP(I : X)$. This is actually the case, as the next proposition shows (an extension to the variable exponent $p \in \mathcal{P}([0,1])$ can be made):

Proposition 2.4.29. *Let $p \in [1, \infty)$, and let $f \in S^pQ - AAP(I : X)$ be uniformly continuous. Then $f \in Q - AAP(I : X)$.*

Proof. The proof of proposition is very similar to the proof of Theorem 2.4.28(ii). Clearly, it suffices to consider the case $p = 1$. Define, for every number $\sigma \in (0,1)$, the function $f_\sigma(\cdot)$ by replacing the function $h(\cdot)$ in (53) with the function $f(\cdot)$. Then the function $f_\sigma(\cdot)$ is bounded and continuous ($\sigma \in (0,1)$). Furthermore, (54) holds with the functions $h_\sigma(\cdot)$ and $h(\cdot)$ replaced therein with the functions $f_\sigma(\cdot)$ and $f(\cdot)$. Due to [**247**, Theorem 2.13(ii)], it suffices to show that the function $f_\sigma(\cdot)$ is quasi-asymptotically almost periodic for each number $\sigma \in (0,1)$. But, this simply follows from the estimate

$$\left\| f_\sigma(t+\tau) - f_\sigma(t) \right\| \leqslant \frac{1}{\sigma} \int_t^{t+1} \| f(s+\tau) - f(s) \| \, ds, \quad t \in I, \ \tau \in I, \ \sigma \in (0,1),$$

which can be proved as on [**62**, p. 80]. $\square$

Remark 2.4.30. The proof of Proposition 2.4.29 considerably shortens the proof of [**353**, Proposition 12]. Therefore, the word "Stepanov" in the formulations of Theorem 2.4.4 and Theorem 2.4.5 can be encompassed with the round brackets.

The following proposition will be important in the sequel:

Proposition 2.4.31. *Suppose that the function $f : I \to X$ is uniformly continuous and (asymptotically) uniformly recurrent. Then there exist a finite number $L \geqslant 1$ and an increasing mapping $g : [0, \infty) \to [1, \infty)$ such that* (38) *holds and $f(\cdot)$ is (asymptotically) $\cdot_g$-almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$.*

Proof. Without loss of generality, we may assume that the equation (19) holds with the sequence (α_n) satisfying $\alpha_{n+1} - \alpha_n \geqslant 1$. It suffices to prove the proposition for uniformly recurrent functions. Let $\varepsilon > 0$ be fixed. Due to the uniform continuity of $f(\cdot)$, we have that there exist an integer $n_0 \in \mathbb{N}$ and a finite real number $\delta > 0$ such that the set $\vartheta(f, \varepsilon)$ contains the union of disjunct intervals $[\alpha_n - \delta, \alpha_n + \delta]$ for $n \geqslant n_0$. Let $g : [0, \infty) \to [1, \infty)$ be any increasing mapping such that $g(n) > \alpha_{n+1}$ for all $n \in \mathbb{N}$. Hence, (38) holds with some finite number

$L \geqslant 1$. Furthermore, if $x \in [n, n+1]$, then the interval $[0, g(x)]$ contains at least $(n-n_0)$ disjunct intervals of length δ whose union belongs to $\vartheta(f, \varepsilon)$. This simply implies that $m([\vartheta(f,\varepsilon)](0, g(x))) \geqslant \delta(n-n_0)$ and therefore $m([\vartheta(f,\varepsilon)](0, g(x)))/x \geqslant \delta(n-n_0)/(n+1)$. This simply implies $\underline{d}_c(\vartheta(f,\varepsilon)) > 0$, so that $f(\cdot)$ is $\underline{d}_{gc}$-almost periodic and therefore $\overline{d}_{gc}$-almost periodic. $\square$

REMARK 2.4.32. The proof of Proposition 2.4.31 does not work for the upper $l; gc$-Banach density $\overline{Bd}_{l;gc}(\cdot)$ and the upper $u; gc$-Banach density $\overline{Bd}_{u;gc}(\cdot)$. In general, these densities differ from the densities

$$\overline{Bd}_{l:gc}(A) := \liminf_{x \to +\infty} \sup_{y \geqslant 0} \frac{m(A(y, y+g(x)))}{x}$$

and

$$\overline{Bd}_{u:gc}(A) := \limsup_{x \to +\infty} \sup_{y \geqslant 0} \frac{m(A(y, y+g(x)))}{x},$$

respectively. Repeating verbatim the above arguments, it can be simply proved that for any uniformly continuous, uniformly recurrent function $f : I \to X$ there exist a finite number $L \geqslant 1$ and an increasing mapping $g : [0, \infty) \to [1, \infty)$ such that (38) holds and $f(\cdot)$ is $\cdot_g$-almost periodic for $\cdot_g \in \{\overline{Bd}_{l:gc}, \overline{Bd}_{u:gc}\}$.

REMARK 2.4.33. By the proof of Proposition 2.4.31, it follows that, for every uniformly continuous, uniformly recurrent functions $f_i : I \to X$ $(1 \leqslant i \leqslant n)$, we can find a finite number $L \geqslant 1$ and an increasing mapping $g : [0, \infty) \to [1, \infty)$ such that (38) holds and $f_i(\cdot)$ is $\cdot_g$-almost periodic for all $1 \leqslant i \leqslant n$ and $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$.

Keeping in mind the corresponding definitions and Proposition 2.4.31, the next result follows immediately (the previous two remarks can be reformulated in this context, as well):

PROPOSITION 2.4.34. *Suppose that $p \in \mathcal{P}([0,1])$, $f : I \to X$ is (asymptotically) Stepanov $p(x)$-uniformly recurrent and $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is uniformly continuous. Then there exist a finite number $L \geqslant 1$ and an increasing mapping $g : [0, \infty) \to [1, \infty)$ such that (38) holds and $f(\cdot)$ is (asymptotically) Stepanov $(p(x), \odot_g)$-almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$.*

It is worth noticing that Proposition 2.4.31 cannot be applied to the compactly almost automorphic functions which are not asymptotically uniformly recurrent, in general. Concerning this problematic, we would like to present the following illustrative example:

EXAMPLE 2.4.35. Any almost periodic function has to be compactly almost automorphic, while the converse statement is not true, however. The first example of a scalar-valued compactly almost automorphic function which is not almost periodic has been constructed by A. M. Fink (see [**171**, p. 521]). Set $a_n :=\text{sign}(\cos(n\pi\sqrt{2}))$, $n \in \mathbb{Z}$ and define after that the function $f : \mathbb{R} \to \mathbb{R}$ by $f(t) := \alpha a_n + (1-\alpha)a_{n+1}$ if $t \in [n, n+1)$ for some integer $n \in \mathbb{Z}$ and $t = \alpha n + (1-\alpha)(n+1)$ for some number $\alpha \in (0, 1]$. As verified in [**171**], this function is compactly almost automorphic (therefore, uniformly continuous) but not almost periodic. We will extend this result by showing that the function $f(\cdot)$ is not asymptotically uniformly recurrent. If we suppose the contraposition, then there exists a strictly increasing sequence (τ_n) of positive real numbers tending to plus infinity such that, for every $\varepsilon > 0$, we have the existence of two finite numbers $M > 0$ and $n_0 \in \mathbb{N}$ such that

$$\big\|f(x+\tau_n) - f(x)\big\| \leqslant 2\varepsilon, \quad |x| \geqslant M,\ n \geqslant n_0.$$

Let $\varepsilon \in (0, 1/2)$ and $n \geqslant n_0$. Then it is clear that there exists $l \in \mathbb{N}$, as large as we want, such that $a_l > 0$ and $a_{l+1} < 0$. Then $f(l + (1/2)) = 0$ and therefore $|f(l + (1/2) + \tau_n)| \leqslant 2\varepsilon$. This clearly implies the existence of an integer $k \in \mathbb{Z}$ such that the number $l + (1/2) + \tau_n$ lies in a certain small neighborhood of number $k + (1/2)$; more precisely, since the linear function connecting the points $(k, -1)$ and $(k + 1, 1)$ is given by $y = 2x - 2k - 1$, we get from the above that $|2(l + (1/2) + \tau_n) - 2k - 1| \leqslant 2\varepsilon$, which simply implies $|\tau_n - (k - l)| \leqslant \varepsilon$ and therefore $\tau_n \in (0, \varepsilon] \cup \bigcup_{k\in\mathbb{N}}[k - \varepsilon, k + \varepsilon]$. Fix now an integer $k \in \mathbb{N}$. We will show that the inclusion $\tau_n \in [k - \varepsilon, k + \varepsilon]$ cannot be true. Otherwise, for each real number $t \in \mathbb{R}$ we have $|f(t + \tau_n) - f(t + k)| \leqslant 2 \cdot \varepsilon = 2\varepsilon$, which can be easily approved, so that

$$\begin{aligned} |f(t+k) - f(t)| &\leqslant \big|f(t+k) - f(t+\tau_n)\big| + \big|f(t+\tau_n) - f(t)\big| \\ &\leqslant 2\varepsilon + \varepsilon = 3\varepsilon, \quad |t| \geqslant M. \end{aligned}$$

This contradicts Lemma 2.4.8. Notice also that the argumentation given above shows that, for every $\varepsilon \in (0, 1)$, we have $\vartheta(f, \varepsilon) \cap (\varepsilon/2, +\infty) = \emptyset$. Furthermore, for every $\varepsilon \in (0, 1)$ and $\tau \in (0, \varepsilon/2]$, we have $|f(t+\tau)-f(t)| \leqslant 2\tau \leqslant \varepsilon$ so that, actually,

$$\forall \varepsilon \in (0, 1) : \quad \vartheta(f, \varepsilon) = (0, \varepsilon/2].$$

Let us recall that A. M. Fink has constructed in [**170**, Example 6.1] an odd almost periodic function $f : \mathbb{R} \to \mathbb{R}$ satisfying that

$$\int_0^t f(s)\, ds \leqslant 0, \quad t \in \mathbb{R}, \quad \int_0^{2^{n-1}} f(s)\, ds \leqslant -n, \quad n \in \mathbb{N}$$

and the function

$$F(t) := e^{\int_0^t f(s)\, ds}, \quad t \in \mathbb{R}$$

is bounded but not almost periodic. The construction goes as follows. For any number $n \in \mathbb{N} \setminus \{1\}$, we define the function $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(t) := (-n)/(2^{n-1}-1)$, $t \in [1, 2^{n-1} - 1]$, $f_n(0) := 0$, $f_n(2^{n-1} - 1) := 0$, $f_n(\cdot)$ is linear on segments $[0, 1]$ and $[2^{n-1} - 1, 2^{n-1}]$; after that, we extend $f_n(\cdot)$ to be odd and periodic of period 2^n. The function $f(t) := \sum_{n=2}^{\infty} f_n(t)$, $t \in \mathbb{R}$ is well defined, odd and satisfies the above-mentioned properties. Further on, we have $F(t) \leqslant 1$ for all $t \in \mathbb{R}$ so that the Lagrange mean value theorem directly yields that the function $F(\cdot)$ is Lipschitzian with the Lipschitz constant $\|f\|_\infty$; in particular, $F(\cdot)$ is uniformly continuous. It could be of some interest to know whether the function $F(\cdot)$ is not uniformly recurrent.

Finally, it should be note that several intriguing examples of functions with almost periodic behaviour have been constructed by D. Bugajewski, A. Nawrocki in [**84**] and M. Vesely in [**343**].

Before providing the proofs of Theorem 2.4.2, Theorem 2.4.4, Theorem 2.4.5 and Theorem 2.4.7, we would like to address one more problem to our readers:

QUESTION 2.4.36. Let us recall that the function $f(\cdot)$, given by (29), is almost automorphic function and not compactly almost automorphic. We would like to ask whether for each number $\varepsilon \in (0,1)$ we have that $\vartheta(f,\varepsilon) \neq \emptyset$ ($\vartheta(f,\varepsilon)$ is unbounded)?

PROOF OF THEOREM 2.4.2. We will first prove that for each fixed number $\tau \in \mathbb{R}$ we have that the function $f(\cdot+\tau)-f(\cdot)$ belongs to the space $ANP(\mathbb{R}:\mathbb{C})$. Towards this end, note that

$$\begin{aligned} f(t+\tau)-f(t) &= \sum_{n=1}^{\infty} \frac{1}{n}\left[\sin^2 \frac{t+\tau}{2^n} - \sin^2 \frac{t}{2^n}\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{2n}\left[\cos \frac{t}{2^{n-1}} - \cos \frac{t+\tau}{2^{n-1}}\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2t+\tau}{2^n} \sin \frac{\tau}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\left[\sin \frac{t}{2^{n-1}} \cos \frac{\tau}{2^n} + \cos \frac{t}{2^{n-1}} \sin \frac{\tau}{2^n}\right] \sin \frac{\tau}{2^n}, \quad t \in \mathbb{R}. \end{aligned}$$

Since the functions $t \mapsto \sin \frac{t}{2^{n-1}}$, $t \in \mathbb{R}$ and $t \mapsto \cos \frac{t}{2^{n-1}}$, $t \in \mathbb{R}$ are anti-periodic of anti-period $T = 2^{n-1}\pi$, it follows that the function

$$f_k(t) := \sum_{n=1}^{k} \frac{1}{n}\left[\sin \frac{t}{2^{n-1}} \cos \frac{\tau}{2^n} + \cos \frac{t}{2^{n-1}} \sin \frac{\tau}{2^n}\right] \sin \frac{\tau}{2^n}, \quad t \in \mathbb{R}$$

belongs to the space $ANP_0(\mathbb{R}:\mathbb{C})$. Moreover, $\lim_{k\to+\infty} f_k(t) = f(t+\tau)-f(t)$ uniformly on $\mathbb{R}$ since

$$\left|\sum_{n=k+1}^{\infty} \frac{1}{n}\left[\sin \frac{t}{2^{n-1}} \cos \frac{\tau}{2^n} + \cos \frac{t}{2^{n-1}} \sin \frac{\tau}{2^n}\right] \sin \frac{\tau}{2^n}\right| \leqslant |\tau| \sum_{n=k+1}^{\infty} \frac{1}{n2^{n-1}}, \quad t \in \mathbb{R}.$$

Especially, due to the fact that $ANP(\mathbb{R}:\mathbb{C}) = AP_{\mathbb{R}\setminus\{0\}}(\mathbb{R}:\mathbb{C})$, we have $0 \notin \sigma(f(\cdot+\tau)-f(\cdot))$, i.e.,

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t |f(s+\tau)-f(s)|\, ds = 0.$$

This readily implies

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t |f(s+\tau)-f(s)|^p\, ds = 0, \quad p \geqslant 1,$$

because

$$|f(s+\tau)-f(s)|^p \leqslant |f(s+\tau)-f(s)| \cdot \left(\sup_{x\geqslant 0} |f(x+\tau)-f(x)|\right)^{p-1}, \quad s \geqslant 0.$$

Taking into account [**234**, Proposition 2.13.4], we easily get that for each numbers $t,\ \tau \in \mathbb{R}$ we have

$$\begin{aligned}\lim_{l\rightarrow+\infty}\frac{1}{2l}\int_{-l}^{l}\bigl|f\bigl(t+\tau+x\bigr)-f(t+x)\bigr|^{p}\,dx\\ =\limsup_{l\rightarrow+\infty}\frac{1}{2l}\int_{-l}^{l}\bigl|f\bigl(t+\tau+x\bigr)-f(t+x)\bigr|^{p}\,dx=0,\end{aligned}$$

so that the function $f(\cdot)$ is Weyl p-almost automorphic with the limit function $f^{*}\equiv f$. This completes the proof of Theorem 2.4.2. $\square$

Proof of Theorem 2.4.4. Suppose that the function $h(\cdot)$ is Stepanov quasi-asymptotically almost periodic. It is clear that the function $h(\cdot)$ is asymptotically Stepanov uniform recurrent, so that Theorem 2.4.24(ii) implies that the function $h(\cdot)$ is asymptotically Stepanov almost periodic. Since $h(\cdot)$ is uniformly continuous, Lemma 2.3.3(i) implies that the function $h(\cdot)$ is asymptotically almost periodic. This cannot be true because the restriction of function $h(\cdot)$ to the non-negative real axis is not asymptotically (Stepanov) almost automorphic by Lemma 2.4.3. $\square$

Proof of Theorem 2.4.5. The function $f(\cdot)$, given by (34), satisfies that for each $\varepsilon>0$ there exists a positive real number $\delta>0$ such that the set $\vartheta(f,\varepsilon)$ contains the set $\bigcup_{n\geqslant\lceil 1/\varepsilon\rceil}[\tau_n-\delta,\tau_n+\delta]$ as well as $f(x)=f_n(x)$ for all $x\in[-\tau_{n-1},\tau_{n-1}]$ $(n\in\mathbb{N})$. Furthermore, the function $f(\cdot)$ equals zero on arbitrarily long intervals and for each number $\varepsilon\in(0,1)$ we have that the sets $\{x\in\mathbb{R}:f(x)\notin[1-\varepsilon,1+\varepsilon]\}$ and $\vartheta(f,\varepsilon)$ are disjunct (see [**172**, Example 8, pp. 31-33] for more details). This essentially implies that the function $f(\cdot)$ cannot be asymptotically Stepanov almost automorphic (we will present a direct proof, without appealing to Lemma 2.3.3(ii) and Proposition 2.4.26(iii)). If we suppose the contraposition, then there exist a Stepanov almost automorphic function $h(\cdot)$ and a function $q\in C_0(\mathbb{R}:L^1([0,1]:\mathbb{C}))$ such that $f(t)=h(t)+q(t)$ for a.e. $t\in\mathbb{R}$. Moreover, we have the existence of disjunct intervals $I_n=[b_n,b_n'']\subseteq[0,\infty)$ whose length is strictly greater than n^2 and which satisfy that $f(x)=0$ for all $x\in I_n$ $(n\in\mathbb{N})$. Define $b_n:=(b_n'+b_n'')/2$ $(n\in\mathbb{N})$. Then there exist a subsequence (a_n) of (b_n) and a function $g^{*}\in L^1_{loc}(\mathbb{R}:\mathbb{C})$ such that

$$\lim_{n\rightarrow+\infty}\int_{t}^{t+1}\bigl|f(x+a_n)-q(x+a_n)-g^{*}(x)\bigr|\,dx=0$$

for all $t\in\mathbb{R}$, and

$$\lim_{n\rightarrow+\infty}\int_{t}^{t+1}\bigl|g^{*}(x-a_n)-[f(x)-q(x)]\bigr|\,dx=0$$

for all $t\geqslant 0$. Let $\varepsilon\in(0,1/2)$ be given. Then there exists $n_0\in\mathbb{N}$ such that $n_0/(n_0-1)>3\varepsilon/2$ and $\int_{\tau_{n_0}}^{1+\tau_{n_0}}|q(x)|\,dx<\varepsilon/8$. Since $1\geqslant f(x)\geqslant f_n(x)\geqslant n_0/(n_0-1)$ for $x=\tau_{n_0}$, $f_n(x)=0$ for $x=\tau_{n_0}+1$ and the function $f_n(\cdot)$ is linear on the interval $[\tau_{n_0},\tau_{n_0}+1]$ (see also [**75**, part I, p. 115]), the second limit equality with $t=\tau_{n_0}$

easily implies the existence of an integer $n_1 \geqslant n_0$ such that

$$\int_{\tau_{n_0}-a_n}^{1+\tau_{n_0}-a_n} \bigl|g^*(x)\bigr|\,dx \geqslant \frac{n_0}{2(n_0-1)} - \frac{\varepsilon}{2} > \frac{\varepsilon}{4}, \quad n \geqslant n_1.$$

Returning to the first limit equation, with $t = \tau_{n_0} - a_{n_1}$, and taking into account that $\lim_{m\to\infty} \int_t^{t+1} |q(x+a_m)|\,dx < \varepsilon/8$ for all $m \in \mathbb{N}$ sufficiently large, we obtain the existence of an integer $m_1 \geqslant n_1$ such that

$$\int_{\tau_{n_0}-a_{n_1}+a_m}^{1+\tau_{n_0}-a_{n_1}+a_m} |f(x)|\,dx = \int_{\tau_{n_0}-a_{n_1}}^{1+\tau_{n_0}-a_{n_1}} \bigl|f\bigl(x+a_m\bigr)\bigr|\,dx > \frac{\varepsilon}{4} - \frac{\varepsilon}{8} > 0$$

for all $m \geqslant m_1$. But, this is simply impossible because for large values of m we have that $[\tau_{n_0} - a_{n_1} + a_m, 1 + \tau_{n_0} - a_{n_1} + a_m]$ is contained in a larger interval where the function $f(\cdot)$ equals zero. If we assume that the function $f(\cdot)$ is Stepanov quasi-asymptotically almost periodic, then the first part of proof of Theorem 2.4.4 yields that the function $f(\cdot)$ is asymptotically Stepanov almost periodic, which cannot be true according to the first part of proof of this theorem. $\square$

EXAMPLE 2.4.37. Without going into full details, let us only note that the function $f(\cdot)$ considered above can be Weyl p-almost automorphic ($p \geqslant 1$) if the sequence (τ_n) marches rapidly to plus infinity. This follows from the fact that the function $f(\cdot)$ is bounded and belongs to the space $PAP_0(\mathbb{R} : \mathbb{C})$. To explain this in more detail, let a_n denote the number of triangles appearing on the graph of function $f_n(\cdot)$. Then $a_1 = 1$ and $a_n = (2n-1)a_{n-1}$, $n \in \mathbb{N} \smallsetminus \{1\}$ so that $a_n = (2n-1)!!$, $n \in \mathbb{N}$. The Lebesgue measure of each such triangle cannot exceed 1 so that $\int_{-\infty}^{+\infty} f_n(x)\,dx \leqslant (2n-1)!!$, $n \in \mathbb{N}$. Suppose, for simplicity, that $\lim_{n\to+\infty}(2n-1)!!/\tau_{n-2} = 0$. If $\tau_{n-1} \geqslant l \geqslant \tau_{n-2}$ for some sufficiently large integer $n \in \mathbb{N}$, then

$$\frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l} f_n(x)\,dx \leqslant \frac{1}{\tau_{n-2}}\int_{-\infty}^{\infty} f_n(x)\,dx \leqslant \frac{(2n-1)!!}{\tau_{n-2}},$$

so that $\lim_{l\to+\infty}(1/2l)\int_{-l}^{l} f(x)\,dx = 0$, as claimed. Needless to say that, due to Proposition 2.4.31, there exists a suitable function $g(\cdot)$ such that the function $f(\cdot)$ is $\cdot_g$-almost periodic for $\cdot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$ (see also [**198**, pp. 477-478]).

PROOF OF THEOREM 2.4.7. It is already known that the function $f(\cdot)$ satisfies $\lim_{i\to+\infty} \|f(\cdot + 2p_i) - f(\cdot)\|_\infty = 0$, so that $f(\cdot)$ is uniformly recurrent. Keeping in mind Proposition 2.4.29 and arguing as in the proof of Theorem 2.4.4, we get that $f(\cdot)$ is (Stepanov) quasi-asymptotically almost periodic if and only if $f(\cdot)$ is asymptotically almost periodic. By Proposition 2.4.26(ii), this would imply that the function $f(\cdot)$ is almost periodic; this is not the case because the function $f(\cdot)$ is not almost automorphic (asymptotically almost automorphic, equivalently, due to Proposition 2.4.26(i)). If we suppose the contrary, then there exist a subsequence (p_{i_k}) of (p_i) and a function $\omega : \mathbb{R} \to \mathbb{R}$ such that $\lim_{k\to+\infty} f(t + p_{i_k}) = \omega(t)$ and $\lim_{k\to+\infty} \omega(t - p_{i_k}) = f(t)$ for all $t \in \mathbb{R}$. Observe that the function $f_i(\cdot)$ satisfies $f_i(t + p_i) \geqslant 1 - \varepsilon$, provided $|t| \leqslant \varepsilon p_i$ and $i \in \mathbb{N}$. Let $t \in \mathbb{R}$ and $\varepsilon > 0$ be given.

Then there exists $i_0 \in \mathbb{N}$ such that $|t| \leqslant \varepsilon p_i$ for all integers $i \geqslant i_0$. Therefore, for any integer $i \geqslant i_0$, we have

$$1 \geqslant f\big(t+p_i\big) \geqslant f_i\big(t+p_i\big) \geqslant 1-\varepsilon,$$

so that $1 = \lim_{i\to+\infty} f(t+p_i) = \lim_{k\to+\infty} f(t+p_{i_k}) = \omega(t)$. Therefore, $\omega(t) \equiv 1$ and returning to the second limit equality we get $f(t) \equiv 1$, which is a contradiction (see also [**129**, Figure 3.7.3, p. 208]). □

We continue by proposing an interesting result closely connected with our previous analysis of uniformly recurrent functions and the recent researches of I. Area, J. Losada and J. J. Nieto [**27**]-[**28**] concerning the quasi-periodic properties of fractional integrals and fractional derivatives of scalar-valued periodic functions (see also I. Area, J. Losada, J. J. Nieto [**29**] and J. M. Jonnalagadda [**216**] for the discrete analogues). In [**234**], we have emphasized that the almost periodic properties and the almost automorphic properties of the Riemann-Liouville integrals are very unexplored in the vector-valued case.

Suppose that $\alpha \in (0,1)$ and $T>0$. In [**27**, Theorem 1], the authors have proved that the Riemann-Liouville integral $J_t^\alpha f(t) := \int_0^t g_\alpha(t-s)f(s)\,ds$, $t \in \mathbb{R}$ of a non-zero essentially bounded $T-$periodic function $f : \mathbb{R} \to \mathbb{R}$ cannot be $T-$periodic. Suppose now that $f : \mathbb{R} \to X$ is a non-zero essentially bounded $T-$periodic function. Then [**28**, Lemma 3] continues to holds for $f(\cdot)$, as it can be simply verified, so that the function $J_t^\alpha f(\cdot)$ is S-asymptotically T-periodic. If we suppose that the function $J_t^\alpha f(\cdot)$ is uniformly recurrent (compactly almost automorphic), this would imply by [**209**, Lemma 3.1] and the arguments used in the proof of [**209**, Proposition 3.4] that the function $J_t^\alpha f(\cdot)$ is T-periodic. This will be used in the proof of the following proper extension of [**28**, Theorem 9]:

THEOREM 2.4.38. *Suppose that $\alpha \in (0,1)$, $T>0$ and $f : \mathbb{R} \to X$ is a non-zero essentially bounded T-periodic function. Then $J_t^\alpha f(\cdot)$ cannot be uniformly recurrent (almost automorphic).*

PROOF. Suppose that $J_t^\alpha f(\cdot)$ is uniformly recurrent (almost automorphic) and $x^* \in X^*$ is an arbitrary functional. Let $\langle x^*, f(\cdot)\rangle = a(\cdot) + ib(\cdot)$, where $a(\cdot)$ and $b(\cdot)$ are real-valued functions. Then it is clear that the function $J_t^\alpha \langle x^*, f(\cdot)\rangle = J_t^\alpha a(\cdot) + iJ_t^\alpha b(\cdot)$ is uniformly recurrent (almost automorphic) because $J_t^\alpha \langle x^*, f(\cdot)\rangle = \langle x^*, J_t^\alpha f(\cdot)\rangle$, which further implies that the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are uniformly recurrent (almost automorphic). Let us assume first that the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are uniformly recurrent. Since $a(\cdot)$ and $b(\cdot)$ are essentially bounded functions of period T, the above discussion implies that $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are periodic functions of period T. Then we can apply [**27**, Theorem 1] in order to see that $a(\cdot) \equiv b(\cdot) \equiv 0$. This implies $\langle x^*, f(\cdot)\rangle \equiv 0$ and therefore $f(\cdot) \equiv 0$. The proof is quite similar if we assume that the function $J_t^\alpha f(\cdot)$ is almost automorphic, when the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are also almost automorphic. Since the function $J_t^\alpha f(\cdot)$ is bounded, repeating verbatim the above arguments we may deduce from [**28**, Theorem 5] that the functions $J_t^\alpha a(\cdot)$ and $J_t^\alpha b(\cdot)$ are asymptotically T-periodic and, in particular, bounded and uniformly continuous. Therefore, the functions $J_t^{\alpha}a(\cdot)$ and $J_t^{\alpha}b(\cdot)$ are compactly almost automorphic. But, then we can argue in the same way as for the uniform recurrence to get that $a(\cdot) \equiv b(\cdot) \equiv 0$. $\square$

Applying the trick used in the first part of the proof and the well known fact that a weakly bounded set in a locally convex space is bounded, we may conclude that the statements of [**27**, Theorem 1, Corollary 2] and [**28**, Lemma 2, Lemma 3; Proposition 1, Proposition 2; Theorem 2, Theorem 3, Theorem 4, Theorem 8] hold in the vector-valued case (concerning the above-mentioned statements from [**28**], it seems very plausible that the continuity of function $f(\cdot)$ in their formulations can be replaced with the essential boundedness). It is clear that [**28**, Corollary 1] cannot be reformulated even for the complex-valued functions and, regarding the main structural results established in [**27**]-[**28**], it remains to be considered whether the statements of [**28**, Theorem 5, Theorem 6, Theorem 7] hold in the vector-valued case. We will analyze this question somewhere else.

We proceed further with some applications of (asymptotically) uniformly recurrent functions and (asymptotically) $\odot_g$-almost periodic functions. We shall mostly be concerned with the invariance of (asymptotical) uniform recurrence and (asymptotical) $\odot_g$-almost periodicity under the actions of convolution products.

Let $f : \mathbb{R} \to X$. We will first investigate the uniformly recurrent and $\odot_g$-almost periodic properties of the function

$$F(t) := \int_{-\infty}^{t} R(t-s)f(s)\, ds, \quad t \in \mathbb{R}, \tag{55}$$

where a strongly continuous operator family $(R(t))_{t>0} \subseteq L(X,Y)$ satisfies certain assumptions. In our recent research studies regarding this question, it is commonly assumed that the function $f(\cdot)$ is Stepanov $p(x)$-bounded for some function $p \in \mathcal{P}([0,1])$. If this is the case, we can simply reformulate the statement of Proposition 2.5.17 in our new framework (cf. also [**359**, Examples 4, 5, 7, 8; pp. 32-34], which can be simply reformulated for the uniform recurrence and $\odot_g$-almost periodicity):

PROPOSITION 2.4.39. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is Stepanov $p(x)$-bounded and Stepanov $p(x)$-uniformly recurrent (Stepanov $(p(x), \odot_g)$-almost periodic), as well as the mapping $t \mapsto \check{f}(\cdot - t) \in L^{p(x)}([0,1] : X)$ is continuous, then the function $F : \mathbb{R} \to Y$, given by (55), is well-defined and uniformly recurrent ($\odot_g$-almost periodic).*

PROOF. The function $F(\cdot)$ is well defined due to the computation carried out in the proof of Proposition 2.5.17. The proof of the above-mentioned proposition also shows that, if $\tau \in \mathbb{R}$ is an ε-period of function $\hat{\check{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$, then the resulting function $F(\cdot)$ satisfies, under given conditions on $(R(t))_{t>0}$, an estimate of the type $\|F(t+\tau) - F(t)\|_Y \leqslant L\varepsilon$, $t \in \mathbb{R}$, where $L \geqslant 1$ is a finite constant independent of t, ε and τ. Hence, the assumption $\odot_g(\vartheta(\hat{\check{f}}, \varepsilon)) > 0$ for all $\varepsilon > 0$ implies that $\odot_g(\vartheta(F, \varepsilon)) > 0$ for all $\varepsilon > 0$. Therefore, it remains to be proved that the function $F(\cdot)$ is continuous. But, this follows similarly as in the proof of [**234**, Proposition 3.5.3] and our assumption that the mapping $t \mapsto \check{f}(\cdot - t) \in L^{p(x)}([0,1] : X)$ is continuous (see also [**143**, Proposition 5.1]). $\square$

REMARK 2.4.40. In general case $p \in \mathcal{P}([0,1])$, the mapping $t \mapsto \check{f}(\cdot - t) \in L^{p(x)}([0,1] : X)$ is not necessarily continuous (see e.g., [**256**, p. 602]). This is always true provided that $p \in D_+([0,1])$.

Basically, case in which the function $f : \mathbb{R} \to X$ is not Stepanov $p(x)$-bounded has not attracted the attention of the authors so far. Keeping in mind our previous results, we would like to state the following proposition with regards to this question (the uniform continuity of function $\hat{\check{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ has not been assumed above):

PROPOSITION 2.4.41. *Suppose that* $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$, $\check{f} : \mathbb{R} \to X$ *is Stepanov* $p(x)$*-uniformly recurrent (Stepanov* $(p(x), \odot_g)$*-almost periodic), there exists a continuous function* $P : \mathbb{R} \to [1,\infty)$ *such that*

$$\|f(t - \cdot)\|_{L^{p(\cdot)}[0,1]} \leqslant P(t), \quad t \in \mathbb{R} \tag{56}$$

and $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that for each* $t \in \mathbb{R}$ *we have*

$$\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} P(t-k) < \infty. \tag{57}$$

If the function $\hat{\check{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ *is uniformly continuous, then the function* $F : \mathbb{R} \to Y$*, given by* (55)*, is well-defined and uniformly recurrent (*$\odot_g$*-almost periodic).*

PROOF. We will only outline the most important details for Stepanov $(p, \odot_g)$-almost periodic functions. The function $F(\cdot)$ is well defined since, due to Lemma 1.1.6(i) and the estimates (56)-(57), we have:

$$\begin{aligned}
\int_0^\infty \|R(s)\|\|f(t-s)\|\,ds &= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(s)\|\|f(t-s)\|\,ds \\
&= \sum_{k=0}^{\infty} \int_0^1 \|R(s+k)\|\|f(t-s-k)\|\,ds \\
&\leqslant 2 \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}([0,1]:X)} \|f(t-k-\cdot)\|_{L^{p(x)}([0,1]:X)} \\
&\leqslant 2 \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}([0,1]:X)} P(t-k) < \infty,
\end{aligned}$$

for any $t \in \mathbb{R}$. It is clear that our assumptions imply

$$M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} < \infty,$$

so that $\vartheta(f,\varepsilon) \subseteq \vartheta(F, M\varepsilon)$. Since we have assumed that the function $\hat{\hat{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ is uniformly continuous, the arguments contained in the proof of [**234**, Proposition 2.6.11] can be repeated verbatim in order to see that the function $F(\cdot)$ is continuous. This completes the proof of proposition. $\square$

Proposition 2.4.39 and Proposition 2.4.41 can be simply incorporated in the study of the existence and uniqueness of uniformly recurrent and $\odot_g$-almost periodic solutions of the fractional Cauchy inclusion

$$D_{t,+}^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), \ t \in \mathbb{R}, \tag{58}$$

where $D_{t,+}^{\gamma}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1]$, $f : \mathbb{R} \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator satisfying condition (P) (see Subsection 2.5.3 and [**234**] for more details).

Taking into account Proposition 2.4.39 and Proposition 2.4.41, we can simply provide extensions of [**234**, Proposition 2.6.13, Theorem 2.9.5, Theorem 2.9.7, Theorem 2.9.15], concerning the asymptotical Stepanov p-uniform recurrence/asymptotical Stepanov $(p,\odot_g)$-almost periodicity of the finite convolution product

$$\mathbf{F}(t) := \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0.$$

These results can be applied in the qualitative analysis of asymptotically uniformly recurrent/asymptotically $\odot_g$-almost periodic solutions (asymptotically Stepanov p-uniformly recurrent/asymptotically Stepanov $(p,\odot_g)$-almost periodic solutions) of the following abstract Cauchy inclusion

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), \ t \geqslant 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^{\gamma}$ denotes the Caputo fractional derivative of order $\gamma \in (0,1]$, $x_0 \in X$, $f : [0,\infty) \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator satisfying condition (P) (see Subsection 2.5.3 and [**234**] for more details).

The sum of two uniformly recurrent ($\odot_g$-almost periodic) functions need not be uniformly recurrent ($\odot_g$-almost periodic), unfortunately. But, it is worth noticing that there exist many concrete situations where this difficulty can be overcomed. For example, it is very simple to extend the assertions of [**234**, Theorem 2.14.7] and [**153**, Theorem 2.3] for the asymptotical Stepanov $(p,\odot_g)$-almost periodicity. To explain this in more detail, let us observe that the equation appearing on [**153**, p. 240, l. 5] can be rewritten as

$$\int_{-\infty}^{t} \Gamma(t,s)f(s)\,ds = \lim_{k\to+\infty} \int_0^k \Gamma(t,t-s)f(t-s)\,ds, \quad t \in \mathbb{R};$$

arguing as in the proof of above-mentioned theorem from [**153**] we may conclude that for each integer $k \in \mathbb{N}$ the function $t \mapsto \int_0^k \Gamma(t,t-s)f(t-s)\,ds$, $t \in \mathbb{R}$ is $\odot_g$-almost periodic, provided that the function $f(\cdot)$ is Stepanov $(p,\odot_g)$-almost periodic and Stepanov p-bounded ($p > 1$), while the case $p = 1$ follows from the same arguments and the proof of [**234**, Theorem 2.14.6], when it is necessary to assume that $f(\cdot)$ is Stepanov $(1,\odot_g)$-almost periodic and Stepanov 1-bounded. In both cases, $p > 1$ and $p = 1$, we need to employ the property (iv) to achieve the final results.

We close the subsection with the observation that the results whose proofs lean heavily on the use of Bochner criterion cannot be really reconsidered for uniformly recurrent and $\odot_g$-almost periodic functions.

2.4.3. Composition principles for almost periodic type functions and applications. In this subsection, we introduce and analyze the classes of two-parameter (asymptotically) uniformly recurrent functions, two-parameter (asymptotically) $\odot_g$-almost periodic functions and their Stepanov generalizations. Several composition principles are established in this context, which enables one to provide certain applications to the abstract semilinear integro-differential Cauchy problems and inclusions. Since the structural results presented in this subsection can be deduced by uncomplicated modifications of results known in the existing literature, we have decided to provide the main details of proofs for only two statements, Theorem 2.4.44 and Theorem 2.4.46.

For every $\varepsilon > 0$ and for every bounded set $B \subseteq Y$, we define $\vartheta(F; \varepsilon, B)$ as the set constituted of all numbers $\tau > 0$ such that

$$\|F(t+\tau, y) - F(t, y)\| \leqslant \varepsilon, \quad t \in I, \ y \in B.$$

The following definition is crucial in our analysis:

DEFINITION 2.4.42. (i) A continuous function $F : I \times Y \to X$ is called uniformly recurrent, resp. $\odot_g$-almost periodic, if and only if for every $\varepsilon > 0$ and every compact $K \subseteq Y$ there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n \to +\infty} \sup_{t \in I} \big\| F(t + \alpha_n, y) - F(t, y) \big\| = 0, \quad y \in K, \tag{59}$$

resp. if and only if for every $\varepsilon > 0$ and every compact $K \subseteq Y$ we have $\odot_g(\vartheta(f; \varepsilon, K)) > 0$.

The collection of all two-parameter uniformly recurrent functions, resp. $\odot_g$-almost periodic functions, will be denoted by $UR(I \times Y : X)$, resp. $AP_{\odot_g}(I \times Y : X)$.

(ii) A continuous function $F : I \times Y \to X$ is called uniformly recurrent on bounded sets, resp. $\odot_g$-almost periodic on bounded sets, if and only if for every $\varepsilon > 0$ and every bounded set $B \subseteq Y$ there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that (59) holds with $K = B$, resp. if and only if for every $\varepsilon > 0$ and every bounded set $B \subseteq Y$ we have $\odot_g(\vartheta(f; \varepsilon, B)) > 0$.

The collection of all two-parameter uniformly recurrent functions on bounded sets, resp. $\odot_g$-almost periodic functions on bounded sets, will be denoted by $UR_b(I \times Y : X)$, resp. $AP_{\odot_g, b}(I \times Y : X)$.

(iii) A continuous function $F : I \times Y \to X$ is said to be asymptotically uniformly recurrent, resp. asymptotically $\odot_g$-almost periodic, if and only if $F(\cdot)$ admits a decomposition $F = G + Q$, where $G \in UR(\mathbb{R} \times Y : X)$, resp. $G \in AP_{\odot_g}(\mathbb{R} \times Y : X)$, and $Q \in C_0(I \times Y : X)$.

Denote by $AUR(I \times Y : X)$, resp. $AAP_{\odot_g}(I \times Y : X)$, the collection consisting of all asymptotically uniformly recurrent functions, resp. asymptotically $\odot_g$-almost periodic functions.

(iv) A continuous function $F : I \times Y \to X$ is said to be asymptotically uniformly recurrent on bounded sets, resp. asymptotically $\odot_g$-almost periodic on bounded sets, if and only if $F(\cdot)$ admits a decomposition $F = G + Q$, where $G \in UR_b(\mathbb{R} \times Y : X)$, resp. $G \in AP_{\odot_g,b}(\mathbb{R} \times Y : X)$, and $Q \in C_0(I \times Y : X)$.

Denote by $AUR_b(I \times Y : X)$, resp. $AAP_{\odot_g,b}(I \times Y : X)$, the collection consisting of all asymptotically uniformly recurrent functions, resp. asymptotically $\odot_g$-almost periodic functions.

In the contrast to the approach of C. Zhang for almost periodic functions depending on the parameter [**366**] (see also [**234**, Definition 2.1.4]), we do not assume a priori the boundedness of function $f(\cdot, \cdot)$ in our approach. This is quite reasonable because uniformly recurrent functions and $\odot_g$-almost periodic functions of one real variable need not be bounded, in general. It is worth noticing that introducing parts (ii) and (iv) is motivated by definition of almost periodicity used by T. Diagana in [**234**, Definition 3.29].

For the Stepanov classes, we will use the following notion (see also [**234**, Definition 2.2.4, Definition 2.2.5; Lemma 2.2.7]):

Definition 2.4.43. Let $p \in \mathcal{P}([0,1])$.

(i) A function $F : I \times Y \to X$ is called Stepanov $p(x)$-uniformly recurrent/Stepanov $p(x)$-uniformly recurrent on bounded sets (Stepanov $(p(x), \odot_g)$-almost periodic/Stepanov $(p(x), \odot_g)$-almost periodic on bounded sets) if and only if the function $\hat{F} : I \times Y \to L^{p(x)}([0,1] : X)$ is uniformly recurrent/uniformly recurrent on bounded sets ($\odot_g$-almost periodic/$\odot_g$-almost periodic on bounded sets).

(ii) We say that $F : I \times Y \to X$ is asymptotically Stepanov $p(x)$-uniformly recurrent/asymptotically Stepanov $p(x)$-uniformly recurrent on bounded sets (asymptotically Stepanov $(p(x), \odot_g)$-almost periodic/asymptotically Stepanov $(p(x), \odot_g)$-almost periodic on bounded sets) if and only if there exist two functions $G : \mathbb{R} \times Y \to X$ and $Q : I \times Y \to X$ satisfying that for each $y \in Y$ the functions $G(\cdot, y)$ and $Q(\cdot, y)$ are locally $p(x)$-integrable, as well as that the following holds:
   (a) $\hat{G} : \mathbb{R} \times Y \to L^{p(x)}([0,1] : X)$ is uniformly recurrent/uniformly recurrent on bounded sets ($\odot_g$-almost periodic/$\odot_g$-almost periodic on bounded sets),
   (b) $\hat{Q} \in C_0(I \times Y : L^{p(x)}([0,1] : X))$,
   (c) $F(t, y) = G(t, y) + Q(t, y)$ for all $t \in I$ and $y \in Y$.

If $p(x) \equiv p \in [1, \infty)$, then we also say that a function $F : I \times Y \to X$ is Stepanov p-uniformly recurrent/Stepanov p-uniformly recurrent on bounded sets etc.

The serious difficulty in our investigations presents the fact that for two given uniformly recurrent functions $f : I \to X$ and $g : I \to X$, the sequence (α_n) for which (19) holds need not have a subsequence (α_{n_k}) for which

$$\lim_{k\to\infty}\sup_{t\in\mathbb{R}}\bigl\|g(t+\alpha_{n_k})-g(t)\bigr\|=0;$$

moreover, for given two $\odot_g$-almost periodic functions $f : I \to X$ and $g : I \to X$, the set consisting of their joint ε-periods can be bounded (this cannot occur for almost periodic functions). Now we will slightly improve [**234**, Theorem 3.30] for uniformly recurrent functions and $\odot_g$-almost periodic functions:

THEOREM 2.4.44. *Suppose that $f : I \to Y$ is uniformly recurrent ($\odot_g$-almost periodic) and the range of $f(\cdot)$ is relatively compact, resp. bounded. If $F : I \times Y \to X$ is uniformly recurrent ($\odot_g$-almost periodic), resp. uniformly recurrent on bounded sets ($\odot_g$-almost periodic on bounded sets), and there exists a finite constant $L > 0$ such that*

$$\|F(t,x)-F(t,y)\| \leqslant L\|x-y\|_Y,\quad t\in I,\ x,\ y\in Y, \tag{60}$$

then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic), providing additionally the following condition: there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity for which (19) *holds and* (59) *holds with $K = \overline{\{f(t) : t \in I\}}$, resp. for each $\varepsilon > 0$ we have that $\odot_g(\vartheta(F;\varepsilon,\overline{\{f(t) : t \in I\}}) \cap \vartheta(f,\varepsilon)) > 0$.*

PROOF. The proof of theorem is very similar to the proof of [**234**, Theorem 3.30] and we will only outline the main details for $\odot_g$-almost periodic functions. Let $\varepsilon > 0$ be given, and let $\tau \in \vartheta(F;\varepsilon/2(1+L),\overline{\{f(t) : t \in I\}}) \cap \vartheta(f,\varepsilon/2(1+L))$. Then $\|f(t+\tau)-f(t)\| \leqslant \varepsilon/2(1+L)$, $t \in I$ and we have

$$\|\mathcal{F}(t+\tau)-\mathcal{F}(t)\| \leqslant L\|f(t+\tau)-f(t)\|_Y+\|F(t+\tau,f(t))-F(t+\tau,f(t))\|,\quad t\in I.$$

Hence,

$$\|\mathcal{F}(t+\tau)-\mathcal{F}(t)\| \leqslant [L\varepsilon/2(1+L)]+\varepsilon/2(1+L)<\varepsilon,\quad t\in I,$$

which completes the proof. ☐

Similarly we can prove the following slight extension of [**234**, Theorem 3.31]:

THEOREM 2.4.45. *Suppose that $f : I \to Y$ is a bounded uniformly recurrent function (bounded $\odot_g$-almost periodic function). If $F : I \times Y \to X$ is uniformly recurrent on bounded sets ($\odot_g$-almost periodic on bounded sets) and uniformly continuous on bounded sets, uniformly for $t \in I$, then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic), providing additionally the following condition: there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity for which* (19) *holds and* (59) *holds with $K = \overline{\{f(t) : t \in I\}}$, resp. for each $\varepsilon > 0$ we have that $\odot_g(\vartheta(F;\varepsilon,\overline{\{f(t) : t \in I\}}) \cap \vartheta(f,\varepsilon)) > 0$.*

Before proceeding further, it should be observed that the statement of [**234**, Theorem 3.32] (see also the proof of [**170**, Theorem 2.11]) can be formulated and slightly extended for uniformly recurrent ($\odot_g$-almost periodic) functions with relatively compact range.

Composition principles for asymptotically almost periodic functions have been analyzed in a great number of research papers. With regards to this question, we will state and give the main details of proof for the following slight extension of [**135**, Theorem 3.49], only (observe, however, that we can similarly reconsider and slightly extend the statements of [**135**, Theorem 3.50-Theorem 3.52]).

THEOREM 2.4.46. *Suppose that $h : I \to Y$ is uniformly recurrent ($\odot_g$-almost periodic), the range of $h(\cdot)$ is relatively compact, resp. bounded, $q \in C_0(I : X)$ and $f(t) = h(t) + q(t)$ for all $t \in I$. Suppose, further, $H : I \times Y \to X$ is uniformly recurrent ($\odot_g$-almost periodic), resp. uniformly recurrent on bounded sets ($\odot_g$-almost periodic on bounded sets), there exists a finite constant $L > 0$ such that* (60) *holds with the function $F(\cdot, \cdot)$ replaced therein with the function $H(\cdot, \cdot)$, and there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity for which* (19) *holds with the function $f(\cdot)$ replaced therein with the function $h(\cdot)$ and* (59) *holds with the function $f(\cdot)$ replaced therein with the function $h(\cdot)$ and set $K = \overline{\{h(t) : t \in I\}}$, resp. for each $\varepsilon > 0$ we have that $\odot_g(\vartheta(H; \varepsilon, \overline{\{h(t) : t \in I\}}) \cap \vartheta(h, \varepsilon)) > 0$. If $f(\cdot)$ has a relatively compact range, $Q \in C_0(I \times Y : X)$ and $F(t, y) = H(t, y) + Q(t, y)$ for all $t \in I$ and $y \in Y$, then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is asymptotically uniformly recurrent (asymptotically $\odot_g$-almost periodic).*

PROOF. Due to Theorem 2.4.44, we have that the mapping $t \mapsto H(t, h(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic). Furthermore, we have the decomposition

$$F(t, f(t)) = H(t, h(t)) + [H(t, f(t)) - H(t, h(t))] + Q(t, f(t)), \quad t \in I.$$

Since the function $H(\cdot, \cdot)$ satisfies (60), we have

$$\|H(t, f(t)) - H(t, h(t))\| \leqslant L\|f(t) - h(t)\|_Y \leqslant L\|q(t)\|_Y \to 0 \text{ as } |t| \to +\infty.$$

The proof of theorem completes the observation that $\lim_{|t|\to+\infty} \|Q(t, f(t))\| = 0$, which follows from definition of space $C_0(I \times Y : X)$ and our assumption that $f(\cdot)$ has a relatively compact range. □

REMARK 2.4.47. The assumption [**135**, (3.13)] is superfluous. Furthermore, we note that the assumption that the range of $h(\cdot)$ is relatively compact, resp. bounded, implies that $f(\cdot)$ is bounded; therefore, if we use the space $C_{0,b}(I \times Y : X)$ in place of $C_0(I \times Y : X)$ here, the assumption that $f(\cdot)$ has a relatively compact range is superfluous, as well.

REMARK 2.4.48. Consider, for simplicity, asymptotically uniformly recurrent functions. The principal part $\mathbf{f}(\cdot)$ of function $\mathcal{F}(t) = F(t, f(t))$, $t \in I$ satisfies (19) with the same sequence (α_n) and the function $\mathbf{f}(\cdot)$ in place of $f(\cdot)$. This holds for all remaining results established in this subsection, and this fact will be of some importance for applications made later on.

Concerning the composition principles for Stepanov almost periodic functions, the most influential paper written by now is the paper [**276**] by W. Long and H.-S. Ding. Repating almost verbatim the arguments given in the proof of [**276**, Lemma 2.1, Theorem 2.2] (see also [**142**, Theorem 2.4]), we can deduce the following result (we feel it is our duty to say that the previously proved results are more appropriate for applications in finite-dimensional spaces because condition on relative compactness of range of function $f(\cdot)$ is almost inevitable to be used; see condition (ii) below):

THEOREM 2.4.49. *Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $p \in \mathcal{P}([0, 1])$. Suppose that the following conditions hold:*

(i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent, resp. Stepanov $(p(x), \odot_g)$-almost periodic and there exist a function $r(\cdot) \geqslant \max(p(\cdot), (p(\cdot)/(p(\cdot) - 1))$ and a function $L_F \in L_S^{r(x)}(I)$ such that (25) holds true.*

(ii) *The function $f : I \to Y$ is Stepanov $p(x)$-uniformly recurrent, resp. Stepanov $(p(x), \odot_g)$-almost periodic, and there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{f(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact in Y.*

(iii) *For every compact set $K \subseteq Y$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\in I} \sup_{u\in K} \|F(t + s + \alpha_n, u) - F(t + s, u)\|_{L^{p(s)}[0,1]} = 0 \tag{61}$$

and (19) holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^{p(x)}([0,1]:X)}$ therein, resp. for every number $\varepsilon > 0$ and for every compact set $K \subseteq Y$, the set consisting of all positive real numbers $\tau > 0$ such that

$$\sup_{t\in I} \sup_{u\in K} \|F(t + s + \tau, u) - F(t + s, u)\|_{L^{p(s)}[0,1]} < \varepsilon \tag{62}$$

and (17) holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^{p(x)}([0,1]:X)}$ therein.

Set $q(x) := p(x)r(x)/(p(x) + r(x)) \in [1, p(x))$ provided $x \in [0, 1]$ and $r(x) < \infty$ and $q(x) := p(x)$ provided $r(x) = +\infty$. Then $q(x) := p(x)r(x)/(p(x) + r(x)) \in [1, p(x))$ provided $x \in [0, 1]$ and $r(x) < \infty$ and $F(\cdot, f(\cdot))$ is Stepanov $q(x)$-uniformly recurrent, resp. Stepanov $(q(x), \odot_g)$-almost periodic. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $q(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $q(x)$-bounded, as well.

In [**234**, Theorem 2.7.2], we have also considered the value $p = 1$ in Theorem 2.4.49 and the usual condition regarding the existence of a Lipschitz constant $L > 0$ such that (60) holds.

Using the foregoing arguments, we can simply deduce the following extension of the above-mentioned theorem:

THEOREM 2.4.50. *Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $p \in \mathcal{P}([0, 1])$. Suppose that the following conditions hold:*

(i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent, resp. Stepanov $(p(x), \odot_g)$-almost periodic, $L > 0$ and (60) holds.*

(ii) *The same as condition* (ii) *of* Theorem 2.4.49.

(iii) *The same as condition* (iii) *of* Theorem 2.4.49.

Then the function $F(\cdot, f(\cdot))$ is Stepanov $p(x)$-uniformly recurrent, resp. Stepanov $(p9x), \odot_g)$-almost periodic. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $p(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $p(x)$-bounded, as well.

Following the analysis of F. Bedouhene, Y. Ibaouene, O. Mellah and P. Raynaud de Fitte [**56**, Theorem 3] for the class of equi-Weyl p-almost periodic functions and the analysis of W. Long and H.-S. Ding [**276**], in [**242**, Theorem 2.1] we have established a new composition principle for the class of Stepanov p-almost periodic functions that is not comparable with [**276**, Theorem 2.2]. Using the proof of the last-mentioned theorem and the proof of [**242**, Theorem 2.1], we can deduce the following generalization of Theorem 2.4.50:

THEOREM 2.4.51. *Suppose that $p(x),\ q(x) \in [1, \infty),\ r(x) \in [1, \infty],\ 1/p(x) = 1/q(x) + 1/r(x)$ and the following conditions hold:*

- (i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent, resp. Stepanov $(p(x), \odot_g)$-almost periodic, and there exists a function $L_F \in L_S^{r(x)}(I)$ such that* (25) *holds.*
- (ii) *The same as condition* (ii) *of* Theorem 2.4.49, *with the function p replaced with the function q therein.*
- (iii) *For every compact set $K \subseteq Y$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that* (61) *holds and* (19) *holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^{q(x)}([0,1]:X)}$ therein, resp. for every number $\varepsilon > 0$ and for every compact set $K \subseteq Y$, the set consisting of all positive real numbers $\tau > 0$ such that* (62) *holds and* (17) *holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^{q(x)}([0,1]:X)}$ therein.*

Then the function $F(\cdot, f(\cdot))$ is Stepanov $p(x)$-uniformly recurrent, resp. Stepanov $(p(x), \odot_g)$-almost periodic. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $p(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $p(x)$-bounded, as well.

It is also straightforward to reformulate the statements of [**234**, Proposition 2.7.3-Proposition 2.7.4], resp. [**242**, Proposition 2.1], for the asymptotical Stepanov $p(x)$-uniform recurrence and the asymptotical Stepanov $(p(x), \odot_g)$-almost periodicity. Details can be left to the interested readers.

Now we will present two interesting applications of established theoretical results in the analysis of the existence and uniqueness of uniformly recurrent type solutions of the abstract semilinear fractional integro-differential inclusions.

1. In the first application, we will consider the finite-dimensional space $X := \mathbb{C}^n$, where $n \geqslant 2$. Suppose that $c > 0,\ A,\ B \in \mathbb{C}^{n,n}$ (the space of all complex matrices of format $n \times n$), the matrix B is not invertible, as well as that the degree of complex polynomial $P(\lambda) := \det(\lambda B - A),\ \lambda \in \mathbb{C}$ is equal to n and its roots lie in the region $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < -c(|\operatorname{Im} \lambda| + 1)\}$. Due to [**236**, Proposition 2.1.2], we have that the region Ψ from the formulation of condition (P) belongs to the

resolvent set of multivalued linear operator $\mathcal{A} = AB^{-1}$ as well as that

$$\big(\lambda - AB^{-1}\big)^{-1} = B\big(\lambda B - A\big)^{-1}, \quad \lambda \in \Psi.$$

Since the degree of complex polynomial $P(\cdot)$ is equal to n, the above formula simply implies that there exists a positive real constant $M > 0$ such that condition (P) holds with $\beta = 1$, so that the operator $\mathcal{A}$ generates an exponentially decaying strongly continuous degenerate semigroup $(T(t))_{t\geqslant 0}$ which can be analytically extented to a sector around positive real axis (cf. [**236**] for more details).

Suppose now that $0 < \gamma < 1$ and $\nu > -1$. Define

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu}\int_0^{\infty} s^{\nu}\Phi_{\gamma}(s)T\big(st^{\gamma}\big)x\,ds, \quad t > 0,\ x \in X, \tag{63}$$

$$S_{\gamma}(t) := T_{\gamma,0}(t) \text{ and } P_{\gamma}(t) := \gamma T_{\gamma,1}(t)/t^{\gamma}, \quad t > 0;$$

see also E. Bazhlekova [**52**] and R.-N. Wang, D.-H. Chen, T.-J. Xiao [**348**]. Recall [**236**] that, in general case $\beta \in (0,1]$, there exists a finite constant $M_1 > 0$ such that

$$\big\|S_{\gamma}(t)\big\| + \big\|P_{\gamma}(t)\big\| \leqslant M_1 t^{\gamma(\beta-1)}, \quad t > 0, \tag{64}$$

as well as

$$\big\|S_{\gamma}(t)\big\| \leqslant M_1 t^{-\gamma},\ t \geqslant 1 \quad \text{and} \quad \big\|P_{\gamma}(t)\big\| \leqslant M_2 t^{-2\gamma},\ t \geqslant 1. \tag{65}$$

Set $R_{\gamma}(t) := t^{\gamma-1}P_{\gamma}(t)$, $t > 0$. Then (64)-(65) yield

$$\|R_{\gamma}(t)\| = O\big(t^{\gamma-1} + t^{-\gamma-1}\big), \quad t > 0. \tag{66}$$

Consider now the following abstract fractional inclusion

$$D_+^{\gamma}\vec{u}(t) \in -\mathcal{A}\vec{u}(t) + F(t,\vec{u}(t)), \quad t \in \mathbb{R}, \tag{67}$$

where $D_+^{\gamma}u(t)$ denotes the Weyl-Liouville fractional derivative of order γ and $F : \mathbb{R} \times X \to X$; after the usual substitution $\vec{v}(t) \in B^{-1}\vec{u}(t)$, $t \in \mathbb{R}$, this inclusion becomes

$$D_+^{\gamma}\big[B\vec{v}(t)\big] = -A\vec{v}(t) + F\big(t, B\vec{v}(t)\big), \quad t \in \mathbb{R}.$$

Following J. Mu, Y. Zhoa and L. Peng [**297**], it will be said that a continuous function $u : \mathbb{R} \to X$ is a mild solution of (67) if and only if

$$\vec{u}(t) = \int_{-\infty}^{t} R_{\gamma}(t-s)F\big(s,\vec{u}(s)\big)\,ds, \quad t \in \mathbb{R}.$$

For the sequel, fix a strictly increasing sequence (α_n) of positive reals tending to plus infinity. Denote

$$BUR_{(\alpha_n)}(\mathbb{R} : X) := \big\{\vec{u} \in UR(\mathbb{R} : X)\,;\ \vec{u}(\cdot) \text{ is bounded and (19) holds with } f = \vec{u}\big\}.$$

Equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_{\infty}$, $BUR_{(\alpha_n)}(\mathbb{R} : X)$ becomes a complete metric space.

Now we are able to state the following result:

THEOREM 2.4.52. *Suppose that the function $F : \mathbb{R} \times X \to X$ satisfies that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\in\mathbb{R}} \sup_{y\in B} \|F(t, y)\| \leqslant M_B$. Suppose, further, that the function $F : \mathbb{R} \times X \to X$ is Stepanov p-uniformly recurrent with $p > 1$, and there exist a number $r \geqslant \max(p, p/(p-1))$ and a function $L_F \in L^r_S(I)$ such that $q := pr/(p + r) > 1$ and (25) holds with $I = \mathbb{R}$. If*

$$\frac{(\gamma-1)q}{q-1} > -1, \tag{68}$$

there exists an integer $n \in \mathbb{N}$ such that $M_n < 1$, where

$$M_n := \sup_{t\geqslant 0} \int_{-\infty}^{t} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \Big\| R_\gamma(t - x_n) \Big\| \times \prod_{i=2}^{n} \Big\| R_\gamma(x_i - x_{i-1}) \Big\| \prod_{i=1}^{n} L_F(x_i)\, dx_1\, dx_2 \cdots dx_n,$$

and for every compact set $K \subseteq Y$, (61) holds, then the abstract fractional Cauchy inclusion (67) has a unique bounded uniformly recurrent solution.

PROOF. Define $\Upsilon : BUR_{(\alpha_n)}(\mathbb{R} : X) \to BUR_{(\alpha_n)}(\mathbb{R} : X)$ by

$$(\Upsilon\vec{u})(t) := \int_{-\infty}^{t} R_\gamma(t - s)F(s, \vec{u}(s))\, ds, \quad t \in \mathbb{R}.$$

Let us firstly show that the mapping $\Upsilon(\cdot)$ is well defined. Suppose that $\vec{u} \in BUR_{(\alpha_n)}(\mathbb{R} : X)$. Then $R(\vec{u}) = B$ is a bounded set so that the mapping $t \mapsto F(t, \vec{u}(t))$, $t \in \mathbb{R}$ is bounded due to the prescribed assumption. Applying Theorem 2.4.49, we have that the function $F(\cdot, \vec{u}(\cdot))$ is Stepanov q-uniformly recurrent. Define $q' := q/(q-1)$. Then (66) and (68) together imply that $\|R_\gamma(\cdot)\| \in L^{q'}[0, 1]$ and $\sum_{k=0}^{\infty} \|R_\gamma(\cdot)\|_{L^{q'}[k,k+1]} < \infty$ due to our analysis from [**234**, Remark 2.6.12]. Applying Proposition 2.4.39, we get that the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t-s)F(s, \vec{u}(s))\, ds$, $t \in \mathbb{R}$ is bounded, continuous and uniformly recurrent, which yields that $\Upsilon\vec{u} \in BUR_{(\alpha_n)}(\mathbb{R} : X)$, as claimed. Furthermore, a simple calculation shows that

$$\Big\| \big(\Upsilon^n \vec{u_1}\big) - \big(\Upsilon^n \vec{u_2}\big) \Big\|_\infty \leqslant M_n \big\| \vec{u_1} - \vec{u_2} \big\|_\infty, \quad \vec{u_1},\ \vec{u_2} \in BUR_{(\alpha_n)}(\mathbb{R} : X),\ n \in \mathbb{N}.$$

Since we have assumed that there exists an integer $n \in \mathbb{N}$ such that $M_n < 1$, the well known extension of the Banach contraction principle shows that the mapping $\Upsilon(\cdot)$ has a unique fixed point, finishing the proof of the theorem. ☐

2. Suppose that a closed multivalued linear operator $\mathcal{A}$ satisfies condition (P) in X, which can be finite-dimensional or infinite-dimensional, with general exponent $\beta \in (0, 1]$. Consider the abstract semilinear fractional differential inclusion

$$(\mathrm{DFP})_{f,\gamma,s} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + F(t, u(t)),\ t > 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^\gamma$ denotes the Caputo fractional derivative of order γ, $x_0 \in X$ and $F : [0,\infty) \times X \to X$. By a mild solution of $(\mathrm{DFP})_{f,\gamma,s}$, we mean any function $u \in C([0,\infty) : X)$ satisfying that

$$u(t) = S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geqslant 0.$$

In what follows, we will assume that $\lim_{t\to 0+} S_\gamma(t)x_0 = x_0$ so that the mapping $t \mapsto S_\gamma(t)x_0$, $t \geqslant 0$ belongs to the space $C_0([0,\infty) : X)$; see the estimate (64). Arguing as in the proof of [**135**, Theorem 3.46], we may conclude that $\mathcal{X} := BUR_{(\alpha_n)}([0,\infty) : X) \oplus C_0([0,\infty) : X)$ is a complete metric space equipped with the distance $d(\cdot,\cdot)$ used above. Set, for every $u \in \mathcal{X}$ and $n \in \mathbb{N}$,

$$\big(\Upsilon_A u\big)(t) := S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geqslant 0;$$

$$A_n := \sup_{t\geqslant 0} \int_0^t \int_0^{x_n} \cdots \int_0^{x_2} \big\|R_\gamma(t-x_n)\big\| \times \prod_{i=2}^n \big\|R_\gamma(x_i - x_{i-1})\big\| \prod_{i=1}^n L_F(x_i)\,dx_1\,dx_2 \cdots dx_n.$$

Then a simple calculation shows that

$$\Big\|\big(\Upsilon_A^n u\big) - \big(\Upsilon_A^n v\big)\Big\|_\infty \leqslant A_n \big\|u - v\big\|_\infty, \quad u,\ v \in \mathcal{X},\ n \in \mathbb{N}.$$

Keeping in mind [**248**, Proposition 3.1], Theorem 2.4.46, Remark 2.4.47-Remark 2.4.48 and the proof of [**234**, Lemma 2.6.3], we can similarly clarify the following result:

THEOREM 2.4.53. *Suppose that the function $F : [0,\infty) \times X \to X$ is continuous and satisfies that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\geqslant 0}\sup_{y\in B}\|F(t,y)\| \leqslant M_B$. Suppose, further, that $H : [0,\infty) \times X \to X$ is uniformly recurrent on bounded sets, there exists a finite constant $L > 0$ such that (60) holds with the function $F(\cdot,\cdot)$ replaced therein with the function $H(\cdot,\cdot)$ and $I = [0,\infty)$. Let (59) hold with any bounded set $B = K$, and let there exist an integer $n \in \mathbb{N}$ such that $A_n < 1$. If $Q \in C_{0,b}(I \times Y : X)$ and $F(t,y) = H(t,y) + Q(t,y)$ for all $t \geqslant 0$ and $y \in Y$, then the abstract fractional Cauchy inclusion (DFP)$_{f,\gamma,s}$ has a unique mild solution.*

Let Ω be a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $X := L^p(\Omega)$. Suppose that the operator $A := \Delta - b$ acts on X with the Dirichlet boundary conditions, and that B is the multiplication operator by the function $m(x)$. As explained in [**234**], we can apply Theorem 2.4.53 with $\mathcal{A} = AB^{-1}$ in the study of existence and uniqueness of asymptotically uniformly recurrent solutions of the semilinear fractional Poisson heat equation

$$\left\{\begin{array}{l} \mathbf{D}_t^\gamma[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t, m(x)v(t,x)), \quad t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega. \end{array}\right.$$

2.5. Generalized almost periodicity in Lebesgue spaces with variable exponents. Part I

The main purpose of this section is to investigate generalized asymptotically almost periodic functions in Lebesgue spaces with variable exponents. Suppose that $f : [a, b] \to \mathbb{R}$ is a non-negative Lebesgue-integrable function, where $a,\ b \in \mathbb{R}$, $a < b$, and $\phi : [0, \infty) \to \mathbb{R}$ is a convex function. Let us recall that the Jensen integral inequality states that

$$\phi\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \leqslant \frac{1}{b-a}\int_a^b \phi(f(x))\,dx.$$

Using this integral inequality, we can simply prove that, for every two sequences (a_k) and (x_k) of non-negative real numbers such that $\sum_{k=0}^{\infty} a_k = 1$, we have

$$\phi\left(\sum_{k=0}^{\infty} a_k x_k\right) \leqslant \sum_{k=0}^{\infty} a_k \phi(x_k). \tag{69}$$

If $\phi : [0, \infty) \to \mathbb{R}$ is a concave function, then the above inequalities reverse.

2.5.1. Almost periodic and asymptotically almost periodic type solutions with variable exponents $L^{p(x)}$. Before proceeding further, we need to recall the recently introduced notions of $S^{p(x)}$-boundedness and (asymptotical) Stepanov $p(x)$-almost periodicity:

Definition 2.5.1. ([**142**]) Let $p \in \mathcal{P}([0, 1])$ and let $I = \mathbb{R}$ or $I = [0, \infty)$. A function $f \in M(I : X)$ is said to be Stepanov $p(x)$-bounded (or $S^{p(x)}$-bounded) if and only if $f(\cdot + t) \in L^{p(x)}([0, 1] : X)$ for all $t \in I$, and the sup norm of Bochner transform satisfies $\sup_{t\in I} \|f(\cdot + t)\|_{p(x)} < \infty$; more precisely,

$$\|f\|_{S^{p(x)}} := \sup_{t\in I} \inf\left\{\lambda > 0 : \int_0^1 \varphi_{p(x)}\left(\frac{\|f(x+t)\|}{\lambda}\right) dx \leqslant 1\right\} < \infty.$$

The collection of such functions will be denoted by $L_S^{p(x)}(I : X)$.

From Definition 2.5.1 it follows that the space $L_S^{p(x)}(I : X)$ is translation invariant in the sense that, for every $f \in L_S^{p(x)}(I : X)$ and $\tau \in I$, we have $f(\cdot + \tau) \in L_S^{p(x)}(I : X)$. This is not the case with the notion introduced by T. Diagana and M. Zitane in [**145**]-[**146**]. In the second part of the following definition, we extend the notion of asymptotical Stepanov $p(x)$-almost periodicity introduced in case $I = [0, \infty)$ to the general case of interval I (see also [**142**, Proposition 4.12]):

Definition 2.5.2. ([**142**])

(i) Let $p \in \mathcal{P}([0, 1])$ and let $I = \mathbb{R}$ or $I = [0, \infty)$. A function $f \in L_S^{p(x)}(I : X)$ is said to be Stepanov $p(x)$-almost periodic (or Stepanov $p(x)$-a.p.) if and only if the function $\hat{f} : I \to L^{p(x)}([0, 1] : X)$ is almost periodic. The collection of such functions will be denoted by $APS^{p(x)}(I : X)$.

(ii) Let $p \in \mathcal{P}([0,1])$. Then a function $f \in L_S^{p(x)}(I:X)$ is said to be asymptotically Stepanov $p(x)$-almost periodic (or asymptotically Stepanov $p(x)$-a.p.) if and only if the function if and only if there exist two locally p-integrable functions $g:\mathbb{R} \rightarrow X$ and $q: I \rightarrow X$ satisfying the following conditions:
   (i) g is $S^{p(x)}$-almost periodic,
   (ii) $\hat{q}$ belongs to the class $C_0(I : L^{p(x)}([0,1] : X))$,
   (iii) $f(t) = g(t) + q(t)$ for all $t \in I$.
   The collection of such functions will be denoted by $AAPS^{p(x)}(I : X)$.

As in the case of Stepanov $p(x)$-boundedness, the space $APS^{p(x)}(I : X)$ is translation invariant in the sense that, for every $f \in APS^{p(x)}(I : X)$ and $\tau \in I$, we have $f(\cdot + \tau) \in APS^{p(x)}(I : X)$. A similar statement holds for the space $AAPS^{p(x)}([0,\infty) : X)$.

We will extend [**145**, Definition 3.10] in the following way (in this paper, the authors have considered the case $I = \mathbb{R}$ and $p \in C_+(\mathbb{R})$; we can extend the notion introduced in [**145**, Definition 3.11] in the same way):

DEFINITION 2.5.3. Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $p \in \mathcal{P}(I)$. Then it is said that a measurable function $f : I \rightarrow X$ belongs to the space $BS^{p(x)}(I : X)$ if and only if

$$\|f\|_{\mathbf{S}^{p(x)}} := \sup_{t\in I} \inf \left\{ \lambda > 0 : \int_t^{t+1} \varphi_{p(x)}(\|f(x)\|/\lambda)\, dx \leqslant 1 \right\} < \infty.$$

Equipped with the norm $\|\cdot\|_{S^{p(x)}}$, the space $L_S^{p(x)}(I : X)$ consisting of all S^p-bounded functions is a Banach space, which is continuously embedded in $L_S^1(I : X)$, for any $p \in \mathcal{P}([0,1])$. Furthermore, it can be easily shown that $APS^{p(x)}(I : X)$ ($AAPS^{p(x)}(I : X)$ with $I = [0,\infty)$) is a closed subspace of $L_S^{p(x)}(I : X)$ and therefore is a Banach space itself, for any $p \in \mathcal{P}([0,1])$.

If $p \in \mathcal{P}([0,1])$, then Lemma 1.1.6(ii) yields $L^{p(x)}([0,1] : X) \hookrightarrow L^1([0,1] : X)$, where the symbol $\hookrightarrow$ stands for a "continuous embedding", so that $L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$, as well.

We have the following:

PROPOSITION 2.5.4. *Suppose* $p \in \mathcal{P}([0,1])$. *Then the following continuous embeddings hold:*

(i) $L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$, *as well as*
(ii) $APS^{p(x)}(I : X) \hookrightarrow APS^1(I : X)$ *and*
$AAPS^{p(x)}([0,\infty) : X) \hookrightarrow AAPS^1([0,\infty) : X)$.

Similarly, the following holds:

PROPOSITION 2.5.5. *Suppose* $p \in D_+([0,1])$ *and* $1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty$ *for a.e.* $x \in [0,1]$. *Then the following continuous embeddings hold:*

(i) $L_S^{p^+}(I : X) \hookrightarrow L_S^{p(x)}(I : X) \hookrightarrow L_S^{p^-}(I : X)$, *as well as*

(ii) $APS^{p^+}(I:X) \hookrightarrow APS^{p(x)}(I:X) \hookrightarrow APS^{p^-}(I:X)$ *and* $AAPS^{p^+}([0,\infty):X) \hookrightarrow AAPS^{p(x)}([0,\infty):X) \hookrightarrow AAPS^{p^-}([0,\infty):X)$.

Now we will prove that any almost periodic function is $S^{p(x)}$-almost periodic, for any $p \in \mathcal{P}([0,1])$.

PROPOSITION 2.5.6. *Let* $p \in \mathcal{P}([0,1])$, *and let* $f : I \to X$ *be almost periodic. Then* $f(\cdot)$ *is* $S^{p(x)}$*-almost periodic.*

PROOF. To prove that $f(\cdot)$ is $S^{p(x)}$-bounded and $\|f\|_{L_S^{p(x)}} \leqslant \|f\|_\infty$, it suffices to show that, for every $t \in \mathbb{R}$, we have:

$$[\|f\|_\infty, \infty) \subseteq \left\{ \lambda > 0 : \int_0^1 \varphi_{p(x)}\left(\frac{\|f(x+t)\|}{\lambda} \right) dx \leqslant 1 \right\}. \tag{70}$$

For $\lambda \geqslant \|f\|_\infty$, we have $\|f(x+t)\|/\lambda \leqslant 1$, $t \in I$. It can be simply perceived that, in this case,

$$\varphi_{p(x)}\left(\frac{\|f(x+t)\|}{\lambda} \right) \leqslant 1, \quad t \in I,$$

so that the integrand does not exceed 1; as a matter of fact, by definition of $\varphi_{p(x)}(\cdot)$, we only need to observe that, for every $x \in [0,1]$ with $p(x) < \infty$, we have $(\|f(t+x)\|/\lambda)^{p(x)} \leqslant 1^{p(x)} = 1$, $t \in I$. Hence, (70) holds. Using the uniform continuity of $f(\cdot)$ and a similar argumentation, we can show that the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is uniform continuous. For direct proof of almost periodicity of function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, we can argue as follows. For $\varepsilon > 0$ given as above, there is a finite number $l > 0$ such that any subinterval I' of I of length l contains a number $\tau \in I'$ such that $\|f(t+\tau) - f(t)\| \leqslant \varepsilon$, $t \in I$. It suffices to observe that, for this $\varepsilon > 0$, we can choose the same length $l > 0$ and the same ε-almost period τ from I' ensuring the validity of inequality $\|\hat{f}(t+\tau+\cdot) - \hat{f}(t+\cdot)\|_{L^{p(x)}([0,1]:X)} \leqslant \varepsilon$, $t \in I$: in order to see that the last inequality holds true, we only need to prove that, for every $t \in I$, we have

$$[\varepsilon, \infty) \subseteq \left\{ \lambda > 0 : \int_0^1 \varphi_{p(x)}\left(\frac{\|f(t+\tau+x) - f(t+x)\|}{\lambda} \right) dx \leqslant 1 \right\}.$$

Indeed, if $\lambda \geqslant \varepsilon$, then $\|f(t+\tau+x) - f(t+x)\|/\lambda \leqslant 1$, $t \in I$ and the integrand cannot exceed 1 : this simply follows from definition of $\varphi_{p(x)}(\cdot)$ and observation that, for every $x \in [0,1]$ with $p(x) < \infty$, we have $(\|f(t+\tau+x) - f(t+x)\|/\lambda)^{p(x)} \leqslant 1^{p(x)} = 1$, $t \in I$. The proof of the proposition is thereby complete. $\square$

We can similarly prove the following proposition:

PROPOSITION 2.5.7. *Let* $p \in \mathcal{P}([0,1])$, *and let* $f : I \to X$ *be asymptotically almost periodic. Then* $f(\cdot)$ *is asymptotically* $S^{p(x)}$*-almost periodic.*

Taking into account Proposition 2.6.19(ii) and the method employed in the proof of Proposition 2.5.6, we can state the following:

PROPOSITION 2.5.8. *Assume that* $p \in \mathcal{P}([0,1])$ *and* $f \in L_S^{p(x)}(I : X)$. *Then the following holds:*

(i) $L^\infty(I : X) \hookrightarrow L_S^{p(x)}(I : X) \hookrightarrow L_S^1(I : X)$.
(ii) $AP(I : X) \hookrightarrow APS^{p(x)}(I : X) \hookrightarrow APS^1(I : X)$ *and* $AAP(I : X) \hookrightarrow AAPS^{p(x)}(I : X) \hookrightarrow AAPS^1(I : X)$.
(iii) *The continuity (uniform continuity) of* $f(\cdot)$ *implies continuity (uniform continuity) of* $\hat{f}(\cdot)$.

In general case, we have the following:

PROPOSITION 2.5.9. *Assume that* $p,\ q \in \mathcal{P}([0,1])$ *and* $p \leqslant q$ *a.e. on* $[0,1]$. *Then we have:*

(i) $L_S^{q(x)}(I : X) \hookrightarrow L_S^{p(x)}(I : X)$.
(ii) $APS^{q(x)}(I : X) \hookrightarrow APS^{p(x)}(I : X)$ *and* $AAPS^{q(x)}(I : X) \hookrightarrow AAPS^{p(x)}(I : X)$.
(iii) *If* $p \in D_+([0,1])$, *then*

$$L^\infty(I : X) \cap APS^{p(x)}(I : X) = L^\infty(I : X) \cap APS^1(I : X)$$

and

$$L^\infty(I : X) \cap AAPS^{p(x)}(I : X) = L^\infty(I : X) \cap AAPS^1(I : X).$$

PROOF. We will prove only (iii) for almost periodicity. Keeping in mind Proposition 2.5.5(ii), it suffices to assume that $p(x) \equiv p > 1$. Then, clearly, $L^\infty(I : X) \cap APS^p(I : X) \subseteq L^\infty(I : X) \cap APS^1(I : X)$ and it remains to be proved the opposite inclusion. So, let $f \in L^\infty(I : X) \cap APS^1(I : X)$. The required conclusion follows from the elementary definitions and the following simple calculation, which is valid for any $t,\ \tau \in \mathbb{R}$:

$$\begin{aligned} &\left[\int_t^{t+1} \|f(\tau+s) - f(s)\|^p\, ds\right]^{1/p} \\ &\leqslant \left[\int_t^{t+1} \bigl(2\|f\|_\infty\bigr)^{p-1} \|f(\tau+s) - f(s)\|\, ds\right]^{1/p} \\ &= \bigl(2\|f\|_\infty\bigr)^{(p-1)/p} \left[\int_t^{t+1} \|f(\tau+s) - f(s)\|\, ds\right]^{1/p}. \end{aligned}$$

□

REMARK 2.5.10. Recall that $APS^{p(x)}(I : X)$ can be strictly contained in $APS^1(I : X)$, even in the case that $p(x) \equiv p > 1$ is a constant function. The already employed example of H. Bohr and E. Følner shows that $AAPS^p(I : X)$ can be strictly contained in $AAPS^1(I : X)$ for $p > 1$ (see e.g. [**205**, Lemma 1]).

REMARK 2.5.11. Proposition 2.5.6 and Proposition 2.5.7 can be simply deduced by using Proposition 2.5.9(ii) and the equalities $AP(I : X) = APS^\infty(I : X) \cap C(I :$

$X)$, $AAP(I:X) = AAPS^{\infty}(I:X) \cap C([0,\infty):X)$, which can be proved almost trivially.

Now we would like to present the following illustrative example:

EXAMPLE 2.5.12. Define $\operatorname{sign}(0) := 0$. Then, for every trigonometric polynomial $f : \mathbb{R} \to \mathbb{R}$, we have that the function $F(\cdot) :=\operatorname{sign}(f(\cdot))$ is Stepanov 1-almost periodic. Since $F \in L^{\infty}(\mathbb{R})$, Proposition 2.5.9(iii) yields that the function $F(\cdot)$ is Stepanov p-almost periodic for any $p \geqslant 1$, while Proposition 2.5.8(i) yields that the function $F(\cdot)$ is Stepanov $p(x)$-bounded for any $p \in \mathcal{P}([0,1])$. Due to Proposition 2.5.5(ii), we have $F \in APS^{p(x)}(\mathbb{R} : \mathbb{C})$ for any $p \in D_{+}([0,1])$.

Consider now the case that $f(x) := \sin x + \sin \sqrt{2}x$, $x \in \mathbb{R}$ and $p(x) := 1 - \ln x$, $x \in [0,1]$. We will prove that $F \notin APS^{p(x)}(\mathbb{R} : \mathbb{C})$. Speaking-matter-of-factly, it is sufficient to show that, for every $\lambda \in (0, 2/e)$ and for every $l > 0$, we can find an interval $I \subseteq \mathbb{R}$ of length $l > 0$ such that, for every $\tau \in I$, there exists $t \in \mathbb{R}$ such that

$$\begin{aligned}\int_0^1 \Big(\frac{1}{\lambda}\Big)^{1-\ln x} \Big| &\operatorname{sign}\big[\sin(x+t+\tau) + \sin\sqrt{2}(x+t+\tau)\big] \\ &- \operatorname{sign}\big[\sin(x+t) + \sin\sqrt{2}(x+t)\big]\Big|^{1-\ln x}\, dx = \infty. \end{aligned} \tag{71}$$

Let $\lambda \in (0, 2/e)$ and $l > 0$ be given. Take arbitrarily any interval $I \subseteq \mathbb{R} \smallsetminus \{0\}$ of length l and after that take arbitrarily any number $\tau \in I$. Since $(1/\lambda)^{1-\ln x} \geqslant 1/x$, $x \in [0,1]$ and $1 - \ln x \geqslant 1$, $x \in [0,1]$, a continuity argument shows that it is enough to prove the existence of a number $t \in \mathbb{R}$ such that

$$\big[\sin(t+\tau) + \sin\sqrt{2}(t+\tau)\big] \cdot \big[\sin t + \sin\sqrt{2}t\big] < 0. \tag{72}$$

If $\sin\tau + \sin\sqrt{2}\tau > 0$ ($\sin\tau + \sin\sqrt{2}\tau < 0$), then we can take $t \sim 0-$ ($t \sim 0+$). Hence, we assume henceforward $\sin\tau + \sin\sqrt{2}\tau = 0$ and $\tau \neq 0$. There exist two possibilities:

$$\tau \in \frac{2\mathbb{Z}\pi}{1+\sqrt{2}} \smallsetminus \{0\} \quad \text{or} \quad \tau \in \frac{(2\mathbb{Z}+1)\pi}{\sqrt{2}-1}.$$

In the first case, take $t_0 = \frac{\pi}{\sqrt{2}-1}$. Then an elementary argumentation shows that $\tau + t_0 \notin \frac{2\mathbb{Z}\pi}{1+\sqrt{2}} \cup \frac{(2\mathbb{Z}+1)\pi}{\sqrt{2}-1}$ so that $\sin(t_0+\tau) + \sin\sqrt{2}(t_0+\tau) \neq 0$. If $\sin(t_0+\tau) + \sin\sqrt{2}(t_0+\tau) > 0$ ($\sin(t_0+\tau) + \sin\sqrt{2}(t_0+\tau) < 0$), then for t satisfying (72) we can take any number belonging to a small left/right interval around t_0 for which $\sin t + \sin\sqrt{2}t < 0$ ($\sin t + \sin\sqrt{2}t > 0$). In the second case, there exists an integer $m \in \mathbb{Z}$ such that $\tau = \frac{(2m+1)\pi}{\sqrt{2}-1}$ and we can take $t_0 = \frac{(-2m+1)\pi}{\sqrt{2}-1}$. Then $\tau + t_0 = \frac{2\pi}{\sqrt{2}-1}$ and $\sin(t_0+\tau) + \sin\sqrt{2}(t_0+\tau) \neq 0$, so that we can use a trick similar to that used in the first case. Let us only mention in passing that, with the notion introduced in [**143**], the function $F(\cdot)$ cannot be $S^{p(x)}$-almost automorphic, as well.

The situation is quite different if we consider the case that $f(x) := \sin x$, $x \in \mathbb{R}$. Then $F(\cdot)$ is Stepanov $p(x)$-almost periodic for any $p \in \mathcal{P}([0,1])$. Speaking-matter-of-factly, it can be easily shown that the mapping $\hat{F} : \mathbb{R} \to L^{p(x)}[0,1]$ is continuous and $\|F(t+\tau+\cdot)-F(t+\cdot)\|_{L^{p(x)}[0,1]}=0$ for all $t\in\mathbb{R}$ and $\tau\in 2\pi\mathbb{Z}$. This, in turn, implies the claimed statement.

2.5.2. Generalized two-parameter almost periodic type functions and composition principles. Assume that $I=\mathbb{R}$ or $I=[0,\infty)$. The notion of (asymptotical) Stepanov $p(x)$-almost periodicity for the functions depending on two parameters is introduced as follows:

DEFINITION 2.5.13. Let $p\in\mathcal{P}([0,1])$.

(i) A function $f:I\times Y\to X$ is called Stepanov $p(x)$-almost periodic, $S^{p(x)}$-almost periodic for short, if and only if $\hat{f}:I\times Y\to L^{p(x)}([0,1]:X)$ is almost periodic. The vector space consisting of all such functions will be denoted by $APS^{p(x)}(I\times Y:X)$.

(ii) A function $f:I\times Y\to X$ is said to be asymptotically $S^{p(x)}$-almost periodic if and only if it admits a decomposition $f(t,y)=g(t,y)+q(t,y)$, $t\in I$, where $g\in APS^{p(x)}(\mathbb{R}\times Y:X)$ and $q\in C_0(I\times Y:X)$. The vector space consisting of all such functions will be denoted by $AAPS^{p(x)}(I\times Y:X)$.

The proof of following proposition is very similar to the proof of [**234**, Lemma 2.2.7] and therefore omitted (for simplicity, we wil not consider case $I=\mathbb{R}$ here).

PROPOSITION 2.5.14. *Let $p\in\mathcal{P}([0,1])$. Suppose that $\hat{f}:[0,\infty)\times Y\to L^{p(x)}([0,1]:X)$ is an asymptotically almost periodic function. Then there are two functions $g:\mathbb{R}\times Y\to X$ and $q:[0,\infty)\times Y\to X$ satisfying that for each $y\in Y$ the functions $g(\cdot,y)$ and $q(\cdot,y)$ are Stepanov $p(x)$-bounded, as well as that the following holds:*

(i) *$\hat{g}:\mathbb{R}\times Y\to L^{p(x)}([0,1]:X)$ is almost periodic,*

(ii) *$\hat{q}\in C_0([0,\infty)\times Y:L^{p(x)}([0,1]:X))$,*

(iii) *$f(t,y)=g(t,y)+q(t,y)$ for all $t\geqslant 0$ and $y\in Y$.*

Moreover, for every compact set $K\subseteq Y$, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals such that $\lim_{n\to\infty}t_n=\infty$ and $g(t,y)=\lim_{n\to\infty}f(t+t_n,y)$ for all $y\in Y$ and a.e. $t\geqslant 0$.

In [**234**, Theorem 2.7.1, Theorem 2.7.2], we have slightly improved the important composition principle attributed to W. Long, S.-H. Ding [**276**, Theorem 2.2]. Further refinements for $S^{p(x)}$-almost periodicity can be deduced similarly, with appealing to Lemma 1.1.6(i)-(iii) and the arguments employed in the proof of [**276**, Theorem 2.2]:

THEOREM 2.5.15. *Let $I=\mathbb{R}$ or $I=[0,\infty)$, and let $p\in\mathcal{P}([0,1])$. Suppose that the following conditions hold:*

(i) *$F\in APS^{p(x)}(I\times Y:X)$ and there exist a function $r\in\mathcal{P}([0,1])$ such that $r(\cdot)\geqslant\max(p(\cdot),p(\cdot)/(p(\cdot)-1))$ and a function $L_F\in L_S^{r(x)}(I)$ such that (25) holds;*

(ii) *$u\in APS^{p(x)}(I:Y)$, and there exists a set $\mathrm{E}\subseteq I$ with $m(\mathrm{E})=0$ such that $K:=\{u(t):t\in I\setminus\mathrm{E}\}$ is relatively compact in Y; here, $m(\cdot)$ denotes the Lebesgue measure.*

Define $q \in \mathcal{P}([0,1])$ *by* $q(x) := p(x)r(x)/(p(x)+r(x))$, *if* $x \in [0,1]$ *and* $r(x) < \infty$, $q(x) := p(x)$, *if* $x \in [0,1]$ *and* $r(x) = +\infty$. *Then* $q(x) \in [1, p(x))$ *for* $x \in [0,1]$, $r(x) < \infty$ *and* $F(\cdot, u(\cdot)) \in APS^{q(x)}(I : X)$.

Concerning asymptotical two-parameter Stepanov $p(x)$-almost periodicity, we can deduce the following composition principles with $X = Y$; the proof is very similar to those of [**234**, Proposition 2.7.3, Proposition 2.7.4] established in the case of constant functions $p,\ q,\ r$ and the interval $I = [0, \infty)$:

PROPOSITION 2.5.16. *Let* $p \in \mathcal{P}([0,1])$. *Suppose that the following conditions hold:*

(i) $g \in APS^{p(x)}(\mathbb{R} \times X : X)$, *there exist a function* $r \in \mathcal{P}([0,1])$ *such that* $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/(p(\cdot)-1))$ *and a function* $L_g \in L_S^{r(x)}(\mathbb{R})$ *such that* (25) *holds with the function* $f(\cdot,\cdot)$ *replaced by the function* $g(\cdot,\cdot)$ *therein.*

(ii) $v \in APS^{p(x)}(\mathbb{R} : X)$, *and there exists a set* $\mathrm{E} \subseteq \mathbb{R}$ *with* $m(\mathrm{E}) = 0$ *such that* $K = \{v(t) : t \in \mathbb{R} \setminus \mathrm{E}\}$ *is relatively compact in* X.

(iii) $f(t,x) = g(t,x) + q(t,x)$ *for all* $t \in I$ *and* $x \in X$, *where* $\hat{q} \in C_0(I \times X : L^{q(x)}([0,1] : X))$ *with* $q(\cdot)$ *defined as above;*

(iv) $u(t) = v(t) + \omega(t)$ *for all* $t \geqslant 0$, *where* $\hat{\omega} \in C_0(I : L^{p(x)}([0,1] : X))$.

(v) *There exists a set* $E' \subseteq I$ *with* $m(E') = 0$ *such that* $K' = \{u(t) : t \in I \setminus E'\}$ *is relatively compact in* X.

Then $f(\cdot, u(\cdot)) \in AAPS^{q(x)}(I : X)$.

2.5.3. Generalized (asymptotical) almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$: action of convolution products. Throughout this subsection, which has also appeared as a part of [**236**], we assume that $p \in \mathcal{P}([0,1])$ and a multivalued linear operator $\mathcal{A}$ fulfills condition (P). Then we know that the degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathcal{A}$ satisfies the estimate $\|T(t)\| \leqslant M_0 e^{-ct} t^{\beta-1}$, $t > 0$ for some finite constant $M_0 > 0$. Furthermore, $(T(t))_{t>0}$ is given by

$$T(t)x = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}\bigl(\lambda - \mathcal{A}\bigr)^{-1} x\, d\lambda, \quad t > 0,\ x \in X,$$

where Γ is the upwards oriented curve $\lambda = -c(|\eta|+1) + i\eta$ $(\eta \in \mathbb{R})$. For any $0 < \gamma < 1$ and $\nu > -\beta$, we define the operator family $(T_{\gamma,\nu}(t))_{t>0}$ through (63). Set, as before, $S_\gamma(t) := T_{\gamma,0}(t)$ and $P_\gamma(t) := \gamma T_{\gamma,1}(t)/t^\gamma$, $\quad t > 0$. Then $(S_\gamma(t))_{t>0}$ is a subordinated (g_γ, I)-regularized resolvent family generated by $\mathcal{A}$, which is generally not strongly continuous at zero. By our analysis from [**236**], we know that there exists a finite constant $M_1 > 0$ such that

$$\bigl\|S_\gamma(t)\bigr\| + \bigl\|P_\gamma(t)\bigr\| \leqslant M_1 t^{\gamma(\beta-1)}, \quad t > 0$$

as well as that there exists a finite constant $M_2 > 0$ such that

$$\bigl\|S_\gamma(t)\bigr\| \leqslant M_2 t^{-\gamma},\ t \geqslant 1 \quad \text{and} \quad \bigl\|P_\gamma(t)\bigr\| \leqslant M_2 t^{-2\gamma},\ t \geqslant 1.$$

Set $R_\gamma(t) := t^{\gamma-1} P_\gamma(t)$, $t > 0$.

We will first investigate infinite convolution products. Keeping in mind composition principles clarified in the previous section, it is almost straightforward to reformulate some known results concerning semilinear analogues of the above inclusions (see e.g. [**234**, Theorem 2.7.6-Theorem 2.7.9; Theorem 2.9.10-Theorem 2.9.11; Theorem 2.9.17-Theorem 2.9.18]); because of that, this question will not be examined here for the sake of brevity.

We start by stating the following generalization of [**239**, Proposition 2.11] (the reflexion at zero keeps the spaces of Stepanov p-almost periodic functions unchanged, which may or may not be the case with the spaces of Stepanov $p(x)$-almost periodic functions):

PROPOSITION 2.5.17. *Suppose that* $q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ *and* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. *If* $\check{g} : \mathbb{R} \to X$ *is* $S^{p(x)}$*-almost periodic, then the function* $G : \mathbb{R} \to Y$, *given by* (55), *is well-defined and almost periodic.*

PROOF. Without loss of generality, we may assume that $X = Y$. It is clear that, for every $t \in \mathbb{R}$, we have that $G(t) = \int_0^{\infty} R(s)g(t-s)\,ds$ and that the last integral is absolutely convergent due to Lemma 1.1.6(i) and $S^{p(x)}$-boundedness of function $\check{g}(\cdot)$:

$$\begin{aligned}
\int_0^{\infty} \|R(s)\|\|g(t-s)\|\,ds &= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(s)\|\|g(t-s)\|\,ds \\
&= \sum_{k=0}^{\infty} \int_0^1 \|R(s+k)\|\|g(t-s-k)\|\,ds \\
&\leqslant 2 \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}([0,1]:X)} \|g(t-k-\cdot)\|_{L^{p(\cdot)}([0,1]:X)} \\
&\leqslant 2M \sup_{t\in\mathbb{R}} \|\check{g}(\cdot - t)\|_{L^{p(\cdot)}([0,1]:X)},
\end{aligned}$$

for any $t \in \mathbb{R}$. Let a number $\varepsilon > 0$ be fixed. Then there is a finite number $l > 0$ such that any subinterval I of $\mathbb{R}$ of length l contains a number $\tau \in I$ such that $\|\check{g}(t-\tau+\cdot) - \check{g}(t+\cdot)\|_{L^{p(x)}([0,1]:X)} \leqslant \varepsilon$, $t \in \mathbb{R}$. Invoking Lemma 1.1.6(i) and this fact, we get

$$\begin{aligned}
&\|G(t+\tau) - G(t)\| \\
&\quad\leqslant \int_0^{\infty} \|R(r)\| \cdot \|g(t+\tau-r) - g(t-r)\|\,dr \\
&\quad= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(r)\| \cdot \|g(t+\tau-r) - g(t-r)\|\,dr \\
&\quad= \sum_{k=0}^{\infty} \int_0^1 \|R(r+k)\| \cdot \|g(t+\tau-r-k) - g(t-r-k)\|\,dr \\
&\quad\leqslant 2 \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} \|g(t+\tau-\cdot-k) - g(t-\cdot-k)\|_{L^{p(x)}[0,1]}
\end{aligned}$$

$$= 2\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} \|\check{g}(\cdot - t - \tau + k) - \check{g}(\cdot - t + k)\|_{L^{p(\cdot)}[0,1]}$$
$$\leqslant 2\varepsilon \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} = 2M\varepsilon, \quad t \in \mathbb{R},$$

which clearly implies that the set of all ε-periods of $G(\cdot)$ is relatively dense in $\mathbb{R}$. It remains to be proved the uniform continuity of $G(\cdot)$. Since $\hat{\check{g}}(\cdot)$ is uniformly continuous, we have the existence of a number $\delta \in (0,1)$ such that

$$\|\check{g}(\cdot - t') - \check{g}(\cdot - t)\|_{L^{p(x)}[0,1]} < \varepsilon, \text{ provided } t,\ t' \in \mathbb{R} \text{ and } |t - t'| < \delta. \tag{73}$$

For any $\delta' \in (0, \delta)$, the above computation with $\tau = \delta' = t' - t$ and (73) together imply that, for every $t \in \mathbb{R}$,

$$\big\|G(t + \delta') - G(t)\big\|$$
$$\leqslant 2\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} \|\check{g}(\cdot - t' + k) - \check{g}(\cdot - t + k)\|_{L^{p(\cdot)}[0,1]}$$
$$\leqslant 2\varepsilon \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} = 2M\varepsilon.$$

This completes the proof of proposition. □

Example 2.5.18. (i) Suppose that $\beta \in (0,1)$ and $(R(t))_{t>0} = (T(t))_{t>0}$ is a degenerate semigroup generated by $\mathcal{A}$. Let us recall that there exists a finite constant $M > 0$ such that $\|T(t)\| \leqslant Mt^{\beta-1}$, $t \in (0,1]$ and $\|T(t)\| \leqslant Me^{-ct}$, $t \geqslant 1$. Let $p_0 > 1$ be such that

$$\frac{p_0}{p_0 - 1}(\beta - 1) \leqslant -1,$$

let $p \in \mathcal{P}([0,1])$, and let $\|T(\cdot)\|_{L^{q(x)}[0,1]} < \infty$. Assume that we have constructed a function $\check{g} \in APS^{p(x)}(\mathbb{R} : X)$ such that $\check{g} \notin APS^{p}(\mathbb{R} : X)$ for all $p \geqslant p_0$ (Question: Could we manipulate here somehow with the construction established in [**77**, Example, p. 70]?) Then, in our concrete situation, [**239**, Proposition 2.11] cannot be applied since

$$\frac{p}{p - 1}(\beta - 1) \leqslant -1, \quad p \in [1, p_0).$$

Now we will briefly explain that $\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$, showing that Proposition 2.5.17 is applicable. Strictly speaking, for $k = 0$, $\|T(\cdot)\|_{L^{q(x)}[0,1]} < \infty$ by our assumption, while, for $k \geqslant 1$, it can be simply shown that $\|R(\cdot{+}k)\|_{L^{q(x)}[0,1]} \leqslant Me^{-ck}$ so that $\sum_{k=0}^{\infty} \|R(\cdot{+}k)\|_{L^{q(x)}[0,1]} < \infty$, as claimed.

(ii) By a mild solution of problem obtained by replacing the MLO $\mathcal{A}$ with the MLO $-\mathcal{A}$ in (58), we mean the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t - s)g(s)\,ds$, $t \in \mathbb{R}$ (cf. also [**297**, Lemma 6]). Let $p \in \mathcal{P}([0,1])$, and let $\|R_\gamma(\cdot)\|_{L^{q(x)}[0,1]} <$

∞. Then, for $k \geqslant 1$, we have $\|R_\gamma(\cdot + k)\|_{L^{q(x)}[0,1]} \leqslant M_2 k^{-1-\gamma}$. Hence, $\sum_{k=0}^{\infty} \|R_\gamma(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$ and we can apply Proposition 2.5.17.

The results obtained for the infinite convolution product can be simply incorporated in the study of existence and uniqueness of almost periodic solutions of the following abstract Cauchy differential inclusion of first order

$$u'(t) \in \mathcal{A}u(t) + g(t), \quad t \in \mathbb{R}$$

and the abstract Cauchy relaxation differential inclusion (58) with the MLO $\mathcal{A}$ replaced therein with $-\mathcal{A}$. I t is also clear that Proposition 2.5.17 can be used to study the existence and uniqueness of almost periodic solutions of the following abstract integral inclusion

$$u(t) \in \mathcal{A} \int_{-\infty}^{t} a(t-s)u(s)\,ds + g(t),\ t \in \mathbb{R}$$

where $a \in L^1_{loc}([0,\infty))$, $a \neq 0$, $\check{g} : \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic and $\mathcal{A}$ is a closed multivalued linear operator on X; see e.g., [**234**].

In the following proposition, whose proof is very similar to that of [**143**, Proposition 3.12], we state some invariance properties of generalized asymptotical almost periodicity in Lebesgue spaces with variable exponents $L^{p(x)}$ under the action of finite convolution products (see also [**234**, Proposition 2.7.5, Lemma 2.9.3] for similar results). This proposition generalizes [**239**, Proposition 2.13] provided that $p > 1$ in its formulation.

Proposition 2.5.19. *Suppose that $p \in \mathcal{P}([0,1])$, $q \in D_+([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that, for every $t \geqslant 0$, we have that $m_t := \sum_{k=0}^{\infty} \|R(\cdot + t + k)\|_{L^{q(x)}[0,1]} < \infty$. Suppose, further, that $\check{g} : \mathbb{R} \to X$ is $S^{p(x)}$-almost periodic, $q \in L_S^{p(x)}([0,\infty) : X)$ and $f(t) = g(t) + q(t)$, $t \geqslant 0$. Let $r_1,\ r_2 \in \mathcal{P}([0,1])$ and the following holds:*

(i) *For every $t \geqslant 0$, the mapping $x \mapsto \int_0^{t+x} R(t+x-s)q(s)\,ds$, $x \in [0,1]$ belongs to the space $L^{r_1(x)}([0,1] : X)$ and we have*

$$\lim_{t \to +\infty} \left\| \int_0^{t+x} R(t+x-s)q(s)\,ds \right\|_{L^{r_1(x)}[0,1]} = 0.$$

(ii) *For every $t \geqslant 0$, the mapping $x \mapsto m_{t+x}$, $x \in [0,1]$ belongs to the space $L^{r_2(x)}[0,1]$ and we have*

$$\lim_{t \to +\infty} \|m_{t+x}\|_{L^{r_2(x)}[0,1]} = 0.$$

Then the function $H(\cdot)$, given by

$$H(t) := \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0,$$

is well-defined, bounded and belongs to the class $APS^{p(x)}(\mathbb{R} : X) + S_0^{r_1(x)}([0,\infty) : X) + S_0^{r_2(x)}([0,\infty) : X)$, with the meaning clear.

REMARK 2.5.20. In [**239**, Remark 2.14], we have examined the conditions under which the function $H(\cdot)$ defined above is asymptotically almost periodic, provided that the function $g(\cdot)$ is S^p-almost periodic for some $p \in [1, \infty)$. The interested reader may try to analyze similar problems with function $\breve{g}(\cdot)$ being $S^{p(x)}$-almost periodic for some $p \in \mathcal{P}([0, 1])$.

In order to describe how Proposition 2.5.19 can be applied in concrete situations, we need the following weakened definition of Caputo fractional derivatives of order $\gamma \in (0, 1)$. The Caputo fractional derivative $\mathbf{D}_t^\gamma u(t)$ is defined for those functions $u : [0, T] \to X$ for which $u_{|(0,T]}(\cdot) \in C((0, T] : X)$, $u(\cdot) - u(0) \in L^1((0, T) : X)$ and $g_{1-\gamma} * (u(\cdot) - u(0)) \in W^{1,1}((0, T) : X)$, by

$$\mathbf{D}_t^\gamma u(t) = \frac{d}{dt}\left[g_{1-\gamma} * \Big(u(\cdot) - u(0)\Big)\right](t), \quad t \in (0, T].$$

We will use the following definition:

DEFINITION 2.5.21. (cf. [**236**, Section 3.5] for more details) By a classical solution of the abstract fractional Cauchy problem

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + f(t), \ t > 0, \\ \quad u(0) = x_0, \end{cases}$$

we mean any function $u \in C([0, \infty) : X)$ satisfying that the function $\mathbf{D}_t^\gamma u(t)$ is well-defined on any finite interval $(0, T]$ and belongs to the space $C((0, T] : X)$, as well as that $u(0) = u_0$ and $\mathbf{D}_t^\gamma u(t) - f(t) \in \mathcal{A}u(t)$ for $t > 0$.

Applying [**236**, Theorem 3.5.3], we have that the unique classical solution of $(\mathrm{DFP})_{f,\gamma}$ is given by the formula

$$u(t) = S_\gamma(t)x_0 + \int_0^t \big(t - s\big)^{\gamma - 1} P_\gamma(t - s)f(s)\, ds, \quad t \geqslant 0.$$

Suppose that $x_0 \in X$ belongs to the domain of continuity of $(S_\gamma(t))_{t>0}$ (by that, we mean that $\lim_{t\to 0+} S_\gamma(t)x_0 = x_0$; this holds in the case that $x \in D((-\mathcal{A})^\theta)$ with $1 \geqslant \theta > 1 - \beta$ or that $x \in X_{\mathcal{A}}^\theta$ with $1 > \theta > 1 - \beta$). Then the function $t \mapsto S_\gamma(t)x_0$, $t \geqslant 0$ is continuous and tends to zero as $t \to +\infty$. Keeping this in mind and imposing some additional conditions of function $f(\cdot)$, we can straightforwardly apply Proposition 2.5.19. This proposition can be also applied in the qualitative properties of solutions to the following inhomogeneous abstract Cauchy problems of third order:

$$\alpha u'''(t) + u''(t) - \beta Au(t) - \gamma Au'(t) = f(t), \quad \alpha,\ \beta,\ \gamma > 0,\ t \geqslant 0,$$

appearing in the theory of dynamics of elastic vibrations of flexible structures [**126**].

Finally, we will present some illustrative applications. Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded subset with smooth boundary $\partial\Omega$ and let $1 < p < \infty$. Among other things, one can make use of Proposition 2.5.19 to establish the existence and uniqueness of asymptotically $S^{p(x)}$-almost automorphic solutions to the damped Poisson-wave type equation, in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$, given by

$$\begin{cases} \frac{\partial}{\partial t}\bigl(m(x)\frac{\partial u}{\partial t}\bigr) + \bigl(2\omega m(x) - \Delta\bigr)\frac{\partial u}{\partial t} + \bigl(A(x;D) - \omega\Delta + \omega^2 m(x)\bigr)u(x,t) = f(x,t), \\ t \geqslant 0,\ x \in \Omega; \\ \\ u = \frac{\partial u}{\partial t} = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ \\ u(0,x) = u_0(x),\ m(x)\bigl[(\frac{\partial u}{\partial t})(x,0) + \omega u_0\bigr] = m(x)u_1(x), \quad x \in \Omega, \end{cases}$$

where $m(x) \in L^\infty(\Omega)$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, Δ is the Dirichlet Laplacian in $L^2(\Omega)$, acting on its maximal domain, $H_0^1(\Omega) \cap H^2(\Omega)$, and $A(x;D)$ is a second-order linear differential operator on Ω with continuous coefficients on $\overline{\Omega}$, see, e.g., [**167**, Example 6.1] and [**234**] for further details.

Notice that we can also consider the existence and uniqueness of asymptotically $S^{p(x)}$-almost periodic solutions to the following fractional damped Poisson-wave type equation, in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$, given by

$$\begin{cases} \mathbf{D}_t^\gamma\bigl(m(x)\mathbf{D}_t^\gamma u\bigr) + \bigl(2\omega m(x) - \Delta\bigr)\mathbf{D}_t^\gamma u + \bigl(A(x;D) - \omega\Delta + \omega^2 m(x)\bigr)u(x,t) = f(x,t), \\ \\ t \geqslant 0,\ x \in \Omega; \\ \\ u = \mathbf{D}_t^\gamma u = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \\ \\ u(0,x) = u_0(x),\ m(x)\bigl[\mathbf{D}_t^\gamma u(x,0) + \omega u_0\bigr] = m(x)u_1(x), \quad x \in \Omega. \end{cases}$$

2.5.4. (p,ϕ,F)-Classes and $[p,\phi,F]$-classes of Weyl almost periodic functions. Throughout this subsection, we assume the following general conditions:

(A): $I = \mathbb{R}$ or $I = [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$, $p \in \mathcal{P}(I)$ and $F : (0,\infty) \times I \to (0,\infty)$.

(B): The same as (A) with the assumption $p \in \mathcal{P}(I)$ replaced by $p \in \mathcal{P}([0,1])$ therein.

We introduce the notions of an (equi-)Weyl-(p,ϕ,F)-almost periodic function and an (equi-)Weyl-$(p,\phi,F)_i$-almost periodic function, where $i = 1, 2$, as follows (see [**234**] for the case that $p(x) \equiv p \in [1,\infty)$, $\phi(x) = x$ and $F(l,t) = l^{(-1)/p}$, when we deal with the usually considered (equi-)Weyl-p-almost periodic functions, as well as to [**142**, Remark 4.13] for the case that $\phi(x) = x$ and and $F(l,t) = l^{(-1)/p(t)}$):

Definition 2.5.22. Suppose that condition (A) holds, $f : I \to X$ and $\phi(\|f(\cdot + \tau) - f(\cdot)\|) \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-(p,ϕ,F)-almost periodic, $f \in e - W_{ap}^{(p,\phi,F)}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{(p,\phi,F,\tau)} := \sup_{t \in I} \Bigl[F(l,t) \Bigl[\phi\bigl(\| f(\cdot + \tau) - f(\cdot) \| \bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Bigr] \leqslant \varepsilon. \tag{74}$$

(ii) It is said that the function $f(\cdot)$ is Weyl-(p, ϕ, F)-almost periodic, $f \in W_{ap}^{(p,\phi,F)}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{(p,\phi,F,\tau)} := \limsup_{l \to \infty} \sup_{t \in I} \Bigl[F(l,t) \Bigl[\phi\bigl(\| f(\cdot + \tau) - f(\cdot) \| \bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Bigr] \leqslant \varepsilon. \tag{75}$$

DEFINITION 2.5.23. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot + \tau) - f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$(p, \phi, F)_1$-almost periodic, $f \in e - W_{ap}^{(p,\phi,F)_1}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{(p,\phi,F,\tau)_1} := \sup_{t \in I} \Bigl[F(l,t) \phi \Bigl[\bigl(\| f(\cdot + \tau) - f(\cdot) \| \bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Bigr] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$(p, \phi, F)_1$-almost periodic, $f \in W_{ap}^{(p,\phi,F)_1}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{(p,\phi,F,\tau)_1} := \limsup_{l \to \infty} \sup_{t \in I} \Bigl[F(l,t) \phi \Bigl[\bigl(\| f(\cdot + \tau) - f(\cdot) \| \bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Bigr] \leqslant \varepsilon.$$

DEFINITION 2.5.24. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot + \tau) - f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$(p, \phi, F)_2$-almost periodic, $f \in e - W_{ap}^{(p,\phi,F)_2}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{(p,\phi,F,\tau)_2} := \sup_{t \in I} \phi \Bigl[F(l,t) \Bigl[\bigl(\| f(\cdot + \tau) - f(\cdot) \| \bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Bigr] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$(p, \phi, F)_2$-almost periodic, $f \in W_{ap}^{(p,\phi,F)_2}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point

$\tau \in I'$ such that

$$\|f\|_{(p,\phi,F,\tau)_2} := \limsup_{l\to\infty} \sup_{t\in I} \phi\Biggl[F(l,t)\Bigl[\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l]}\Bigr]\Biggr] \leqslant \varepsilon.$$

Before we go any further, we would like to that the above definitions are incapable of being compared to each other: for example, in Definition 2.5.22, we calculate the value of $\phi(\|f(\cdot+\tau)-f(\cdot)\|)_{L^{p(\cdot)}[t,t+l]}$, while in Definition 2.5.23 we first calculate the value of $\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(\cdot)}[t,t+l]}$ and after that we apply the function $\phi(\cdot)$.

If $i = 1, 2$ and $F(l,t) = \psi(l)^{(-1)/p(t)}$ for some function $\psi : (0,\infty) \to (0,\infty)$ and all $t \in I$, then we also say that the function $f(\cdot)$ is (equi-)Weyl-(p,ϕ,ψ)-almost periodic, resp. (equi-)Weyl-$(p,\phi,\psi)_i$-almost periodic, when the corresponding class of functions is also denoted by $(e-)W_{ap}^{(p,\phi,\psi)}(I:X)$, resp. $(e-)W_{ap}^{(p,\phi,\psi)_i}(I:X)$. There is no need to say that the above classes coincide in the case that $\phi(x) \equiv x$.

EXAMPLE 2.5.25. (i) If $\phi(0) = 0$, then any continuous periodic function $f : I \to X$ belongs to any of the above introduced function spaces. If $\phi(0) > 0$, then a constant function cannot belong to any of the function spaces introduced in Definition 2.5.24, while the function spaces introduced in Definition 2.5.22-Definition 2.5.23 can contain constant functions (see also Remark 2.5.27(iii)).

(ii) If $\phi(x) = x$ and $p(x) \equiv p \in [1,\infty)$, then any Stepanov p-bounded function $f : I \to X$ belongs to any of the above introduced function spaces with $F(l,t) \equiv l^{-\sigma}$, where $\sigma > 1/p$; in particular, if $f(\cdot)$ is Stepanov $p(x)$-bounded and $p \in D_+(I)$, then $f(\cdot)$ belongs to any of the above introduced function spaces with $F(l,t) \equiv l^{-\sigma}$, where $\sigma > 1/p^+$. This simply follows from the inequality

$$\left(\int_t^{t+l} \|f(s+\tau)-f(s)\|^p\,ds\right)^{1/p} \leqslant \sum_{k=0}^{\lfloor l\rfloor}\left(\int_{t+k}^{t+k+1} \|f(s+\tau)-f(s)\|^p\,ds\right)^{1/p},$$

which is valid for any $t,\ \tau \in I$, $l > 0$, and a simple argumentation. Suppose now that $I = \mathbb{R}$ or $I = [0,\infty)$, $p \in \mathcal{P}(I)$ and $f \in BS^{p(x)}(I:X)$. A similar line of reasoning shows that $f(\cdot)$ belongs to any of the above introduced function spaces provided that

(a) $p \in D_+(I)$ and $F(l,t) \equiv l^{-\sigma}$, where $\sigma > 1/p^+$, or

(b) $F(l,t) \equiv l^{-\sigma}$, where $\sigma > 1$, in general case. For this, it is only worth noting that we have $\varphi_{p(x)}(t/l^\sigma) \leqslant (1/l^\sigma)\varphi_{p(x)}(t)$ for any $t \geqslant 0$ and $l \geqslant 1$.

(iii) If X does not contain an isomorphic copy of the sequence space c_0, $\phi(x) = x$ and $F(l,t) \equiv F(t)$, where $\lim_{t\to+\infty} F(t) = +\infty$, then there is no trigonometric polynomial $f(\cdot)$ and function $p \in \mathcal{P}(\mathbb{R})$ such that $f \in e - W_{ap}^{(p,x,F)}(\mathbb{R}:X)$. If we suppose the contrary, then using the fact that the space $L^{p(x)}[t,t+l]$ is continuously embedded into the space $L^1[t,t+l]$ with the constant of embeddings less than or equal to $2(1+l)$

(see, e.g., [**147**, Corollary 3.3.4]), where $t \in \mathbb{R}$ and $l > 0$, we get that for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq \mathbb{R}$ of length L contains a point $\tau \in I'$ such that

$$\sup_{t\in\mathbb{R}} \Bigl[F(t) \bigl\| f(\cdot+\tau) - f(\cdot) \bigr\|_{L^1[t,t+l]} \Bigr] \leqslant 2\varepsilon(1+l). \tag{76}$$

Let such numbers $l > 0$ and $\tau \in \mathbb{R}$ be fixed. By (76), we get that the mapping $t \mapsto f_1(t) \equiv \int_t^{t+l} \|f(s+\tau) - f(s)\|\,ds$, $t \geqslant 0$ belongs to the space $C_0([0,\infty):\mathbb{C})$. On the other hand, the mapping $s \mapsto \|f(s+\tau)-f(s)\|$, $s \in \mathbb{R}$ is almost periodic and satisfies that $\int_0^t \|f(s+\tau)-f(s)\|\,ds < \infty$, so that the mapping $t \mapsto f_2(t) \equiv \int_0^t \|f(s+\tau) - f(s)\|\,ds$, $t \in \mathbb{R}$ is almost periodic by Theorem 2.1.1(vi). By the translation invariance, the same holds for the mapping $f_1(\cdot) = f_2(\cdot+\tau) - f_2(\cdot)$. Since $f_1 \in C_0([0,\infty):\mathbb{C})$, we get that $f_1 \equiv 0$, so that $\|f(s+\tau)-f(s)\| = 0$ for all $s \geqslant 0$ and $f(\cdot)$ is periodic, which is a contradiction. Based on the conclusion obtained in this part, we will not examine the question whether, for a given number $\varepsilon > 0$ and an equi-Weyl-(p,ϕ,F)-almost periodic function or an equi-Weyl-$(p,\phi,F)_i$-almost periodic function ($i = 1,2$), we can find a trigonometric polynomial $P(\cdot)$ such that $\|P - f\|_{(p,\phi,F)} < \varepsilon$ or $\|P - f\|_{(p,\phi,F)_i} < \varepsilon$ ($i = 1,2$), where

$$e - \|f\|_{(p,\phi,F)} := \sup_{t\in I} \Bigl[F(l,t) \Bigl[\phi\bigl(\|f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Bigr],$$

$$e - \|f\|_{(p,\phi,F)_1} := \sup_{t\in I} \Bigl[F(l,t) \phi\Bigl[\bigl(\|f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Bigr]$$

and

$$e - \|f\|_{(p,\phi,F)_2} := \sup_{t\in I} \phi \Bigl[F(l,t) \Bigl[\bigl(\|f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Bigr].$$

For the usually considered class of equi-Weyl-p-almost periodic functions, where $1 \leqslant p < \infty$, the answer to the above question is affirmative (see, e.g., [**234**, Theorem 2.3.2]). Observe also that the sub-additivity of function $\phi(\cdot)$ implies the sub-additivity of functions $e - \|\cdot\|_{(p,\phi,F)}$ and $e - \|\cdot\|_{(p,\phi,F)_i}$, where $i = 1,2$; since the limit superior is also a sub-additive operation, the same holds for the functions $\|\cdot\|_{(p,\phi,F)}$ and $\|\cdot\|_{(p,\phi,F)_i}$, where $i = 1,2$, defined as above (cf. the second parts of Definition 2.5.22-Definition 2.5.24, as well as Definition 2.5.28-Definition 2.5.30 below).

In the case that the function $\phi(\cdot)$ is convex and $p(x) \equiv 1$, we have the following result:

PROPOSITION 2.5.26. *Suppose that $p(x) \equiv 1$, $f : I \to X$, $\|f(\cdot+\tau) - f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I, as well as condition*

(C): *$\phi(\cdot)$ is convex and there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\phi(lx) \leqslant \varphi(l)\phi(x)$ for all $l > 0$ and $x \geqslant 0$*

holds. Set $F_1(l,t) := F(l,t)l[\varphi(l)]^{-1}$, $l > 0$, $t \in I$ *and* $F_2(l,t) := l^{-1}\varphi(F(l,t)l)$, $l > 0$, $t \in I$. *Then we have:*

(i) $f \in (e-)W_{ap}^{(1,\phi,F)} \Rightarrow f \in (e-)W_{ap}^{(1,\phi,F_1)_1}$.
(ii) $f \in (e-)W_{ap}^{(1,\phi,F_2)} \Rightarrow f \in (e-)W_{ap}^{(1,\phi,F)_2}$.

PROOF. To prove (i), suppose that $f \in (e-)W_{ap}^{(1,\phi,F)}$. Then the assumption (C) and the Jensen integral inequality together imply

$$\begin{aligned}\phi\Big(\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]}\Big) &= \phi\Big(l\cdot l^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]}\Big)\\ &\leqslant \varphi(l)\phi\Big(l^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]}\Big) \leqslant \varphi(l)l^{-1}\Big[\phi\big(\|f(\cdot+\tau)-f(\cdot)\|\big)\Big]_{L^1[t,t+l]}.\end{aligned}$$

This simply yields $f \in (e-)W_{ap}^{(1,\phi,F_1)_1}$. To prove (ii), suppose that $f \in (e-)W_{ap}^{(1,\phi,F_2)}$. Then the assumption (C) and the Jensen integral inequality together imply

$$\begin{aligned}\phi\Big(F(l,t)\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]}\Big) &= \phi\Big(F(l,t)l\cdot l^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[t,t+l]}\Big)\\ &\leqslant \varphi(F(t,l)l)l^{-1}\Big[\phi\big(\|f(\cdot+\tau)-f(\cdot)\|\big)\Big]_{L^1[t,t+l]}.\end{aligned}$$

This simply yields $f \in (e-)W_{ap}^{(1,\phi,F)_2}$. $\square$

Before we go any further, let us recall that any equi-Weyl-p-almost periodic function needs to be Weyl p-almost periodic, while the converse statement does not hold in general. On the other hand, it is not true that an equi-Weyl-(p,ϕ,ψ)-almost periodic function, resp. equi-Weyl-$(p,\phi,\psi)_i$-almost periodic function, is Weyl-(p,ϕ,ψ)-almost periodic, resp. Weyl-$(p,\phi,\psi)_i$-almost periodic; moreover, an unrestrictive choice of function $\psi(\cdot)$ allows us to work with a substantially large class of quasi-almost periodic functions: As it can be simply approved, any Stepanov p-almost periodic function $f(\cdot)$ is equi-Weyl-(p,ϕ,ψ)-almost periodic with $p(x) \equiv p \in [1,\infty)$, $\psi(l) \equiv 1$, $\phi(x) = x$; on the other hand, any continuous Stepanov p-almost periodic function $f(\cdot)$ which is not periodic cannot be Weyl-$(p,x,1)$-almost periodic, for example. Let us explain the last fact in more detail. If we suppose the contrary, then for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (75) holds with $p(x) \equiv p \in [1,\infty)$, $\psi(l) \equiv 1$ and $\phi(x) = \varphi(x) = x$. This simply implies that for each $\varepsilon > 0$ we can find a strictly increasing sequence (l_n) of positive real numbers tending to infinity such that for each $t \in I$ and $n \in \mathbb{N}$ we have $\int_{t+l_n}^{t} \|f(x+\tau)-f(x)\|^p\,dx \leqslant \varepsilon$ for each $\varepsilon > 0$; hence, $\int_I \|f(x+\tau)-f(x)\|^p\,dx \leqslant \varepsilon$ and therefore $\int_I \|f(x+\tau)-f(x)\|^p\,dx = 0$. This yields $f(x+\tau) = f(x)$, $x \in I$, which is a contradiction with our preassumption.

REMARK 2.5.27. (i) It is clear that, if $f(\cdot)$ is an (equi-)Weyl-(p,ϕ,F)-almost periodic function, resp. (equi-)Weyl-$(p,\phi,F)_1$-almost periodic function, and $F(l,t) \geqslant F_1(l,t)$ for every $l > 0$ and $t \in I$, then $f(\cdot)$ is (equi-)Weyl-(p,ϕ,F_1)-almost periodic, resp. (equi-)Weyl-$(p,\phi,F_1)_1$-almost periodic. Furthermore, if $f(\cdot)$ is an (equi-)Weyl-$(p,\phi,F)_2$-almost periodic function, then $f(\cdot)$ is an (equi-)Weyl-$(p,\phi,F_1)_2$-almost periodic function provided that $F(l,t) \geqslant F_1(l,t)$ for every $l > 0$, $t \in I$ and $\phi(\cdot)$ is monotonically increasing, or $F(l,t) \leqslant F_1(l,t)$ for every $l > 0$, $t \in I$ and $\phi(\cdot)$ is monotonically decreasing.

(ii) If $f(\cdot)$ is an (equi-)Weyl-(p,ϕ,F)-almost periodic function, resp. (equi-)Weyl-$(p,\phi,F)_i$-almost periodic function, $\phi_1(\cdot)$ is measurable and $0 \leqslant \phi_1 \leqslant \phi$, then Lemma 1.1.6(iii) yields that $f(\cdot)$ is (equi-)Weyl-(p,ϕ_1,F)-almost periodic, resp. (equi-)Weyl-$(p,\phi_1,F)_i$-almost periodic, where $i = 1, 2$.

(iii) Regarding the first parts in the above definitions, it is worth noticing that we do not allow the number $l > 0$ to be sufficiently large: in some concrete situations, it is crucial to allow the number $l > 0$ to be sufficiently small; we will explain this fact by two illustrative examples. First, let us consider Definition 2.5.22(i). Suppose that $p(x) \equiv p \in [1,\infty)$ and there exists an absolute constant $c > 0$ such that for each $l > 0$ and $\tau \in I$ we have

$$\sup_{t \in I} \phi\big(\big\|f(\cdot+\tau) - f(\cdot)\big\|\big)_{L^{p(x)}[t,t+l]} \leqslant c.$$

Then it simply follows that the function $f(\cdot)$ is equi-Weyl-(p,ϕ,ψ)-almost periodic provided that $\lim_{l\to 0+}\psi(l) = +\infty$. Second, suppose that $f \in L^\infty(I : X)$. Then $f(\cdot)$ is equi-Weyl-$(p,x,1)$-almost periodic for any $p \in \mathcal{D}(I)$, which can be simply approved by considering the case of constant coefficient $p(x) \equiv p^+$ and the choice $l = l(\varepsilon) = \varepsilon$.

In order to ensure the translation invariance of Weyl spaces with variable exponent, we need to follow a slightly different approach ([**142**]-[**143**]):

Definition 2.5.28. Suppose that condition (B) holds, $f : I \to X$ and $\phi(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|) \in L^{p(x)}([0,1])$ for any $\tau \in I$, $t \in I$ and $l > 0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$[p,\phi,F]$-almost periodic, $f \in e - W_{ap}^{[p,\phi,F]}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e - \|f\|_{[p,\phi,F,\tau]} := \sup_{t\in I}\Bigg[F(l,t)\Big[\phi\big(\big\|f(\cdot l + t + \tau) - f(t + \cdot l)\big\|\big)_{L^{p(\cdot)}[0,1]}\Big]\Bigg] \leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$[p,\phi,F]$-almost periodic, $f \in W_{ap}^{[p,\phi,F]}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{[p,\phi,F,\tau]} := \limsup_{l\to\infty}\sup_{t\in I}\Bigg[F(l,t)\Big[\phi\big(\big\|f(\cdot l + t + \tau) - f(t + \cdot l)\big\|\big)_{L^{p(\cdot)}[0,1]}\Big]\Bigg] \leqslant \varepsilon.$$

Definition 2.5.29. Suppose that condition (B) holds, $f : I \to X$ and $\|f(\cdot l + t + \tau) - f(t + \cdot l)\| \in L^{p(x)}([0,1])$ for any $\tau \in I$, $t \in I$ and $l > 0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$[p,\phi,F]_1$-almost periodic, $f \in e-W_{ap}^{[p,\phi,F]_1}(I:X)$ for short, if and only if for each $\varepsilon>0$ we can find two real numbers $l>0$ and $L>0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e-\|f\|_{[p,\phi,F,\tau]_1}:=\sup_{t\in I}\Biggl[F(l,t)\phi\Bigl[\bigl(\bigl\|f(\cdot l+t+\tau)-f(t+\cdot l)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]\leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$[p,\phi,F]_2$-almost periodic, $f \in W_{ap}^{[p,\phi,F]_2}(I:X)$ for short, if and only if for each $\varepsilon>0$ we can find a real number $L>0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{[p,\phi,F,\tau]_1}:=\limsup_{l\to\infty}\sup_{t\in I}\Biggl[F(l,t)\phi\Bigl[\bigl(\bigl\|f(\cdot l+t+\tau)-f(t+\cdot l)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]\leqslant \varepsilon.$$

DEFINITION 2.5.30. Suppose that condition (B) holds, $f:I\to X$ and $\|f(\cdot l+t+\tau)-f(t+\cdot l)\| \in L^{p(x)}([0,1])$ for any $\tau \in I$, $t\in I$ and $l>0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$[p,\phi,F]_2$-almost periodic, $f \in e-W_{ap}^{[p,\phi,F]_2}(I:X)$ for short, if and only if for each $\varepsilon>0$ we can find two real numbers $l>0$ and $L>0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$e-\|f\|_{[p,\phi,F,\tau]_2}:=\sup_{t\in I}\phi\Biggl[F(l,t)\Bigl[\bigl(\bigl\|f(\cdot l+t+\tau)-f(t+\cdot l)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]\leqslant \varepsilon.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$[p,\phi,F]_2$-almost periodic, $f \in W_{ap}^{[p,\phi,F]_2}(I:X)$ for short, if and only if for each $\varepsilon>0$ we can find a real number $L>0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\|f\|_{[p,\phi,F,\tau]_2}:=\limsup_{l\to\infty}\sup_{t\in I}\phi\Biggl[F(l,t)\Bigl[\bigl(\bigl\|f(\cdot l+t+\tau)-f(t+\cdot l)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]\leqslant \varepsilon.$$

REMARK 2.5.31. (i) Let $p \in \mathcal{P}([0,1])$, let $I=\mathbb{R}$ or $I=[0,\infty)$, and let a function $f \in L_S^{p(x)}(I:X)$ be Stepanov $p(x)$-almost periodic. Then it readily follows that $f(\cdot)$ is equi-Weyl-$[p,\phi,F]$-almost periodic with $\phi(x)\equiv x$ and $F(l,t)\equiv 1$.

(ii) In the case that $p(x)\equiv p\in[1,\infty)$, it can be simply verified that the class of (equi)-Weyl-$[p,\phi,[l/\psi(l)]^{1/p}]$-almost periodic functions, resp. (equi)-Weyl-$[p,\phi,[l/\psi(l)]^{1/p}]_2$-almost periodic functions, coincides with the class of (equi)-Weyl-(p,ϕ,ψ)-almost periodic functions, resp. (equi)-Weyl-$(p,\phi,\psi)_2$-almost periodic functions. It is clear that the class of (equi)-Weyl-$[p,\phi,[l/\psi(l)]^{1/p}]_1$-almost periodic functions and the class of (equi)-Weyl-$(p,\phi,\psi)_1$-almost periodic functions coincide provided that $\phi(cx)=c\phi(x)$ for all $c,\ x\geqslant 0$.

(iii) It can be simply verified that the validity of condition

(D): For any $\tau_0 \in I$ there exists $c > 0$ such that

$$\frac{F(l,t)}{F(l,t+\tau_0)} \leqslant c, \quad t \in I,\ l > 0$$

implies that the spaces $(e-)W_{ap}^{[p,\phi,F]}(I : X)$ and $(e-)W_{ap}^{[p,\phi,F]_1}(I : X)$ are translation invariant; this particularly holds provided the function $F(l,t)$ does not depend on the variable t. Furthermore, the space $(e-)W_{ap}^{[p,\phi,F]_2}(I : X)$ is translation invariant provided condition

(D)': For any $\tau_0 \in I$ there exists $c > 0$ such that

$$\phi(F(l,t)x) \leqslant c\phi\bigl(F(l,t+\tau_0)x\bigr), \quad x \geqslant 0,\ t \in I,\ l > 0.$$

(iv) If $p,\ q \in \mathcal{P}([0,1])$ and $q(x) \leqslant p(x)$ for a.e. $x \in [0,1]$, then Lemma 1.1.6(ii) yields that any (equi)-Weyl-$[p,\phi,F]$-almost periodic function is (equi)-Weyl-$[q,\phi,F]$-almost periodic. Furthermore, condition $x,\ y \geqslant 0$ and $x \leqslant cy$ implies $\phi(x) \leqslant c\phi(y)$, resp. $x,\ y \geqslant 0$ and $x \leqslant cy$ implies $\phi(F(l,t)x) \leqslant c\phi(F(l,t)y)$ for all $l > 0$ and $t \in I$, ensures that any (equi)-Weyl-$[p,\phi,F]_1$-almost periodic function is (equi)-Weyl-$[q,\phi,F]_1$-almost periodic, resp. any (equi)-Weyl-$[p,\phi,F]_2$-almost periodic function is (equi)-Weyl-$[q,\phi,F]_2$-almost periodic.

(v) It is clear that, if $f(\cdot)$ is an (equi)-Weyl-$[p,\phi,F]$-almost periodic function, resp. (equi)-Weyl-$[p,\phi,F]_1$-almost periodic function, and $F(l,t) \geqslant F_1(l,t)$ for every $l > 0$ and $t \in I$, then $f(\cdot)$ is (equi)-Weyl-$[p,\phi,F_1]$-almost periodic, resp. (equi)-Weyl-$[p,\phi,F_1]_1$-almost periodic. Furthermore, any (equi)-Weyl-$[p,\phi,F]_2$-almost periodic function is (equi)-Weyl-$[p,\phi,F]_2$-almost periodic provided that $F(l,t) \geqslant F_1(l,t)$ for every $l > 0,\ t \in I$ and $\phi(\cdot)$ is monotonically increasing, or $F(l,t) \leqslant F_1(l,t)$ for every $l > 0$, $t \in I$ and $\phi(\cdot)$ is monotonically decreasing.

(vi) If $f(\cdot)$ is an (equi-)Weyl-$[p,\phi,F]$-almost periodic function, $\phi_1(\|f(\cdot l + t + \tau) - f(t + \cdot l)\|)$ is measurable for any $\tau \in I$, $t \in I$, $l > 0$, and $0 \leqslant \phi_1 \leqslant \phi$, then Lemma 1.1.6(iii) yields that $f(\cdot)$ is an (equi)-Weyl-$[p,\phi_1,F]$-almost periodic. Furthermore, if $0 \leqslant \phi_1 \leqslant \phi$, only, and $f(\cdot)$ is an (equi-)Weyl-$[p,\phi,F]_i$-almost periodic function, then $f(\cdot)$ is an (equi-)Weyl-$[p,\phi_1,F]_i$-almost periodic function, where $i = 1, 2$.

In the case that the function $\phi(\cdot)$ is convex and $p(x) \equiv 1$, we have the following proposition which can be shown following the lines of the proof of Proposition 2.5.26:

Proposition 2.5.32. *Suppose that $\phi(\cdot)$ is convex, $p(x) \equiv 1$, $f : I \to X$ and $\|f(\cdot l + t + \tau) - f(t + \cdot l)\| \in L^{p(x)}([0,1])$ for any $\tau \in I$, $t \in I$ and $l > 0$. Then the following holds:*

(i) $f \in (e-)W_{ap}^{[1,\phi,F]} \Rightarrow f \in (e-)W_{ap}^{[1,\phi,F]_1}$.

(ii) *If condition* (C) *holds, then* $f \in (e-)W_{ap}^{[1,\phi,\varphi\circ F]} \Rightarrow f \in (e-)W_{ap}^{[1,\phi,F]_2}$.

Regarding Proposition 2.5.26 and Proposition 2.5.32, it should be observed that the reverse inclusions and inequalities can be obtained assuming condition

(C)': $\phi(\cdot)$ is concave and there exists a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\phi(lx) \geqslant \varphi(l)\phi(x)$ for all $l > 0$ and $x \geqslant 0$.

It is clear that any (equi-)Weyl-p-almost periodic function $f(\cdot)$ is (equi-)Weyl-(p, ϕ, ψ)-almost periodic with $p(x) \equiv p \in [1,\infty)$, $\phi(x) = x$, $\psi(l) = l$. Concerning this observation, we wish to present two illustrative examples:

Example 2.5.33. Let us recall (see e.g., Example 4.27 in the survey article [**23**] by J. Andres, A. M. Bersani, R. F. Grande) that the function $g(\cdot) := \chi_{[0,1/2]}(\cdot)$ is equi-Weyl-p-almost periodic for any $p \in [1,\infty)$ but not Stepanov almost periodic. Since for each l, $\tau \in \mathbb{R}$ we have

$$\left(\sup_{t\in\mathbb{R}} \int_t^{t+l} |f(x+\tau) - f(x)|^p \, dx \right)^{1/p} \leqslant 1,$$

it can be easily seen that the function $g(\cdot)$ is equi-Weyl-(p, x, ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$ such that $\lim_{l\to+\infty} \psi(l) = +\infty$; moreover, for each $\varepsilon \in (0, 1/2)$ we can always find $t \in \mathbb{R}$ such that

$$\int_t^{t+1} |f(x+\tau) - f(x)|^p \, dx > \varepsilon, \quad \tau > \varepsilon.$$

Hence, the function $g(\cdot)$ cannot be equi-Weyl-(p, x, l^0)-almost periodic. Taking into account Remark 2.5.27(iii) and the above conclusions, we get that $g(\cdot)$ is equi-Weyl-(p, x, l^σ)-almost periodic if and only if $\sigma \neq 0$.

Example 2.5.34. Let us recall ([**23**, Example 4.29], [**234**]) that the Heaviside function $g(\cdot) := \chi_{[0,\infty)}(\cdot)$ is not equi-Weyl-1-almost periodic but it is Weyl-p-almost periodic for any number $p \in [1,\infty)$. Furthermore, it is not difficult to see that for each real number $\tau \in \mathbb{R}$ we have

$$\sup_{t\in\mathbb{R}} \left(\int_t^{t+l} |f(x+\tau) - f(x)|^p \, dx \right)^{1/p} = |\tau|^{1/p}$$

for any real number $l > |\tau|$. This implies that the function $g(\cdot)$ is Weyl-(p, x, ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$ such that $\lim_{l\to+\infty} \psi(l) = +\infty$ as well as that $g(\cdot)$ cannot be Weyl-(p, x, ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$ such that $\limsup_{l\to+\infty} [\psi(l)]^{-1} > 0$; in particular, $g(\cdot)$ is Weyl-(p, x, l^σ)-almost periodic if and only if $\sigma > 0$. On the other hand, the function $g(\cdot)$ cannot be equi-Weyl-(p, x, ψ)-almost periodic for any function $\psi : (0,\infty) \to (0,\infty)$; in actual fact, if we suppose contrary, then the equation (74) is violated with $|\tau|^{1/p} > \varepsilon\psi(l)^{1/p}$. See also [**234**, Example 2.11.15-Example 2.11.17].

Before we switch to the next subsection, we feel it is our duty to say that the approach used for definition of functions spaces introduced in Definition 2.5.22-Definition 2.5.24 and Definition 2.5.28-Definition 2.5.30 will be exploited mutiple times in the remainder of book, for various types of generalized almost periodicity.

2.5.5. Weyl ergodic components with variable exponents. Unless stated otherwise, in this subsection we assume that $p \in \mathcal{P}([0,\infty))$, $\phi : [0,\infty) \to [0,\infty)$ and $F : (0,\infty) \times [0,\infty) \to (0,\infty)$. In the following three definitions, we extend the notion of an (equi-)Weyl-p-vanishing function introduced in [**246**], where the case $p(x) \equiv p \in [1,\infty)$, $F(l,t) \equiv l^{(-1)/p}$ and $\phi(x) \equiv x$ has been considered:

DEFINITION 2.5.35. (i) It is said that a function $q : [0,\infty) \to X$ is equi-Weyl-(p,ϕ,F)-vanishing if and only if $\phi(\|q(t+\cdot)\|) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l > 0$ and

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \sup_{x\geqslant 0}\Bigl[F(l,t)\bigl\|\phi(\|q(t+v)\|)\bigr\|_{L^{p(v)}[x,x+l]}\Bigr] = 0. \tag{77}$$

(ii) It is said that a function $q : [0,\infty) \to X$ is Weyl-(p,ϕ,F)-vanishing if and only if $\phi(q(t+\cdot)) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l > 0$ and

$$\lim_{t\to+\infty} \limsup_{l\to+\infty} \sup_{x\geqslant 0}\Bigl[F(l,t)\bigl\|\phi(\|q(t+v)\|)\bigr\|_{L^{p(v)}[x,x+l]}\Bigr] = 0. \tag{78}$$

DEFINITION 2.5.36. (i) It is said that a function $q : [0,\infty) \to X$ is equi-Weyl-$(p,\phi,F)_1$-vanishing if and only if $q(t+\cdot) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l > 0$ and

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \sup_{x\geqslant 0}\Bigl[F(l,t)\phi\Bigl(\bigl\|q(t+v)\bigr\|_{L^{p(v)}[x,x+l]}\Bigr)\Bigr] = 0. \tag{79}$$

(ii) It is said that a function $q : [0,\infty) \to X$ is Weyl-$(p,\phi,F)_1$-vanishing if and only if $q(t+\cdot) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l > 0$ and

$$\lim_{t\to+\infty} \limsup_{l\to+\infty} \sup_{x\geqslant 0}\Bigl[F(l,t)\phi\Bigl(\bigl\|q(t+v)\bigr\|_{L^{p(v)}[x,x+l]}\Bigr)\Bigr] = 0. \tag{80}$$

DEFINITION 2.5.37. (i) It is said that a function $q : [0,\infty) \to X$ is equi-Weyl-$(p,\phi,F)_2$-vanishing if and only if $q(t+\cdot) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l > 0$ and

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \sup_{x\geqslant 0}\phi\Bigl[F(l,t)\bigl\|q(t+v)\bigr\|_{L^{p(v)}[x,x+l]}\Bigr] = 0. \tag{81}$$

(ii) It is said that a function $q : [0,\infty) \to X$ is Weyl-$(p,\phi,F)_2$-vanishing if and only if $q(t+\cdot) \in L^{p(\cdot)}[x,x+l]$ for all $t,\ x,\ l > 0$ and

$$\lim_{t\to+\infty} \limsup_{l\to+\infty} \sup_{x\geqslant 0}\phi\Bigl[F(l,t)\bigl\|q(t+v)\bigr\|_{L^{p(v)}[x,x+l]}\Bigr] = 0. \tag{82}$$

Denote by $W^{p(x)}_{\phi,F,0}([0,\infty):X)$ and $e-W^{p(x)}_{\phi,F,0}([0,\infty):X)$ [$W^{p(x);1}_{\phi,F,0}([0,\infty):X)$ and $e-W^{p(x);1}_{\phi,F,0}([0,\infty):X)/W^{p(x);2}_{\phi,F,0}([0,\infty):X)$ and $e-W^{p(x);2}_{\phi,F,0}([0,\infty):X)$] the sets consisting of all Weyl-(p,ϕ,F)-vanishing functions and equi-Weyl-(p,ϕ,F)-vanishing functions [Weyl-$(p,\phi,F)_1$-vanishing functions and equi-Weyl-$(p,\phi,F)_1$-vanishing functions/Weyl-$(p,\phi,F)_2$-vanishing functions and equi-Weyl-$(p,\phi,F)_2$-vanishing functions], respectively. In the case that $p(x) \equiv p \in [1,\infty)$, $F(l,t) \equiv l^{(-1)/p}$ and $\phi(x) \equiv x$, the above classes coincide and we denote them by $W^p_0([0,\infty):X)$ and $e-W^p_0([0,\infty):X)$. These classes are very general and we want only to recall that, for instance, an equi-Weyl-p-vanishing function $q(\cdot)$ need not be bounded as $t \to +\infty$ ([**246**]).

A great number of very simple examples can be constructed in order to show that, in general case, the limit

$$\lim_{t\to+\infty} \sup_{x\geqslant 0}\Big[F(l,t)\big\|\phi(\|q(t+v)\|)\big\|_{L^{p(v)}[x,x+l]}\Big]$$

in the equation (77) does not exist for any fixed number $l > 0$; the same holds for the equations (78)-(82). The question when these limits exist is meaningful but it will not be analyzed here.

Further on, we have the following observation:

REMARK 2.5.38. (i) Suppose that the function $\phi(\cdot)$ is monotonically increasing and satisfies that for each scalars $\alpha,\ \beta \geqslant 0$ there exists a finite real number $\pi(\alpha,\beta) > 0$ such that, for every non-negative real numbers $x,\ y \geqslant 0$, we have

$$\phi(\alpha x + \beta y) \leqslant \pi(\alpha,\beta)[\phi(x) + \phi(y)].$$

Then (equi-)Weyl-(p,ϕ,F)-vanishing functions and (equi-)Weyl-$(p,\phi,F)_i$-vanishing functions, where $i = 1, 2$, form a vector space.

(ii) If the function $F(l,t)$ satisfies condition (D), resp. (D)', then the space of (equi-)Weyl-(p,ϕ,F)-vanishing functions and the space of (equi-)Weyl-$(p,\phi,F)_1$-vanishing functions, resp. the space of (equi-)Weyl-$(p,\phi,F)_2$-vanishing functions, are translation invariant.

In this section, we will not follow the approach obeyed in [**142**] and previous section, with the principal assumption $p \in \mathcal{P}([0,1])$. With regards to this question, we will present only one illustrative example:

EXAMPLE 2.5.39. Suppose that $p \in \mathcal{P}([0,1])$. Let us recall that the space of Stepanov $p(\cdot)$-vanishing functions (see [**142**]), denoted by $S_0^{p(x)}([0,\infty) : X)$, is consisting of those functions $q \in L_S^{p(x)}([0,\infty) : X)$ such that $\hat{q} \in C_0([0,\infty) : L^{p(x)}([0,1] : X))$. The notion of space $S_0^{p(x)}([0,\infty) : X)$ can be extended in many other ways; for example:

(i) Let $\phi : (0,\infty) \to (0,\infty)$ and $G : (0,\infty) \to (0,\infty)$. Then we say that a function $q(\cdot)$ belongs to the space $S^{p(\cdot)}_{\phi,G,0}([0,\infty) : X)$ if and only if $\phi(\|q(t+\cdot)\|) \in L^{p(\cdot)}[0,1]$ for all $t \geqslant 0$ and

$$\lim_{t\to+\infty} G(t)\big\|\phi(\|q(t+v)\|)\big\|_{L^{p(v)}[0,1]} = 0.$$

In this part, as well as in parts (ii) and (iii), we will use the 1-periodic extension of function $p(\cdot)$ to the non-negative real axis, denoted henceforth by $p_1(\cdot)$. Then the class $S^{p(\cdot)}_{\phi,G,0}([0,\infty) : X)$ is contained in the class of equi-Weyl-(p_1,ϕ,F)-vanishing functions with a suitable chosen function $F(l,t)$. More precisely, let a number $\varepsilon > 0$ be fixed. Then there exists a sufficiently large real number $t_0 > 0$ such that $\|\phi(q(t+v))\|_{L^{p(v)}[0,1]} < \varepsilon G(t)^{-1}$ for all numbers $t \geqslant t_0$. This implies that, for every $t \geqslant t_0$, $x \geqslant 0$ and $m \in \mathbb{N}_0$, we have

$$\int_0^1 \varphi_{p(v)}\Big(\phi(\|q(t+v+\lfloor x\rfloor+m)\|)/\big[\varepsilon G(t)^{-1}\big]\Big)\,dv \leqslant 1.$$

Using the inequality ($x \geqslant 0$, $l > 0$)

$$\int_x^{x+l} \varphi_{p_1(v)}\Big(\phi(\|q(t+v)\|)/\big[\varepsilon G(t)^{-1}\big]\Big)\,dv$$

$$\leqslant \sum_{k=0}^{l} \int_{\lfloor x\rfloor+k}^{\lfloor x\rfloor+k+1} \varphi_{p_1(v)}\Big(\phi(\|q(t+v)\|)/\big[\varepsilon G(t)^{-1}\big]\Big)\,dv,$$

the above yields

$$\int_x^{x+l} \varphi_{p_1(v)}\Big(\phi(\|q(t+v)\|)/\big[\varepsilon G(t)^{-1}\big]\Big)\,dv \leqslant l+1, \text{ i.e.,}$$

$$\int_x^{x+l} \frac{1}{l+1}\varphi_{p_1(v)}\Big(\phi(\|q(t+v)\|)/\big[\varepsilon G(t)^{-1}\big]\Big)\,dv \leqslant 1.$$

Since

$$\varphi_{p_1(v)}\Big(\phi(\|q(t+v)\|)/\big[\varepsilon (l+1) G(t)^{-1}\big]\Big) \leqslant \frac{1}{l+1}\varphi_{p_1(v)}\Big(\phi(\|q(t+v)\|)/\big[\varepsilon G(t)^{-1}\big]\Big),$$

the above implies $\|\phi(\|q(t+v)\|)\|_{L^{p(v)}[x,x+l]} < \varepsilon G(t)^{-1}(1+l)$ for all $t \geqslant t_0$, $x \geqslant 0$ and $l > 0$. Hence, the required conclusion holds provided that there exists a finite real constant $C > 0$ such that

$$\Big|F(l,t)G(t)^{-1}(1+l)\Big| \leqslant C, \quad l > 0, \ t > 0.$$

(ii) Let $\phi : (0,\infty) \to (0,\infty)$ and $G : (0,\infty) \to (0,\infty)$. Then we say that a function $q(\cdot)$ belongs to the space $S^{p(\cdot)}_{\phi,G,0;1}([0,\infty) : X)$ if and only if $q(t+\cdot) \in L^{p(\cdot)}[0,1]$ for all $t \geqslant 0$ and

$$\lim_{t\to+\infty} G(t)\phi\Big(\big\|q(t+v)\big\|_{L^{p(v)}[0,1]}\Big) = 0.$$

Then the class $S^{p(\cdot)}_{\phi,G,0;1}([0,\infty) : X)$ is contained in the class of equi-Weyl-$(p_1,\phi,F)_1$-vanishing functions with a suitable chosen function $F(l,t)$. Arguing as in (i), this holds provided that, for example, $\sup \phi^{-1}([0,G(t)^{-1}]) < \infty$ and

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \ F(l,t)(l+1)\sup \phi^{-1}\Big(\big[0,G(t)^{-1}\big]\Big) = 0.$$

(iii) Let $\phi : (0,\infty) \to (0,\infty)$ and $G : (0,\infty) \to (0,\infty)$. Then we say that a function $q(\cdot)$ belongs to the space $S^{p(\cdot)}_{\phi,G,0;2}([0,\infty) : X)$ if and only if $q(t+\cdot) \in L^{p(\cdot)}[0,1]$ for all $t \geqslant 0$ and

$$\lim_{t\to+\infty} \phi\Big(G(t)\big\|\phi(q(t+v))\big\|_{L^{p(v)}[0,1]}\Big) = 0.$$

Then the class $S^{p(\cdot)}_{\phi,G,0;2}([0,\infty) : X)$ is contained in the class of equi-Weyl-$(p_1,\phi,F)_2$-vanishing functions with a suitable chosen function $F(l,t)$. Arguing as in (i), this holds provided that, for example, the function $\phi(\cdot)$ is monotonically increasing, $\sup \phi^{-1}([0,1]) < +\infty$ and

$$\lim_{l\to+\infty}\limsup_{t\to+\infty}\ \phi\Bigl(F(l,t)G(t)^{-1}(1+l)\sup\phi^{-1}([0,1])\Bigr)=0.$$

An analogue of Proposition 2.5.26 can be proved for (equi-)Weyl-(p,ϕ,F)-vanishing functions and (equi-)Weyl-$(p,\phi,F)_i$-vanishing functions, provided that the function $\phi(\cdot)$ is convex and $q(v)\equiv 1$. Furthermore, an analogue of Remark 2.5.27(i)-(ii) can be formulated for (equi-)Weyl-(p,ϕ,F)-vanishing functions and (equi-)Weyl-$(p,\phi,F)_i$-vanishing functions. Concerning Lemma 1.1.6(ii) and Remark 2.5.31(v), it should be noted that the embedding type result established in already mentioned [**147**, Corollary 3.3.4] for scalar-valued functions (see also Lemma 1.1.6(ii)) enables one to see that the following expected result holds true:

PROPOSITION 2.5.40. *Suppose $r,\ p\in\mathcal{P}([0,\infty))$ and $1\leqslant r(x)\leqslant p(x)$ for a.e. $x\geqslant 0$. Let $F_1(l,t)=2\max(l^{essinf(1/r(x)-1/p(x))},l^{esssup(1/r(x)-1/p(x))})F(l,t)$ or $F_1(l,t)=2(1+l)F(l,t)$ for all $l>0$ and $t\geqslant 0$. Then we have:*

(i) *If the function $q(\cdot)$ is (equi-)Weyl-(r,ϕ,F)-vanishing provided that $q(\cdot)$ is (equi-)Weyl-(p,ϕ,F_1)-vanishing.*

(ii) *Suppose that there exists a function $\varphi:[0,\infty)\to[0,\infty)$ such that $\phi(cx)\leqslant\varphi(c)\phi(x)$ for all $c\geqslant 0$ and $x\geqslant 0$. Let $F_2(l,t)=\varphi(2(1+l))F(l,t)$ or $F_1(l,t)=\varphi(2\max(l^{essinf(1/r(x)-1/p(x))},l^{esssup(1/r(x)-1/p(x))}))F(l,t)$ for $l>0$ and $t\geqslant 0$. Then the function $q(\cdot)$ is (equi-)Weyl-$(r,\phi,F)_1$-vanishing provided that $q(\cdot)$ is (equi-)Weyl-$(p,\phi,F_2)_1$-vanishing.*

(iii) *If $\phi(\cdot)$ is monotonically increasing, then the function $q(\cdot)$ is (equi-)Weyl-$(r,\phi,F)_2$-vanishing provided that $q(\cdot)$ is (equi-)Weyl-$(p,\phi,F_1)_2$-vanishing.*

The case of constant coefficients $1\leqslant r\leqslant p$ also deserves attention, when the choices $F_1(l,t)=l^{1/r-1/p}F(l,t)$ in (i), (iii) and $F_1(l,t)=\varphi(l^{1/r-1/p})F(l,t)$ in (ii) can be made.

We continue by reexaming the conclusions established in [**246**, Example 4.5, Example 4.6]:

EXAMPLE 2.5.41. Define

$$q(t):=\sum_{n=0}^{\infty}\chi_{[n^2,n^2+1]}(t),\quad t\geqslant 0.$$

Then we know that $\hat{q}\notin C_0([0,\infty):L^p([0,1]:\mathbb{C}))$ and the function $q(\cdot)$ is equi-Weyl-p-almost periodic for any exponent $p\geqslant 1$; see [**246**, Example 4.5]. In this example, we have proved the estimate

$$\left(\int_x^{x+l}\bigl\|q(t+v)\bigr\|^p\,dv\right)^{1/p}\leqslant\left(2+\frac{l}{\sqrt{t}+\sqrt{l}}\right)^{1/p}\leqslant 2+\left(\frac{l}{\sqrt{t}+\sqrt{l}}\right)^{1/p},$$

for any $x \geqslant 0,\ t \geqslant 0,\ l > 0$, so that the function $q(\cdot)$ is equi-Weyl-(p, x, F)-vanishing provided that

$$\lim_{l \to +\infty} \limsup_{t \to +\infty} \ F(l,t)\left[2 + \left(\frac{l}{\sqrt{t}+\sqrt{l}}\right)^{1/p}\right] = 0.$$

In particular, this holds for function $F(l,t) = l^{\sigma}$, where $\sigma < 0$.

EXAMPLE 2.5.42. Define

$$q(t) := \sum_{n=0}^{\infty} \sqrt{n}\chi_{[n^2,n^2+1]}(t), \quad t \geqslant 0.$$

Then we know that the function $q(\cdot)$ is not equi-Weyl-p-vanishing for any exponent $p \geqslant 1$ as well as that the function $q(\cdot)$ is Weyl-p-vanishing for any exponent $p \geqslant 1$; see [**246**, Example 4.6]. In this example, we have proved the estimate

$$\left(\int_x^{x+l} \bigl\|q(t+v)\bigr\|^p \, dv\right)^{1/p} \leqslant \bigl(l+t\bigr)^{1/2p}, \quad x \geqslant 0,\ t \geqslant 0,\ l > 0,$$

so that the function $q(\cdot)$ is Weyl-(p, x, F)-vanishing provided that

$$\lim_{t \to +\infty} \limsup_{l \to +\infty} \ F(l,t)\bigl(l+t\bigr)^{1/2p} = 0.$$

In particular, this holds for function $F(l,t) = l^{\sigma}$, where $\sigma < (-1)/2p$.

We will present one more illustrative example:

EXAMPLE 2.5.43. Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are two sequences of positive real numbers such that $(a_n)_{n\in\mathbb{N}}$ is strictly monotonically increasing, $\lim_{n\to+\infty}(a_{n+1} - a_n) = +\infty$ and $\lim_{n\to+\infty} \phi(b_n) = 0$. Let $q : [0,\infty) \to (0,\infty)$ be defined by $q(t) := b_n$ if and only if $t \in [a_{n-1}, a_n)$ for some $n \in \mathbb{N}$, where $a_0 := 0$. If $p \in D_+([0,\infty))$, $l > 0$ and $t > 0$, then we have

$$\begin{aligned}
&\sup_{x \geqslant 0}\Bigl[F(l,t)\bigl\|\phi(q(t+v))\bigr\|_{L^{p(v)}[x,x+l]}\Bigr] \\
&\qquad \leqslant \sup_{x \geqslant 0}\Bigl[2(1+l)F(l,t)\bigl\|\phi(q(t+\cdot))\bigr\|_{L^{p^+}[x,x+l]}\Bigr] \\
&\qquad = \sup_{x \geqslant 0}\Bigl[2(1+l)F(l,t)\bigl\|\phi(q(\cdot))\bigr\|_{L^{p^+}[t+x,t+x+l]}\Bigr].
\end{aligned}$$

Assume, additionally, that there exists a function $G : (0,\infty) \to (0,\infty)$ such that $F(l,t) \leqslant G(l)$ for all $l > 0$ and $t > 0$. Since we have assumed that $\lim_{n\to+\infty}(a_{n+1} - a_n) = +\infty$, for each number $l > 0$ we have

$$\limsup_{t \to +\infty} \sup_{x \geqslant 0}\Bigl[2(1+l)F(l,t)\bigl\|\phi(q(\cdot))\bigr\|_{L^{p^+}[t+x,t+x+l]}\Bigr] = 0,$$

because $\lim_{n\to+\infty} \phi(b_n) = 0$ and

$$\bigl\|\phi(q(\cdot))\bigr\|_{L^{p^+}[t+x,t+x+l]} \leqslant l \max\bigl(\phi(b_n),\ \phi(b_{n+1})\bigr),$$

where $n \in \mathbb{N}$ is such that $x+t \leqslant a_n$ and $x+t+l \leqslant a_{n+1}$. Therefore, the function $q(\cdot)$ is equi-Weyl-(p,ϕ,F)-vanishing.

In [**246**], we have introduced a great number of various types of asymptotically Weyl almost periodic function spaces with constant exponent $p \geqslant 1$. In order to relax our exposition, we will introduce here only one general definition of an asymptotically Weyl almost periodic function with variable exponent, which extends the notion introduced in Definition 2.6.1(ii):

Definition 2.5.44. Let $h : I \to X$. Then we say that $h(\cdot)$ is asymptotically Weyl almost periodic with variable exponent if and only if there exist two functions $g : \mathbb{R} \to X$ and $q : I \to X$ such that $h(t) = g(t) + q(t)$ for a.e. $t \in I$, $g(\cdot)$ belongs to some of function spaces introduced in Definition 2.5.22-Definition 2.5.24 or Definition 2.5.28-Definition 2.5.30 and $q(\cdot)$ belongs to some of function spaces introduced in Definition 2.5.35-Definition 2.5.37 (with possibly different functions $p,\ p_1;\ \phi,\ \phi_1;\ F,\ F_1$ and the meaning clear).

Observe that we can also extend the notion of Weyl p-pseudo ergodic component ($p \geqslant 1$) following the approach obeyed in the previous part of section and provide certain extensions of [**246**, Proposition 4.11] in this context. Details can be left to the interested reader.

2.5.6. Weyl almost periodicity with variable exponent and convolution products. In the analyses of (equi-)Weyl-(p,ϕ,F)-almost periodic functions and (equi-)Weyl-$[p,\phi,F]$-almost periodic functions, we will use the following conditions:

- (A1): $I=\mathbb{R}$ or $I=[0,\infty)$, $\psi : (0,\infty) \to (0,\infty)$, $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$, $p \in \mathcal{P}(I)$.
- (B1): The same as (A) with the assumption $p \in \mathcal{P}(I)$ replaced by $p \in \mathcal{P}([0,1])$ therein.

Theorem 2.5.45. *Suppose that condition* (A1) *holds with* $I=\mathbb{R}$, $\check{g} : \mathbb{R} \to X$ *is (equi-)Weyl-*(p,ϕ,F)*-almost periodic and measurable,* $F_1 : (0,\infty) \times I \to (0,\infty)$, $p,\ q \in \mathcal{P}(\mathbb{R})$, $1/p(x)+1/q(x)=1$, $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* (a_k) *is a sequence of positive real numbers such that* $\sum_{k=0}^{\infty} a_k = 1$. *If for every real numbers* $x,\ \tau \in \mathbb{R}$ *we have*

$$\int_{-x}^{\infty} \|R(v+x)\| \|\check{g}(v)\| \, dv < \infty, \tag{83}$$

and if, for every $t \in \mathbb{R}$ *and* $l>0$, *we have*

(84)

$$H(l,x) := \sum_{k=0}^{\infty} a_k \varphi(l a_k^{-1}) \big\| \varphi\big(\|R(v+x)\|\big) \big\|_{L^{q(v)}[-x+kl,-x+(k+1)l]} F(l,-x+lk)^{-1} < \infty,$$

$$\int_t^{t+l} \varphi_{p(x)}\Big(2l^{-1}H(l,x)F_1(l,t)^{-1}\Big)\, dx \leqslant 1, \tag{85}$$

resp. if (84) *holds and there exists* $l_0 > 0$ *such that for all* $l \geqslant l_0$ *and* $t \in \mathbb{R}$ *we have* (85), *then the function* $G : \mathbb{R} \to Y$, *given by* (55), *is well-defined and (equi-)Weyl-*(p, ϕ, F_1)*-almost periodic.*

PROOF. We will prove the theorem only for the class of equi-Weyl-(p, ϕ, F)-almost periodic functions. Since $G(x) = \int_{-x}^{\infty} R(v+x)\breve{g}(v)\,dv$, $x \in \mathbb{R}$, the estimate in (117) shows that the function $G(\cdot)$ is well-defined and that the integral in definition of $G(x)$ converges absolutely ($x \in \mathbb{R}$). Furthermore, the same estimate shows that for each real number τ we have $\int_{-x}^{\infty} \|R(v+x)\|\|\breve{g}(v+\tau)\|\,dv = \int_{-(x-\tau)}^{\infty} \|R(v+(x-\tau))\|\|\breve{g}(v)\|\,dv < \infty$, so that the integral in definition of $G(x+\tau) - G(x)$ converges absolutely ($x \in \mathbb{R}$). Let $\varepsilon > 0$ be a fixed real number. Then we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (74) holds for the function $\breve{g}(\cdot)$, with the number τ replaced by the number $-\tau$ therein. Using our assumptions from condition (A1), the Jensen integral inequality applied to the function $\phi(\cdot)$ (see also condition (117)), the fact that the functions $\phi(\cdot)$ and $\varphi_{p(x)}(\cdot)$ are monotonically increasing, (69) and Lemma 1.1.6(i), we get that for each real number $x \in \mathbb{R}$ the following holds:

$$
\begin{aligned}
&\varphi_{p(x)}\Big(\phi(\|G(x+\tau)-G(x)\|)/\lambda\Big)\\
&\leqslant \varphi_{p(x)}\Bigg(\phi\Big(\int_{-x}^{\infty}\|R(v+x)\|\|\breve{g}(v+\tau)-\breve{g}(v)\|\,dv\Big)/\lambda\Bigg)\\
&= \varphi_{p(x)}\Bigg(\phi\Big(\sum_{k=0}^{\infty}a_k\int_{-x+kl}^{-x+(k+1)l}a_k^{-1}\|R(v+x)\|\|\breve{g}(v+\tau)-\breve{g}(v)\|\,dv\Big)/\lambda\Bigg)\\
&\leqslant \varphi_{p(x)}\Bigg(\sum_{k=0}^{\infty}a_k\phi\Big(\int_{-x+kl}^{-x+(k+1)l}a_k^{-1}\|R(v+x)\|\|\breve{g}(v+\tau)-\breve{g}(v)\|\,dv\Big)/\lambda\Bigg)\\
&\leqslant \varphi_{p(x)}\Bigg(\sum_{k=0}^{\infty}a_k\phi\Big(la_k^{-1}\cdot l^{-1}\int_{-x+kl}^{-x+(k+1)l}\|R(v+x)\|\|\breve{g}(v+\tau)-\breve{g}(v)\|\,dv\Big)/\lambda\Bigg)\\
&\leqslant \varphi_{p(x)}\Bigg(l^{-1}\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\int_{-x+kl}^{-x+(k+1)l}\phi\Big(\|R(v+x)\|\|\breve{g}(v+\tau)-\breve{g}(v)\|\Big)\,dv/\lambda\Bigg)\\
&\leqslant \varphi_{p(x)}\Bigg(l^{-1}\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\int_{-x+kl}^{-x+(k+1)l}\varphi(\|R(v+x)\|)\phi\Big(\|\breve{g}(v+\tau)-\breve{g}(v)\|\Big)\,dv/\lambda\Bigg)\\
&\leqslant \varphi_{p(x)}\Bigg(2l^{-1}\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\big\|\varphi(\|R(v+x)\|)\big\|_{L^{q(v)}[-x+kl,-x+(k+1)l]}\\
&\times\phi\Big(\|\breve{g}(v+\tau)-\breve{g}(v)\|\Big)_{L^{p(v)}[-x+kl,-x+(k+1)l]}\Big)/\lambda\Bigg)\\
&\leqslant \varphi_{p(x)}\Bigg(2l^{-1}\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\big\|\varphi(\|R(v+x)\|)\big\|_{L^{q(v)}[-x+kl,-x+(k+1)l]}\varepsilon F(l,-x+kl)^{-1}/\lambda\Bigg).
\end{aligned}
$$

Let $K \subseteq \mathbb{R}$ be an arbitrary compact set. Since the above computation holds for every real number $\tau \in \mathbb{R}$ and for every arbitrarily large real number $l > 0$, we can find $t \in \mathbb{R}$ such that $K \subseteq [t, t+l]$. Now we get from (85) that the function

$\phi(\|G(\cdot+\tau)-G(\cdot)\|)$ belongs to the space $L^{p(x)}(K)$ by definition. Condition (85) and the above computation also imply that for each real number $t \in \mathbb{R}$ we have

$$\int_t^{t+l} \varphi_{p(x)}\Bigl(\phi(\|G(x+\tau)-G(x)\|)/\lambda\Bigr)\,dx \leqslant 1,$$

with $\lambda=\varepsilon F_1(l,t)$, which simply implies the final conclusion. □

REMARK 2.5.46. (i) Suppose that $p(x) \equiv p \in [1,\infty)$. Then condition (85) can be weakened to

$$\int_t^{t+l} \varphi_{p(x)}\Bigl(l^{-1}H(l,x)F_1(l,t)^{-1}\Bigr)\,dx \leqslant 1, \tag{86}$$

resp. there exists $l_0>0$ such that for all $l \geqslant l_0$ and $t \in \mathbb{R}$ we have (86).

(ii) Suppose that $\phi(x)=\varphi(x)=\psi(x)=x$. Then condition (85), resp. (86), holds provided that $l \geqslant 1$ and the term in the large brackets in this equation does not exceed $1/l$ or that $0<l<1$ and the term in the large brackets in this equation does not exceed 1. Similar comments can be made in the case of consideration of Theorem 2.5.48 below (see also Corollary 2.3.3).

COROLLARY 2.5.47. *Suppose that condition* (A1) *holds with* $I=\mathbb{R}$, $p(x) \equiv p \geqslant 1$, $1/p+1/q=1$, $\breve{g}:\mathbb{R}\rightarrow X$ *is (equi-)Weyl-*(p,ϕ,F)*-almost periodic and measurable,* $F_1:(0,\infty)\times I \rightarrow (0,\infty)$, $(R(t))_{t>0}\subseteq L(X,Y)$ *is a strongly continuous operator family and* (a_k) *is a sequence of positive real numbers such that* $\sum_{k=0}^{\infty}a_k=1$. *If for every real numbers* $x,\ \tau\in\mathbb{R}$ *we have* (117) *and if, for every* $t\in\mathbb{R}$ *and* $l>0$, *we have*

$$H_p(l,x):=\sum_{k=0}^{\infty}a_k\varphi(la_k^{-1})\bigl\|\varphi\bigl(\|R(\cdot)\|\bigr)\bigr\|_{L^q[kl,(k+1)l]}F(l,-x+lk)^{-1}<\infty \tag{87}$$

and

$$\int_t^{t+l}\Bigl(l^{-1}H_p(l,x)F_1(l,t)^{-1}\Bigr)^p\,dx \leqslant 1, \tag{88}$$

resp. if (87) *holds and there exists* $l_0>0$ *such that for all* $l\geqslant l_0$ *and* $t\in\mathbb{R}$ *we have* (88), *then the function* $G:\mathbb{R}\rightarrow Y$, *given by* (55), *is well-defined and (equi-)Weyl-*(p,ϕ,F_1)*-almost periodic.*

Now we will state and prove the following result with regards to the class of (equi-)Weyl-$[p,\phi,F]$-almost periodic functions:

THEOREM 2.5.48. *Suppose that condition* (B1) *holds with* $I=\mathbb{R}$, $g:\mathbb{R}\rightarrow X$ *is measurable,* $\omega:(0,\infty)\rightarrow(0,\infty)$, $F:(0,\infty)\times I\rightarrow(0,\infty)$, (a_k) *is a sequence of positive real numbers such that* $\sum_{k=0}^{\infty}a_k=1$, $(b_k)_{k\geqslant 0}$ *is a sequence of positive real numbers,* $S:(0,\infty)\times\mathbb{R}\rightarrow(0,\infty)$ *is a given function, as well as for each* $\varepsilon>0$ *we can find two real numbers* $l>0$ *and* $L>0$ *such that any interval* $I'\subseteq I$ *of length*

L contains a point $\tau \in I'$ such that

(89)
$$\sup_{x\in[0,1]}\Bigl[\phi\bigl(\bigl\|g(xl+t-r-k+\tau)-g(xl+t-r-k)\bigr\|\bigr)_{L^{p(r)}[0,1]}\Bigr] \leqslant \omega(\varepsilon)b_k S(l,t)$$

for any integer $k \geqslant 0$ and real number $t \in \mathbb{R}$. Suppose, further, that the second inequality in (117) holds, $p,\ q \in \mathcal{P}([0,1])$, $1/p(x)+1/q(x)=1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family. If for every real numbers $t,\ \tau \in \mathbb{R}$, every positive real number $l>0$ and every real number $x \in [0,1]$ we have

$$\int_0^\infty \|R(v)\|\|g(xl+t+\tau-v)-g(xl+t-v)\|\,dv<\infty, \tag{90}$$

and if, for every $t \in \mathbb{R}$, $x \in [0,1]$ and $l,\ \varepsilon>0$, we have

$$W(x):=\sum_{k=0}^{\infty} a_k\varphi\bigl(a_k^{-1}\bigr)\bigl\|\varphi\bigl(\|R(v+x)\|\bigr)\bigr\|_{L^{q(v)}[0,1]}b_k<\infty, \tag{91}$$

$$\int_0^1 \varphi_{p(x)}\Bigl(2\varepsilon^{-1}F_1(l,t)^{-1}\omega(\varepsilon)S(l,t)W(x)\Bigr)\,dx \leqslant 1, \tag{92}$$

resp. if (91) holds and there exists $l_0>0$ such that for all $l \geqslant l_0$, $\varepsilon>0$ and $t \in \mathbb{R}$ we have (92), then the function $G:\mathbb{R}\to Y$, given by (55), is well-defined and (equi-)Weyl-$[p,\phi,F_1]$-almost periodic.

PROOF. We will prove the theorem only for the class of equi-Weyl-$[p,\phi,F]$-almost periodic functions. As above, the function $G(\cdot)$ is well-defined. Let $\varepsilon>0$ be a fixed real number. Then we can find two real numbers $l>0$ and $L>0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (89) holds for any integer $k \geqslant 0$ and any real number $t \in \mathbb{R}$. Using our assumptions from condition (B1), the Jensen integral inequality applied to the function $\phi(\cdot)$ (see also condition (90)), the fact that the functions $\phi(\cdot)$ and $\varphi_{p(x)}(\cdot)$ are monotonically increasing, (69) and Lemma 1.1.6(i), we get that, for every real numbers $x \in [0,1]$ and $t \in \mathbb{R}$, the following holds:

$$\begin{aligned}
&\varphi_{p(x)}\Bigl(\phi(\|G(xl+t+\tau)-G(xl+t)\|)/\lambda\Bigr)\\
\leqslant&\varphi_{p(x)}\left(\phi\Bigl(\int_0^\infty \|R(v)\|\|g(xl+t+\tau-v)-g(xl+t-v)\|\,dv\Bigr)/\lambda\right)\\
=&\varphi_{p(x)}\left(\phi\Bigl(\sum_{k=0}^{\infty}a_k\int_0^1 a_k^{-1}\|R(v+k)\|\|g(xl+t+\tau-v-k)-g(xl+t-v-k)\|\,dv\Bigr)/\lambda\right)\\
\leqslant&\varphi_{p(x)}\left(\sum_{k=0}^{\infty}a_k\int_0^1 \phi\Bigl(a_k^{-1}\|R(v+k)\|\|g(xl+t+\tau-v-k)-g(xl+t-v-k)\|\,dv\Bigr)/\lambda\right)\\
\leqslant&\varphi_{p(x)}\left(\sum_{k=0}^{\infty}a_k\varphi\bigl(a_k^{-1}\bigr)\int_0^1 \varphi\bigl(\|R(v+k)\|\bigr)\right.\\
\times&\left.\phi\Bigl(\bigl\|g(xl+t+\tau-v-k)-g(xl+t-v-k)\bigr\|\Bigr)\,dv/\lambda\right)
\end{aligned}$$

$$\leqslant\varphi_{p(x)}\Biggl(\frac{2}{\lambda}\sum_{k=0}^{\infty}a_k\varphi\bigl(a_k^{-1}\bigr)\varphi\bigl(\|R(v+k)\|\bigr)_{L^{q(v)}[0,1]}$$

$$\times\phi\Bigl(\|g(xl+t+\tau-v-k)-g(xl+t-v-k)\|\Bigr)_{L^{p(v)}[0,1]}\Biggr)$$

$$\leqslant\varphi_{p(x)}\Biggl(\frac{2}{\lambda}\sum_{k=0}^{\infty}a_k\varphi\bigl(a_k^{-1}\bigr)\varphi\bigl(\|R(v+k)\|\bigr)_{L^{q(v)}[0,1]}\omega(\varepsilon)b_kS(l,t)\Biggr).$$

Arguing as in the proof of Theorem 2.5.45, we get from condition (92) that the function $\phi(\|G(\cdot l+t+\tau)-G(t+\cdot l)\|)$ belongs to the space $L^{p(\cdot)}([0,1])$ for arbitrary real numbers $\tau,\ t\in\mathbb{R}$ and $l>0$. Condition (92) implies that for each real numbers $t\in\mathbb{R}$ and $x\in[0,1]$ we have

$$\int_0^1\varphi_{p(x)}\Bigl(\phi(\|G(xl+t+\tau)-G(xl+t)\|)/\lambda\Bigr)\,dx\leqslant 1,$$

with $\lambda=\varepsilon F_1(l,t)^{-1}$, which simply implies the final conclusion. □

COROLLARY 2.5.49. *Suppose that condition* (B1) *holds with* $I=\mathbb{R}$ *and* $p(x)\equiv p\in[1,\infty)$, $1/p+1/q=1$, $g:\mathbb{R}\to X$ *is measurable,* $\omega:(0,\infty)\to(0,\infty)$, $F:(0,\infty)\times I\to(0,\infty)$, (a_k) *is a sequence of positive real numbers such that* $\sum_{k=0}^{\infty}a_k=1$, $(b_k)_{k\geqslant 0}$ *is a sequence of positive real numbers,* $S:(0,\infty)\times\mathbb{R}\to(0,\infty)$ *is a given function, as well as for each* $\varepsilon>0$ *we can find real numbers* $l>0$ *and* $L>0$ *such that any interval* $I'\subseteq I$ *of length* L *contains a point* $\tau\in I'$ *such that* (89) *holds with* $p(r)\equiv p$, *for any integer* $k\geqslant 0$ *and any real number* $t\in\mathbb{R}$. *Suppose, further, that the second inequality in* (117) *holds, and* $(R(t))_{t>0}\subseteq L(X,Y)$ *is a strongly continuous operator family. If for every real numbers* $t,\ \tau\in\mathbb{R}$, *every positive real number* $l>0$ *and every real number* $x\in[0,1]$ *we have* (90), *and if, for every* $t\in\mathbb{R}$, $x\in[0,1]$ *and* $l>0$, *we have*

$$W_p(x):=\sum_{k=0}^{\infty}a_k\varphi\bigl(a_k^{-1}\bigr)\bigl\|\varphi\bigl(\|R(\cdot)\|\bigr)\bigr\|_{L^q[x,x+1]}b_k<\infty \tag{93}$$

and

$$\int_0^1\varphi_{p(x)}\Bigl(2F_1(l,t)^{-1}S(l,t)W_p(x)\Bigr)\,dx\leqslant 1, \tag{94}$$

resp. if (93) *holds and there exists* $l_0>0$ *such that for all* $l\geqslant l_0$ *and* $t\in\mathbb{R}$ *we have* (94), *then the function* $G:\mathbb{R}\to Y$, *given by* (55), *is well-defined and (equi-)Weyl-*$[p,\phi,F_1]$*-almost periodic.*

Concerning Theorem 2.5.48, it should be noted that, in [**142**, Proposition 6.1], we have analyzed the situation in which the function $\check{g}:\mathbb{R}\to X$ is $S^{p(x)}$-almost periodic and $\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(\cdot)}[0,1]}<\infty$. Then the resulting function $G(\cdot)$ is almost periodic, which cannot be derived from the above-mentioned theorem.

For the class of (equi-)Weyl-$(p,\phi,F)_1$-almost periodic functions, we will state the following result:

THEOREM 2.5.50. *Suppose that* $\breve{g} : \mathbb{R} \to X$ *is (equi-)Weyl-*$(p, \phi, F)_1$*-almost periodic and measurable,* $F_1 : (0, \infty) \times I \to (0, \infty)$, $p,\ q \in \mathcal{P}(\mathbb{R})$, $1/p(x)+1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X, Y)$ *is a strongly continuous operator family and for every real numbers* $x,\ \tau \in \mathbb{R}$ *we have* (117). *Suppose that, for every real number* $t \in \mathbb{R}$ *and positive real numbers* $l,\ \varepsilon > 0$, *there exist two positive real numbers* $a > 0$ *and* $\lambda > 0$ *such that* $\lambda \leqslant a$, $[0, a] \subseteq \phi^{-1}([0, \varepsilon F(l, t)^{-1}])$,

$$\sum_{k=0}^{\infty} \big\| R(v + x) \big\|_{L^{q(v)}[-x+kl, -x+(k+1)l]} \sup \phi^{-1}\Big(\big[0, \varepsilon F(l, -x + kl)^{-1}\big]\Big) < \infty \tag{95}$$

and the term

(96)

$$\int_t^{t+l} \varphi_{p(x)}\left(2 \frac{\sum_{k=0}^{\infty} \big\| R(v + x) \big\|_{L^{q(v)}[-x+kl, -x+(k+1)l]} \sup \phi^{-1}\Big(\big[0, \varepsilon F(l, -x + kl)^{-1}\big]\Big)}{\lambda}\right) dx$$

does not exceed 1, *resp.* (95) *holds and there exists* $l_0 > 0$ *such that for all* $l \geqslant l_0$, $\varepsilon > 0$ *and* $t \in \mathbb{R}$ *we have that the term in* (96) *does not exceed* 1, *then the function* $G : \mathbb{R} \to Y$, *given by* (55), *is well-defined and (equi-)Weyl-*$(p, \phi, F_1)_1$*-almost periodic.*

PROOF. As in the proof of Theorem 2.5.45, we have that the function $G(\cdot)$ is well-defined and the integrals in definitions of $G(x)$ and $G(x + \tau) - G(x)$ converge absolutely ($x,\ \tau \in \mathbb{R}$). By Lemma 1.1.6(ii), we get that the function $G(\cdot+\tau)-G(\cdot)$ belongs to the space $L^{p(x)}(K)$ for each compact set $K \subseteq \mathbb{R}$. The remainder follows similarly as in the proof of Theorem 2.5.45, by using condition (95), as well as the estimates

$$\begin{aligned} \|G(x + \tau) - G(x)\| \leqslant 2 \sum_{k=0}^{\infty} \big\| R(v + x) \big\|_{L^{q(v)}[-x+kl, -x+(k+1)l]} \\ \times \big\| \breve{g}(v + \tau) - \breve{g}(v) \big\|_{L^{p(v)}[-x+kl, -x+(k+1)l]} \end{aligned}$$

and

$$\big\| \breve{g}(v + \tau) - \breve{g}(v) \big\|_{L^{p(v)}[-x+kl, -x+(k+1)l]} \leqslant \sup \phi^{-1}\Big(\big[0, \varepsilon F(l, -x + kl)^{-1}\big]\Big),$$

and the equivalence relation

$$\begin{aligned} &\phi\Big(\big\| G(\cdot + \tau) - G(\cdot) \big\|_{L^{p(x)}[t, t+l]}\Big) \leqslant \varepsilon F_1(l, t)^{-1} \\ &\qquad \Leftrightarrow \big\| G(\cdot + \tau) - G(\cdot) \big\|_{L^{p(x)}[t, t+l]} \leqslant \phi^{-1}\Big(\big[0, \varepsilon F_1(l, t)^{-1}\big]\Big), \end{aligned}$$

for any $x,\ t,\ \tau \in \mathbb{R}$ and $l > 0$. □

Concerning the class of (equi-)Weyl-$[p, \phi, F]_1$-almost periodic functions, we can state the following result; the proof can be deduced as above and therefore omitted (we can similarly formulate analogues of Corollary 2.5.47 and Corollary 2.5.49, as well as the conclusions from Remark 2.5.46):

THEOREM 2.5.51. *Suppose that $g : \mathbb{R} \to X$ is measurable, $\omega : (0,\infty) \to (0,\infty)$, $F : (0,\infty) \times I \to (0,\infty)$, $(b_k)_{k\geqslant 0}$ is a sequence of positive real numbers, $S : (0,\infty) \times \mathbb{R} \to (0,\infty)$ is a given function, as well as for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that*

$$\sup_{x\in[0,1]} \Bigl[\bigl\| g(xl + t - r - k + \tau) - g(xl + t - r - k) \bigr\|_{L^{p(r)}[0,1]} \Bigr] \leqslant \omega(\varepsilon) b_k S(l,t)$$

for any integer $k \geqslant 0$ and real number $t \in \mathbb{R}$. Suppose, further, that the second inequality in (117) *holds, $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family. If for every real numbers $t,\ \tau \in \mathbb{R}$, every positive real number $l > 0$ and every real number $x \in [0,1]$ we have* (90), *if*

$$W_2(x) := \sum_{k=0}^{\infty} \bigl\| R(v + x) \bigr\|_{L^{q(v)}[0,1]} b_k < \infty, \quad x \in [0,1], \tag{97}$$

and if, for every $t \in \mathbb{R}$ and $l,\ \varepsilon > 0$, we have the existence of two positive real numbers $a > 0$ and $\lambda > 0$ such that $\lambda \leqslant a$, $[0,a] \subseteq \phi^{-1}([0, \varepsilon F_1(l,t)^{-1}])$ and

$$\int_0^1 \varphi_{p(x)} \left(2 \frac{\omega(\varepsilon) S(l,t) W_2(x)}{\lambda} \right) dx \leqslant 1, \tag{98}$$

resp. if (97) *holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ we have* (98), *then the function $G : \mathbb{R} \to Y$, given by* (55), *is well-defined and (equi-)Weyl-$[p, \phi, F_1]$-almost periodic.*

REMARK 2.5.52. The assertions of Theorem 2.5.50, resp. Theorem 2.5.51, can be much simpler formulated provided that:

- (A2): The function $\phi : [0,\infty) \to [0,\infty)$ is a monotonically increasing bijection and $p \in \mathcal{P}(\mathbb{R})$, resp.
- (B2): The function $\phi : [0,\infty) \to [0,\infty)$ is a monotonically increasing bijection and $p \in \mathcal{P}([0,1])$.

Any of these conditions implies that the function $\phi^{-1} : [0,\infty) \to [0,\infty)$ is a monotonically increasing bijection, as well. If condition (A2), resp. (B2), holds, then the class of (equi-)Weyl-$(p,\phi,F)_2$-almost periodic functions, resp. (equi-)Weyl-$[p,\phi,F]_2$-almost periodic functions, coincides with the class of (equi-)Weyl-$(p,x,F)_2$-almost periodic functions, resp. (equi-)Weyl-$[p,x,F]_2$-almost periodic functions.

Regarding the invariance of (equi-)Weyl-$(p,\phi,F)_2$-almost periodicity and (equi-)Weyl-$[p,\phi,F]_2$-almost periodicity under the actions of infinite convolution products, we will only state the following analogues of Theorem 2.5.50 and Theorem 2.5.51:

THEOREM 2.5.53. *Suppose that $\check{g} : \mathbb{R} \to X$ is (equi-)Weyl-$(p,\phi,F)_2$-almost periodic and measurable, $F_1 : (0,\infty) \times I \to (0,\infty)$, $p,\ q \in \mathcal{P}(\mathbb{R})$, $1/p(x)+1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and for every real numbers $x,\ \tau \in \mathbb{R}$ we have* (117). *Suppose that, for every real number $t \in \mathbb{R}$ and* *positive real numbers l, $\varepsilon > 0$, there exist two positive real numbers $a > 0$ and $\lambda > 0$ such that $\lambda \leqslant a$, $[0, a] \subseteq F(l, t)^{-1}\phi^{-1}([0, \varepsilon])$,*

$$\sum_{k=0}^{\infty} \|R(v + x)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]} F(l, -x + kl)^{-1} < \infty \tag{99}$$

and the term

$$\int_t^{t+l} \varphi_{p(x)}\left(2\frac{\sum_{k=0}^{\infty}\|R(v + x)\|_{L^{q(v)}[-x+kl,-x+(k+1)l]} F(l, -x + kl)^{-1} \sup \phi^{-1}([0, \varepsilon])}{\lambda}\right) dx \tag{100}$$

does not exceed 1, resp. (99) holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ we have that the term in (100) does not exceed 1, then the function $G : \mathbb{R} \to Y$, given by (55), is well-defined and (equi-)Weyl-$(p, \phi, F_1)_2$-almost periodic.

Theorem 2.5.54. *Suppose that, with the exception of equation (98), all remaining assumptions from the formulation of* Theorem 2.5.51 *hold. If for every $t \in \mathbb{R}$ and l, $\varepsilon > 0$ we have the existence of two positive real numbers $a > 0$ and $\lambda > 0$ such that $\lambda \leqslant a$, $[0, a] \subseteq F_1(l, t)^{-1}\phi^{-1}([0, \varepsilon])$ and*

$$\int_0^1 \varphi_{p(x)}\left(2\frac{\omega(\varepsilon)S(l, t)W_2(x)}{\lambda}\right) dx \leqslant 1, \tag{101}$$

resp. if (97) holds and there exists $l_0 > 0$ such that for all $l \geqslant l_0$, $\varepsilon > 0$ and $t \in \mathbb{R}$ we have (101), then the function $G : \mathbb{R} \to Y$, given by (55), is well-defined and (equi-)Weyl-$[p, \phi, F_2]$-almost periodic.

The invariance of asymptotical Weyl-p-almost periodicity under the action of finite convolution product, where the exponent $p \in [1, \infty)$ has a constant value, has been examined in [**56**], [**246**, Proposition 5.3, Example 5.4-5.6] and [**168**, Proposition 1, Remark 2-Remark 5]. Concerning the invariance of asymptotical Weyl-$p(x)$-almost periodicity under the action of finite convolution product, we will state and prove only one proposition. In order to do that, suppose that (see also Definition 2.5.44, where the domain of function $g(\cdot)$ is the non-negative real axis) there exist two functions $g : \mathbb{R} \to X$ and $q : [0, \infty) \to X$ such that $h(t) = g(t) + q(t)$ for a.e. $t \geqslant 0$, $g(\cdot)$ belongs to some of function spaces introduced in Definition 2.5.22-Definition 2.5.24 or Definition 2.5.28-Definition 2.5.30, with $I = \mathbb{R}$, and $q(\cdot)$ belongs to some of function spaces introduced in Definition 2.5.35-Definition 2.5.37, with $I = [0, \infty)$. The study of qualitative properties of the function

$$t \mapsto H(t) \equiv \int_0^t R(t - s)[g(s) + q(s)]\, ds, \quad t \geqslant 0$$

is based on the decomposition

$$H(t) = \int_0^t R(t - s)q(s)\, ds + \left[\int_{-\infty}^t R(t - s)g(s)\, ds - \int_t^{\infty} R(s)g(t - s)\, ds\right], \quad t \geqslant 0$$

and the use of corresponding results for infinite convolution product. In the following proposition, we will consider the qualitative properties of functions

$$t \mapsto H_1(t) \equiv \int_t^{\infty} R(s)g(t-s)\,ds, \quad t \geqslant 0 \tag{102}$$

and

$$t \mapsto H_2(t) \equiv \int_0^{t} R(t-s)q(s)\,ds, \quad t \geqslant 0$$

separately. In the first part of proposition, we continue our analysis from [**143**, Proposition 5.2]; our previous results show that the case $p(x) \equiv p > 1$ is not simple in the analysis of asymptotical Weyl-p-almost periodicity so that we will consider the case $p(x) \equiv 1$ in the second part, with the notion introduced in Definition 2.5.35(i) only (cf. also [**246**, Proposition 5.3(i)]).

PROPOSITION 2.5.55. (i) *Suppose that* $p,\ q \in \mathcal{P}([0,1])$, $1/p(x)+1/q(x) = 1$ *and* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family. Let the function* $\check{g} : \mathbb{R} \to X$ *be Stepanov* $p(x)$*-bounded and let for each* $t \geqslant 0$ *the series* $\sum_{k=0}^{\infty} \|R(\cdot+t+k)\|_{L^{q(\cdot)}[0,1]} \equiv S(t)$ *be convergent. Then the function* $H_1(\cdot)$*, given by* (102)*, is well-defined. Furthermore, this function is continuous provided that the Bochner transform* $\hat{\check{g}} : \mathbb{R} \to L^{p(x)}([0,1])$ *is uniformly continuous, while the function* $H_1(\cdot)$ *satisfies* $\lim_{t\to+\infty} H_1(t) = 0$ *provided that* $\lim_{t\to+\infty} S(t) = 0$.

(ii) *Suppose that* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family such that* $\int_0^{\infty} \|R(s)\|_{L(X,Y)}\,ds < \infty$*. Let the function* $q : [0,\infty) \to Y$ *be equi-Weyl-*$(1,x,F)$*-vanishing and let* $F_1 : (0,\infty)\times[0,\infty) \to (0,\infty)$*. If for each* $\varepsilon > 0$ *there exists* $l_0 > 0$ *such that for each* $l > l_0$ *there exists* $t_{0,l} > 0$ *such that for each* $t \geqslant t_{0,l}$ *we have*

$$\sup_{x\geqslant 0}\left[F_1(l,t)\int_0^{x+t}\left[\int_{x+t}^{x+t+l}\big\|R(s-r)\big\|_{L(X,Y)}\,ds\right]\big\|q(r)\big\|_Y\,dr\right] < \varepsilon,$$

and if, additionally, there exists a finite constant $M > 0$ *such that*

$$\frac{F_1(l,t)}{F(l,t)} \leqslant M, \quad l > 0,\ t \geqslant 0, \tag{103}$$

then the function $H_2(\cdot)$ *is equi-Weyl-*$(1,x,F_1)$*-vanishing.*

PROOF. (i): The first part follows from the Stepanov $p(x)$-boundedness of function $\check{g}(\cdot)$ and the following simple computation

$$\left\|\int_t^{\infty} R(s)\check{g}(s-t)\,ds\right\| = \left\|\sum_{k=0}^{\infty}\int_0^1 R(s+t+k)\check{g}(s+k)\,ds\right\|$$

$$\leqslant 2\sum_{k=0}^{\infty}\big\|R(\cdot+t+k)\big\|_{L^{q(\cdot)}[0,1]}\sup_{k\in\mathbb{N}_0}\big\|\check{g}(\cdot+k)\big\|_{L^{p(\cdot)}[0,1]}.$$

This computation also shows that $\lim_{t\to+\infty} H_1(t) = 0$ provided that $\lim_{t\to+\infty} S(t) = 0$. For remainder, let us suppose that the function $\hat{\check{g}} : \mathbb{R} \to L^{p(x)}([0,1])$ is uniformly continuous. Let (t_n) be a sequence of non-negative reals converging to a number $t \geqslant 0$. Then we can use the Hölder inequality and the decomposition

$$\int_t^{\infty} R(s)g(t-s)\,ds - \int_{t_n}^{\infty} R(s)g\bigl(t_n - s\bigr)\,ds$$
$$= \int_t^{\infty} R(s)\bigl[\check{g}(s-t) - \check{g}\bigl(s-t_n\bigr)\bigr]\,ds + \int_t^{t_n} R(s)\check{g}(s-t)\,ds, \quad n \in \mathbb{N}$$

in order to see that

$$\left\| \int_t^{\infty} R(s)g(t-s)\,ds - \int_{t_n}^{\infty} R(s)g\bigl(t_n - s\bigr)\,ds \right\|$$
$$\leqslant 2\sum_{k=0}^{\infty} \|R(\cdot + t + k)\|_{L^{q(\cdot)}[0,1]} \sup_{k\in\mathbb{N}_0} \bigl\|\check{g}(\cdot + k) - \check{g}(\cdot + k + (t-t_n))\bigr\|_{L^{p(\cdot)}[0,1]}$$
$$+ 2\bigl\|R(\cdot)\bigr\|_{L^{q(\cdot)}[0,|t_n - t|]} \bigl\|\check{g}(\cdot)\bigr\|_{L^{p(\cdot)}[0,1]}, \quad n \in \mathbb{N}.$$

Since $\bigl\|R(\cdot)\bigr\|_{L^{q(\cdot)}[0,|t_n-t|]} \to 0$ as $n \to +\infty$ (see, e.g., [**147**, Lemma 3.2.8(c)]) and the function $\hat{\check{g}} : \mathbb{R} \to L^{p(x)}([0,1])$ is uniformly continuous, the proof of the first part is completed.

(ii): By the proof of [**246**, Proposition 5.3(i)], we have

$$F_1(l,t)\int_{x+t}^{x+t+l} \bigl\|H_2(s)\bigr\|_Y\,ds \leqslant F_1(l,t)\int_0^{x+t}\left[\int_{x+t}^{x+t+l} \bigl\|R(s-r)\bigr\|_{L(X,Y)}\,ds\right]\bigl\|q(r)\bigr\|_Y\,dr$$
$$+ F_1(l,t)\left[\int_0^{\infty} \bigl\|R(v)\bigr\|_{L(X,Y)}\,dv\right] \cdot \int_{x+t}^{x+t+l} \bigl\|q(r)\bigr\|_Y\,dr,$$

for any $x \geqslant 0$ and $l > 0$. Our preassumption shows that the first addend is equi-Weyl-$(1, x, F_1)$-vanishing. The second addend is likewise equi-Weyl-$(1, x, F_1)$-vanishing because we have assumed that the function $q(\cdot)$ is equi-Weyl-$(1, x, F)$-vanishing and condition (103). $\square$

We round off this subsection by examing the convolution invariance of Weyl almost periodic functions with variable exponent. In order to do that, we shall basically follow the method proposed in the proof of Theorem 2.5.45.

PROPOSITION 2.5.56. *Suppose that* $I = \mathbb{R}$, $\psi \in L^1(\mathbb{R})$, $(a_k)_{k\in\mathbb{Z}}$ *is a sequence of positive real numbers satisfying* $\sum_{k\in\mathbb{Z}} a_k = 1$ *and condition* (A1) *holds true. Let* $f \in (e-)W_{ap}^{(p,\phi,F)}(\mathbb{R} : X) \cap L^{\infty}(\mathbb{R} : X)$. *Then the function*

$$x \mapsto (\psi * f)(x) := \int_{-\infty}^{+\infty} \psi(x-y)f(y)\,dy, \quad x \in \mathbb{R} \tag{104}$$

is well-defined and belongs to the space $L^{\infty}(\mathbb{R} : X)$. *Furthermore, if* $p_1 \in \mathcal{P}(\mathbb{R})$, $F_1 : (0,\infty) \times \mathbb{R} \to (0,\infty)$ *and if, for every* $t \in \mathbb{R}$ *and* $l > 0$, *we have*

(105)
$$\int_t^{t+l} \varphi_{p_1(x)}\left(2l^{-1}F_1(l,t)\varphi(l)\sum_{k\in\mathbb{Z}} \frac{a_k\bigl\|\varphi\bigl(a_k^{-1}\psi(x-z)\bigr)\bigr\|_{L^{q(z)}[x-(k+1)l,x-kl]}}{F(l,x-(k+1)l)}\right)dx \leqslant 1,$$

then we have $\psi * f \in (e-)W_{ap}^{(p_1,\phi,F_1)}(\mathbb{R} : X)$.

PROOF. The proof can be deduced by using the arguments contained in the proof of Theorem 2.5.45, the equalities

$$\Big\|\phi\big(\|(\psi * f)(\cdot + \tau) - (\psi * f)(\cdot)\|\big)\Big\|_{L^{p_1(\cdot)}[t,t+l]}$$
$$= \inf\Bigg\{\lambda > 0 : \int_t^{t+l} \varphi_{p_1(x)}\Bigg(\frac{\phi\big(\|(\psi * f)(x + \tau) - (\psi * f)(x)\|\big)}{\lambda}\Bigg)\, dx \leqslant 1\Bigg\}$$
$$= \inf\Bigg\{\lambda > 0 : \int_t^{t+l} \varphi_{p_1(x)}\Bigg(\frac{\phi\big(\| \int_{-\infty}^{+\infty} \psi(y)[f(x+\tau-y) - f(x-y)]\, dy\|\big)}{\lambda}\Bigg)\, dx \leqslant 1\Bigg\}$$

and the following computation:

$$\int_t^{t+l} \varphi_{p_1(x)}\Bigg(\frac{\phi\big(\| \int_{-\infty}^{+\infty} \psi(y)[f(x+\tau-y) - f(x-y)]\, dy\|\big)}{\lambda}\Bigg)\, dx$$
$$\leqslant \int_t^{t+l} \varphi_{p_1(x)}\Bigg(\frac{\phi\big(\sum_{k\in\mathbb{Z}} a_k \| \int_{kl}^{(k+1)l} a_k^{-1}\psi(y)[f(x+\tau-y) - f(x-y)]\, dy\|\big)}{\lambda}\Bigg)\, dx$$
$$\leqslant \int_t^{t+l} \varphi_{p_1(x)}\Bigg(\frac{\sum_{k\in\mathbb{Z}} a_k \phi\big(l^{-1}l\| \int_{kl}^{(k+1)l} a_k^{-1}\psi(y)[f(x+\tau-y) - f(x-y)]\, dy\|\big)}{\lambda}\Bigg)\, dx$$
$$\leqslant \int_t^{t+l} \varphi_{p_1(x)}\Bigg(\frac{\sum_{k\in\mathbb{Z}} a_k \varphi(l) l^{-1} \int_{kl}^{(k+1)l} \phi\big(a_k^{-1}\psi(y)\|f(x+\tau-y) - f(x-y)\|\big)\, dy}{\lambda}\Bigg)\, dx$$
$$\leqslant \int_t^{t+l} \varphi_{p_1(x)}\Bigg(\frac{\sum_{k\in\mathbb{Z}} a_k \varphi(l) l^{-1} \int_{kl}^{(k+1)l} \varphi\big(a_k^{-1}\psi(y)\big)\phi\big(\|f(x+\tau-y) - f(x-y)\|\big)\, dy}{\lambda}\Bigg)\, dx$$
$$= \int_t^{t+l} \varphi_{p_1(x)}\Bigg(\frac{\sum_{k\in\mathbb{Z}} a_k \varphi(l) l^{-1} \int_{x-(k+1)l}^{x-kl} \varphi\big(a_k^{-1}\psi(x-z)\big)\phi\big(\|f(z+\tau) - f(z)\|\big)\, dz}{\lambda}\Bigg)\, dx$$
$$\leqslant \int_t^{t+l} \varphi_{p_1(x)}\Bigg(2\sum_{k\in\mathbb{Z}} a_k \varphi(l) l^{-1} \big\|\varphi\big(a_k^{-1}\psi(x-z)\big)\big\|_{L^{q(z)}[x-(k+1)l,x-kl]}$$
$$\times \frac{\big\|\phi\big(\|f(z+\tau) - f(z)\|\big)\big\|_{L^{p(z)}[x-(k+1)l,x-kl]}}{\lambda}\Bigg)\, dx,$$

which is valid for every $t,\ \tau \in \mathbb{R}$ and $l > 0$. □

We can similarly prove the following result for the class of (equi-)Weyl-$[p, \phi, F]$-almost periodic functions:

PROPOSITION 2.5.57. *Suppose that* $I = \mathbb{R}$, $\psi \in L^1(\mathbb{R})$, $(a_k)_{k\in\mathbb{Z}}$ *is a sequence of positive real numbers satisfying* $\sum_{k\in\mathbb{Z}} a_k = 1$ *and condition* (B1) *holds true. Let* $f \in (e-)W_{ap}^{[p,\phi,F]}(\mathbb{R} : X) \cap L^\infty(\mathbb{R} : X)$. *Then the function* $(\psi * f)(\cdot)$ *defined by* (104) *belongs to the space* $L^\infty(\mathbb{R} : X)$. *Furthermore, if* $p_1 \in \mathcal{P}([0,1])$, $F_1 : (0,\infty) \times \mathbb{R} \to (0,\infty)$ *and if, for every* $t \in \mathbb{R}$ *and* $l > 0$, *we have*

$$\int_0^1 \varphi_{p_1(x)}\Bigg(2F_1(l,t)\sum_{k\in\mathbb{Z}} \frac{\big\|\varphi\big(la_k^{-1}\psi(xl - (z+k)l)\big)\big\|_{L^{q(z)}[0,1]}}{F(l, t+kl)}\Bigg)\, dx \leqslant 1, \tag{106}$$

then we have $\psi * f \in (e-)W_{ap}^{[p_1,\phi,F_1]}(\mathbb{R} : X)$.

In the case of consideration of constant coefficients, the coefficient 2 in the equations (105) and (106) can be neglected. The interested reader may try to formulate the corresponding results for the classes of (equi-)Weyl-$(p,\phi,F)_i$-almost periodic functions and (equi-)Weyl-$[p,\phi,F]_i$-almost periodic functions, where $i = 1,2$, as well as to formulate an extension of [**246**, Proposition 4.3] for Weyl almost periodic functions with variable exponent.

2.5.7. Growth order of $(R(t))_{t>0}$**.** In this subsection, we will analyze solution operator families $(R(t))_{t>0} \subseteq L(X,Y)$ which satisfies condition

(107)
$$\|R(t)\|_{L(X,Y)} \leqslant \frac{Mt^{\beta-1}}{1+t^{\gamma}},\ t>0 \text{ for some finite constants } \gamma>1,\ \beta\in(0,1],\ M>0,$$

or condition

(108)
$$\|R(t)\|_{L(X,Y)} \leqslant Mt^{\beta-1}e^{-ct},\quad t>0 \text{ for some finite constants } \beta\in(0,1] \text{ and } c>0.$$

For simplicity, we will analyze only the constant exponents $p(x) \equiv p \in [1,\infty)$ as well the class of (equi-)Weyl-(p,ϕ,F)-almost periodic functions and the class of (equi-)Weyl-$(p,\phi,F)_i$-almost periodic functions, where $i = 1,2$. So, let $1/p+1/q = 1$ and let $(R(t))_{t>0} \subseteq L(X,Y)$ satisfy (107) or (108). We will additionally assume that $q(\beta-1) > -1$ provided that $p > 1$, resp. $\beta = 1$, provided that $p = 1$.

In [**234**, Proposition 2.11.1, Theorem 2.11.4], the author has investigated the estimate (107) and case $p(x) \equiv p \in [1,\infty)$, where the resulting function $G(\cdot)$ is also bounded and continuous (see also [**168**] and [**245**]). We would like to note that Theorem 2.5.45 provides a new way of looking at the invariance of the (equi-)Weyl-p-almost periodicity under the action of infinite convolution product as well as that the (equi-)Weyl-p-almost periodicity in [**234**, Theorem 2.11.4] can be proved directly from Corollary 2.5.47. Let us explain this in more detail. Let a function $g : \mathbb{R} \to X$ be (equi-)Weyl-p-almost periodic. Then the function $G : \mathbb{R} \to Y$, defined through (55), is (equi-)Weyl-p-almost periodic and we can show this in the following way. It is clear that the function $\check{g}(\cdot)$ is also (equi-)Weyl-p-almost periodic. By Corollary 2.5.47, with an arbitrary sequence of positive real numbers such that $\sum_{k=0}^{\infty} a_k = 1$ and the function $\varphi(x) \equiv x$, observing also that the class of (equi-)Weyl-p-almost periodic functions is closed under pointwise multiplications with scalars, it suffices to show, by considering the function $(M^{-1}R(t))_{t>0}$ for a sufficiently large real number $M > 0$, that for every real numbers $t \in \mathbb{R}$ and $l > 0$ we have

$$\int_t^{t+l}\Biggl(\sum_{k=0}^{\infty}\Bigl(\int_{kl}^{(k+1)l}\frac{t^{(\beta-1)q}\,dt}{(1+t^{\gamma})^q}\Bigr)^{1/q}\Biggr)^p dx \leqslant \text{Const.},\tag{109}$$

provided that $p > 1$, resp.

$$\int_t^{t+l}\sum_{k=0}^{\infty}\Bigl\|\frac{\cdot^{\beta-1}}{1+\cdot^{\gamma}}\Bigr\|_{L^{\infty}[kl,(k+1)l]}\,dx \leqslant \text{Const.},\tag{110}$$

provided that $p=1$. As

$$\int_{kl}^{(k+1)l}\frac{t^{(\beta-1)q}\,dt}{(1+t^{\gamma})^{q}}\leqslant\frac{1}{1+k^{q\gamma}l^{q\gamma}}(k+1)^{(\beta-1)q}l^{(\beta-1)q+1},\quad k\in\mathbb{N}_0,$$

the estimate (109) follows from the inequality $(\beta-1+(1/q)-\gamma)p+1\leqslant 0$, which is true. The estimate (110) is much simpler and follows from the inequality $\gamma>1$.

Concerning Theorem 2.5.50 and Theorem 2.5.53, we will provide two illustrative examples:

EXAMPLE 2.5.58. Suppose that $\phi(x)=x^{\alpha}$, $x\geqslant 0$, where $\alpha>0$. If the estimate (107) holds, then condition (95) holds provided that, for every $x\in\mathbb{R}$ and $l>0$, we have

$$\sum_{k=0}^{\infty}k^{\beta-1-\gamma}\big[F(l,-x+kl)\big]^{(-1)/\alpha}<\infty,$$

while condition (96) holds provided that, for every $t\in\mathbb{R}$ and $l>0$, we have

$$\int_t^{t+l}\Biggl(\Biggl(\frac{1}{1+k^{q\gamma}l^{q\gamma}}(k+1)^{(\beta-1)q}l^{(\beta-1)q+1}\Biggr)^{1/q}\Biggl(\frac{F(l,t)}{F(l,-x+kl)}\Biggr)^{1/\alpha}\Biggr)^{p}dx\leqslant 1,$$

if $p>1$, resp.

$$\int_t^{t+l}\frac{(kl)^{\beta-1}}{1+k^{\gamma}l^{\gamma}}\Biggl(\frac{F(l,t)}{F(l,-x+kl)}\Biggr)^{1/\alpha}dx\leqslant 1,$$

if $p=1$. If the estimate (108) holds, then condition (95) holds provided that, for every $x\in\mathbb{R}$ and $l>0$, we have

$$\sum_{k=0}^{\infty}e^{-ck}k^{\beta-1}\big[F(l,-x+kl)\big]^{(-1)/\alpha}<\infty,$$

while condition (96) holds provided that, for every $t\in\mathbb{R}$ and $l>0$, we have

$$\int_t^{t+l}\Biggl(e^{-ck}(kl)^{\beta-1}\Biggl(\frac{F(l,t)}{F(l,-x+kl)}\Biggr)^{1/\alpha}\Biggr)^{p}dx\leqslant 1.$$

EXAMPLE 2.5.59. Suppose that condition (A2) holds. If the estimate (107) holds, then condition (99) holds provided that

$$\sum_{k=0}^{\infty}k^{\beta-1-\gamma}F(l,-x+kl)^{-1}<\infty,$$

while condition (100) holds provided that, for every $t\in\mathbb{R}$ and $l>0$, we have

$$\int_t^{t+l}\Biggl(\Biggl(\frac{1}{1+k^{q\gamma}l^{q\gamma}}(k+1)^{(\beta-1)q}l^{(\beta-1)q+1}\Biggr)^{1/q}\frac{F(l,t)}{F(l,-x+kl)}\Biggr)^{p}dx\leqslant 1,$$

if $p>1$, resp.

$$\int_t^{t+l}\frac{(kl)^{\beta-1}}{1+k^{\gamma}l^{\gamma}}\frac{F(l,t)}{F(l,-x+kl)}\,dx\leqslant 1,$$

if $p = 1$. If the estimate (108) holds, then (99) holds provided that, for every $x \in \mathbb{R}$ and $l > 0$, we have

$$\sum_{k=0}^{\infty} e^{-ck} k^{\beta-1} \big[F(l, -x + kl)\big]^{(-1)} < \infty,$$

while condition (100) holds provided that, for every $t \in \mathbb{R}$ and $l > 0$, we have

$$\int_t^{t+l} \left(e^{-ck} (kl)^{\beta-1} \frac{F(l,t)}{F(l, -x + kl)} \right)^p dx \leqslant 1.$$

At the end of this section, let us only note that we can incorporate our results in the study of the abstract fractional Cauchy inclusions (58) and $(\mathrm{DFP})_{f,\zeta}$, provided that the multivalued linear operator $\mathcal{A}$ satisfies condition (P). Then there exists a strongly continuous operator family $(S_\zeta(t))_{t>0}$ satisfying the estimate of type (107), in the case $\zeta \in (0,1)$, or estimate of type (108), in the case $\zeta = 1$, such that the unique mild solution of problem $(\mathrm{DFP})_{f,\zeta}$ is given by

$$t \mapsto u(t) \equiv S_\zeta(t) u_0 + \int_0^t S_\zeta(t-s) f(s), \quad t \geqslant 0,$$

where u_0 belongs to the continuity set of $(S_\zeta(t))_{t>0}$, i.e., $\lim_{t\to 0+} S_\zeta(t)u_0 = u_0$. Moreover, $\lim_{t\to+\infty} S_\zeta(t)u_0 = 0$ and Proposition 2.5.55 can be straightforwardly applied.

2.6. Generalized almost periodicity in Lebesgue spaces with variable exponents. Part II

In this section, we introduce and analyze Stepanov uniformly recurrent functions, Doss uniformly recurrent functions and Doss almost periodic functions in Lebesgue spaces with variable exponents. We investigate the invariance of these types of generalized almost periodicity in Lebesgue spaces with variable exponents under the actions of convolution products, providing also some illustrative applications to the abstract semilinear integro-differential inclusions in Banach spaces.

The organization of section can be briefly described as follows. Subsection 2.6.1 investigates the Stepanov uniformly recurrent functions in Lebesgue spaces with variable exponents. The proofs of structural results in this section can be given by employing the slight modifications of the corresponding results from [**142**] (see also [**248**]) and therefore omitted. Our main contributions are given in Subsection 2.6.2 and Subsection 2.6.3, where we introduce and analyze several various classes of Doss almost periodic (uniformly recurrent) functions in Lebesgue spaces with variable exponents and the invariance of generalized Doss almost periodicity under the actions of convolution products. The final subsection is reserved for applications of our abstract theoretical results to the abstract semilinear integro-differential inclusions in Banach spaces. In addition to the above, we provide several illustrative examples, remarks and comments about the material presented.

2.6.1. Stepanov uniform recurrence in Lebesgue spaces with variable exponents. First of all, we will introduce the concept of (asymptotical) $S^{p(x)}$-uniform recurrence:

DEFINITION 2.6.1. Let $p \in \mathcal{P}([0,1])$, and let $f : I \to X$ be such that $f \in L^{p(x)}(K : X)$ for any compact set $K \subseteq I$.

(i) We say that $f(\cdot)$ is Stepanov $p(x)$-uniformly recurrent if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is uniformly recurrent. The collection of such functions will be denoted by $URS^{p(x)}(I : X)$ ($URS^{p}(I : X)$, if $p(x) \equiv p \in [1,\infty)$).

(ii) We say that $f(\cdot)$ is asymptotically Stepanov $p(x)$-uniformly recurrent if and only if there exist a Stepanov $p(x)$-uniformly recurrent function $h : \mathbb{R} \to X$ and a function $q \in L_S^{p(x)}(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^{p(x)}([0,1] : X))$. The collection of such functions will be denoted by $AURS^{p(x)}(I : X)$ ($AURS^{p}(I : X)$, if $p(x) \equiv p \in [1,\infty)$).

The spaces $URS^{p(x)}(I : X)$ and $AURS^{p(x)}(I : X)$ are translation invariant, as it can be easily approved. Furthermore, we have the following proposition which can be deduced by using the same argumentation as in the proofs of corresponding structural results concerning Stepanov almost periodicity with variable exponent:

PROPOSITION 2.6.2. (i) *Suppose* $p \in \mathcal{P}([0,1])$. *Then* $URS^{p(x)}(I : X) \subseteq URS^{1}(I : X)$, $AURS^{p(x)}(I : X) \subseteq AURS^{1}(I : X)$, $UR(I : X) \subseteq URS^{p(x)}(I : X) \subseteq URS^{1}(I : X)$ *and* $AUR(I : X) \subseteq AURS^{p(x)}(I : X) \subseteq AURS^{1}(I : X)$.

(ii) *Suppose* $p \in D_+([0,1])$ *and* $1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty$ *for a.e.* $x \in [0,1]$. *Then we have* $URS^{p^+}(I : X) \subseteq URS^{p(x)}(I : X) \subseteq URS^{p^-}(I : X)$ *and* $AURS^{p^+}(I : X) \subseteq AURS^{p(x)}(I : X) \subseteq AURS^{p^-}(I : X)$.

(iii) *Assume that* $p,\ q \in \mathcal{P}([0,1])$ *and* $p \leqslant q$ *a.e. on* $[0,1]$. *Then we have:* $URS^{q(x)}(I : X) \subseteq URS^{p(x)}(I : X)$ *and* $AURS^{q(x)}(I : X) \subseteq AURS^{p(x)}(I : X)$.

(iv) *If* $p \in D_+([0,1])$, *then*

$$L^{\infty}(I : X) \cap URS^{p(x)}(I : X) = L^{\infty}(I : X) \cap URS^{1}(I : X)$$

and

$$L^{\infty}(I : X) \cap AURS^{p(x)}(I : X) = L^{\infty}(I : X) \cap AURS^{1}(I : X).$$

We continue by providing two illustrative examples.

EXAMPLE 2.6.3. Let us recall that H. Bohr and E. Følner have constructed, for any given number $p > 1$, a Stepanov almost periodic function defined on the whole real axis that is Stepanov p-bounded and not Stepanov p-almost automorphic (see [**77**, Example, pp. 70-73]). We want to observe here that the function $f(\cdot)$ cannot be Stepanov p-uniformly recurrent. Strictly speaking, let us consider case $h_1 = 2$ in the afore-mentioned example. If we suppose the contrary, then the mapping $\hat{f} : \mathbb{R} \to L^{p}([0,1] : X)$ is uniformly recurrent, which in particular implies that for each number $\varepsilon > 0$ there exists an arbitrarily large positive real number $\tau > 0$ such that

$$\int_{-3/2}^{3/2} \bigl|f(s+\tau) - f(s)\bigr|^p \, ds < 2\varepsilon^p,$$

which is in contradiction with the estimate $\int_{-3/2}^{3/2} \bigl|f(s+\tau) - f(s)\bigr|^p \, ds \geqslant 2^{-p}$ (see [**77**, p. 73, l.-9 - l.-4]).

Example 2.6.4. Define $\mathrm{sign}(0) := 0$, $f(x) := \sin x + \sin \sqrt{2}x$, $x \in \mathbb{R}$ and $p(x) := 1 - \ln x$, $x \in [0,1]$. We know that the function $\mathrm{sign} f(\cdot)$ is neither Stepanov $p(x)$-almost periodic nor Stepanov $p(x)$-almost automorphic ([**142**]-[**143**]). Moreover, we have already proved that for every real numbers $\lambda \in (0, 2/e)$, $l > 0$, every interval $I \subseteq \mathbb{R} \setminus \{0\}$ of length l and every number $\tau \in I$, there exists a number $t \in \mathbb{R}$ such that

$$\int_0^1 \Bigl(\frac{1}{\lambda}\Bigr)^{1-\ln x} \Bigl|\mathrm{sign}\bigl[\sin(x+t+\tau) + \sin \sqrt{2}(x+t+\tau)\bigr] \\ - \mathrm{sign}\bigl[\sin(x+t) + \sin \sqrt{2}(x+t)\bigr]\Bigr|^{1-\ln x} \, dx = \infty.$$

This implies that the function $\mathrm{sign} f(\cdot)$ cannot be Stepanov $p(x)$-uniformly recurrent, as well.

Now we will state two results about the invariance of uniform recurrence under the actions of infinite convolution products. The first result slightly extends [**234**, Proposition 2.6.11]; the proof can be given by using the same arguments as in the proof of above-mentioned proposition, with the appealing to the Hölder inequality in Lemma 1.1.6(i):

Proposition 2.6.5. *Suppose that $q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is $S^{p(x)}$-bounded, $S^{p(x)}$-uniformly recurrent and the function $\hat{\check{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ is uniformly continuous, then the function $F : \mathbb{R} \to Y$, given by* (55)*, is well-defined and uniformly recurrent.*

Using a similar argumentation, we can clarify the following result in which we do not require that the function $\check{f} : \mathbb{R} \to X$ is $S^{p(x)}$-bounded:

Proposition 2.6.6. *Suppose that $q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is $S^{p(x)}$-uniformly recurrent, the function $\hat{\check{f}} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ is uniformly continuous,*

$$\bigl\|\check{f}(\cdot - t)\bigr\|_{L^{p(x)}[0,1]} \leqslant P(t), \quad t \in \mathbb{R}$$

and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that for each $t \in \mathbb{R}$ we have

$$\sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} P(t-k) < \infty,$$

then the function $F : \mathbb{R} \to Y$, given by (55), *is well-defined and uniformly recurrent.*

Now we will introduce the notion of (asymptotical) Stepanov $p(x)$-uniform recurrence for the functions depending on two parameters; this notion extends the notion introduced in Definition 2.4.42 and Definition 2.4.43, where we have considered the constant coefficient $p(x) \equiv p \in [1, \infty)$:

DEFINITION 2.6.7. Let $p \in \mathcal{P}([0,1])$.

(i) A function $f : I \times Y \to X$ is called Stepanov $p(x)$-uniformly recurrent if and only if $\hat{f} : I \times Y \to L^{p(x)}([0,1] : X)$ is uniformly recurrent.
(ii) A function $f : I \times Y \to X$ is said to be asymptotically $S^{p(x)}$-uniformly recurrent if and only if there exist a Stepanov $p(x)$-uniformly recurrent function $g : [0, \infty) \times Y \to X$ and a function $q \in C_0(I \times Y \to X)$ such that $f(t, y) = g(t, y) + q(t, y)$ for all $t \in I$ and $y \in Y$. $\hat{f} : [0, \infty) \times Y \to L^{p(x)}([0,1] : X)$ is asymptotically uniformly recurrent.

A great number of composition principles established for Stepanov $p(x)$-almost periodic functions can be straightforwardly extended for Stepanov $p(x)$-uniformly recurrent functions. For example, we have:

THEOREM 2.6.8. *Let $p \in \mathcal{P}([0,1])$. Suppose that the following conditions hold:*

(i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent, and there exist a function $r \in \mathcal{P}([0,1])$ and a function $L_f \in L_S^{r(x)}(I)$ such that $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/(p(\cdot) - 1))$ and* (25) *holds;*
(ii) *The function $f : I \to Y$ is Stepanov $p(x)$-uniformly recurrent and there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that $K := \{f(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact in Y.*
(iii) *For every compact set $K \subseteq Y$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n \to +\infty} \sup_{t \in I} \sup_{u \in K} \|F(t + s + \alpha_n, u) - F(t + s, u)\|_{L^{p(s)}([0,1]:X)} = 0 \tag{111}$$

and (19) *holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^{p(x)}([0,1]:X)}$ therein.*

Then $q(x) := p(x)r(x)/(p(x) + r(x)) \in [1, p(x))$ and $F(\cdot, f(\cdot))$ is Stepanov $q(x)$-uniformly recurrent. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $q(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $q(x)$-bounded, as well.

It is not so difficult to reformulate the statements of [**234**, Proposition 2.7.3-Proposition 2.7.4] for the asymptotical Stepanov $p(x)$-uniform recurrence. Details can be left to the interested readers.

2.6.2. Doss almost periodicity and Doss uniform recurrence in Lebesgue spaces with variable exponents. Throughout this subsection, we assume that condition (A) holds true. The notion of Doss-$p(x)$-almost periodicity has not been introduced so far. Following the approach obeyed for the classes of (equi-)Weyl-(p, ϕ, F)-almost periodic functions and (equi-)Weyl-$(p, \phi, F)_i$-almost periodic functions $(i = 1, 2)$, we introduce the following notion for Doss classes:

DEFINITION 2.6.9. Suppose that condition (A) holds, $f : I \rightarrow X$ and $\phi(\|f(\cdot+\tau)-f(\cdot)\|) \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) A function $f(\cdot)$ is said to be Doss-(p,ϕ,F)-almost periodic if and only if for every $\varepsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t\rightarrow+\infty}\Biggl[F(t)\Bigl[\phi\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)_{L^{p(x)}[-t,t]}\Bigr]\Biggr] < \varepsilon, \tag{112}$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t\rightarrow+\infty}\Biggl[F(t)\Bigl[\phi\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)_{L^{p(x)}[0,t]}\Bigr]\Biggr] < \varepsilon,$$

in the case that $I = [0,\infty)$, is relatively dense in I.

(ii) A function $f(\cdot)$ is said to be Doss-(p,ϕ,F)-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\rightarrow+\infty}\alpha_n = +\infty$ and

$$\lim_{n\rightarrow+\infty}\limsup_{t\rightarrow+\infty}\Biggl[F(t)\Bigl[\phi\bigl(\|f(\cdot+\alpha_n)-f(\cdot)\|\bigr)_{L^{p(x)}[-t,t]}\Bigr]\Biggr] = 0,$$

in the case that $I = \mathbb{R}$, resp.,

$$\lim_{n\rightarrow+\infty}\limsup_{t\rightarrow+\infty}\Biggl[F(t)\Bigl[\phi\bigl(\|f(\cdot+\alpha_n)-f(\cdot)\|\bigr)_{L^{p(x)}[0,t]}\Bigr]\Biggr] = 0,$$

in the case that $I = [0,\infty)$, is relatively dense in I.

DEFINITION 2.6.10. Suppose that condition (A) holds, $f : I \rightarrow X$ and $\|f(\cdot+\tau)-f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) A function $f(\cdot)$ is said to be Doss-$(p,\phi,F)_1$-almost periodic if and only if for every $\varepsilon > 0$, the set of numbers $\tau \in I$ for which

$$\limsup_{t\rightarrow+\infty}\Biggl[F(t)\phi\Bigl[\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)_{L^{p(x)}[-t,t]}\Bigr]\Biggr] < \varepsilon,$$

in the case that $I = \mathbb{R}$, resp.,

$$\limsup_{t\rightarrow+\infty}\Biggl[F(t)\phi\Bigl[\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)_{L^{p(x)}[0,t]}\Bigr]\Biggr] < \varepsilon,$$

in the case that $I = [0,\infty)$, is relatively dense in I.

(ii) A function $f(\cdot)$ is said to be Doss-$(p,\phi,F)_1$-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\rightarrow+\infty}\alpha_n = +\infty$ and

$$\lim_{n\rightarrow+\infty}\limsup_{t\rightarrow+\infty}\Biggl[F(t)\phi\Bigl[\bigl(\|f(\cdot+\alpha_n)-f(\cdot)\|\bigr)_{L^{p(x)}[-t,t]}\Bigr]\Biggr] = 0,$$

in the case that $I=\mathbb{R}$, resp.,

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\Bigl[F(t)\Bigl[\phi\bigl(\|f(\cdot+\alpha_n)-f(\cdot)\|\bigr)_{L^{p(x)}[0,t]}\Bigr]\Bigr]=0,$$

in the case that $I=[0,\infty)$, is relatively dense in I.

DEFINITION 2.6.11. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot+\tau)-f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau\in I$ and any compact subset K of I.

(i) A function $f(\cdot)$ is said to be Doss-$(p,\phi,F)_2$-almost periodic if and only if for every $\varepsilon>0$, the set of numbers $\tau\in I$ for which

$$\limsup_{t\to+\infty}\Bigl[\phi\Bigl[F(t)\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)_{L^{p(x)}[-t,t]}\Bigr]\Bigr]<\varepsilon,$$

in the case that $I=\mathbb{R}$, resp.,

$$\limsup_{t\to+\infty}\Bigl[\phi\Bigl[F(t)\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)_{L^{p(x)}[0,t]}\Bigr]\Bigr]<\varepsilon,$$

in the case that $I=[0,\infty)$, is relatively dense in I.

(ii) A function $f(\cdot)$ is said to be Doss-$(p,\phi,F)_2$-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\Bigl[\phi\Bigl[F(t)\bigl(\|f(\cdot+\alpha_n)-f(\cdot)\|\bigr)_{L^{p(x)}[-t,t]}\Bigr]\Bigr]=0,$$

in the case that $I=\mathbb{R}$, resp.,

$$\lim_{n\to+\infty}\limsup_{t\to+\infty}\Bigl[\phi\Bigl[F(t)\bigl(\|f(\cdot+\alpha_n)-f(\cdot)\|\bigr)_{L^{p(x)}[0,t]}\Bigr]\Bigr]=0,$$

in the case that $I=[0,\infty)$, is relatively dense in I.

Case in which $\phi(x)\equiv x$ and $\psi(t)\equiv(2t)^{(-1)/p}$, $t>0$ if $I=\mathbb{R}$, resp. $\psi(t)\equiv t^{(-1)/p}$, $t>0$ if $I=[0,\infty)$, leads to the usual class of Doss p-almost periodic functions ([**234**], [**245**]). The notion introduced in the above three definitions is rather general; for example, in the case that $p(x)\equiv p\in[1,\infty)$ and $\sigma>0$, then any essentially bounded function $f(\cdot)$ is Doss-$(p,x,t^{-(1+\sigma)/p})$-almost periodic.

EXAMPLE 2.6.12. (i) Suppose that $\phi(0)=0$. Then any continuous periodic function $f : I\to X$ is Doss-$(p,\phi,F)_i$-almost periodic for $i=1,2$; furthermore, if $\phi(\cdot)$ is locally bounded, then the function $f(\cdot)$ is Doss-(p,ϕ,F)-almost periodic.

(ii) Suppose that $f : I\to X$ is almost periodic. Then $f(\cdot)$ is Doss-(p,ϕ,F)-almost periodic [Doss-$(p,\phi,F)_1$-almost periodic/Doss-$(p,\phi,F)_2$-almost periodic] if $\phi(\cdot)$ is continuous, monotonically increasing and $F(\cdot)\|1\|_{L^{p(x)}[-\cdot,\cdot]}\in L^\infty((0,\infty))$ [$\phi(\cdot)$ is monotonically increasing, there exists a continuous function $\varphi : [0,\infty)\to[0,\infty)$ such that $\phi(xy)\leqslant\varphi(x)\phi(y)$, $x,\ y\geqslant 0$ and

$F(\cdot)\|1\|_{L^{p(x)}[-\cdot,\cdot]} \in L^{\infty}((0,\infty))/\phi(\cdot)$ is monotonically increasing, there exists a continuous function $\varphi : [0,\infty) \to [0,\infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$, $x,\ y \geqslant 0$ and $\phi(F(\cdot)\|1\|_{L^{p(x)}[-\cdot,\cdot]}) \in L^{\infty}((0,\infty))$].

EXAMPLE 2.6.13. We have alrady clarified that the function $f : \mathbb{R} \to \mathbb{R}$, given by (33), is uniformly continuous, uniformly recurrent and Besicovitch unbounded. Furthermore, we have proved that for each number $\tau \in \mathbb{R}$ we have

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t |f(s+\tau) - f(s)|^p \, ds = 0, \quad p \geqslant 1,$$

so that the function $f(\cdot)$ is Doss $p(x)$-almost periodic for any function $p \in D_+(\mathbb{R})$.

EXAMPLE 2.6.14. Let $\zeta \geqslant 1$ and $0^{\zeta} := 0$. Define the complex-valued function

$$f_{\zeta}(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^{\zeta}\Bigl(\frac{t}{2^n}\Bigr), \quad t \in \mathbb{R}. \tag{113}$$

Then the function $f_{\zeta}(\cdot)$ is Lipschitz continuous and uniformly recurrent. To prove the Lipschitz continuity of function $f_{\zeta}(\cdot)$, it suffices to observe that the function $t \mapsto \sin^{\zeta}(t)$, $t \in \mathbb{R}$ is continuous and that

$$\Bigl|\sin^{\zeta} x - \sin^{\zeta} y\Bigr| \leqslant \zeta|x-y|, \quad x,\ y \in \mathbb{R}. \tag{114}$$

To see that the function $f_{\zeta}(\cdot)$ is uniformly recurrent, it suffices to see that for each integer $k \in \mathbb{N} \setminus \{1\}$ we have

$$\begin{aligned}
\Bigl|f_{\zeta}\bigl(t+2^k\pi\bigr) - f_{\zeta}(t)\Bigr| &= \left|\sum_{n=1}^{\infty} \frac{1}{n}\left[\sin^{\zeta}\Bigl(\frac{t+2^k\pi}{2^n}\Bigr) - \sin^{\zeta}\Bigl(\frac{t}{2^n}\Bigr)\right]\right| \\
= \left|\sum_{n=1}^{k-1} \frac{1}{n}\left[\sin^{\zeta}\Bigl(\frac{t+2^k\pi}{2^n}\Bigr) - \sin^{\zeta}\Bigl(\frac{t}{2^n}\Bigr)\right]\right| &+ \left|\sum_{n=k}^{\infty} \frac{1}{n}\left[\sin^{\zeta}\Bigl(\frac{t+2^k\pi}{2^n}\Bigr) - \sin^{\zeta}\Bigl(\frac{t}{2^n}\Bigr)\right]\right| \\
= \left|\sum_{n=k}^{\infty} \frac{1}{n}\left[\sin^{\zeta}\Bigl(\frac{t+2^k\pi}{2^n}\Bigr) - \sin^{\zeta}\Bigl(\frac{t}{2^n}\Bigr)\right]\right| &\leqslant \sum_{n=k}^{\infty} \frac{1}{n}\left|\sin^{\zeta}\Bigl(\frac{t+2^k\pi}{2^n}\Bigr) - \sin^{\zeta}\Bigl(\frac{t}{2^n}\Bigr)\right| \\
\leqslant \sum_{n=k}^{\infty} \frac{\zeta}{n} 2^{k-n}\pi = \frac{2\pi\zeta}{k}, \quad t \in \mathbb{R},
\end{aligned}$$

where we have applied (114) in the last line of computation. In the case that $\zeta = 2l$ for some integer $l \in \mathbb{N}$, we have that the function $f_{\zeta}(\cdot)$ is Besicovitch unbounded. This can be inspected as in the proof of [**202**, Theorem 1.1], with the additional observation that

$$\int_0^{2^{k-n}\pi} \sin^{2l} t \, dt = \frac{2}{3} \frac{(2l-1)!!}{(2l)!!} \int_0^{2^{k-n}\pi} \sin^2 t \, dt \quad (k \in \mathbb{N} \setminus \{1\},\ 1 \leqslant n \leqslant k);$$

here, we have used the well known recurrent formula

$$\int_0^{2^{k-n}\pi} \sin^{2l} t \, dt = \frac{2l-1}{2l} \int_0^{2^{k-n}\pi} \sin^{2l-2} t \, dt$$

which can be deduced with the help of the partial integration (take $u = \sin^{2l-1} t$ and $dv = \sin t \cdot dt$). We would like to ask whether the function $f_\zeta(\cdot)$ is Besicovitch unbounded in general case and for which functions $p \in D_+(\mathbb{R})$ we have that $f(\cdot)$ is Doss $p(x)$-almost periodic (see also Example 2.9.24).

In order to ensure the translation invariance of generalized Weyl spaces of almost periodic functions, we have analyzed the classes of (equi-)Weyl-$[p,\phi,F]$-almost periodic functions and (equi-)Weyl-$[p,\phi,F]_i$-almost periodic functions ($i = 1,2$). In this subsection, we will follow a slightly different approach. First of all, for any $\tau_0 \in I$ we set $p_{\tau_0}(\cdot) := p(\cdot + \tau_0)$. Then we have the following:

THEOREM 2.6.15. *Suppose that $F_1(\cdot)$ is monotonically decreasing, there exists a function $F_0 : (0,\infty) \to (0,\infty)$ such that $F(xy) \leqslant F_0(x) \cdot F(y)$, $x,\ y > 0$, $\tau_0 \in I$ and*

$$\limsup_{t\to+\infty} F_0\Big(\frac{t}{t+\tau_0}\Big) < \infty. \tag{115}$$

Define $f_{\tau_0}(\cdot) := f(\cdot + \tau_0)$. Then the following holds:

(i) *Suppose that $f(\cdot)$ is Doss-(p,ϕ,F)-almost periodic, resp. Doss-(p,ϕ,F)-uniformly recurrent. Then $f_{\tau_0}(\cdot)$ is Doss-(p_{τ_0},ϕ,F_1)-almost periodic, resp. Doss-(p_{τ_0},ϕ,F_1)-uniformly recurrent.*

(ii) *Suppose that $f(\cdot)$ is Doss-$(p,\phi,F)_1$-almost periodic, resp. Doss-$(p,\phi,F)_1$-uniformly recurrent, and $\phi(\cdot)$ is monotonically increasing. Then $f_{\tau_0}(\cdot)$ is Doss-$(p_{\tau_0},\phi,F_1)_1$-almost periodic, resp. Doss-$(p_{\tau_0},\phi,F_1)_1$-uniformly recurrent.*

(iii) *Suppose that $f(\cdot)$ is Doss-$(p,\phi,F)_2$-almost periodic, resp. Doss-$(p,\phi,F)_2$-uniformly recurrent, $\phi(\cdot)$ is monotonically increasing, there exists a function $\phi_0 : [0,\infty) \to [0,\infty)$ such that $\phi(xy) \leqslant \phi_0(x) \cdot \phi(y)$, $x,\ y \geqslant 0$ and, in place of condition* (115),

$$\limsup_{t\to+\infty} \big(\phi_0 \circ F_0\big)\Big(\frac{t}{t+\tau_0}\Big) < \infty. \tag{116}$$

Then $f_{\tau_0}(\cdot)$ is Doss-$(p_{\tau_0},\phi_1,F_1)_2$-almost periodic, resp. Doss-$(p_{\tau_0},\phi_1,F_1)_2$-uniformly recurrent.

PROOF. We will consider only Doss almost periodic functions with variable exponent. Suppose that $\tau \in I$ and (112) holds. We need to prove first that $\phi(\|f(\cdot+\tau+\tau_0) - f(\cdot+\tau_0)\|) \in L^{p_{\tau_0}(x)}(K)$ for any $\tau \in I$ and any compact subset K of I. But, this directly follows from the corresponding definitions of the space $L^{p_{\tau_0}(x)}(K)$, the function $p_{\tau_0}(\cdot)$ and an elementary substitution $\cdot \mapsto \cdot + \tau_0$. The statement (i) then follows from the next computation:

$$\limsup_{t\to+\infty}\Bigg[F_1(t)\inf\Bigg\{\lambda>0 : \int_0^t \varphi_{p_{\tau_0}(x)}\Bigg(\frac{\phi\big(\|f(x+\tau+\tau_0)-f(x+\tau_0)\|\big)}{\lambda}\Bigg)\,dx \leqslant 1\Bigg\}\Bigg]$$
$$= \limsup_{t\to+\infty}\Bigg[F_1(t)\inf\Bigg\{\lambda>0 : \int_{\tau_0}^{t+\tau_0} \varphi_{p_{\tau_0}(x-\tau_0)}\Bigg(\frac{\phi\big(\|f(x+\tau)-f(x)\|\big)}{\lambda}\Bigg)\,dx \leqslant 1\Bigg\}\Bigg]$$

$$
\begin{aligned}
&= \lim_{t\to+\infty} \sup_{y\geqslant t}\Biggl[F_1(y)\inf\Biggl\{\lambda>0:\int_{\tau_0}^{y+\tau_0}\varphi_{p(x)}\Biggl(\frac{\phi\bigl(\|f(x+\tau)-f(x)\|\bigr)}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr]\\
&\leqslant \limsup_{t\to+\infty}\sup_{y\geqslant t}\Biggl[F_1\Bigl(\frac{t}{t+\tau_0}(y+\tau_0)\Bigr)\\
&\times \inf\Biggl\{\lambda>0:\int_{\tau_0}^{y+\tau_0}\varphi_{p(x)}\Biggl(\frac{\phi\bigl(\|f(x+\tau)-f(x)\|\bigr)}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr]\\
&\leqslant \limsup_{t\to+\infty}\sup_{y\geqslant t}\Biggl[F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\\
&\times F(y+\tau_0)\inf\Biggl\{\lambda>0:\int_{\tau_0}^{y+\tau_0}\varphi_{p(x)}\Biggl(\frac{\phi\bigl(\|f(x+\tau)-f(x)\|\bigr)}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr]\\
&\leqslant \limsup_{t\to+\infty}F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\times\\
&\limsup_{t\to+\infty}\sup_{y\geqslant t}\Biggl[F(y+\tau_0)\inf\Biggl\{\lambda>0:\int_{\tau_0}^{y+\tau_0}\varphi_{p(x)}\Biggl(\frac{\phi\bigl(\|f(x+\tau)-f(x)\|\bigr)}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr]\\
&\leqslant \limsup_{t\to+\infty}F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\times\\
&\limsup_{t\to+\infty}\sup_{y\geqslant t}\Biggl[F(y+\tau_0)\inf\Biggl\{\lambda>0:\int_{0}^{y+\tau_0}\varphi_{p(x)}\Biggl(\frac{\phi\bigl(\|f(x+\tau)-f(x)\|\bigr)}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr]\\
&\leqslant \limsup_{t\to+\infty}F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\times\\
&\limsup_{t\to+\infty}\sup_{y\geqslant t+\tau_0}\Biggl[F(y)\inf\Biggl\{\lambda>0:\int_{0}^{y}\varphi_{p(x)}\Biggl(\frac{\phi\bigl(\|f(x+\tau)-f(x)\|\bigr)}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr]\\
&= \limsup_{t\to+\infty}F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\times\\
&\limsup_{t\to+\infty}\sup_{y\geqslant t}\Biggl[F(y)\inf\Biggl\{\lambda>0:\int_{0}^{y}\varphi_{p(x)}\Biggl(\frac{\phi\bigl(\|f(x+\tau)-f(x)\|\bigr)}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr]
\end{aligned}
$$

$$
\begin{aligned}
&= \limsup_{t\to+\infty}F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\times\\
&\limsup_{t\to+\infty}\Biggl[F(t)\inf\Biggl\{\lambda>0:\int_{0}^{t}\varphi_{p(x)}\Biggl(\frac{\phi\bigl(\|f(x+\tau)-f(x)\|\bigr)}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr]\\
&\leqslant \limsup_{t\to+\infty}F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\cdot\varepsilon.
\end{aligned}
$$

The proof of (ii) is similar because then we can start from the term

$$\limsup_{t\to+\infty}\Biggl[F_1(t)\phi\Biggl(\inf\Biggl\{\lambda>0:\int_0^t\varphi_{p_{\tau_0}(x)}\Biggl(\frac{\|f(x+\tau+\tau_0)-f(x+\tau_0)\|}{\lambda}\Biggr)\,dx\leqslant 1\Biggr\}\Biggr)\Biggr],$$

use the same computation and the assumption that $\phi(\cdot)$ is monotonically increasing. The proof of (iii) is also similar bacause, with the obvious change of computation caused by the use of different notion, we can use the same computation and the inequality (see also (116))

$$\phi\Biggl(F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\cdot F(y+\tau_0)\Biggr)\leqslant\phi_0\Biggl(F_0\Bigl(\frac{t}{t+\tau_0}\Bigr)\Biggr)\cdot\phi_1\Bigl(F(y+\tau_0)\Bigr).$$

□

We will include the proof of the next proposition for the sake of completeness.

PROPOSITION 2.6.16. *Suppose that* $p(x)\equiv 1$, $f:I\to X$, $\|f(\cdot+\tau)-f(\cdot)\|\in L^1(K)$ *for any* $\tau\in I$ *and any compact subset* K *of* I, *as well as condition*

(B)': $\phi(\cdot)$ *is convex and there exists a function* $\varphi:[0,\infty)\to(0,\infty)$ *such that* $\phi(tx)\leqslant\varphi(t)\phi(x)$ *for all* $t\geqslant 0$ *and* $x\geqslant 0$.

Set $F_1(t):=F(t)t[\varphi(t)]^{-1}$, $t>0$, $F_2(t):=(2t)^{-1}\varphi(2F(t)t)$, $t>0$ *provided that* $I=\mathbb{R}$, *and* $F_2(t):=t^{-1}\varphi(F(t)t)$, $t>0$ *provided that* $I=[0,\infty)$. *Then we have:*

(i) *If* $f(\cdot)$ *is Doss-*$(1,\phi,F)$*-almost periodic, resp. Doss-*$(1,\phi,F)$*-uniformly recurrent, then* $f(\cdot)$ *is Doss-*$(1,\phi,F_1)_1$*-almost periodic, resp. Doss-*$(1,\phi,F_1)_1$*-uniformly recurrent.*

(ii) *If* $f(\cdot)$ *is Doss-*$(1,\phi,F_2)$*-almost periodic, resp. Doss-*$(1,\phi,F_2)$*-uniformly recurrent, then* $f(\cdot)$ *is Doss-*$(1,\phi,F)_2$*-almost periodic, resp. Doss-*$(1,\phi,F)_2$*-uniformly recurrent.*

PROOF. We will consider only Doss almost periodic functions with variable exponent and case $I=[0,\infty)$. To prove (i), we can use the assumption (B)' and the Jensen integral inequality ($\tau>0$):

$$\begin{aligned}\phi\Bigl(\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]}\Bigr)&=\phi\Bigl(t\cdot t^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]}\Bigr)\\ &\leqslant\varphi(t)\phi\Bigl(t^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]}\Bigr)\leqslant\varphi(t)t^{-1}\Bigl[\phi\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)\Bigr]_{L^1[0,t]}.\end{aligned}$$

This simply yields that $f(\cdot)$ is Doss-$(1,\phi,F_1)_1$-almost periodic. To prove (ii), suppose that $f(\cdot)$ is Doss-$(1,\phi,F_2)$-almost periodic. Then the assumption (B)' and the Jensen integral inequality together imply ($\tau>0$):

$$\begin{aligned}\phi\Bigl(F(t)\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]}\Bigr)&=\phi\Bigl(F(t)t\cdot t^{-1}\|f(\cdot+\tau)-f(\cdot)\|_{L^1[0,t]}\Bigr)\\ &\leqslant\varphi(F(t))t^{-1}\Bigl[\phi\bigl(\|f(\cdot+\tau)-f(\cdot)\|\bigr)\Bigr]_{L^1[0,t]}.\end{aligned}$$

This simply yields that $f(\cdot)$ is Doss-$(1,\phi,F)_2$-almost periodic. □

REMARK 2.6.17. (i) It is clear that, if $f(\cdot)$ is Doss-(p,ϕ,F)-almost periodic [Doss-(p,ϕ,F)-uniformly recurrent], resp. Doss-$(p,\phi,F)_1$-almost periodic [Doss-$(p,\phi,F)_1$-uniformly recurrent], and $F(t)\geqslant F_1(t)$ for every $t\in I$, then $f(\cdot)$ is Doss-(p,ϕ,F_1)-almost periodic [Doss-(p,ϕ,F_1)–uniformly recurrent], resp. Doss-$(p,\phi,F_1)_1$-almost periodic [Doss-$(p,\phi,F_1)_1$-uniformly recurrent]. Furthermore, if $f(\cdot)$ is Doss-$(p,\phi,F)_2$-almost periodic [Doss-$(p,\phi,F)_2$-uniformly recurrent], then $f(\cdot)$ is Doss-$(p,\phi,F_1)_2$-almost periodic [Doss-$(p,\phi,F_1)_2$-uniformly recurrent] provided that $F(t)\geqslant F_1(t)$ for every $t\in I$ and $\phi(\cdot)$ is monotonically increasing, or $F(t)\leqslant F_1(t)$ for every $t\in I$ and $\phi(\cdot)$ is monotonically decreasing.

(ii) If $f(\cdot)$ is Doss-(p,ϕ,F)-almost periodic [Doss-(p,ϕ,F)-uniformly recurrent], resp. Doss-$(p,\phi,F)_i$-almost periodic [Doss-$(p,\phi,F)_i$-uniformly recurrent], $\phi_1(\cdot)$ is measurable and $0\leqslant\phi_1\leqslant\phi$, then Lemma 1.1.6(iii) yields that $f(\cdot)$ is Doss-(p,ϕ_1,F)-almost periodic [Doss-(p,ϕ_1,F)-uniformly recurrent], resp. Doss-$(p,\phi_1,F)_i$-almost periodic [Doss-$(p,\phi_1,F)_i$-uniformly recurrent], where $i=1,2$.

EXAMPLE 2.6.18. (i) Let $p(x)\equiv p\in[1,\infty)$ and $f(x):=\chi_{[0,1/2]}(x)$, $x\in\mathbb{R}$. Then it can be simply shown that for each real number τ such that $|\tau|>1$ we have

$$\int_{-t}^{t}|f(x+\tau)-f(x)|^p\,dx\leqslant\frac{1}{2}+2\int_0^{1/2}|f(x)|^p\,dx,\quad t\in\mathbb{R}.$$

This implies that $f(\cdot)$ is Doss-$(p,x,t^{-\sigma})$-almost periodic for each real number $\sigma>0$.

(ii) Let $p(x)\equiv p\in[1,\infty)$ and $f(x):=\chi_{[0,\infty)}(x)$, $x\in\mathbb{R}$. Then it can be simply shown that for each real number τ we have

$$\int_{-t}^{t}|f(x+\tau)-f(x)|^p\,dx=\int_{-t+\tau}^{\tau}|f(x)|^p\,dx+\int_{\tau}^{t+\tau}|f(x)|^p\,dx,\quad t\in\mathbb{R}.$$

Hence,

$$\int_{-t}^{t}|f(x+\tau)-f(x)|^p\,dx\leqslant 2|\tau|,\ \text{provided}\ \tau\in\mathbb{R},\ t\geqslant|\tau|,$$

and $f(\cdot)$ is Doss-$(p,x,t^{-\sigma})$-almost periodic for each real number $\sigma>0$.

Concerning embeddings between different Doss almost periodic type spaces with variable exponent, we would like to state the following result:

PROPOSITION 2.6.19. *Let $p,\ q\in\mathcal{P}(I)$ and let $1\leqslant q(x)\leqslant p(x)$ for a.e. $x\in I$.*

(i) *Suppose that a function $f(\cdot)$ is Doss-(p,ϕ,F)-almost periodic, resp. Doss-(p,ϕ,F)-uniformly recurrent, and $F_1(t):=F(t)/t$, $t>0$. Then $f(\cdot)$ is Doss-(q,ϕ,F_1)-almost periodic, resp. Doss-(q,ϕ,F_1)-uniformly recurrent.*

(ii) *Suppose that a function $f(\cdot)$ is Doss-$(p,\phi,F)_1$-almost periodic, resp. Doss-$(p,\phi,F)_1$-uniformly recurrent, $\phi(\cdot)$ is monotonically increasing, there exists a function $\varphi:[0,\infty)\to(0,\infty)$ such that $\phi(xy)\leqslant\varphi(x)\phi(y)$, $x,\ y\geqslant 0$ and $F_1(t):=F(t)/\varphi(2(1+2t))$, $t>0$ provided $I=\mathbb{R}$, resp. $F_1(t):=$*

$F(t)/\varphi(2(1+t))$, $t>0$ provided $I=[0,\infty)$. Then $f(\cdot)$ is Doss-$(q,\phi,F_1)_1$-almost periodic, resp. Doss-$(q,\phi,F_1)_1$-uniformly recurrent.

(iii) *Suppose that a function $f(\cdot)$ is Doss-$(p,\phi,F)_2$-almost periodic, resp. Doss-$(p,\phi,F)_2$-uniformly recurrent, there exists a function $\varphi:[0,\infty)\rightarrow[0,\infty)$ such that $\phi(xy)\leqslant\varphi(x)\phi(y)$, $x,\ y\geqslant 0$ and*

$$\varphi\left(\frac{2F_1(\cdot)(1+2\cdot)}{F(\cdot)}\right)\in L^{\infty}((0,\infty)),\quad \textit{if } I=\mathbb{R},$$

$$\textit{resp. } \varphi\left(\frac{2F_1(\cdot)(1+\cdot)}{F(\cdot)}\right)\in L^{\infty}((0,\infty)),\quad \textit{if } I=[0,\infty).$$

Then $f(\cdot)$ is Doss-$(q,\phi,F_1)_2$-almost periodic, resp. Doss-$(q,\phi,F_1)_2$-uniformly recurrent.

Proof. We will prove only (iii), for the class of Doss-$(p,\phi,F)_2$-almost periodic functions defined on the interval $I=[0,\infty)$. Let the numbers $t,\ \tau>0$ be given. Then the conclusion simply follows from the calculation

$$\begin{aligned}
&\phi\left(F_1(t)\|f(\cdot+\tau)-f(\cdot)\|_{L^{q(x)}[0,t]}\right)\\
&\leqslant\phi\left(2F_1(t)(1+t)\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(x)}[0,t]}\right)\\
&=\phi\left(\frac{2F_1(t)(1+t)}{F(t)}F(t)\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(x)}[0,t]}\right)\\
&\leqslant\varphi\left(\frac{2F_1(t)(1+t)}{F(t)}\right)\phi\left(F(t)\|f(\cdot+\tau)-f(\cdot)\|_{L^{p(x)}[0,t]}\right),
\end{aligned}$$

where we have used Lemma 1.1.6(ii), and the corresponding definition of Doss-$(q,\phi,F_1)_2$-almost periodicity. $\square$

2.6.3. Invariance of generalized Doss almost periodicity with variable exponent under the actions of convolution products. In this subsection, we will investigate the invariance of three types of generalized Doss almost periodicity introduced above under the actions of infinite convolution products (for the sake of simplicity, we will not consider here the finite convolution products).

In [**245**, Theorem 2.1], we have analyzed the invariance of Doss p-almost periodicity under the actions of infinite convolution products, provided that the function $f(\cdot)$ in (55) is Stepanov p-bounded ($1\leqslant p<\infty$). In the formulation of the subsequent result, which is not satisfactory to a certain extent (let us only note that the above mentioned theorem, which is a unique result in the existing literature concerning this problematic, cannot be deduced from Theorem 2.6.20), we will not use this condition:

Theorem 2.6.20. *Suppose that $\varphi:[0,\infty)\rightarrow[0,\infty)$, $\phi:[0,\infty)\rightarrow[0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy)\leqslant\varphi(x)\phi(y)$ for all $x,\ y\geqslant$*

0 and $p \in \mathcal{P}(\mathbb{R})$. Suppose, further, $\check{f} : \mathbb{R} \to X$ is Doss-(p, ϕ, F)-almost periodic, resp. Doss-(p, ϕ, F)-uniformly recurrent, and measurable, $F_1 : (0, \infty) \to (0, \infty)$, $q \in \mathcal{P}(\mathbb{R})$, $1/p(x) + 1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family and for every real number $x \in \mathbb{R}$ we have

$$\int_{-x}^{\infty} \|R(v+x)\| \|\check{f}(v)\| \, dv < \infty. \tag{117}$$

Suppose that for each $\varepsilon > 0$ there exist an increasing sequence (a_m) of positive real numbers tending to plus infinity and a number $t_0(\varepsilon) > 0$ satisfying that, for every $t \geqslant t_0(\varepsilon)$, we have

(118)

$$\int_{-t}^{t} \varphi_{p(x)} \Biggl(2\varphi(a_m) a_m^{-1} F_1(t) \limsup_{m \to +\infty} \Bigl[\bigl[\varphi(\|R(\cdot + x)\|) \bigr]_{L^{q(\cdot)}[-x, -x+a_m]} F\bigl(t + a_m\bigr)^{-1} \Bigr] \Biggr) \, dx \leqslant 1.$$

Then the function $F : \mathbb{R} \to Y$, given by (55), is well-defined and Doss-(p, ϕ, F_1)-almost periodic, resp. Doss-(p, ϕ, F_1)-uniformly recurrent.

PROOF. We will consider only the class of Doss-(p, ϕ, F)-almost periodic functions because the proof for the class of Doss-(p, ϕ, F)-uniformly recurrent functions can be deduced quite analogously. Since $F(x) = \int_{-x}^{\infty} R(v+x)\check{f}(v)\, dv$, $x \in \mathbb{R}$, the validity of condition (117) yields that the function $F(\cdot)$ is well-defined as well as that the integrals in definitions of $F(x)$ and $F(x+\tau) - F(x)$ converge absolutely ($x \in \mathbb{R}$). Let $\varepsilon > 0$ be fixed, and let the sequences (t_n), (t_n') and (a_m) satisfy the prescribed requirements. Using the fact that the function $\phi(\cdot)$ is continuous and the function $\varphi_{p(x)}(\cdot)$ is monotonically increasing, we have ($x \in \mathbb{R}$, $\lambda, \ \tau > 0$):

$$\begin{aligned}
&\varphi_{p(x)}\left(\frac{\phi(\|F(x+\tau) - F(x)\|)}{\lambda} \right) \\
&\leqslant \varphi_{p(x)}\left(\frac{\phi\bigl(\int_{-x}^{\infty} \|R(v+x)\| \|\check{f}(v+\tau) - \check{f}(v)\| \, dv\bigr)}{\lambda} \right) \\
&\leqslant \varphi_{p(x)}\left(\lim_{m \to +\infty} \frac{\phi\bigl(\int_{-x}^{-x+a_m} \|R(v+x)\| \|\check{f}(v+\tau) - \check{f}(v)\| \, dv\bigr)}{\lambda} \right) \\
&= \varphi_{p(x)}\left(\lim_{m \to +\infty} \frac{\phi\bigl(\int_{-x}^{-x+a_m} a_m a_m^{-1} \|R(v+x)\| \|\check{f}(v+\tau) - \check{f}(v)\| \, dv\bigr)}{\lambda} \right) \\
&\leqslant \varphi_{p(x)}\left(\limsup_{m \to +\infty} \frac{\varphi(a_m) a_m^{-1} \int_{-x}^{-x+a_m} \phi(\|R(v+x)\| \|\check{f}(v+\tau) - \check{f}(v)\|) \, dv}{\lambda} \right) \\
&\leqslant \varphi_{p(x)}\Biggl(\limsup_{m \to +\infty} \frac{2\varphi(a_m) a_m^{-1} \bigl[\varphi(\|R(v+x)\|)\bigr]_{L^{q(v)}[-x, -x+a_m]}}{\lambda} \\
&\times \frac{\bigl[\phi(\|\check{f}(v+\tau) - \check{f}(v)\|)\bigr]_{L^{p(v)}[-x, -x+a_m]}}{\lambda} \Biggr),
\end{aligned}$$

where we have also used the Jensen integral inequality and the Hölder inequality. Let $\varepsilon > 0$ be fixed and let $\tau > 0$ be such that (112) holds, i.e., there exists $t_1(\varepsilon, \tau) \geqslant 0$ such that

$$\left[F(t)\Big[\phi\big(\|\check{f}(\cdot+\tau)-\check{f}(\cdot)\|\big)_{L^{p(x)}[-t,t]}\Big]\right] < \varepsilon, \quad t \geqslant t_1(\varepsilon, \tau). \tag{119}$$

Suppose that $t \geqslant \max(t_0(\varepsilon), t_1(\varepsilon, \tau))$. Then for each $x \in [-t, t]$ and $m \in \mathbb{N}$ we have $[-x, -x + a_m] \subseteq [-(t + a_m), t + a_m]$ so that the above calculation and (119) give

$$\varphi_{p(x)}\left(\frac{\phi(\|F(x+\tau)-F(x)\|)}{\lambda}\right)$$
$$\leqslant \varphi_{p(x)}\left(\limsup_{m\to+\infty}\frac{2\varphi(a_m)a_m^{-1}\big[\varphi(\|R(v+x)\|)\big]_{L^{q(v)}[-x,-x+a_m]}\varepsilon/F(t+a_m)}{\lambda}\right).$$

Integrating this estimate over the interval $[-t, t]$ and using (118) we get that the inequality

$$\int_{-t}^{t}\varphi_{p(x)}\left(\frac{\phi(\|F(x+\tau)-F(x)\|)}{\lambda}\right)dx \leqslant 1$$

holds with $\lambda = \varepsilon/F_1(t)$, which completes the proof in a routine manner. $\square$

We can similarly prove the following results for Doss-$(p, \phi, F)_1$-almost periodic functions, resp. Doss-$(p, \phi, F)_1$-uniformly recurrent functions, and Doss-$(p, \phi, F)_2$-almost periodic functions, resp. Doss-$(p, \phi, F)_2$-uniformly recurrent functions; for the sake of brevity, we will only provide descriptions of the proofs since they are very similar to the proof of Theorem 2.6.20 above:

THEOREM 2.6.21. *Suppose that $\phi : [0, \infty) \to [0, \infty)$ is a continuous monotonically increasing bijection and $p \in \mathcal{P}(\mathbb{R})$. Suppose, further, $\check{f} : \mathbb{R} \to X$ is Doss-$(p, \phi, F)_1$-almost periodic, resp. Doss-$(p, \phi, F)_1$-uniformly recurrent, and measurable, $F_1 : (0, \infty) \to (0, \infty)$, $q \in \mathcal{P}(\mathbb{R})$, $1/p(x) + 1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family and, for every real number $x \in \mathbb{R}$, we have (117). Suppose that for each $\varepsilon > 0$ there exist an increasing sequence (a_m) of positive real numbers tending to plus infinity and a number $t_0(\varepsilon) > 0$ satisfying that, for every $t \geqslant t_0(\varepsilon)$, we have*

$$\int_{-t}^{t}\varphi_{p(x)}\left(\frac{\limsup_{m\to+\infty}\Big[2\big[\varphi(\|R(\cdot+x)\|)\big]_{L^{q(\cdot)}[-x,-x+a_m]}\phi^{-1}(\varepsilon/F(t+a_m))\Big]}{\phi^{-1}(\varepsilon/F_1(t))}\right)dx \leqslant 1. \tag{120}$$

Then the function $F : \mathbb{R} \to Y$, given by (55), is well-defined and Doss-$(p, \phi, F_1)_1$-almost periodic, resp. Doss-$(p, \phi, F_1)_1$-uniformly recurrent.

PROOF. As in the proof of Theorem 2.6.20 above, we have that the function $F(\cdot)$ is well-defined as well as that the integrals in definitions of $F(x)$ and $F(x + \tau) - F(x)$ converge absolutely ($x \in \mathbb{R}$). Let $\varepsilon > 0$ be fixed. Then it suffices to show that, for every $t \geqslant t_0(\varepsilon)$, we have $(x \in \mathbb{R},\ \lambda,\ \tau > 0)$

$$\Big\| R(s)[F(x+t+\tau-s)-F(x+t-s)] \Big\|_{L^{p(x)}[-t,t]} \leqslant \phi^{-1}\big(\varepsilon/F_1(t)\big).$$

But, we can repeat the arguments used in the proof of the above-mentioned theorem, with $\phi(x) \equiv x$, in order to see that:

$$\begin{aligned}
&\varphi_{p(x)}\left(\frac{\|F(x+\tau)-F(x)\|}{\lambda}\right)\\
&\quad\leqslant \frac{2\big[\varphi(\|R(v+x)\|)\big]_{L^{q(v)}[-x,-x+a_m]}\big[\|\check{f}(v+\tau)-\check{f}(v)\|\big]_{L^{p(v)}[-x,-x+a_m]}}{\lambda}\Bigg)\\
&\quad\leqslant \frac{2\big[\varphi(\|R(v+x)\|)\big]_{L^{q(v)}[-x,-x+a_m]}\phi^{-1}(\varepsilon/F(t+a_m))}{\lambda}\Bigg).
\end{aligned}$$

The rest of proof is clear because we can take $\lambda = \phi^{-1}(\varepsilon/F_1(t))$ and use condition (120). $\square$

THEOREM 2.6.22. *Suppose that $\phi : [0,\infty) \to [0,\infty)$ is a continuous monotonically increasing bijection and $p \in \mathcal{P}(\mathbb{R})$. Suppose, further, $\check{f} : \mathbb{R} \to X$ is Doss-$(p,\phi,F)_2$-almost periodic, resp. Doss-$(p,\phi,F)_2$-uniformly recurrent, and measurable, $F_1 : (0,\infty) \to (0,\infty)$, $q \in \mathcal{P}(\mathbb{R})$, $1/p(x)+1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and, for every real number $x \in \mathbb{R}$, we have (117). Suppose that for each $\varepsilon > 0$ there exist an increasing sequence (a_m) of positive real numbers tending to plus infinity and a number $t_0(\varepsilon) > 0$ satisfying that, for every $t \geqslant t_0(\varepsilon)$, we have*

$$\int_{-t}^{t} \varphi_{p(x)}\left(2F_1(t)\limsup_{m\to+\infty}\frac{\big[\varphi(\|R(\cdot+x)\|)\big]_{L^{q(\cdot)}[-x,-x+a_m]}}{F\big(t+a_m\big)}\right)dx \leqslant 1. \tag{121}$$

Then the function $F : \mathbb{R} \to Y$, given by (55), is well-defined and Doss-$(p,\phi,F_1)_2$-almost periodic, resp. Doss-$(p,\phi,F_1)_2$-uniformly recurrent.

PROOF. We can use the same trick as above, with $\lambda = \phi^{-1}(\varepsilon)/F_1(t)$ and use condition (121). $\square$

REMARK 2.6.23.
(i) Suppose that $p(x) \equiv p \in [1,\infty)$. Then we can use the usual Hölder inequality in order to see that the estimates (118)-(121) can be modified by removing the multiplication with the number 2 therein.
(ii) Although we will not define the notion of Besicovitch-Doss almost periodicity with variable exponent here, we would like to note that the statement of [**234**, Theorem 2.13.7] and the corresponding part of this result which considers the Doss almost periodicity cannot be so easily reexamined in our framework.

Concerning the convolution invariance of generalized almost periodicity introduced in this subsection, we will only state and prove the following result (see also [**234**, Theorem 3.11.26]):

Proposition 2.6.24. *Suppose that $\psi \in L^1(\mathbb{R})$, $-\infty < a < b < +\infty$, $supp(\psi) \subseteq [a,b]$, $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$, $p,\ q \in \mathcal{P}(\mathbb{R})$ and $1/p(x) + 1/q(x) = 1$. Suppose, further, that the function $f : \mathbb{R} \to X$ is Doss-(p,ϕ,F)-almost periodic, resp. Doss-(p,ϕ,F)-uniformly recurrent, and essentially bounded. Then the function*

$$x \mapsto (\psi * f)(x) := \int_{-\infty}^{+\infty} \psi(x-y)f(y)\,dy, \quad x \in \mathbb{R} \tag{122}$$

is well-defined and essentially bounded. Furthermore, if $p_1 \in \mathcal{P}(\mathbb{R})$, $F_1 : (0,\infty) \to (0,\infty)$ and if, for every $\varepsilon > 0$ there exists a positive real number $t_1(\varepsilon) > 0$ such that

$$\int_{-t}^{t} \varphi_{p_1(x)}\left(2F_1(t)\varphi(b-a)\frac{\big\|\varphi\big(|\psi(x-z)|\big)\big\|_{L^{q(z)}[x-b,x-a]}}{(b-a)F(t+c)}\right)dx \leqslant 1, \tag{123}$$

where $c = \max(|a|,|b|)$, then the function $\psi * f(\cdot)$ is Doss-(p_1,ϕ,F_1)-almost periodic, resp. Doss-(p_1,ϕ,F_1)-uniformly recurrent.

Proof. We will give the main details of proof for the class of Doss-(p,ϕ,F)-almost periodic functions, only. For every $x \in \mathbb{R}$ and $\tau \in \mathbb{R}$, we have

$$\begin{aligned}
&\phi\Big(\|(\psi * f)(x+\tau) - (\psi * f)(x)\|\Big)\\
&\leqslant \phi\left((b-a)(b-a)^{-1}\int_a^b |\psi(y)| \cdot \|f(x+\tau-y) - f(x-y)\|\,dy\right)\\
&\leqslant \frac{\varphi(b-a)}{b-a}\int_a^b \phi\Big(|\psi(y)| \cdot \|f(x+\tau-y) - f(x-y)\|\Big)\,dy\\
&= \frac{\varphi(b-a)}{b-a}\int_{x-b}^{x-a} \phi\Big(|\psi(x-z)| \cdot \|f(z+\tau) - f(z)\|\Big)\,dz\\
&\leqslant 2\frac{\varphi(b-a)}{b-a}\big\|\varphi\big(|\psi(x-z)|\big)\big\|_{L^{q(z)}[x-b,x-a]}\|f(z+\tau)-f(z)\|_{L^{p(z)}[x-b,x-a]},
\end{aligned}$$

where we have used the Jensen integral inequality and the Hölder inequality. The proof can be completed as it has been done in the final part of the proof of Theorem 2.6.20. ☐

Composition principles for Besicovitch almost periodic functions have been investigated by M. Ayachi and J. Blot in [**36**]. We will consider composition principles for Doss almost periodic functions with variable exponents somewhere else.

Fix now a strictly increasing sequence (α_n) of positive reals tending to plus infinity, and set

$$BUR_{(\alpha_n)}(\mathbb{R} : X) := \big\{\vec{u} \in UR(\mathbb{R} : X)\,;\ \vec{u}(\cdot) \text{ is bounded and (19) holds with } f = \vec{u}\big\}.$$

Equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_\infty$, $BUR_{(\alpha_n)}(\mathbb{R} : X)$ is a complete metric space.

Now we are able to state the following result, which is very similar to Theorem 2.4.52:

THEOREM 2.6.25. *Suppose that the function* $F : \mathbb{R} \times X \to X$ *satisfies that for each bounded subset* B *of* X *there exists a finite real constant* $M_B > 0$ *such that* $\sup_{t\in\mathbb{R}} \sup_{y\in B} \|F(t,y)\| \leqslant M_B$. *Suppose, further, that* $p,\ r \in \mathcal{P}([0,1])$, *the function* $F : \mathbb{R} \times X \to X$ *is Stepanov* $p(x)$*-uniformly recurrent,* $r(\cdot) \geqslant \max(p(\cdot), p(\cdot)/(p(\cdot)-1))$ *and there exists a function* $L_F \in L_S^{r(x)}(I)$ *is such that* $q(x) := p(x)r(x)/(p(x)+r(x)) > 1$ *for a.e.* $x \in \mathbb{R}$ *and* (25) *holds with* $I = \mathbb{R}$. *If there exist a positive real number* $q' > 0$ *and an integer* $n \in \mathbb{N}$ *such that* $(\gamma-1)q' > -1$ *and* $q(x)/(q(x)-1) \leqslant q'$ *for a.e.* $x \in \mathbb{R}$, *and* $M_n < 1$, *where*

$$\begin{aligned} M_n := \sup_{t\geqslant 0} \int_{-\infty}^{t} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \Big\| R_\gamma(t-x_n) \Big\| \\ \times \prod_{i=2}^{n} \Big\| R_\gamma(x_i - x_{i-1}) \Big\| \prod_{i=1}^{n} L_F(x_i)\, dx_1\, dx_2 \cdots\, dx_n, \end{aligned}$$

and for every compact set $K \subseteq Y$, (111) *holds, then the abstract fractional Cauchy inclusion* (67) *has a unique bounded uniformly recurrent solution.*

PROOF. We will only outline the main details of proof. Define $\Upsilon : BUR_{(\alpha_n)}(\mathbb{R} : X) \to BUR_{(\alpha_n)}(\mathbb{R} : X)$ by

$$(\Upsilon \vec{u})(t) := \int_{-\infty}^{t} R_\gamma(t-s) F(s, \vec{u}(s))\, ds, \quad t \in \mathbb{R}.$$

Suppose that $\vec{u} \in BUR_{(\alpha_n)}(\mathbb{R} : X)$. Then $R(\vec{u}) = B$ is a bounded set, so that the mapping $t \mapsto F(t, \vec{u}(t))$, $t \in \mathbb{R}$ is bounded. Applying Theorem 2.6.8, we have that the function $F(\cdot, \vec{u}(\cdot))$ is Stepanov $q(x)$-uniformly recurrent. Define $q'(x) := q(x)/(q(x)-1)$ for a.e. $x \in \mathbb{R}$. Then (66) and the prescribed assumptions imply that $\|R_\gamma(\cdot)\| \in L^{q'(x)}[0,1]$ and $\sum_{k=0}^{\infty} \|R_\gamma(\cdot)\|_{L^{q'(x)}[k,k+1]} < \infty$. Applying Proposition 2.6.5, we get that the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t-s) F(s, \vec{u}(s))\, ds$, $t \in \mathbb{R}$ is uniformly recurrent. It can be simply verified that this function is also bounded continuous so that $\Upsilon \vec{u} \in BUR_{(\alpha_n)}(\mathbb{R} : X)$ and the mapping $\Upsilon(\cdot)$ is well defined. A simple calculation shows that

$$\Big\| \big(\Upsilon^n \vec{u_1}\big) - \big(\Upsilon^n \vec{u_2}\big) \Big\|_\infty \leqslant M_n \big\| \vec{u_1} - \vec{u_2} \big\|_\infty, \quad \vec{u_1},\ \vec{u_2} \in BUR_{(\alpha_n)}(\mathbb{R} : X),\ n \in \mathbb{N}.$$

Since we have assumed that $M_n < 1$, the Bryant fixed point theorem shows that the mapping $\Upsilon(\cdot)$ has a unique fixed point. This completes the proof of theorem. □

2.7. Generalized almost periodicity in Lebesgue spaces with variable exponents. Part III

In this section, we consider the Stepanov and Weyl classes of generalized almost periodic type functions and generalized uniformly recurrent type functions. We investigate the invariance of generalized almost periodicity and generalized uniform recurrence with variable exponents under the actions of convolution products, providing also certain applications.

2.7.1. Generalized Weyl uniform recurrence in Lebesgue spaces with variable exponents $L^{p(x)}$. Throughout this subsection, we will occasionally use conditions (A) and (B). We will first extend the notion introduced in Definition 2.5.22-Definition 2.5.24:

DEFINITION 2.7.1. Suppose that condition (A) holds, $f : I \to X$, and $\phi(\|f(\cdot+\tau) - f(\cdot)\|) \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$(p(x), \phi, F)$-uniformly recurrent, $f \in e-W_{ur}^{(p(x),\phi,F)}(I : X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \sup_{t\in I} \Biggl[F\bigl(l_n, t\bigr) \Bigl[\phi\bigl(\|f(\cdot+\alpha_n) - f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l_n]} \Bigr] \Biggr] = 0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$(p(x), \phi, F)$-uniformly recurrent, $f \in W_{ur}^{(p(x),\phi,F)}(I : X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \limsup_{l\to\infty} \sup_{t\in I} \Biggl[F(l, t) \Bigl[\phi\bigl(\|f(\cdot+\alpha_n) - f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Biggr] = 0. \tag{124}$$

DEFINITION 2.7.2. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot+\tau) - f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$(p(x), \phi, F)_1$-uniformly recurrent, $f \in e-W_{ur}^{(p(x),\phi,F)_1}(I : X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \sup_{t\in I} \Biggl[F\bigl(l_n, t\bigr) \phi \Bigl[\bigl(\|f(\cdot+\alpha_n) - f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l_n]} \Bigr] \Biggr] = 0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$(p(x), \phi, F)_1$-uniformly recurrent, $f \in W_{ur}^{(p(x),\phi,F)_1}(I : X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \limsup_{l\to\infty} \sup_{t\in I} \Biggl[F(l, t) \phi \Bigl[\bigl(\|f(\cdot+\alpha_n) - f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l]} \Bigr] \Biggr] = 0.$$

DEFINITION 2.7.3. Suppose that condition (A) holds, $f : I \to X$ and $\|f(\cdot+\tau) - f(\cdot)\| \in L^{p(x)}(K)$ for any $\tau \in I$ and any compact subset K of I.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$(p(x), \phi, F)_2$-uniformly recurrent, $f \in e-W_{ur}^{(p(x),\phi,F)_2}(I : X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \sup_{t\in I} \phi \Biggl[F\bigl(l_n, t\bigr) \Bigl[\bigl(\|f(\cdot+\alpha_n) - f(\cdot)\|\bigr)_{L^{p(\cdot)}[t,t+l_n]} \Bigr] \Biggr] = 0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$(p(x),\phi,F)_2$-uniformly recurrent, $f\in W_{ur}^{(p(x),\phi,F)_2}(I:X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\limsup_{l\to\infty}\sup_{t\in I}\phi\Biggl[F(l,t)\Bigl[\bigl(\bigl\|f(\cdot+\alpha_n)-f(\cdot)\bigr\|\bigr)_{L^{p(\cdot)}[t,t+l]}\Bigr]\Biggr]=0.$$

It is clear that the class of (equi-)Weyl-$(p(x),\phi,F)$-uniformly recurrent functions, resp. (equi-)Weyl-$(p(x),\phi,F)_i$-uniformly recurrent functions, extends the class of (equi-)Weyl-$(p(x),\phi,F)$-almost periodic functions, resp. (equi-)Weyl-$(p(x),\phi,F)_i$-almost periodic functions ($i=1,2$). Case $p(x)\equiv p$, $\phi(x)\equiv x$ and $F(l,t)=l^{(-1)/p}$ is the most indicative, when we say that the function $f(\cdot)$ is (equi-)Weyl-p-uniformly recurrent. The class of (equi-)Weyl-p-uniformly recurrent functions has not been considered elsewhere by now.

We have already shown that an equi-Weyl-(p,ϕ,ψ)-almost periodic function, resp. equi-Weyl-$(p,\phi,\psi)_i$-almost periodic function, does not need to be Weyl-(p,ϕ,ψ)-almost periodic, resp. Weyl-$(p,\phi,\psi)_i$-almost periodic ($i=1,2$). This statement continues to hold for generalized uniformly recurrent functions introduced above. For example, any continuous Stepanov p-almost periodic function $f(\cdot)$ which is not periodic cannot be Weyl-$(p,x,1)$-uniformly recurrent, while it is always equi-Weyl-$(p,x,1)$-almost periodic.

Example 2.7.4. If X does not contain an isomorphic copy of the sequence space c_0, $\phi(x)=x$ and $F(l,t)\equiv F(t)$, where $\lim_{t\to+\infty}F(t)=+\infty$, then there does not exist a non-periodic trigonometric polynomial $f(\cdot)$ and function $p\in\mathcal{P}(\mathbb{R})$ such that $f\in e-W_{ur}^{(p,x,F)}(\mathbb{R}:X)$. This can be verified based on the argumentation contained in Example 2.5.25(iii).

Further on, the statement of Proposition 2.5.26 and the conclusions established in Remark 2.5.27 can be reformulated for the introduced classes of generalized Weyl uniformly recurrent functions. In order to ensure the translation invariance of generalized Weyl spaces of uniformly recurrent functions with variable exponents, we will follow a slightly different approach based on the idea from [**142**]:

Definition 2.7.5. Suppose that condition (B) holds, $f:I\to X$ and $\phi(\|f(\cdot l+t+\tau)-f(t+\cdot l)\|)\in L^{p(x)}([0,1])$ for any $\tau\in I$, $t\in I$ and $l>0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$[p(x),\phi,F]$-uniformly recurrent, $f\in e-W_{ur}^{[p(x),\phi,F]}(I:X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\to+\infty}\alpha_n=+\infty$ and

$$\lim_{n\to+\infty}\sup_{t\in I}\Biggl[F\bigl(l_n,t\bigr)\Bigl[\phi\bigl(\bigl\|f(\cdot l_n+t+\alpha_n)-f(t+\cdot l_n)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]=0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$[p(x),\phi,F]$-uniformly recurrent, $f\in W_{ur}^{[p(x),\phi,F]}(I:X)$ for short, if and only if we can find a sequence (α_n)

of positive real numbers such that $\lim_{n\rightarrow+\infty}\alpha_n=+\infty$ and

$$\lim_{n\rightarrow+\infty}\limsup_{l\rightarrow\infty}\sup_{t\in I}\Biggl[F(l,t)\Bigl[\phi\bigl(\bigl\|f(\cdot l+t+\alpha_n)-f(t+\cdot l)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]=0.$$

DEFINITION 2.7.6. Suppose that condition (B) holds, $f:I\rightarrow X$ and $\|f(\cdot l+t+\tau)-f(t+\cdot l)\|\in L^{p(x)}([0,1])$ for any $\tau\in I$, $t\in I$ and $l>0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$[p(x),\phi,F]_1$-uniformly recurrent, $f\in e-W_{ur}^{[p(x),\phi,F]_1}(I:X)$ for short, if and only if we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\rightarrow+\infty}\alpha_n=+\infty$ and

$$\lim_{n\rightarrow+\infty}\sup_{t\in I}\Biggl[F\bigl(l_n,t\bigr)\phi\Bigl[\bigl(\bigl\|f(\cdot l_n+t+\alpha_n)-f(t+\cdot l_n)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]=0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$[p(x),\phi,F]_2$-uniformly recurrent, $f\in W_{ur}^{[p(x),\phi,F]_2}(I:X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\rightarrow+\infty}\alpha_n=+\infty$ and

$$\lim_{n\rightarrow+\infty}\limsup_{l\rightarrow\infty}\sup_{t\in I}\Biggl[F(l,t)\phi\Bigl[\bigl(\bigl\|f(\cdot l+t+\alpha_n)-f(t+\cdot l)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]=0.$$

DEFINITION 2.7.7. Suppose that condition (B) holds, $f:I\rightarrow X$ and $\|f(\cdot l+t+\tau)-f(t+\cdot l)\|\in L^{p(x)}([0,1])$ for any $\tau\in I$, $t\in I$ and $l>0$.

(i) It is said that the function $f(\cdot)$ is equi-Weyl-$[p(x),\phi,F]_2$-uniformly recurrent, $f\in e-W_{ur}^{[p(x),\phi,F]_2}(I:X)$ for short, if and only we can find two sequences (l_n) and (α_n) of positive real numbers such that $\lim_{n\rightarrow+\infty}\alpha_n=+\infty$ and

$$\lim_{n\rightarrow+\infty}\sup_{t\in I}\phi\Biggl[F\bigl(l_n,t\bigr)\Bigl[\bigl(\bigl\|f(\cdot l_n+t+\alpha_n)-f(t+\cdot l_n)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]=0.$$

(ii) It is said that the function $f(\cdot)$ is Weyl-$[p(x),\phi,F]_2$-uniformly recurrent, $f\in W_{ur}^{[p(x),\phi,F]_2}(I:X)$ for short, if and only if we can find a sequence (α_n) of positive real numbers such that $\lim_{n\rightarrow+\infty}\alpha_n=+\infty$ and

$$\lim_{n\rightarrow+\infty}\limsup_{l\rightarrow\infty}\sup_{t\in I}\phi\Biggl[F(l,t)\Bigl[\bigl(\bigl\|f(\cdot l+t+\alpha_n)-f(t+\cdot l)\bigr\|\bigr)_{L^{p(\cdot)}[0,1]}\Bigr]\Biggr]=0.$$

The statement of Proposition 2.5.32 and the conclusions established in Remark 2.5.31 can be reformulated for generalized Weyl uniformly recurrent functions introduced in the above three definitions. All statements regarding the convolution invariance of the generalized Weyl almost periodicity with variable exponents can be straightforwardly reformulated for generalized Weyl uniformly recurrent functions introduced in this section; we leave readers to make this precise.

2.7.2. Quasi-asymptotically uniformly recurrent type functions with variable exponents. In the following definition, we will extend the notion of quasi-asymptotical almost periodicity:

DEFINITION 2.7.8. We say that a continuous function $f : I \to X$ is quasi-asymptotically uniformly recurrent if and only if there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} \bigl\|f(t+\alpha_n)-f(t)\bigr\| = 0. \tag{125}$$

Denote by $Q-AUR(I:X)$ the set consisting of all quasi-asymptotically uniformly recurrent functions from I into X.

It is expected that the class of quasi-asymptotically uniformly recurrent functions extends the class of asymptotically uniformly recurrent functions. For completeness, we will include all details of the proof of the following proposition:

PROPOSITION 2.7.9. *Suppose that $f : I \to X$ is asymptotically uniformly recurrent. Then $f(\cdot)$ is quasi-asymptotically uniformly recurrent.*

PROOF. Let $h \in UR(\mathbb{R}:X)$, $q \in C_0(I:X)$ and $f = h+q$. By our assumption, we have the existence of a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that (19) holds with the function $f(\cdot)$ replaced therein with the function $h(\cdot)$. Let $n \in \mathbb{N}$ be fixed. Then we can find a sufficiently large real number $M_n' > 0$ and a sufficiently large integer $n_0 \in \mathbb{N}$ such that $\|q(t)\| \leqslant 1/n$ for $|t| \geqslant M_n'$ and $\|h(t+\alpha_n)-h(t)\| \leqslant 1/n$, $n \geqslant n_0$. Then, for every $t \in \mathbb{R}$ such that $|t| \geqslant M_n := M_n' + \alpha_n$, we have $|t|,\ |t+\alpha_n| \geqslant M_n'$ and

$$\Bigl\|\bigl[h(t+\alpha_n)-h(t)\bigr]+\bigl[q(t+\alpha_n)-q(t)\bigr]\Bigr\| \leqslant \frac{1}{n} + \bigl\|q(t+\alpha_n)-q(t)\bigr\| \leqslant \frac{2}{n},\ n \geqslant n_0.$$

This simply implies the required assertion. $\square$

Applying the same arguments, we can deduce the following

PROPOSITION 2.7.10. *Suppose that $f : I \to X$ is quasi-asymptotically uniformly recurrent and $q \in C_0(I:X)$. Then $(f+q)(\cdot)$ is likewise quasi-asymptotically uniformly recurrent.*

The proof of following proposition is simple and can be omitted, as well:

PROPOSITION 2.7.11. *Suppose that $I = \mathbb{R}$ and $f : I \to X$. Then $f(\cdot)$ is quasi-asymptotically uniformly recurrent (quasi-asymptotically almost periodic, S-asymptotically ω-periodic) if and only if $\check{f}(\cdot)$ is quasi-asymptotically uniformly recurrent (quasi-asymptotically almost periodic, S-asymptotically ω-periodic).*

If $f \in Q-AUR(\mathbb{R}:X)$ and $\varphi \in L^1(\mathbb{R})$ has a compact support, then it can be easily seen that the convolution $\varphi * f(\cdot) := \int_{\mathbb{R}} \varphi(\cdot - y)f(y)\,dy$ belongs to the class $Q-AUR(\mathbb{R}:X)$. Further on, any quasi-asymptotically almost periodic function is bounded by definition, and this is no longer true for quasi-asymptotically uniformly recurrent functions. In connection with this, we would like to present the following illustrative example:

EXAMPLE 2.7.12. Let the function $f(\cdot)$ be defined by (33). We know that for each real number $c > 0$ the function $h(t) := \min(c, f(t))$, $t \in \mathbb{R}$ is bounded uniformly continuous, uniformly recurrent, and not (Stepanov) p-quasi-asymptotically almost periodic ($p \geqslant 1$). On the other hand, Proposition 2.7.9 shows that the function $h(\cdot)$ is quasi-asymptotically uniformly recurrent.

Further on, if $f \in C^1(I : X)$ and $f' \in C_0(I : X)$, then the Lagrange mean value theorem implies that the function $f(\cdot)$ is quasi-asymptotically uniformly recurrent. In particular, the function $f(t) := \ln(1+t)$, $t \geqslant 0$ is quasi-asymptotically uniformly recurrent; on the other hand, it can be simply verified that the function $f(\cdot)$ is not asymptotically uniformly recurrent. The notion of quasi-asymptotical uniform anti-recurrence can be also introduced and analyzed (see also [**247**, Example 2.3, Remark 2.4]).

EXAMPLE 2.7.13. The function $f : [0, \infty) \to \mathbb{R}$ given by $f(t) := \sin(\ln(1 + t))$, $t \geqslant 0$ is quasi-asymptotically almost periodic but not asymptotically almost periodic (see [**247**] and [**325**, Example 4.1, Theorem 4.2]). Now we will prove that this function cannot be asymptotically uniformly recurrent. Suppose the contrary, and fix a sufficiently small number $\varepsilon > 0$. Then an elementary argumentation shows that there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a number $t_0(\varepsilon) > 0$ such that $|\sin(\ln(t + \alpha_n)) - \sin(\ln t)| \leqslant 2\varepsilon$ for all $t \geqslant t_0(\varepsilon)$ and $n \in \mathbb{N}$. Hence,

$$\left|\sin \frac{\ln(1 + (\alpha_n/t))}{2} \cos \frac{\ln(t(t + \alpha_n))}{2}\right| \leqslant \varepsilon, \quad t \geqslant t_0(\varepsilon),\ n \in \mathbb{N}.$$

Let $n_0(\varepsilon) \in \mathbb{N}$ be such that $\alpha_n \geqslant t_0(\varepsilon)$ for $n \geqslant n_0(\varepsilon)$. Plugging $t = k\alpha_n$, where $1 \leqslant k \leqslant 5$, the above estimate simply implies that there exists a finite constant $c > 0$ such that

$$\left|\cos \frac{\ln(a\alpha_n^2)}{2}\right| \leqslant c\varepsilon, \quad 2 \leqslant a \leqslant 30,\ n \geqslant n_0(\varepsilon).$$

Then we get the existence of a real number $c_\varepsilon > 0$ such that $\lim_{\varepsilon \to 0+} c_\varepsilon = 0$ and

$$\operatorname{dist}\Bigl(a\alpha_n^2, \bigl\{\exp((2k+1)\pi) : k \in \mathbb{N}_0\bigr\}\Bigr) \leqslant e^{2c_\varepsilon}, \quad 2 \leqslant a \leqslant 30,\ n \geqslant n_0(\varepsilon).$$

It can be simply verified that this estimate cannot be satisfied simultaneously for $a = 2$ and $a = e^\pi$, which yields a contradiction.

In [**247**, Theorem 2.5], we have proved that any asymptotically almost automorphic function which is also quasi-asymptotically almost periodic is always asymptotically almost periodic. The arguments contained in the proof of the above-mentioned theorem also show that any asymptotically uniformly recurrent function which is quasi-asymptotically almost periodic is always asymptotically almost periodic as well as that the following result holds true:

THEOREM 2.7.14. *Let* $\mathrm{F}(I : X)$ *be any space of functions* $h : I \to X$ *satisfying that for each* $\tau \in I$ *the supremum formula holds for the function* $h(\cdot + \tau) - h(\cdot)$, *that is*

$$\sup_{t\in I}\|h(\cdot+\tau)-h(\cdot)\| = \sup_{t\in I,\ t\geqslant a}\|h(\cdot+\tau)-h(\cdot)\|, \quad a\in I.$$

Then we have:

(i) $[\mathrm{F}(I:X)+C_0(I:X)]\cap Q-AUR(I:X)\subseteq AUR(I:X)$.
(ii) $\mathrm{F}(I:X)\cap Q-AUR(I:X)\subseteq UR(I:X)$.

Proof. We will include the main details of the proof for the sake of completeness. Let $h\in \mathrm{F}(I:X)$, $q\in C_0(I:X)$ and $f=h+q\in Q-AUR(I:X)$. By our assumptions, there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that, for every integer $n\in\mathbb{N}$, there exists an integer $n_0\in\mathbb{N}$ with

$$\Bigl\|\bigl[h(t+\alpha_n)-h(t)\bigr]+\bigl[q(t+\alpha_n)-q(t)\bigr]\Bigr\|\leqslant 1/n, \text{ for } t\in I,\ |t|\geqslant M_n,\ n\geqslant n_0.$$

Let $n\in\mathbb{N}$ be fixed. Since $q\in C_0(I:X)$, we have that there exists a finite number $M_n'\geqslant M_n$ such that

$$\Bigl\|h(t+\alpha_n)-h(t)\Bigr\|\leqslant 2/n, \text{ provided } t\in I \text{ and } |t|\geqslant M_n',\ n\geqslant n_0.$$

Define the function $H_n: I\to X$ by $H_n(t):=h(t+\alpha_n)-h(t)$, $t\in I$. Since the supremum formula holds for the function $H_n(\cdot)$, we get

$$\sup_{t\in I}\bigl\|H_n(t)\bigr\| = \sup_{t\geqslant M_n'}\bigl\|H_n(t)\bigr\|\leqslant 2/n.$$

Hence, $\lim_{n\to+\infty}\sup_{t\in I}\|h(t+\alpha_n)-h(t)\|=0$ and $h(\cdot)$ is thus uniformly recurrent, which immediately implies part (i). Part (ii) can be deduced similarly. ☐

In the following illustrative application of Theorem 2.7.14, we will consider case in which $I=\mathbb{R}$ and $\mathrm{F}(I:X)=AA(I:X)$, the space of all almost automorphic functions from I into X (see [**234**] for more details):

Example 2.7.15. Set $a_n:=\mathrm{sign}(\cos(n\pi\sqrt{2}))$, $n\in\mathbb{Z}$ and define after that the function $f:\mathbb{R}\to\mathbb{R}$ by $f(t):=\alpha a_n+(1-\alpha)a_{n+1}$ if $t\in[n,n+1)$ for some integer $n\in\mathbb{Z}$ and $t=\alpha n+(1-\alpha)(n+1)$ for some number $\alpha\in(0,1]$. This function is compactly almost automorphic but not almost periodic; furthermore, we have proved that the function $f(\cdot)$ is not asymptotically uniformly recurrent. Using this fact and Theorem 2.7.14, it readily follows that the function $f(\cdot)$ is not quasi-asymptotically uniformly recurrent, as well.

2.7.3. Stepanov classes of quasi-asymptotically uniformly recurrent type functions. Throughout this subsection, we use the following conditions:

$(A)_S$: $I=\mathbb{R}$ or $I=[0,\infty)$, $\phi:[0,\infty)\to[0,\infty)$, $p\in\mathcal{P}(I)$, $\mathsf{F}: I\times(0,\infty)\times I\to(0,\infty)$, $F: I\times\mathbb{N}\to(0,\infty)$, $\mathtt{F}: I\to(0,\infty)$ and $\omega\in I$.

$(B)_S$: The same as $(A)_S$ with the assumption $p\in\mathcal{P}(I)$ replaced by $p\in\mathcal{P}([0,1])$ therein.

We first introduce the Stepanov-(p, ϕ, F)-classes of quasi-asymptotically uniformly recurrent functions and the Stepanov-$(p, \phi, F)_i$-classes of quasi-asymptotically uniformly recurrent functions, where $i = 1, 2$ and $p \in \mathcal{P}(I)$. In this approach, we may loose the information about the translation invariance of introduced spaces:

Definition 2.7.16. Let $(A)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov-(p, ϕ, F)-quasi-asymptotically almost periodic, resp. Stepanov-(p, ϕ, F)-quasi-asymptotically uniformly recurrent, if and only if $\phi(\|f(\cdot + \tau) - f(\cdot)\|) \in L^{p(\cdot)}(K)$ for any $\tau \in I$ and any compact set $K \subseteq I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t| \geqslant M(\varepsilon,\tau)} \mathsf{F}(t, \varepsilon, \tau)\phi\big(\big\|f(\cdot + \tau) - f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+1]} \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} F(t, n)\phi\big(\big\|f(\cdot + \alpha_n) - f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+1]} = 0. \tag{126}$$

(ii) We say that a function $f : I \to X$ is Stepanov-$(p, \phi, \mathtt{F})$-asymptotically ω-periodic if and only if $\phi(\|f(\cdot + \omega) - f(\cdot)\|) \in L^{p(\cdot)}(K)$ for any compact set $K \subseteq I$ and

$$\lim_{|t|\to\infty} \mathtt{F}(t)\phi\big(\big\|f(\cdot + \omega) - f(\cdot)\big\|\big)_{L^{p(\cdot)}[t,t+1]} = 0.$$

Definition 2.7.17. Let $(A)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov-$(p, \phi, \mathsf{F})_1$-quasi-asymptotically almost periodic, resp. Stepanov-$(p, \phi, F)_1$-quasi-asymptotically uniformly recurrent, if and only if $\|f(\cdot+\tau) - f(\cdot)\| \in L^{p(\cdot)}(K)$ for any $\tau \in I$ and any compact set $K \subseteq I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)} \mathsf{F}(t, \varepsilon, \tau)\phi\Big(\big\|f(\cdot + \tau) - f(\cdot)\big\|_{L^{p(\cdot)}[t,t+1]}\Big) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} F(t, n)\phi\Big(\big\|f(\cdot + \alpha_n) - f(\cdot)\big\|_{L^{p(\cdot)}[t,t+1]}\Big) = 0.$$

(ii) We say that a function $f : I \to X$ is Stepanov-$(p, \phi, \mathtt{F})_1$-asymptotically ω-periodic if and only if $\|f(\cdot+\omega) - f(\cdot)\| \in L^{p(\cdot)}(K)$ for any compact set $K \subseteq I$ and

$$\lim_{|t|\to\infty} \mathtt{F}(t)\phi\Big(\big\|f(\cdot+\omega) - f(\cdot)\big\|_{L^{p(\cdot)}[t,t+1]}\Big) = 0.$$

Definition 2.7.18. Let $(A)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov-$(p, \phi, \mathsf{F})_2$-quasi-asymptotically almost periodic, resp. Stepanov-$(p, \phi, F)_2$-quasi-asymptotically uniformly recurrent, if and only if $\|f(\cdot+\tau) - f(\cdot)\| \in L^{p(\cdot)}(K)$ for any $\tau \in I$ and any compact set $K \subseteq I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)} \phi\Big(\mathsf{F}(t, \varepsilon, \tau)\big\|f(\cdot+\tau) - f(\cdot)\big\|_{L^{p(\cdot)}[t,t+1]}\Big) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} \phi\Big(F(t, n)\big\|f(\cdot+\alpha_n) - f(\cdot)\big\|_{L^{p(\cdot)}[t,t+1]}\Big) = 0.$$

(ii) Then we say that a function $f : I \to X$ is Stepanov-$(p, \phi, \mathtt{F})_2$-asymptotically ω-periodic if and only if $\|f(\cdot+\omega) - f(\cdot)\| \in L^{p(\cdot)}(K)$ for any compact set $K \subseteq I$ and

$$\lim_{|t|\to\infty} \phi\Big(\mathtt{F}(t)\big\|f(\cdot+\omega) - f(\cdot)\big\|_{L^{p(\cdot)}[t,t+1]}\Big) = 0.$$

In the second approach, we will employ condition $(B)_S$ and assume therefore that $p \in \mathcal{P}([0, 1])$. Using the substitution $\cdot \to \cdot + t$, the translation invariance of considered function spaces can be achieved (see e.g., Remark 2.5.31(iii)):

Definition 2.7.19. Let $(B)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov-$[p, \phi, \mathsf{F}]$-quasi-asymptotically almost periodic, resp. Stepanov-$[p, \phi, F]$-quasi-asymptotically uniformly recurrent, if and only if $\phi(\|f(\cdot+t+\tau) - f(\cdot+t)\|) \in L^{p(\cdot)}[0, 1]$ for any $\tau,\ t \in I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)} \mathsf{F}(t, \varepsilon, \tau)\phi\big(\big\|f(\cdot+t+\tau) - f(\cdot+t)\big\|\big)_{L^{p(\cdot)}[0,1]} \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} F(t, n)\phi\big(\big\|f(\cdot+t+\alpha_n) - f(\cdot+t)\big\|\big)_{L^{p(\cdot)}[0,1]} = 0.$$

(ii) Then we say that a function $f : I \to X$ is Stepanov-$[p, \phi, \mathtt{F}]$-asymptotically ω-periodic if and only if $\phi(\|f(\cdot + t + \omega) - f(\cdot + t)\|) \in L^{p(\cdot)}[0,1]$ for any $t \in I$ and

$$\lim_{|t|\to\infty} \mathtt{F}(t)\phi\big(\big\|f(\cdot + t + \omega) - f(\cdot + t)\big\|\big)_{L^{p(\cdot)}[0,1]} = 0.$$

DEFINITION 2.7.20. Let $(B)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov-$[p, \phi, \mathsf{F}]_1$-quasi-asymptotically almost periodic, resp. Stepanov-$[p, \phi, F]_1$-quasi-asymptotically uniformly recurrent, if and only if $\|f(\cdot+t+\tau)-f(\cdot+t)\| \in L^{p(\cdot)}[0,1]$ for any $\tau,\ t \in I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)} \mathsf{F}(t, \varepsilon, \tau)\phi\Big(\big\|f(\cdot + t + \tau) - f(\cdot + t)\big\|_{L^{p(\cdot)}[0,1]}\Big) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} F(t, n)\phi\Big(\big\|f(\cdot + t + \alpha_n) - f(\cdot + t)\big\|_{L^{p(\cdot)}[0,1]}\Big) = 0.$$

(ii) Then we say that a function $f : I \to X$ is Stepanov-$[p, \phi, \mathtt{F}]_1$-asymptotically ω-periodic if and only if $\|f(\cdot+t+\omega) - f(\cdot+t)\| \in L^{p(\cdot)}[0,1]$ for any $t \in I$ and

$$\lim_{|t|\to\infty} \mathtt{F}(t)\phi\Big(\big\|f(\cdot + t + \omega) - f(\cdot + t)\big\|_{L^{p(\cdot)}[0,1]}\Big) = 0.$$

DEFINITION 2.7.21. Let $(B)_S$ hold.

(i) A function $f : I \to X$ is called Stepanov-$[p, \phi, \mathsf{F}]_2$-quasi-asymptotically almost periodic, resp. Stepanov-$[p, \phi, F]_2$-quasi-asymptotically uniformly recurrent, if and only if $\|f(\cdot+t+\tau)-f(\cdot+t)\| \in L^{p(\cdot)}[0,1]$ for any $\tau,\ t \in I$ as well as for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\sup_{|t|\geqslant M(\varepsilon,\tau)} \phi\Big(\mathsf{F}(t, \varepsilon, \tau)\big\|f(\cdot + t + \tau) - f(\cdot + t)\big\|_{L^{p(\cdot)}[0,1]}\Big) \leqslant \varepsilon,$$

resp. there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{|t|\geqslant M_n} \phi\Big(F(t, n)\big\|f(\cdot + t + \alpha_n) - f(\cdot + t)\big\|_{L^{p(\cdot)}[0,1]}\Big) = 0.$$

(ii) Then we say that a function $f : I \to X$ is Stepanov-$[p, \phi, \mathtt{F}]_2$-asymptotically ω-periodic if and only if $\|f(\cdot + t + \omega) - f(\cdot + t)\| \in L^{p(\cdot)}[0,1]$ for any $t \in I$ and

$$\lim_{|t| \to \infty} \phi\Big(\mathtt{F}(t)\big\|f(\cdot + t + \omega) - f(\cdot + t)\big\|_{L^{p(\cdot)}[0,1]}\Big) = 0.$$

REMARK 2.7.22. The notion introduced in the above definitions is rather general. Let us only say the following: suppose that $I = \mathbb{R}$, the function $\phi(\cdot)$ is locally bounded, $\omega \in \mathbb{R}$ and

$$\sup_{t \in \mathbb{R}} \Big[\|f(\cdot)\|_{L^{p(\cdot - \omega)}[t,t+1]} + \|f(\cdot)\|_{L^{p(\cdot)}[t,t+1]}\Big] < \infty.$$

Then it readily follows that $f(\cdot)$ is Stepanov-$(p, \phi, \mathtt{F})$-asymptotically ω-periodic for any function $\mathtt{F} \in C_0(\mathbb{R} : X)$.

The notion introduced in the above definitions extends the notion of Stepanov p-quasi-asymptotical almost periodicity and the notion of Stepanov p-asymptotical ω-periodicity ($1 \leqslant p < \infty$). In case that $p(x) \equiv p \in [1, \infty)$, the Stepanov-$(p, \phi, F)$-classes coincide with the corresponding Stepanov-$[p, \phi, F]$-classes of functions. The most intriguing case, without any doubt, is that in which the functions F, F, $\mathtt{F}$ are identically equal to one and $\phi(x) \equiv x$; if this is the case and $p \in \mathcal{P}([0,1])$ (see Definition 2.7.19-Definition 2.7.21), then we also say that the function $f : I \to X$ is Stepanov $p(x)$-quasi-asymptotically almost periodic (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent, Stepanov $p(x)$-asymptotically ω-periodic). In what follows, by $S^{p(x)}Q - AAP(I : X)$ ($S^{p(x)}Q - AUR(I : X)$, $S^{p(x)}SAP_\omega(I : X)$) we denote the collection of all such functions. It can be easily verified that the function $f : I \to X$ is Stepanov $p(x)$-quasi-asymptotically almost periodic (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent, Stepanov $p(x)$-asymptotically ω-periodic) if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$ is quasi-asymptotically almost periodic (quasi-asymptotically uniformly recurrent, S-asymptotically ω-periodic). This enables one to transfer the statements of Proposition 2.7.11 and Theorem 2.7.14 to the Stepanov classes (see also [**247**, Theorem 2.10, Proposition 2.11]) as well as to conclude that $S^{p(x)}SAP_\omega(I : X) \subseteq S^{p(x)}Q - AAP(I : X) \subseteq S^{p(x)}Q - AUR(I : X)$ for any $p \in \mathcal{P}([0,1])$; see also [**247**, Proposition 2.7].

Unfortunately, the spaces of (Stepanov $p(x)$-) quasi-asymptotically uniformly recurrent type functions are not closed under the operations of pointwise addition and multiplication. For instance, the consideration from [**247**, Example 2.16-Example 2.18] enables one to see that the following holds:

(i) There exist a continuous periodic function $f : \mathbb{R} \to \mathbb{R}$ and a function $g \in SAP_2(\mathbb{R} : \mathbb{R})$ such that the function $(f \cdot g)(\cdot)$ is not quasi-asymptotically uniformly recurrent.

(ii) There exist a Stepanov p-almost periodic function $f : \mathbb{R} \to \mathbb{R}$, where the exponent $p \geqslant 1$ can be chosen arbitrarily, and a function $g \in SAP_4(\mathbb{R} : \mathbb{R})$ such that $(f \cdot g)(\cdot)$ does not belong to the class $S^1Q - AUR(\mathbb{R} : \mathbb{R})$.

(iii) There exist a continuous periodic function $f : [0, \infty) \to \mathbb{R}$ and a function $g \in SAP_4([0, \infty) : \mathbb{R})$ such that the function $(f + g)(\cdot)$ does not belong to the class $S^1Q - AUR([0, \infty) : \mathbb{R})$.

We continue by stating the following:

PROPOSITION 2.7.23. *Suppose that $\phi(\cdot)$ is continuous for $t = 0$, $\phi(0) = 0$ and any of the functions F, F, $\boldsymbol{F}$ is bounded. Then any quasi-asymptotically uniformly recurrent function $f : I \to X$ is Stepanov-(p, ϕ, F)-quasi-asymptotically uniformly recurrent, Stepanov-$[p, \phi, F]$-quasi-asymptotically uniformly recurrent as well as Stepanov-$(p, \phi, F)_i$-quasi-asymptotically uniformly recurrent and Stepanov-$[p, \phi, F]_i$-quasi-asymptotically uniformly recurrent ($i = 1, 2$). The same statement holds for the corresponding classes of quasi-asymptotically almost periodic functions and S-asymptotically ω-periodic functions.*

PROOF. We will provide the main details of the proof for the class of Stepanov-$[p, \phi, F]$-quasi-asymptotically uniformly recurrent functions. Let (α_n) and (M_n) be the sequences from Definition 2.7.8, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|\phi(t)| = |\phi(t) - \phi(0)| < \varepsilon$, $|t| \leqslant \delta$. Hence, $\sup \phi([0, \delta]) \leqslant \varepsilon$. By our assumption, we have the existence of an integer $n_0 \in \mathbb{N}$ such that

$$\sup_{|t| \geqslant M_n} \big\| f(t + \alpha_n) - f(t) \big\| \leqslant \delta, \quad n \geqslant n_0.$$

Let $n \in \mathbb{N}$ with $n \geqslant n_0$ be fixed. Then, for every $t \geqslant M_n' := M_n + 1$, we have $|t + x| \geqslant |t| - 1 \geqslant M_n$, $x \in [0, 1]$. This implies that, for every $t \geqslant M_n'$, $x \in [0, 1]$ and $\lambda \geqslant \varepsilon$, we have $\phi(\|f(t + \alpha_n + x) - f(t + x)\|)/\lambda \leqslant 1$, $\varphi_{p(x)}(\phi(\|f(t + \alpha_n + x) - f(t + x)\|)/\lambda) \leqslant 1$ and therefore

$$\int_0^1 \varphi_{p(x)}\Big(\phi(\|f(t + \alpha_n + x) - f(t + x)\|)/\lambda\Big)\, dx \leqslant 1.$$

Thus,

$$[\varepsilon, \infty) \subseteq \left\{ \lambda > 0 : \int_0^1 \varphi_{p(x)}\Big(\phi(\|f(t + \alpha_n + x) - f(t + x)\|)/\lambda\Big)\, dx \leqslant 1 \right\},$$

which yields that

$$\phi\big(\big\| f(\cdot + t + \alpha_n) - f(\cdot + t) \big\|\big)_{L^{p(\cdot)}[0,1]} \leqslant \varepsilon, \quad n \geqslant n_0.$$

This completes the proof by the boundedness of function $F(\cdot, \cdot)$. $\square$

As an immediate consequence, we have the following statement:

COROLLARY 2.7.24. *Suppose that $\omega \in I$ and $p \in \mathcal{P}([0, 1])$. Then any quasi-asymptotically almost periodic (quasi-asymptotically uniformly recurrent, S-asymptotically ω-periodic) function $f : I \to X$ is Stepanov $p(x)$-quasi-asymptotically almost periodic (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent, Stepanov $p(x)$-asymptotically ω-periodic).*

Using the trivial inequalities and Lemma 1.1.6, we can clarify numerous inclusions for the introduced classes of functions. For instance, we can simply deduce the following:

(i) $S^{p(x)}SAP_\omega(I:X) \subseteq S^1SAP_\omega(I:X)$, $S^{p(x)}Q-AAP(I:X) \subseteq S^1Q-AAP(I:X)$ and $S^{p(x)}Q-AUR(I:X) \subseteq S^1Q-AUR(I:X)$.

(ii) Suppose $p \in D_+([0,1])$ and $1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty$ for a.e. $x \in [0,1]$. Then we have $S^{p^+}SAP_\omega(I:X) \subseteq S^{p(x)}SAP_\omega(I:X) \subseteq S^{p^-}SAP_\omega(I:X)$, $S^{p^+}Q-AAP(I:X) \subseteq S^{p(x)}Q-AAP(I:X) \subseteq S^{p^-}Q-AAP(I:X)$, and $S^{p^+}Q-AUR(I:X) \subseteq S^{p(x)}Q-AUR(I:X) \subseteq S^{p^-}Q-AUR(I:X)$.

(iii) Suppose $p,\ q \in \mathcal{P}([0,1])$ and $p \leqslant q$ a.e. on $[0,1]$. Then we have $S^{q(x)}SAP_\omega(I:X) \subseteq S^{p(x)}SAP_\omega(I:X)$, $S^{q(x)}Q-AAP(I:X) \subseteq S^{p(x)}Q-AAP(I:X)$ and $S^{q(x)}Q-AUR(I:X) \subseteq S^{p(x)}Q-AUR(I:X)$.

These inclusions can be simply transferred and reformulated for the general classes of functions introduced in Definition 2.7.16-Definition 2.7.18 and Definition 2.7.19-Definition 2.7.21; details can be left to the interested readers.

The first part of subsequent result is very similar to Proposition 2.5.26; the proof is based on the use of Jensen integral inequality and therefore omitted.

PROPOSITION 2.7.25. (i) *Suppose that $\phi(\cdot)$ is convex, $p(x) \equiv 1$ and $f \in L^1_{loc}(I:X)$. If $f(\cdot)$ is Stepanov $(p,1,F)$-quasi asymptotically uniformly recurrent, then $f(\cdot)$ is Stepanov $(p,1,F)_1$-quasi asymptotically uniformly recurrent. If the function $\phi(\cdot)$ is concave, then the above inclusion reverses.*

(ii) *Suppose that there exists a function $\varphi : [0,\infty) \rightarrow [0,\infty)$ such that $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$. If $f(\cdot)$ is Stepanov $(p,\phi,F)_1$-quasi asymptotically uniformly recurrent, resp. Stepanov $[p,\phi,F]_1$-quasi asymptotically uniformly recurrent, then $f(\cdot)$ is Stepanov $(p,\phi,F_1)_2$-quasi asymptotically uniformly recurrent, resp. Stepanov $[p,\phi,F_1]_2$-quasi asymptotically uniformly recurrent, provided that $F = \varphi \circ F_1$.*

Furthermore, the same statements hold for the corresponding classes of quasi-asymptotically almost periodic functions and S-asymptotically ω-periodic functions.

The basic structural properties of quasi-asymptotically almost periodic functions clarified in [**247**, Theorem 2.13] can be formulated in our framework, for the general classes of functions introduced in this subsection, as well. We leave readers to make this explicit.

If $p \in [1,\infty)$, then any Stepanov p-quasi-asymptotically almost periodic function is Weyl p-almost periodic (see [**247**, Proposition 2.12]). The argumentation used in the proof of this result also shows that any Stepanov p-quasi-asymptotically uniformly recurrent function is Weyl-p-uniformly recurrent. In general case, we will state and prove only one result of this type regarding the notion introduced in Definition 2.7.1 and Definition 2.7.16. Before doing that, observe that if $p \in \mathcal{P}(I)$, $a,\ b,\ c \in I$, $a<b<c$ and $f \in L^{p(x)}[a,c]$, then $f \in L^{p(x)}[a,b]$, $f \in L^{p(x)}[b,c]$ and

$$\|f\|_{L^{p(x)}[a,c]} \leqslant \|f\|_{L^{p(x)}[a,b]} + \|f\|_{L^{p(x)}[b,c]}. \tag{127}$$

PROPOSITION 2.7.26. *Suppose that the function $f : I \rightarrow X$ is Stepanov-(p,ϕ,F)-quasi-asymptotically uniformly recurrent. If $F_1 : (0,\infty) \times I \rightarrow (0,\infty)$*

satisfies

$$\lim_{n\to+\infty}\limsup_{l\to+\infty}\sup_{t\in I}F_1(l,t)\left[\frac{1}{F(t,n)}+\cdots+\frac{1}{F(\lfloor t+l\rfloor,n)}\right]<\infty \tag{128}$$

and

$$\lim_{l\to+\infty}\sup_{t\in I}F_1(l,t)=0, \tag{129}$$

then the function $f(\cdot)$ is Weyl-$(p(x),\phi,F_1)$-uniformly recurrent.

Proof. By our assumption, we have $\phi(\|f(\cdot+\tau)-f(\cdot)\|)\in L^{p(\cdot)}(K)$ for any $\tau\in I$ and any compact set $K\subseteq I$; furthermore, we know that there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that (126) holds. We will prove that (124) holds with the function $F(\cdot,\cdot)$ replaced therein with the function $F_1(\cdot,\cdot)$. Let $n\in\mathbb{N}$ and $l>0$ be fixed. If $t\in I$, then there exist four possibilities:

1. $|t|\geqslant M_n$ and $|t+l|\geqslant M_n$;
2. $|t|\geqslant M_n$ and $|t+l|\leqslant M_n$;
3. $|t|\leqslant M_n$ and $|t+l|\geqslant M_n$;
4. $|t|\leqslant M_n$ and $|t+l|\leqslant M_n$.

The consideration is similar for all these cases and we will give the proof for case [1.], only. If $t\geqslant 0$, then we have $t\geqslant M_n$, $t+l\geqslant M_n$ and therefore

$$\begin{aligned}&\Bigl[F_1(l,t)\Bigl[\phi\bigl(\bigl\|f(\cdot+\alpha_n)-f(\cdot)\bigr\|\bigr)_{L^{p(\cdot)}[t,t+l]}\Bigr]\Bigr]\\&\leqslant F_1(l,t)\left[\frac{\varepsilon}{F(t,n)}+\cdots+\frac{\varepsilon}{F(\lfloor t+l\rfloor,n)}\right];\end{aligned}$$

see also (127). Employing condition (128), we immediately get (124). If $t\leqslant 0$, then we have $t\leqslant -M_n$ and $t+l\geqslant M_n$ for a sufficiently large $l>0$ (it suffices to consider only this case because, in (124), we operate with $\limsup_{l\to+\infty}\cdot$). We have

$$\begin{aligned}&\Bigl[F_1(l,t)\Bigl[\phi\bigl(\bigl\|f(\cdot+\alpha_n)-f(\cdot)\bigr\|\bigr)_{L^{p(\cdot)}[t,t+l]}\Bigr]\Bigr]\\&\leqslant\Bigl[F_1(l,t)\Bigl[\phi\bigl(\bigl\|f(\cdot+\alpha_n)-f(\cdot)\bigr\|\bigr)_{L^{p(\cdot)}[t,-M_n]}+\phi\bigl(\bigl\|f(\cdot+\alpha_n)-f(\cdot)\bigr\|\bigr)_{L^{p(\cdot)}[-M_n,M_n]}\\&+\phi\bigl(\bigl\|f(\cdot+\alpha_n)-f(\cdot)\bigr\|\bigr)_{L^{p(\cdot)}[M_n,t+l]}\Bigr]\Bigr]\\&\leqslant F_1(l,t)\left[\left(\frac{\varepsilon}{F(t,n)}+\cdots+\frac{\varepsilon}{F(t+\lfloor -t-M_n\rfloor,n)}\right)\right.\\&\left.+\left(\frac{\varepsilon}{F(M_n,n)}+\cdots+\frac{\varepsilon}{F(M_n+\lfloor t+l-M_n\rfloor,n)}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&+ F_1(l,t)\phi\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[-M_n,M_n]}\\
&\leqslant 2F_1(l,t)\left[\frac{\varepsilon}{F(t,n)}+\cdots+\frac{\varepsilon}{F(\lfloor t+l\rfloor,n)}\right]\\
&+ F_1(l,t)\phi\big(\|f(\cdot+\alpha_n)-f(\cdot)\|\big)_{L^{p(\cdot)}[-M_n,M_n]}.
\end{aligned}
$$

Using (128)-(129), we get (124). □

2.7.4. Composition principles for the class of quasi-asymptotically uniformly recurrent functions. In this subsection, we will briefly consider quasi-asymptotically uniformly recurrent functions depending on two parameters and related composition theorems (for the sake of brevity, we will say only a few words about the Stepanov classes). In order to unify several different approaches used in the existing literature (see also Definition 2.4.42-Definition 2.4.43 and Theorem 2.4.44), in this subsection we will assume that $\mathbf{B} \subseteq P(Y)$, where $P(Y)$ denotes the power set of Y; usually, $\mathbf{B}$ denotes the collection of all bounded subsets of Y or all compact subsets of Y.

Definition 2.7.27. (i) By $C_{0,\mathbf{B}}(I\times Y : X)$ we denote the space of all continuous functions $H : I\times Y \to X$ such that $\lim_{|t|\to+\infty} H(t,y) = 0$ uniformly for y in any subset $B\in\mathbf{B}$.

(ii) A continuous function $F : I\times Y\to X$ is said to be uniformly continuous on $\mathbf{B}$, uniformly for $t\in I$ if and only if for every $\varepsilon>0$ and for every $B\in\mathbf{B}$ there exists a number $\delta_{\varepsilon,B}>0$ such that $\|F(t,x)-F(t,y)\|\leqslant\varepsilon$ for all $t\in I$ and all $x,\ y\in B$ satisfying that $\|x-y\|\leqslant\delta_{\varepsilon,B}$.

We continue by introducing the following definition:

Definition 2.7.28. Suppose that $F : I\times Y\to X$ is a continuous function and $\mathbf{B}\subseteq P(Y)$. Then we say that $F(\cdot,\cdot)$ is quasi-asymptotically uniformly recurrent, uniformly on $\mathbf{B}$ if and only if for every $B\in\mathbf{B}$ there exist a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity and a sequence (M_n) of positive real numbers such that:

$$
\lim_{n\to+\infty}\sup_{|t|\geqslant M_n}\big\|F(t+\alpha_n,x)-F(t,x)\big\| = 0,\quad x\in B. \tag{130}
$$

Denote by $Q-AUR_{\mathbf{B}}(I\times Y : X)$ the set consisting of all quasi-asymptotically uniformly recurrent, uniformly on $\mathbf{B}$ functions from $I\times Y$ into X.

Using the argumentation employed in the proofs of [**135**, Theorem 3.30, Theorem 3.31], we may deduce the following results:

Theorem 2.7.29. *Suppose that* $\mathbf{B}\subseteq P(Y)$, $R(f)\in\mathbf{B}$, $F\in Q-AUR_{\mathbf{B}}(I\times Y : X)$ *and* $f\in Q-AUR(I:Y)$. *If there exist a finite number* $L>0$ *such that* (60) *holds a strictly increasing sequence* (α_n) *of positive real numbers tending to plus infinity and a sequence* (M_n) *of positive real numbers such that* (130) *holds with* $B=R(f)$ *and* (19) *holds, then the function* $t\mapsto F(t,f(t))$, $t\in I$ *belongs to the class* $Q-AUR(I:X)$.

THEOREM 2.7.30. *Suppose that* $\mathbf{B} \subseteq P(Y)$, $R(f) \in \mathbf{B}$, $F \in Q - AUR_{\mathbf{B}}(I \times Y : X)$ *and* $f \in Q - AUR(I : Y)$. *If* $F : I \times Y \to X$ *is uniformly continuous on* $\mathbf{B}$, *uniformly for* $t \in I$ *and there exist a strictly increasing sequence* (α_n) *of positive real numbers tending to plus infinity and a sequence* (M_n) *of positive real numbers such that* (130) *holds with* $B = R(f)$ *and* (19) *holds, then the function* $t \mapsto F(t, f(t))$, $t \in I$ *belongs to the class* $Q - AUR(I : X)$.

Similarly as in Definition 2.9.92, we can introduce the notion of a quasi-asymptotically almost periodic, uniformly on $\mathbf{B}$ function and the notion of a S-asymptotically ω-periodic, uniformly on $\mathbf{B}$ function. It is worth noticing that Theorem 2.7.29 and Theorem 2.7.30 continue to hold in this framework.

In [**247**, Definition 2.22], we have introduced the notion of a Stepanov p-quasi-asymptotically almost periodic function depending on two parameters ($1 \leqslant p < \infty$); the notion of a Stepanov $p(x)$-quasi-asymptotically almost periodic function (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent function, Stepanov $p(x)$-asymptotically ω-periodic function) can be introduced in a similar fashion. The interested reader may try to extend [**247**, Theorem 2.23, Theorem 2.24] in this context.

2.7.5. Invariance of generalized quasi-asymptotical uniform recurrence under the actions of convolution products. This subsection investigates the invariance of generalized quasi-asymptotical uniform recurrence under the actions of finite and infinite convolution products. Using the same arguments as in the proofs of [**247**, Proposition 3.1, Proposition 3.2], we can deduce the validity of the following statement:

PROPOSITION 2.7.31. (i) *Suppose that* $(R(t))_{t>0} \subseteq L(X, Y)$ *is a strongly continuous operator family and* $\int_0^\infty \|R(s)\|\, ds < \infty$. *If* $f \in Q - AUR([0, \infty) : X) \cap L^\infty([0, \infty) : X)$, *then the function* $F(\cdot)$, *defined by*

$$\mathrm{F}(t) := \int_0^t R(t-s) f(s)\, ds,\ t \geqslant 0, \tag{131}$$

belongs to the class $Q - AUR([0, \infty) : Y) \cap L^\infty([0, \infty) : Y)$.

(ii) *Suppose that* $(R(t))_{t>0} \subseteq L(X, Y)$ *is a strongly continuous operator family and* $\int_0^\infty \|R(s)\|\, ds < \infty$. *If* $f \in Q - AUR(\mathbb{R} : X) \cap L^\infty(\mathbb{R} : X)$, *then the function* $\mathbf{F}(t)$, *defined by* (55), *with the function* $F(\cdot)$ *replaced therein with the function* $\mathbf{F}(\cdot)$, *belongs to the class* $Q - AUR(\mathbb{R} : Y) \cap L^\infty(\mathbb{R} : Y)$.

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We would like to illustrate Proposition 2.7.31 by the following example:

EXAMPLE 2.7.32. Suppose that $X = H$ is an infinite-diemnsional Hilbert space with inner product $\langle \cdot, \cdot \rangle$. In [**287**], R. K. Miller and R. L. Wheeler have investigated the wellposedness of the following abstract Cauchy problem of non-scalar type

$$x'(t) = Au(t) + \int_0^t b(t-s)(A + aI)u(s)\, ds + f(t), \quad x(0) = x_0; \tag{132}$$

here, $b(t)$ is a scalar-valued kernel, $b \in C^1([0,\infty))$, $a \in \mathbb{C}$, $f : [0,\infty) \to H$ is continuous and A is a densely defined, self-adjoint closed linear operator in H. If the assumptions [**287**, (A1)-(A5)] hold with the coefficients $\alpha = \beta_0 = \beta_1 = 0$, then [**287**, Theorem 7] implies that there exists a unique residual resolvent $(R(t))_{t\geqslant 0}$ for (132) such that $\|R(\cdot)\| \in L^p([0,\infty))$ for $2 \leqslant p < \infty$. Furthermore, if the assumptions [**287**, (A1)-(A5)] hold with the coefficients $\alpha = \beta_0 = \beta_1 = 0$ and the assumption [**287**, (A6)] holds provided that $B\sigma(L) \neq \emptyset$ (see [**287**, p. 273] for the notion), then [**287**, Theorem 8] implies that there exists a unique residual resolvent $(R(t))_{t\geqslant 0}$ for (132) such that $\|R(\cdot)\| \in L^p([0,\infty))$ for $1 \leqslant p < \infty$; if this is the case, then Proposition 2.7.31 is applicable since, due to [**287**, Theorem 2], the unique solution of (132) for all $x_0 \in D(A)$ and $f \in C^1([0,\infty) : X)$ is given by

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0.$$

For some other foundational papers concerning integrability of solution operator families appearing in the theory of abstract Volterra integro-differential equations, we can recommend for the reader [**176**], [**190**], [**263**] and [**288**]. Compehensive survey of non-updated results can be found in [**319**, Section 10].

Concerning the invariance of Stepanov quasi-asymptotically almost periodic properties analyzed in the previous subsection, it would be really difficult and rather long to examine all introduced classes. Primarily from this reason, we will focus our attention on the notion introduced in Definition 2.7.19, only.

The following result admits a simple reformulation for the corresponding classes of quasi-asymptotically almost periodic functions and S-asymptotically ω-periodic functions:

Proposition 2.7.33. *Suppose that (a_k) is a sequence of positive real numbers such that $\sum_{k=0}^{\infty} a_k = 1$, $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$, $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that*

$$M := \sum_{k=0}^{\infty} a_k \varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]} < \infty. \tag{133}$$

Suppose, further, that for every $x \in \mathbb{R}$ we have $\int_{-x}^{\infty} \|R(v+x)\| \|\check{f}(v)\|\,dv < \infty$, as well as that $\check{f}(\cdot)$ is Stepanov-$[p,\phi,F]$-quasi-asymptotically uniformly recurrent, $M_1 := \sup_{t\in\mathbb{R}} [\phi(\|f(t-s)\|)]_{L^{p(s)}[0,1]} < \infty$, $F_1 : (0,\infty) \times \mathbb{N} \to (0,\infty)$ is bounded and satisfies that there exists a finite real constant $c > 0$ such that $F_1(t,n) \leqslant cF(t,n)$ for all $t > 0$ and $n \in \mathbb{N}$. Then the function $\mathbf{F} : \mathbb{R} \to Y$, given by (55), with the function $F(\cdot)$ replaced therein with the function $\mathbf{F}(\cdot)$, is well-defined and Stepanov-$[\infty,\phi,F_1]$-quasi-asymptotically uniformly recurrent.

Proof. Since for every $x \in \mathbb{R}$ we have $\int_{-x}^{\infty} \|R(v+x)\| \|\check{f}(v)\|\,dv < \infty$, it can be easily verified that the function $\mathbf{F}(\cdot)$ is well defined as well as that the integral which defines $\mathbf{F}(x+\tau) - \mathbf{F}(x)$ is absolutely convergent for every $x \in \mathbb{R}$ and $\tau \in \mathbb{R}$. For the rest, let (α_n) and (M_n) be the sequences from Definition 2.7.19, for the function $f(\cdot)$ replaced therein with the function $\check{f}(\cdot)$. Let $\varepsilon > 0$ be given, and let $n_0 \in \mathbb{N}$ be such that $\phi(\|f(t+\alpha_n+\cdot) - f(t+\cdot)\|)_{L^{p(\cdot)}[0,1]} < \varepsilon/F(t,n)$, $n \geqslant n_0$, $|t| \geqslant M_n$. Clearly, there exists $k_0(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=k_0(\varepsilon)}^{\infty} a_k \varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]} < \varepsilon. \tag{134}$$

Let $\alpha_n \geqslant k_0(\varepsilon)$ for $n \geqslant n_1$. Let $n \in \mathbb{N}$ with $n \geqslant \max(n_0, n_1)$ be fixed, and let $|t| \geqslant M_n' := M_n + \alpha_n + 2$. Then for each $x \in [0,1]$ we have (apply the Jensen inequality, (69) and the Hölder inequality)

$$\begin{aligned}
&\bigl\|\mathbf{F}\bigl(t+x+\alpha_n\bigr) - \mathbf{F}(t+x)\bigr\| \\
&\leqslant \phi\Biggl(\int_0^{\infty} \|R(s)\|\|f(x+t+\alpha_n-s) - f(x+t-s)\|\,ds\Biggr) \\
&= \phi\Biggl(\sum_{k=0}^{\infty} a_k \int_0^1 a_k^{-1}\|R(s+k)\|\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|\,ds\Biggr) \\
&\leqslant \sum_{k=0}^{\infty} a_k \phi\Biggl(\int_0^1 a_k^{-1}\|R(s+k)\|\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|\,ds\Biggr) \\
&\leqslant \sum_{k=0}^{\infty} a_k \varphi\bigl(a_k^{-1}\bigr) \int_0^1 \phi\Bigl(\|R(s+k)\|\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|\Bigr)ds \\
&\leqslant 2\sum_{k=0}^{\infty} a_k \varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]} \\
&\times \Bigl[\phi\bigl(\|f(x+t+\alpha_n-k-\cdot) - f(x+t-k-\cdot)\|\bigr)\Bigr]_{L^{p(\cdot)}[0,1]},
\end{aligned}$$

which implies that for $t \leqslant -M_n'$ we have

$$\begin{aligned}
\bigl\|\mathbf{F}\bigl(t+x+\alpha_n\bigr) - \mathbf{F}(t+x)\bigr\| &\leqslant 2\sum_{k=0}^{\infty} a_k \varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]} \frac{\varepsilon}{F(t,n)} \\
&\leqslant 2c\sum_{k=0}^{\infty} a_k \varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]} \frac{\varepsilon}{F_1(t,n)}.
\end{aligned}$$

If $t \geqslant M_n'$, then we have $\lfloor t - M_n \rfloor \geqslant k_0(\varepsilon)$ and (134) implies

$$\begin{aligned}
&\bigl\|\mathbf{F}\bigl(t+x+\alpha_n\bigr) - \mathbf{F}(t+x)\bigr\| \\
&\leqslant 2\sum_{k=0}^{\lfloor t-M_n \rfloor} a_k \varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]} \\
&\times \Bigl[\phi\bigl(\|f(x+t+\alpha_n-k-s) - f(x+t-k-s)\|\bigr)\Bigr]_{L^{p(s)}[0,1]}
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{k=\lfloor t-M_n\rfloor}^{\lceil t+M_n\rceil} a_k\varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]}\\
&\quad\times\Bigl[\phi\bigl(\|f(x+t+\alpha_n-k-s)-f(x+t-k-s)\|\bigr)\Bigr]_{L^{p(s)}[0,1]}\\
&+2\sum_{k=\lceil t+M_n\rceil}^{\infty} a_k\varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]}\\
&\quad\times\Bigl[\phi\bigl(\|f(x+t+\alpha_n-k-s)-f(x+t-k-s)\|\bigr)\Bigr]_{L^{p(s)}[0,1]}\\
&\leqslant 2\frac{\varepsilon}{F(t,n)}\Biggl(\sum_{k=0}^{\lfloor t-M_n\rfloor}+\sum_{k=\lceil t+M_n\rceil}^{\infty}\Biggr)a_k\varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]}+\varepsilon\cdot\varphi(2)\cdot M_1\\
&\leqslant 2c\frac{\varepsilon}{F_1(t,n)}\Biggl(\sum_{k=0}^{\lfloor t-M_n\rfloor}+\sum_{k=\lceil t+M_n\rceil}^{\infty}\Biggr)a_k\varphi\bigl(a_k^{-1}\bigr)\Bigl[\varphi\bigl(\|R(\cdot+k)\|\bigr)\Bigr]_{L^{q(\cdot)}[0,1]}+\varepsilon\cdot\varphi(2)\cdot M_1,
\end{aligned}
$$

since

$$
\begin{aligned}
&\Bigl[\phi\bigl(\|f(x+t+\alpha_n-k-s)-f(x+t-k-s)\|\bigr)\Bigr]_{L^{p(s)}[0,1]}\\
&\leqslant\Bigl[\phi\bigl(\|f(x+t+\alpha_n-k-s)\|+\|f(x+t-k-s)\|\bigr)\Bigr]_{L^{p(s)}[0,1]}\\
&\leqslant\varphi(2)\left[\frac{1}{2}\phi\bigl(\|f(x+t+\alpha_n-k-s)\|\bigr)+\frac{1}{2}\phi\bigl(\|f(x+t-k-s)\|\bigr)\right]_{L^{p(s)}[0,1]}\\
&\leqslant\varphi(2)\cdot M_1.
\end{aligned}
$$

This simply completes the proof. □

We will also state the following special corollary, which generalizes [**247**, Proposition 3.4]:

Proposition 2.7.34. *Suppose that $q\in\mathcal{P}([0,1])$, $1/p(x)+1/q(x)=1$ and $(R(t))_{t>0}\subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M:=\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]}<\infty$. If $\check{f}:\mathbb{R}\to X$ is Stepanov $p(x)$-quasi-asymptotically uniformly recurrent (Stepanov $p(x)$-quasi-asymptotically almost periodic, Stepanov $p(x)$-asymptotically ω-periodic) and $S^{p(x)}$-bounded, then the function $\mathbf{F}:\mathbb{R}\to Y$, given by* (55)*, with the function $F(\cdot)$ replaced therein with the function $\mathbf{F}(\cdot)$, is well defined, bounded and quasi-asymptotically uniformly recurrent (quasi-asymptotically almost periodic, S-asymptotically ω-periodic).*

Proof. We will consider the Stepanov $p(x)$-quasi-asymptotically uniformly recurrent functions, only. Since $\sum_{k=0}^{\infty}\|R(\cdot+k)\|_{L^{q(x)}[0,1]}<\infty$ and $\check{f}(\cdot)$ is $S^{p(x)}$-bounded, we can apply the same arguments as in the proofs of [**142**, Proposition 6.1] and [**143**, Proposition 5.1] in order to see that the function $\mathbf{F}(\cdot)$ is bounded and continuous. The remainder of proof follows from the computations carried out in the proof of Proposition 2.7.35, with $p(x) = \varphi(x) = \phi(x) = x$ and $F(t, n) = F_1(t, n) = 1$. $\square$

The following result regarding the finite convolution product admits a reformulation for the corresponding classes of quasi-asymptotically almost periodic functions and S-asymptotically ω-periodic functions, as well:

PROPOSITION 2.7.35. *Suppose that (a_k) is a sequence of positive real numbers such that $\sum_{k=0}^{\infty} a_k = 1$, $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leqslant \varphi(x)\phi(y)$ for all $x,\ y \geqslant 0$, $p,\ q \in \mathcal{P}([0,1])$, $1/p(x)+1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that* (133) *holds. Suppose, further, that the mapping* $\mathrm{F} : [0,\infty) \to Y$, *given by* (131), *is well-defined as well as that $f(\cdot)$ is Stepanov-$[p, \phi, F]$-quasi-asymptotically uniformly recurrent,*

$$M_1 := \sup_{t\geqslant 0} \sup_{t\in[0,s]} \Big[\phi\big(\|f(t-s)\|\big)\Big]_{L^{p(s)}[0,1]} < \infty,$$

$F_1 : (0,\infty) \times \mathbb{N} \to (0,\infty)$ is bounded and satisfies that there exists a finite real constant $c > 0$ such that $F_1(t,n) \leqslant cF(t,n)$ for all $t > 0$ and $n \in \mathbb{N}$. Then the function $\mathrm{F}(\cdot)$ *is Stepanov-$[\infty, \phi, F_1]$-quasi-asymptotically uniformly recurrent.*

PROOF. The proof is very similar to the proof of Proposition 2.7.35 and we will only outline two details. Let $\varepsilon > 0$ be fixed, and let the numbers $M_n > 0$ and $k_0(\varepsilon),\ n_0,\ n_1 \in \mathbb{N}$ be as above. Then for each $x \in [0,1]$, $|t| \geqslant M_n' + \alpha_n + 2$ and $n \in \mathbb{N}$ with $n \geqslant \max(n_0, n_1)$ we have

$$\begin{aligned}
&\phi\big(\|F(x+t+k+\alpha_n) - F(x+t+k)\|\big)\\
&\quad\leqslant \sum_{k=0}^{\lceil t\rceil} a_k \varphi\big(a_k^{-1}\big)\Big[\varphi\big(\|R(\cdot+k)\|\big)\Big]_{L^{q(\cdot)}[0,1]}\\
&\quad\times \Big[\phi\big(\|f(x+t+k+\alpha_n-s) - f(x+t+k-s)\|\big)\Big]_{L^{p(s)}[0,1]}.
\end{aligned}$$

After that, we can decompose the sum $\sum_{k=0}^{\lceil t\rceil}\cdot$ into two parts:

$$\sum_{k=0}^{\lceil t\rceil}\cdot = \sum_{k=0}^{k_0(\varepsilon)}\cdot + \sum_{k=k_0(\varepsilon)}^{\lceil t\rceil}\cdot,$$

and apply the similar arguments. This completes the proof in a routine manner. $\square$

Similarly we can deduce the following extension of [**247**, Proposition 3.3] (see also the proof of [**143**, Proposition 5.1]):

PROPOSITION 2.7.36. *Suppose that $q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(x)}[0,1]} < \infty$. If $f : [0,\infty) \to X$ is Stepanov $p(x)$-quasi-asymptotically* *almost periodic (Stepanov $p(x)$-quasi-asymptotically uniformly recurrent, Stepanov $p(x)$-asymptotically ω-periodic), $f(t-\cdot) \in L^{p(x)}[0,t]$ for $0 < t \leqslant 1$ and*

$$\sup_{k\in\mathbb{N}_0}\sup_{t\geqslant 0}\left\|f(t+k-\cdot)\right\|_{L^{p(\cdot)}[0,1]} < \infty,$$

then the function $\mathrm{F} : [0,\infty) \to Y$*, given by* (131)*, is well-defined, bounded and quasi-asymptotically almost periodic (quasi-asymptotically uniformly recurrent, S-asymptotically ω-periodic).*

Remark 2.7.37. We would like to note that it is very difficult to remove the assumption on the boundedness of function $f(\cdot)$ in Proposition 2.7.31, resp. the Stepanov $p(x)$-boundedness of functions in Proposition 2.7.34-Proposition 2.7.36, in contrast to our recent research study [**248**].

2.7.6. Applications to the abstract Volterra integro-differential equations. Concerning possible applications of our theoretical results to the abstract Volterra integro-differential equations in Banach spaces, we would like to say first a few words about the abstract nonautonomous differential equations of first order. In the first part of [**247**, Section 4], we have investigated the generalized almost periodic properties of the mild solutions to the abstract Cauchy problems

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{135}$$

$$u'(t) = A(t)u(t) + f(t), \quad t > 0;\ u(0) = x, \tag{136}$$

where the operator family $A(\cdot)$ satisfies certain conditions. In [**247**, Subsection 4.1], we have investigated the generalized almost periodic properties of the semilinear analogues to the abstract Cauchy problems (135)-(136).

The statement of [**247**, Theorem 4.1] can be straightforwardly extended for the inhomogeneities $f \in S^{p(x)}Q - AAP([0,\infty) : X)$ by replacing the number q in the equation [**247**, (4.1)] with the function $q(x)$ and using the translation $\cdot \mapsto \cdot + k$ $(1/p(x) + 1/q(x) = 1)$ therein; we can also consider the inhomogeneities $f \in S^{p(x)}Q - AUR([0,\infty) : X)$ which are Stepanov $p(x)$-bounded, by slightly modifying the equation [**247**, (4.1)] in the formulation of this result. Similar comments can be made for [**247**, Theorem 4.3]. Concerning semilinear problems, the statements of [**247**, Theorem 4.6, Theorem 4.7] can be reformulated by replacing the space $Q - AAP(I : X)$ with the space $BQ - AUR_{(\alpha_n)}(I : X)$ consisting of all bounded functions $f : I \to X$ which are quasi-asymptotically uniformly recurrent and for which there exists a fixed sequence (α_n) of positive real numbers such that (125) holds; equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_\infty$, this space becomes a complete metric space. The conclusions established in [**247**, Example 2.8] can be reexamined in this context, as well.

By a mild solution of the abstract semilinear Cauchy inclusion

$$(DFP)_{F,\gamma,s} : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + F(t,u(t)),\ t > 0, \\ u(0) = x_0, \end{cases}$$

we mean any function $u \in C([0,\infty) : X)$ satisfying that

$$u(t) = S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geqslant 0.$$

Now we are in a position to state the following result:

THEOREM 2.7.38. *Suppose that the function $F : \mathbb{R} \times X \to X$ is continuous and satisfies that for each bounded subset B of X there exist a finite real constant $M_B > 0$ and a sequence (M_n) of positive real numbers such that* (130) *holds and* $\sup_{t\in\mathbb{R}} \sup_{x\in B} \|F(t,x)\| \leqslant M_B$. *Let there exist a finite number $L > 0$ such that* (60) *holds, and let there exist an integer $n \in \mathbb{N}$ such that $A_n < 1$, where*

$$A_n := \sup_{t\geqslant 0} \int_0^t \int_0^{x_n} \cdots \int_0^{x_2} L^n \big\|R_\gamma(t-x_n)\big\| \times \prod_{i=2}^n \big\|R_\gamma(x_i - x_{i-1})\big\|\,dx_1\,dx_2 \cdots dx_n.$$

Then the abstract fractional Cauchy inclusion (DFP)$_{F,\gamma,s}$ has a unique solution which belongs to the space $BQ-AUR_{(\alpha_n)}([0,\infty) : X)$.

PROOF. Set, for every $u \in C_b([0,\infty) : X)$,

$$(\Upsilon u)(t) := S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geqslant 0.$$

Suppose that $u \in BQ-AUR_{(\alpha_n)}([0,\infty) : X)$. Then $R(u) = B$ is a bounded set and our assumption implies that the mapping $t \mapsto F(t,u(t))$, $t \in \mathbb{R}$ is bounded. Applying Theorem 2.7.29, we have that the function $F(\cdot,u(\cdot))$ is quasi-asymptotically uniformly recurrent. After that, we can employ Proposition 2.7.31(i) and Proposition 2.7.10 (there is no need to say that we can retain the same sequence (α_n) after applying the above-mentioned statements, with the meaning clear) in order to see that $\Upsilon u \in BQ-AUR_{(\alpha_n)}([0,\infty) : X)$. Hence, the mapping $\Upsilon(\cdot)$ is well defined. Since

$$\Big\|\big(\Upsilon^n u\big) - \big(\Upsilon^n v\big)\Big\|_\infty \leqslant A_n \big\|u - v\big\|_\infty, \quad u,\ v \in C_b([0,\infty) : X),\ n \in \mathbb{N},$$

the well known extension of the Banach contraction principle yields that the mapping $\Upsilon(\cdot)$ has a unique fixed point. This completes the proof. □

Let Ω be a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geqslant 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $X := L^p(\Omega)$. Suppose that the operator $A := \Delta - b$ acts on X with the Dirichlet boundary conditions, and that B is the multiplication operator by the function $m(x)$. Then we can apply Theorem 2.7.38 with $\mathcal{A} = AB^{-1}$ in the study of existence and uniqueness of bounded quasi-asymptotically uniformly recurrent solutions of the semilinear fractional Poisson heat equation

$$\begin{aligned} \mathbf{D}_t^\gamma[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,m(x)v(t,x)), \quad t \geqslant 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega. \end{aligned}$$

It is also worth noting that we can apply Theorem 2.7.38 in the analysis of existence and uniqueness of bounded quasi-asymptotically uniformly recurrent solutions of the following fractional semilinear equation with higher order differential operators in the Hölder space $X = C^{\alpha}(\overline{\Omega})$:

$$\begin{cases} \mathbf{D}_t^{\gamma} u(t,x) = - \sum\limits_{|\beta| \leqslant 2m} a_{\beta}(t,x) D^{\beta} u(t,x) - \sigma u(t,x) + f(t, u(t,x)), \ t \geqslant 0, \ x \in \Omega; \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\alpha \in (0,1)$, $m \in \mathbb{N}$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary of class C^{4m}, $D^{\beta} = \prod_{i=1}^{n} (\frac{1}{i} \frac{\partial}{\partial x_i})^{\beta_i}$, the functions $a_{\beta} : \overline{\Omega} \to \mathbb{C}$ satisfy certain conditions and $\sigma > 0$ is sufficiently large. For more details, see [**234**].

Basically, our results on the invariance of generalized quasi-asymptotical almost periodicity and uniform recurrence, established in Subsection 2.7.5, can be applied at any place where the variation of parameters formula takes effect. For our purposes, it will be very important to reexamine [**359**, Example 5]. It is well known that the unique regular solution of the wave equation $u_{xx}(x,t) = u_{tt}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial conditions $u(x,0) = f(x)$, $x \in \mathbb{R}$, $u_t(x,0) = g(x)$, $x \in \mathbb{R}$, is given by the famous d'Alembert formula

$$u(x,t) := \frac{1}{2}\big[f(x+t) + f(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\, ds, \quad x \in \mathbb{R}, \ t \geqslant 0.$$

Let $t_0 > 0$ be a fixed real number. If the function $f(\cdot)$ is quasi-asymptotically uniformly recurrent, resp. $g(\cdot)$ is quasi-asymptotically uniformly recurrent, then the function $x \mapsto 1/2[f(x+t_0) + f(x-t_0)]$, $x \in \mathbb{R}$, resp.

$$H_{t_0}(x) := \frac{1}{2}\int_{x-t_0}^{x+t_0} g(s)\, ds, \quad x \in \mathbb{R},$$

is likewise quasi-asymptotically uniformly recurrent; this can be shown as in [**359**]. Their sum will be quasi-asymptotically uniformly recurrent provided that these functions share the same sequence (α_n) in Definition 2.7.8.

2.8. (ω, c)-Almost periodic type functions and applications

The following notion has recently been introduced and analyzed in the case that $I = \mathbb{R}$; see [**17**]-[**16**]. Let $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$. A continuous function $f : I \to X$ is said to be (ω, c)-periodic if and only if $f(t+\omega) = cf(t)$ for all $t \in I$. The number ω is called c-period of f. The space of all (ω, c)-periodic functions $f : I \to X$ will be denoted with $P_{\omega,c}(I : X)$. Let we note that, by putting $c = 1$, we obtain the space of all ω-periodic functions $f : I \to X$; by putting $c = -1$, we obtain the space of all ω-antiperiodic functions $f : I \to X$; by putting $c = e^{ik\omega}$ we obtain the space of all Bloch (ω, k)-periodic functions.

The following facts about the (ω, c)-periodic functions should be stated at the very beginning (see also [**17**]-[**16**]):

(i) If $f \in P_{\omega,c}([0,\infty) : X)$, $f(\cdot)$ is not identically equal to zero and $|c| > 1$, then $\limsup_{t \to +\infty} \|f(t)\| = +\infty$; if $f \in P_{\omega,c}(\mathbb{R} : X)$ and $|c| > 1$, then $\lim_{t\to-\infty} f(t) = 0$ and, if $f(\cdot)$ is not identically equal to zero, then $\limsup_{t\to+\infty} \|f(t)\| = +\infty$.

(ii) If $f \in P_{\omega,c}(I : X)$ and $f(x) \neq 0$ for all $x \in I$, then the function $(1/f)(\cdot)$ belongs to the class $P_{\omega,1/c}(I : X)$.

(iii) If $f \in P_{\omega,c}(I : X)$ and $|c| = 1$, then the function is almost periodic. To see this, observe that there exists a real number $k \in \mathbb{R}$ such that $f(x + \omega) = e^{ik\omega} f(x)$, $x \in I$, so that the function $f(\cdot)$ is Bloch (ω, k)-periodic. After that, the conclusion simply follows because the function $e^{-ik\cdot} f(\cdot)$ is periodic. In this case, we can simply compute the Bohr spectrum by using the computation:

$$\begin{aligned} P_r(f) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} f(s)\, ds = \lim_{n\to+\infty} \frac{1}{n\omega} \int_0^{n\omega} e^{-irs} f(s)\, ds \\ &= \lim_{n\to+\infty} \frac{1}{n\omega} \sum_{j=0}^{n-1} \int_{j\omega}^{(j+1)\omega} e^{-irs} f(s)\, ds \\ &= \lim_{n\to+\infty} \frac{1}{n\omega} \sum_{j=0}^{n-1} \int_0^{\omega} e^{-ir(s+j\omega)} c^j f(s)\, ds \\ &= \frac{1}{\omega} \int_0^{\omega} e^{-irs} f(s)\, ds \times \lim_{n\to+\infty} \frac{\sum_{j=0}^{n-1} \bigl(ce^{-ir\omega}\bigr)^j}{n}. \end{aligned}$$

Therefore, if $c = e^{ir\omega}$, then $P_r = 1$; otherwise, we have $P_r = 0$ because:

$$\left| \sum_{j=0}^{n-1} \bigl(ce^{-ir\omega}\bigr)^j \right| = \left| \frac{c^n e^{-irn\omega} - 1}{ce^{-ir\omega} - 1} \right| \leqslant \frac{2}{ce^{-ir\omega} - 1}, \quad n \in \mathbb{N}.$$

Furthermore, arguing as in the above-mentioned remark, we may deduce that for each $k \in \mathbb{R}$ the existence of a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \bigl\| f(\cdot + \alpha_n) - e^{ik\alpha_n} f(\cdot) \bigr\|_\infty = 0$$

is equivalent to saying that the function $F(\cdot) := e^{-ik\cdot} f(\cdot)$ is uniformly recurrent.

Due to the argumentation given in the proof of [**17**, Proposition 2.2], with $I = \mathbb{R}$, we have that the function $f(\cdot)$ is (ω, c)-periodic if and only if the function $c^{-\frac{\cdot}{\omega}} f(\cdot)$ belongs to the space $P_\omega(I : X)$. This statement will play an important role in our further work.

In this section, we will consider three different approaches for introducing the spaces of (ω, c)-almost periodic type functions and their Stepanov generalizations. The first approach is the simplest one and (in the case of consideration of (ω, c)-almost automorphic functions and their Stepanov generalizations, we will always tactily assume that $I = \mathbb{R}$):

DEFINITION 2.8.1. Let $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$. Then it is said that a continuous function $f : I \to X$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(ω, c)-almost automorphic/compactly (ω, c)-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by

$$f_{\omega,c}(t) := c^{-(t/\omega)} f(t), \quad t \in I, \tag{137}$$

is uniformly recurrent (almost periodic/almost automorphic/compactly almost automorphic). By $UR_{\omega,c}(I : X)$, $AP_{\omega,c}(I : X)$, $AA_{\omega,c}(I : X)$ and $AA_{\omega,c;\mathbf{c}}(I : X)$ we denote the space of all (ω, c)-uniformly recurrent functions, the space of all (ω, c)-almost periodic functions, the space of all (ω, c)-almost automorphic functions and the space of all compactly (ω, c)-almost automorphic, respectively.

It is clear that the space $P_{\omega,c}(I : X)$ is contained in any of the above introduced spaces. The class of $(\omega, c, \odot_g)$-almost periodic functions can be also introduced and analyzed but we will skip all related details concerning this class of functions for simplicity.

REMARK 2.8.2. If the function $f_{\omega,c}(\cdot)$ is bounded and $|c| < 1$, then we have $\lim_{t\to+\infty} f(t) = 0$; moreover, if $I = \mathbb{R}$, the function $f_{\omega,c}(\cdot)$ is bounded and $|c| > 1$, then we have $\lim_{t\to-\infty} f(t) = 0$.

REMARK 2.8.3. In (137), one can consider an arbitrary function $c(\cdot)$ in place of function $c^{-(\cdot/\omega)}$ but then the things become much more complicated. For example, following the examination from the previous remark, it seems reasonable to use the function $c^{-(|\cdot|/\omega)}$ in place of function $c^{-(\cdot/\omega)}$. We will not follow this approach for simplicity and we will consider here only the asymptotically (ω, c)-almost periodic type functions defined on the non-negative real axis.

It is clear that any (ω, c)-almost periodic function is (ω, c)-uniformly recurrent and compactly (ω, c)-almost automorphic, as well as that any compactly (ω, c)-almost automorphic function is (ω, c)-almost automorphic. Even in the case that $c = 1$ and $\omega > 0$ is arbitrary, there exists a compactly almost automorphic function which is not uniformly recurrent and therefore not almost periodic.

DEFINITION 2.8.4. Let $c \in \mathbb{C}$, $|c| \geqslant 1$ and $\omega > 0$. Then it is said that a continuous function $f : [0, \infty) \to X$ is asymptotically (ω, c)-uniformly recurrent (asymptotically (ω, c)-almost periodic, asymptotically (compactly) (ω, c)-almost automorphic) if and only if there exist an (ω, c)-uniformly recurrent ((ω, c)-almost periodic, (compactly) (ω, c)-almost automorphic) function $h : \mathbb{R} \to X$ and a function $q \in C_0([0, \infty) : X)$ such that $f(t) = h(t) + q(t)$ for all $t \geqslant 0$.

The following facts concerning the introduced classes of functions should be stated:

1. Suppose that $|c| = 1$ and $\omega > 0$. Then we can use Theorem 2.1.1(ii) and Proposition 2.3.1 in order to see that the function $f : I \to X$ is (ω, c)-almost periodic ((compactly) (ω, c)-almost automorphic) if and only if $f(\cdot)$ is almost periodic ((compactly) almost automorphic). In the case that $I = [0, \infty)$, the same assertion holds for the asymptotically (ω, c)-almost

periodic functions and the asymptotically (compactly) (ω, c)-almost automorphic functions.

2. Suppose that $|c| > 1$, $\omega > 0$ and $f : I \to X$ is (ω, c)-uniformly recurrent or (ω, c)-almost automorphic. If $f(\cdot)$ is not identically equal to zero, then the supremum formula yields that $f(\cdot)$ is unbounded; moreover, in the case of consideration of (ω, c)-almost automorphicity, the function $f(\cdot)$ is unbounded as $t \to +\infty$ due to Remark 2.8.2. In the case that $I = [0, \infty)$, the same assertion holds for the asymptotically (ω, c)-uniformly recurrent functions and the asymptotically (compactly) (ω, c)-almost automorphic functions. In particular, a constant non-zero function cannot be asymptotically (ω, c)-uniformly recurrent or asymptotically (ω, c)-almost automorphic.
3. Suppose $c \in \mathbb{C}\backslash\{0\}$, $\omega > 0$ and $f : [0, \infty) \to X$ is (ω, c)-almost periodic. Then it is well known that there exists a unique almost periodic function $F_{\omega,c} : \mathbb{R} \to X$ such that $F_{\omega,c}(t) = f_{\omega,c}(t)$, $t \geqslant 0$. Define $F(t) := c^{t/\omega} F_{\omega,c}(t)$, $t \in \mathbb{R}$. Then it simply follows that the function $F(\cdot)$ is a unique (ω, c)-almost periodic function which extends the function $f(\cdot)$ to the whole real line.
4. Let $c \in \mathbb{R}$ and $\omega > 0$. Then, for every (ω, c)-uniformly recurrent ((compactly) (ω, c)-almost automorphic) function $f(\cdot)$, we have that the function $\|f(\cdot)\|$ is (ω, c)-uniformly recurrent ((compactly) (ω, c)-almost automorphic). In the case that $I = [0, \infty)$, then the same assertion holds for the asymptotically (ω, c)-uniformly recurrent functions and the asymptotically (compactly) (ω, c)-almost automorphic functions.
5. The spaces $UR_{\omega,c}(I : X)$, $AP_{\omega,c}(I : X)$, $AA_{\omega,c}(I : X)$ and $AA_{\omega,c;\mathbf{c}}(I : X)$ are invariant under pointwise multiplications with scalars. In the case that $I = [0, \infty)$, the same holds for the corresponding spaces of asymptotically (ω, c)-almost periodic type functions.
6. The spaces $UR_{\omega,c}(I : X)$, $AP_{\omega,c}(I : X)$, $AA_{\omega,c}(I : X)$ and $AA_{\omega,c;\mathbf{c}}(I : X)$ are translation invariant. In the case that $I = [0, \infty)$, the same holds for the corresponding spaces of asymptotically (ω, c)-almost periodic type functions.
7. If $I = [0, \infty)$, $|c| \geqslant 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $UR_{\omega,c}(I : X)$ $(AP_{\omega,c}(I : X)/AA_{\omega,c}(I : X)/AA_{\omega,c;\mathbf{c}}(I : X))$ converges uniformly to a function $f : I \to X$, then the function $f(\cdot)$ belongs to the space $UR_{\omega,c}(I : X)$ $(AP_{\omega,c}(I : X)/AA_{\omega,c}(I : X)/AA_{\omega,c;\mathbf{c}}(I : X))$. In the case that $I = [0, \infty)$, then the same assertion holds for the asymptotically (ω, c)-almost periodic type function spaces.

For completeness, we will include the most relevant details of the proofs of the following two propositions:

PROPOSITION 2.8.5. *Suppose* $X = \mathbb{C}$, $c \in \mathbb{C} \smallsetminus \{0\}$, $\omega > 0$, $f : I \to \mathbb{C}$ *and* $\inf_{x \in I} |f(x)| > m > 0$. *Then the following holds:*

(i) *If* $|c| = 1$ *and the function* $f(\cdot)$ *is* (ω, c)*-uniformly recurrent* ((ω, c)*-almost periodic/*(ω, c)*-almost automorphic/compactly* (ω, c)*-almost automorphic), then the function* $(1/f)(\cdot)$ *is* $(\omega, 1/c)$*-uniformly recurrent* ($(\omega, 1/c)$*-almost periodic/*$(\omega, 1/c)$*-almost automorphic/compactly* $(\omega, 1/c)$*-almost automorphic).*

(ii) *If* $|c| \leqslant 1$, $I = [0, \infty)$ *and* $f(\cdot)$ *is* (ω, c)*-uniformly recurrent* ((ω, c)*-almost periodic), then the function* $(1/f)(\cdot)$ *is* $(\omega, 1/c)$*-uniformly recurrent* ($(\omega, 1/c)$*-almost periodic).*

PROOF. The proof of (i) essentially follows from the simple argumentation and the conclusions obtained in the point [1.], while the proof of (ii) can be deduced as follows. Suppose that the function $f(\cdot)$ is (ω, c)-almost periodic, i.e., the function $f_{\omega,c}(\cdot)$ is almost periodic. This implies that for each number $\varepsilon > 0$ there exists a finite number $l > 0$ such that any subinterval I' of I contains at least one point τ such that

$$\left|c^{-\frac{t+\tau}{\omega}} f(t+\tau) - c^{-\frac{t}{\omega}} f(t)\right| \leqslant \varepsilon, \quad t \geqslant 0.$$

This implies

$$\left|f(t+\tau) - c^{-\frac{\tau}{\omega}} f(t)\right| \leqslant \varepsilon \left|c^{\frac{t+\tau}{\omega}}\right|, \quad t \geqslant 0.$$

Then the final conclusion is a consequence of the following simple calculation:

$$\begin{aligned}\left|\frac{c^{\frac{t+\tau}{\omega}}}{f(t+\tau)} - \frac{c^{\frac{t}{\omega}}}{f(t)}\right| &= \left|c^{\frac{t}{\omega}}\right| \cdot \left|\frac{f(t+\tau) - c^{-\frac{\tau}{\omega}} f(t)}{f(t+\tau) \cdot f(t)}\right| \\ &\leqslant \frac{\varepsilon}{m^2}\left|c^{\frac{2t+\tau}{\omega}}\right| \leqslant \frac{\varepsilon}{m^2}, \quad t \geqslant 0.\end{aligned}$$

The proof for (ω, c)-uniform recurrence is similar and therefore omitted. □

PROPOSITION 2.8.6. *Suppose that* $I = \mathbb{R}$, $f : \mathbb{R} \to X$ *satisfies that the function* $f_{\omega,c}(\cdot)$ *is a bounded uniformly recurrent (almost periodic, (compactly) almost automorphic) and* $c^{-\frac{\cdot}{\omega}}\psi(\cdot) \in L^1(\mathbb{R})$*. Then the function* $c^{-\frac{\cdot}{\omega}}(\psi * f)(\cdot)$ *is bounded uniformly continuous and the function* $(\psi * f)(\cdot)$ *is* (ω, c)*-uniformly recurrent* ((ω, c)*-almost periodic/(compactly)* (ω, c)*-almost automorphic).*

PROOF. For every $x \in \mathbb{R}$, the convolution $(\psi * f)(x)$ is well defined and we have

$$c^{-\frac{x}{\omega}}(\psi * f)(x) = \int_{-\infty}^{\infty} \left[c^{-\frac{x-y}{\omega}} \psi(x-y)\right] \cdot \left[c^{-\frac{y}{\omega}} f(y)\right] dy, \quad x \in \mathbb{R}.$$

Then the corresponding statement follows from the fact that the space of all almost periodic ((compactly) almost automorphic) functions and the space of all bounded uniformly recurrent functions are convolution invariant. □

The following definitons are logical analogues of Definition 2.8.1-Definition 2.8.4 for Stepanov classes:

DEFINITION 2.8.7. Let $p \in \mathcal{P}([0,1])$, $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$. Then it is said that a function $f \in L^{p(x)}_{loc}(I : X)$ is Stepanov $(p(x), \omega, c)$-uniformly recurrent (Stepanov $(p(x), \omega, c)$-almost periodic/Stepanov $(p(x), \omega, c)$-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by (137), is Stepanov $p(x)$-uniformly recurrent (Stepanov $p(x)$-almost periodic/Stepanov $p(x)$-almost automorphic).

By $S^{p(x)}UR_{\omega,c}(I : X)$, $S^{p(x)}AP_{\omega,c}(I : X)$ and $S^{p(x)}AA_{\omega,c}(I : X)$ we denote the space of all Stepanov $(p(x), \omega, c)$-uniformly recurrent functions, the space of all Stepanov $(p(x), \omega, c)$-almost periodic functions and the space of all Stepanov $(p(x), \omega, c)$-almost automorphic functions, respectively. If $p(x) \equiv p \in [1, \infty)$, then by $S^pUR_{\omega,c}(I : X)$, $S^pAP_{\omega,c}(I : X)$ and $S^pAA_{\omega,c}(I : X)$ we denote the space of all Stepanov (p, ω, c)-uniformly recurrent functions, the space of all Stepanov (p, ω, c)-almost periodic functions and the space of all Stepanov (p, ω, c)-almost automorphic functions, respectively.

Definition 2.8.8. Let $p \in \mathcal{P}([0, 1])$, $c \in \mathbb{C}$, $|c| \geqslant 1$ and $\omega > 0$. Then it is said that a function $f \in L^{p(x)}_{loc}([0, \infty) : X)$ is asymptotically Stepanov $(p(x), \omega, c)$-uniformly recurrent (asymptotically Stepanov $(p(x), \omega, c)$-almost periodic, asymptotically Stepanov $(p(x), \omega, c)$-almost automorphic) if and only if the function $f_{\omega,c}(\cdot)$, defined by (137), is asymptotically Stepanov $p(x)$-uniformly recurrent (asymptotically Stepanov $p(x)$-almost periodic, asymptotically Stepanov $p(x)$-almost automorphic).

By $AS^{p(x)}UR_{\omega,c}(I : X)$, $AS^{p(x)}AP_{\omega,c}(I : X)$ and $AS^{p(x)}AA_{\omega,c}(I : X)$ we denote the space of all asymptotically Stepanov $(p(x), \omega, c)$-uniformly recurrent functions, the space of all asymptotically Stepanov $(p(x), \omega, c)$-almost periodic functions and the space of all asymptotically Stepanov $(p(x), \omega, c)$-almost automorphic functions, respectively. If $p(x) \equiv p \in [1, \infty)$, then by $AS^pUR_{\omega,c}(I : X)$, $AS^pAP_{\omega,c}(I : X)$ and $AS^pAA_{\omega,c}(I : X)$ we denote the space of all asymptotically Stepanov (p, ω, c)-uniformly recurrent functions, the space of all asymptotically Stepanov (p, ω, c)-almost periodic functions and the space of all asymptotically Stepanov (p, ω, c)-almost automorphic functions, respectively.

The conclusions established in the points [1.-2., 4.-7.] can be simply reformulated for the Stepanov classes. For example, if we consider the point [2.], then we may conclude the following: Suppose that $|c| > 1$, $\omega > 0$ and $f : I \to X$ is Stepanov $(p(x), \omega, c)$-uniformly recurrent or Stepanov $(p(x), \omega, c)$-almost automorphic. If $f(\cdot)$ is not almost everywhere equal to zero, then the function $f(\cdot)$ is not Stepanov $p(x)$-bounded; moreover, in the case of consideration of Stepanov $(p(x), \omega, c)$-almost automorphicity, the function $\hat{f}(\cdot)$ is unbounded as $t \to +\infty$ so that a constant non-zero function cannot be Stepanov $(p(x), \omega, c)$-uniformly recurrent or Stepanov $(p(x), \omega, c)$-almost automorphic.

Essentially, any established result for almost periodic type functions and their Stepanov generalizations can be straightforwardly reformulated for (ω, c)-almost periodic type functions and their Stepanov generalizations (in the sequel, we will try not to consider such statements). For example, using the corresponding statement for the uniformly recurrent functions we can immediately deduce the following:

Proposition 2.8.9. $p \in \mathcal{P}([0, 1])$, *If $f : [0, \infty) \to X$ satisfies that the function $f_{\omega,c}(\cdot)$ is uniformly continuous and asymptotically Stepanov $p(x)$-uniformly recurrent, then the function $f(\cdot)$ is asymptotically (ω, c)-uniformly recurrent.*

Let us only note that the uniform continuity of function $f_{\omega,c}(\cdot)$ is ensured provided that $|c| \geqslant 1$ and $f(\cdot)$ is a bounded uniformly continuous function. This follows from the fact that, for every two non-negative real numbers $t_1,\ t_2 \geqslant 0$ such that $t_1 < t_2$, the Darboux inequality yields

$$\begin{aligned}\Big\|c^{-\frac{t_1}{\omega}} f(t_1) - c^{-\frac{t_2}{\omega}} f(t_2)\Big\| &\leqslant \Big\|c^{-\frac{t_1}{\omega}} \big[f(t_1) - f(t_2)\big]\Big\| + \Big\|\big[c^{-\frac{t_1}{\omega}} - c^{-\frac{t_2}{\omega}}\big] f(t_2)\Big\| \\ &\leqslant \big\|f(t_1) - f(t_2)\big\| + \frac{1}{\omega}(\ln|c| + \pi) \cdot \big|t_1 - t_2\big| \cdot \|f\|_\infty.\end{aligned}$$

Now we would like to endow the introduced spaces of (asymptotically) (ω, c)-almost periodic type functions with certain norms. We start with the notion introduced in Definition 2.8.1 and Definition 2.8.4. Define

$$\|f\|_{\omega,c} := \sup_{t\in I}\Big\|c^{-\frac{t}{\omega}} f(t)\Big\|.$$

Proposition 2.8.10. *The spaces* $AP_{\omega,c}(I : X)$, $AA_{\omega,c}(I : X)$, $AA_{\omega,c;\mathbf{c}}(I : X)$, $AAP_{\omega,c}([0,\infty) : X)$, $AAA_{\omega,c}([0,\infty) : X)$ *and* $AAA_{\omega,c;\mathbf{c}}([0,\infty) : X)$, *equipped with the norm* $\|\cdot\|_{\omega,c}$, *are Banach spaces.*

Proof. Denote by $\mathcal{X}$ any of the above spaces. Suppose that $(f_n)_n$ is a Cauchy sequence in $\mathcal{X}$. Hence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m,\ n \geqslant N$, we have $\|f_n - f_m\|_{\omega,c} < \varepsilon$. So, there exist $u_m,\ u_n \in c^{-\frac{\cdot}{\omega}}\mathcal{X}$ (with the meaning clear) such that $f_m(t) = c^{\frac{t}{\omega}} u_m(t)$ and $f_n(t) = c^{\frac{t}{\omega}} u_n(t)$ for all $t \in I$. For $m,\ n \geqslant N$, we have

$$\begin{aligned}\big\|u_m - u_n\big\|_\infty &= \sup_{t\in I}\big\|u_m(t) - u_n(t)\big\| \\ &= \sup_{t\in I}\Big\|c^{-\frac{t}{\omega}} f_m(t) - c^{-\frac{t}{\omega}} f_n(t)\Big\| \\ &= \sup_{t\in I}\Big\||c|^{-\frac{t}{\omega}}\big[f_m(t) - f_n(t)\big]\Big\| \\ &= \big\|f_n - f_m\big\|_{\omega,c} < \varepsilon.\end{aligned}$$

Hence, $(u_n)_n$ is a Cauchy sequence in $c^{-\frac{\cdot}{\omega}}\mathcal{X}$, which is a complete space. Then, there exists $u \in c^{-\frac{\cdot}{\omega}}\mathcal{X}$ such that $\lim_{n\to+\infty} u_n = u$. Define $f(t) := c^{\frac{t}{\omega}} u(t)$, $t \in I$. Thus,

$$\begin{aligned}\big\|f_n - f\big\|_{\omega,c} &= \sup_{t\in I}\Big\||c|^{-\frac{t}{\omega}}\big[f_n(t) - f(t)\big]\Big\| \\ &= \sup_{t\in I}\Big\||c|^{-\frac{t}{\omega}} c^{\frac{t}{\omega}} u_n(t) - |c|^{-\frac{t}{\omega}} c^{\frac{t}{\omega}} u(t)\Big\| \\ &= \sup_{t\in I}\big\|u_n(t) - u(t)\big\| \to 0,\end{aligned}$$

when $n \to \infty$. Hence, $\mathcal{X}$ is a Banach space. $\square$

For any $c \in \mathbb{C} \smallsetminus \{0\}$ and $p \in [1, \infty)$, we denote by $L^p_{S,c}(I : X)$ the space of all functions $f \in L^p_{loc}(I : X)$ such that

$$\|f\|_{p,\omega,c} := \sup_{t \in I} \left(\int_t^{t+1} |c|^{-\frac{s}{\omega}} f(s)\, ds \right)^{1/p}.$$

Then $(L^p_{S,c}(I : X), \|\cdot\|_{p,\omega,c})$ is a Banach space. Arguing as above, we may conclude that $S^pAP_{\omega,c}(I : X)$ $(S^pAA_{\omega,c}(I : X)/AS^pAP_{\omega,c}(I : X)$, $AS^pAA_{\omega,c}(I : X))$ is a closed subspace of $L^p_{S,c}(I : X)$ and therefore a Banach space itself.

2.8.1. (ω, c)-Uniform recurrence of type i and (ω, c)-almost periodicity of type i $(i = 1, 2)$. Suppose temporarily that $f \in P_{\omega,c}(I : X)$ and $n \in \mathbb{N}$. Then we have $f(t + n\omega) = c^n f(t)$, $t \in I$. Setting $\alpha_n = n\omega$, we get that for each $\varepsilon > 0$ and $t \in I$ we have

$$\left\| f(t + \alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t) \right\| \leqslant \varepsilon \quad \text{and} \quad \left\| c^{\frac{-\alpha_n}{\omega}} f(t + \alpha_n) - f(t) \right\| \leqslant \varepsilon. \tag{138}$$

The equation (138) motivates us to introduce the following concepts of (ω, c)-uniform recurrence and (ω, c)-almost periodicity [it is not clear how we can do that for (compact) (ω, c)-almost automorphicity in a satisfactory way].

Definition 2.8.11. Suppose that $f : I \to X$ is continuous, $c \in \mathbb{C} \smallsetminus \{0\}$ and $\omega > 0$.

(i) We say that $f(\cdot)$ is (ω, c)-uniformly recurrent of type 1 (type 2) if and only if there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n \to +\infty} \sup_{t \in I} \left\| f(t + \alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t) \right\| = 0 \quad \left(\lim_{n \to +\infty} \sup_{t \in I} \left\| c^{\frac{-\alpha_n}{\omega}} f(t + \alpha_n) - f(t) \right\| = 0 \right).$$

(ii) We say that $f(\cdot)$ is (ω, c)-almost periodic of type 1 (type 2) if and only if for each $\varepsilon > 0$ the set

$$\left\{ \tau > 0 : \sup_{t \in I} \left\| f(t + \tau) - c^{\frac{\tau}{\omega}} f(t) \right\| < \varepsilon \right\} \quad \left(\left\{ \tau > 0 : \sup_{t \in I} \left\| c^{\frac{-\tau}{\omega}} f(t + \tau) - f(t) \right\| < \varepsilon \right\} \right)$$

is relatively dense in $[0, \infty)$.

By $UR_{\omega,c,i}(I : X)$ and $AP_{\omega,c,i}(I : X)$, we denote the space of all (ω, c)-uniformly recurrent functions of type i and the space of all (ω, c)-almost periodic functions of type i, respectively $(i = 1, 2)$.

It is clear that the set $\{n\omega : n \in \mathbb{N}\}$ is relatively dense in $[0, \infty)$. Taking into account this observation, it follows that the space $P_{\omega,c}(I : X)$ is contained in the spaces $UR_{\omega,c,i}(I : X)$ and $AP_{\omega,c,i}(I : X)$, for $i = 1, 2$; moreover, $UR_{\omega,c,i}(I : X) \supseteq AP_{\omega,c,i}(I : X)$ for $i = 1, 2$ and it is clear that for any $t \in I$ and $\tau \geqslant 0$ we have

$$\begin{aligned} \left\| c^{\frac{-\tau}{\omega}} f(t + \tau) - f(t) \right\| &= \left\| c^{\frac{-\tau}{\omega}} \left[f(t + \tau) - c^{\frac{\tau}{\omega}} f(t) \right] \right\| \\ &= |c|^{\frac{-\tau}{\omega}} \left\| f(t + \tau) - c^{\frac{\tau}{\omega}} f(t) \right\|. \end{aligned}$$

Therefore, in the case that $|c| = 1$, it simply follows that the (ω, c)-almost periodicity of type 1 (type 2) is equivalent with the usual almost periodicity as well as that the notion of (ω, c)-uniform recurrence of type 1 is equivalent with the notion of (ω, c)-uniform recurrence of type 2.

But, in the case that $|c| \neq 1$, the concepts introduced in Definition 2.8.11 are not satisfactory to a great extent. Before stating the corresponding result which justifies this fact, let us denote by $M_{\omega,c}(I : X)$ the space consisting of all functions $f : I \to X$ such that $c^{-\cdot/\omega} f(\cdot) \in P(I : X)$. It is clear that $M_{\omega,c}(I : X)$ is not a vector space together with the usual operations.

THEOREM 2.8.12. *Let $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$.*
- (i) *Suppose that $|c| > 1$. Then $UR_{\omega,c,i}(I : X) = AP_{\omega,c,i}(I : X) = M_{\omega,c}(I : X)$ for $i = 1, 2$.*
- (ii) *Suppose that $|c| < 1$ and $I = \mathbb{R}$. Then $UR_{\omega,c,i}(I : X) = AP_{\omega,c,i}(I : X) = M_{\omega,c}(I : X)$ for $i = 1, 2$.*

Before giving the proof of Theorem 2.8.12, we will state two lemmas. The first one is simple and follows almost immediately from Definition 2.8.11:

LEMMA 2.8.13. *Suppose that $f : I \to X$ is continuous, $c \in \mathbb{C} \setminus \{0\}$ and $\omega > 0$.*
- (i) *If $|c| \geqslant 1$, then $UR_{\omega,c,1}(I : X) \subseteq UR_{\omega,c,2}(I : X)$ and $AP_{\omega,c,1}(I : X) \subseteq AP_{\omega,c,2}(I : X)$.*
- (ii) *If $|c| \leqslant 1$, then $UR_{\omega,c,1}(I : X) \supseteq UR_{\omega,c,2}(I : X)$ and $AP_{\omega,c,1}(I : X) \supseteq AP_{\omega,c,2}(I : X)$.*
- (iii) *In the case that $I = [0, \infty)$ and $|c| \geqslant 1$, then $UR_{\omega,c,2}(I : X) \subseteq UR_{\omega,c}(I : X)$ and $AP_{\omega,c,2}(I : X) \subseteq AP_{\omega,c}(I : X)$.*

LEMMA 2.8.14. *Suppose that $I = \mathbb{R}$ and $f : \mathbb{R} \to X$. Then $f(\cdot)$ is (ω, c)-uniformly recurrent of type 1 (type 2) [(ω, c)-almost periodic of type 1 (type 2)] if and only if the function $\check{f}(\cdot)$ is $(\omega, 1/c)$-uniformly recurrent of type 2 (type 1) [(ω, c)-almost periodic of type 2 (type 1)].*

PROOF. The proof simply follows by observing that, for every $\tau > 0$ and $\varepsilon > 0$, we have:

$$\sup_{t\in I}\Big\|f(t+\tau) - c^{\frac{\tau}{\omega}} f(t)\Big\| < \varepsilon \Leftrightarrow \sup_{t\in I}\Big\|f(-t+\tau) - c^{\frac{\tau}{\omega}} f(-t)\Big\| < \varepsilon$$
$$\Updownarrow$$
$$\sup_{t\in I}\Big\|\check{f}(t-\tau) - c^{\frac{\tau}{\omega}} \check{f}(t)\Big\| < \varepsilon \Leftrightarrow \sup_{t\in I}\Big\|\check{f}(t) - c^{\frac{\tau}{\omega}} \check{f}(t+\tau)\Big\| < \varepsilon$$
$$\Updownarrow$$
$$\sup_{t\in I}\Big\|(1/c)^{-\frac{\tau}{\omega}} \check{f}(t+\tau) - \check{f}(t)\Big\| < \varepsilon.$$

□

Proof of Theorem 2.8.12. Keeping in mind Lemma 2.8.14, it suffices to prove (i). Towards this end, we recognize two cases: $I = [0, \infty)$ and $I = \mathbb{R}$. In the first case, it suffices to show that $UR_{\omega,c,2}([0,\infty) : X) \subseteq M_{\omega,c}([0,\infty) : X)$ and $M_{\omega,c}([0,\infty) : X) \subseteq AP_{\omega,c,1}([0,\infty) : X)$. So, let $f \in UR_{\omega,c,2}([0,\infty) : X)$. This implies that there exist a finite constant $M > 0$ and a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\sup_{t\in I, n\in\mathbb{N}} \Bigl\| c^{\frac{-\alpha_n}{\omega}} f(t+\alpha_n) - f(t) \Bigr\| \leqslant M.$$

Since $f(t) = c^{t/\omega} f_{\omega,c}(t)$, $t \geqslant 0$, the above implies

$$\Bigl\| f_{\omega,c}(t+\alpha_n) - f_{\omega,c}(t) \Bigr\| \leqslant |c|^{-(t/\omega)} M, \quad t \geqslant 0, \ n \in \mathbb{N}.$$

Hence, for every $n \in \mathbb{N}$, we have $\lim_{t\to+\infty}[f_{\omega,c}(t+\alpha_n) - f_{\omega,c}(t)] = 0$. On the other hand, Lemma 2.8.13(iii) yields that, for every $n \in \mathbb{N}$, we have that the function $f_{\omega,c}(\cdot+\alpha_n) - f_{\omega,c}(\cdot)$ is uniformly recurrent; hence, for every $n \in \mathbb{N}$, we have $f_{\omega,c}(\cdot+\alpha_n) \equiv f_{\omega,c}(\cdot)$ and therefore $f_{\omega,c}(\cdot)$ belongs to the space $P([0,\infty) : X)$, as claimed. To see that $M_{\omega,c}([0,\infty) : X) \subseteq AP_{\omega,c,1}([0,\infty) : X)$, suppose that $f_{\omega,c}(t+T) = f_{\omega,c}(t)$ for all $t \geqslant 0$ and some $T > 0$. This simply implies that $f(t+nT) = c^{nT/\omega} f(t)$ for all $n \in \mathbb{N}$ so that $f \in AP_{\omega,c,1}([0,\infty) : X)$ because the set $\{nT : n \in \mathbb{N}\}$ is relatively dense in $[0,\infty)$. Suppose now that $I = \mathbb{R}$. Similarly as above, it follows that $UR_{\omega,c,i}(\mathbb{R} : X) \supseteq AP_{\omega,c,i}(\mathbb{R} : X) \supseteq M_{\omega,c}(\mathbb{R} : X)$ for $i = 1, 2$. Therefore, it suffices to show that $UR_{\omega,c,2}(\mathbb{R} : X) \subseteq M_{\omega,c}(\mathbb{R} : X)$. Let $f \in UR_{\omega,c,2}(\mathbb{R} : X)$. Since the restriction of $f(\cdot)$ on $[0,\infty)$ belongs to the space $UR_{\omega,c,2}([0,\infty) : X)$, it readily follows that there exists a number $T > 0$ such that $f_{\omega,c}(t+T) = f_{\omega,c}(t)$ for all $t \geqslant 0$. To complete the proof, it suffices to prove that this equality holds for all real numbers $t < 0$. Let $\varepsilon > 0$ be fixed. Due to our assumption, we have the existence of an integer $n_0 \in \mathbb{N}$ such that $t + \alpha_n > 0$ as well as that

$$\Bigl\| c^{t/\omega} f_{\omega,c}(t+\alpha_n) - c^{t/\omega} f_{\omega,c}(t) \Bigr\| \leqslant \varepsilon$$
$$\text{and } \Bigl\| c^{(t+T)/\omega} f_{\omega,c}(t+T+\alpha_n) - c^{(t+T)/\omega} f_{\omega,c}(t+T) \Bigr\| \leqslant \varepsilon,$$

i.e.,

$$\Bigl\| c^{t/\omega} f_{\omega,c}(t+\alpha_n) - c^{t/\omega} f_{\omega,c}(t) \Bigr\| \leqslant \varepsilon$$
$$\text{and } \Bigl\| c^{t/\omega} f_{\omega,c}(t+\alpha_n) - c^{t/\omega} f_{\omega,c}(t+T) \Bigr\| \leqslant \varepsilon |c|^{-T/\omega}.$$

This implies

$$\Bigl\| c^{t/\omega} \bigl[f_{\omega,c}(t+T) - f_{\omega,c}(t) \bigr] \Bigr\| \leqslant \varepsilon \Bigl(1 + |c|^{-T/\omega} \Bigr).$$

Letting $\varepsilon \to 0+$, we get $f_{\omega,c}(t+T) = f_{\omega,c}(t)$, as claimed. $\square$

COROLLARY 2.8.15. *Suppose that* $i = 1, 2$, $|c| < 1$, $\omega > 0$ *and* $f \in AP_{\omega,c,i}([0,\infty) : X)$. *Then there exists a function* $F \in AP_{\omega,c,i}(\mathbb{R} : X)$ *such that* $F(t) = f(t)$ *for all* $t \geqslant 0$ *if and only if* $f \in M_{\omega,c}([0,\infty) : X)$.

Further on, the points [4., 5., 6., 7.] from the beginning of this section can be restated as follows:

4'. Let $c \in \mathbb{R}$ and $\omega > 0$. Then, for every (ω, c)-uniformly recurrent function $f(\cdot)$ of type 1 (type 2), we have that the function $\|f(\cdot)\|$ is $(\omega, |c|)$-uniformly recurrent of type 1 (type 2).

5'. The spaces $UR_{\omega,c,i}(I : X)$ and $AP_{\omega,c,i}(I : X)$ are invariant under pointwise multiplications with scalars $(i = 1, 2)$.

6'. The spaces $UR_{\omega,c,i}(I : X)$ and $AP_{\omega,c,i}(I : X)$ are translation invariant $(i = 1, 2)$.

7'. If $I = [0, \infty)$, $|c| \geqslant 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $UR_{\omega,c,2}(I : X)$ converges uniformly to a function $f : I \to X$, then the function $f(\cdot)$ belongs to the space $UR_{\omega,c,2}(I : X)$. Furthermore, if $I = [0, \infty)$, $|c| \leqslant 1$, $\omega > 0$ and the sequence $(f_n(\cdot))$ in $UR_{\omega,c,1}(I : X)$ $(AP_{\omega,c,1}(I : X))$ converges uniformly to a function $f : I \to X$, then the function $f(\cdot)$ belongs to the space $UR_{\omega,c,1}(I : X)$ $(AP_{\omega,c,1}(I : X))$.

Now we will prove the following

PROPOSITION 2.8.16. *Suppose that* $i = 1, 2$, $|c| < 1$, $\omega > 0$ *and* $f \in AP_{\omega,c,i}(I : X)$. *Then the function* $f_{\omega,c}(\cdot)$ *is bounded and* $\lim_{t\to+\infty} f(t) = 0$.

PROOF. By Theorem 2.8.12(ii) and Lemma 2.8.13(iv) it suffices to consider the case $I = [0, \infty)$ and the class $AP_{\omega,c,1}([0, \infty) : X)$. Let $\varepsilon = 1$. Then there exists a finite number $l > 0$ such that any subinterval I' of $[0, \infty)$ contains a point τ such that $\|c^{\frac{-\tau}{\omega}} f(t + \tau) - f(t)\| < 1$ for all $t \geqslant 0$. Suppose that $t \in [nl, (n + 1)l]$ for some $n \in \mathbb{N}$. Then there exists $\tau_n \in [(n - 1)l, nl]$ such that $\|c^{\frac{-\tau_n}{\omega}} f(t' + \tau_n) - f(t')\| < 1$ for all $t' \geqslant 0$. In particular, $t - \tau_n = t' \in [0, 2l]$ and the above implies $\|f(t)\| \leqslant (1+M)|c|^{\tau_n/\omega} \leqslant (1+M)[\max_{t''\in[0,2l]} |c|^{-t''/\omega}]|c|^{t/\omega}$, where $M := \sup_{x\in[0,2l]} \|f(x)\|$. This yields the required limit equality. □

EXAMPLE 2.8.17. Denote the restriction of the function $f(\cdot)$ given by (33) to the non-negative real axis by the same symbol. Then Proposition 2.8.16 implies that the function $c^{-\cdot/\omega} f(\cdot)$ cannot belong to the space $AP_{\omega,c,i}([0, \infty) : \mathbb{C})$ for $i = 1, 2$. On the other hand, it is clear that $c^{-\cdot/\omega} f(\cdot) \in UR_{\omega,c}([0, \infty) : \mathbb{C}) \subseteq UR_{\omega,c,i}([0, \infty) : \mathbb{C})$ for $i = 1, 2$.

COROLLARY 2.8.18. *Suppose that* $|c| < 1$ *and* $\omega > 0$. *Then* $f \in AP_{\omega,c,1}([0, \infty) : X)$ *if and only if the function* $f_{\omega,c}(\cdot)$ *is bounded and continuous.*

PROOF. Due to Proposition 2.8.16, it suffices to show that the boundedness of function $f_{\omega,c}(\cdot)$ implies $f \in AP_{\omega,c,1}([0, \infty) : X)$. If so, then we need to prove that for each $\varepsilon > 0$ the set consisting of all positive reals $t > 0$ such that

$$\left\| c^{(t+\tau)/\omega} f_{\omega,c}(t + \tau) - c^{(t+\tau)/\omega} f_{\omega,c}(t) \right\| \leqslant \varepsilon, \quad t \geqslant 0$$

is relatively dense in $[0, \infty)$. But, this simply follows from the fact that this set contains a ray $[a(\varepsilon), \infty)$ for a sufficiently large real number $a(\varepsilon) > 0$, which can be proved by using the boundedness of $f_{\omega,c}(\cdot)$ and the simple inequality $|c|^{t/\omega} \leqslant 1$, $t \geqslant 0$. □

REMARK 2.8.19. Suppose that $|c| < 1$ and $\omega > 0$. Using Corollary 2.8.18, we can simply prove that $f \in AP_{\omega,c,1}([0,\infty) : X)$ if and only if for every (there exists) strictly increasing sequence (α_n) of positive reals tending to plus infinity such that $\lim_{n\to+\infty} \sup_{t\geqslant 0} \|f(t+\alpha_n) - c^{\frac{\alpha_n}{\omega}} f(t)\| = 0$.

EXAMPLE 2.8.20. Suppose that $f(t) := 2^{-t}[1 + (1/\ln(2+t))]$, $t \geqslant 0$. Due to Corollary 2.8.18, this function belongs to the space $AP_{1,1/2,1}([0,\infty) : \mathbb{C}) \subseteq UR_{1,1/2,1}([0,\infty) : \mathbb{C})$. On the other hand, $f(\cdot)$ does not belong to the space $UR_{1,1/2,2}([0,\infty) : \mathbb{C})$. Otherwise, we would have the existence of an arbitrarily large positive real number $\alpha > 0$ such that

$$\sup_{t\geqslant 0}\left|2^{-t}\frac{\ln(1+(\alpha/(1+t)))}{\ln(2+t)\cdot\ln(2+t+\alpha)}\right| \leqslant \varepsilon.$$

Taking $t = 0$, this simply leads us to a contradiction.

The class $UR_{\omega,c,1}([0,\infty) : X)$ is also extremely non-interesting due to the following characterization:

PROPOSITION 2.8.21. *Suppose* $c \in \mathbb{C} \smallsetminus \{0\}$, $|c| < 1$ *and* $\omega > 0$. *Then* $UR_{\omega,c,1}([0,\infty) : X) = C_0([0,\infty) : X)$.

PROOF. If $f \in C_0([0,\infty) : X)$, then for each strictly increasing sequence (α_n) tending to plus infinity and for each real number $\varepsilon > 0$ we can always find an integer $n_0 \in \mathbb{N}$ such that $\|f(t+\alpha_n) - c^{\alpha_n/\omega} f(t)\| \leqslant (\varepsilon/2) + |c|^{\alpha_n/\omega}\|f(t)\| \leqslant (\varepsilon/2) + |c|^{\alpha_n/\omega}\|f\|_\infty \leqslant \varepsilon$, $t \geqslant 0$, $n \geqslant n_0$, which implies $f \in UR_{\omega,c,1}([0,\infty) : X)$. To prove the converse, let us first show that the assumption $f \in UR_{\omega,c,1}([0,\infty) : X)$ implies the boundedness of $f(\cdot)$. If (α_n) satisfies the requirements of definition of space $UR_{\omega,c,1}([0,\infty) : X)$, then we may assume without loss of generality that $\alpha_{n+1} - \alpha_n > 3$ for all $n \in \mathbb{N}$ and

$$\left\|f(t+\alpha_n)\right\| \leqslant 1 + |c|^{\alpha_n/\omega}\|f(t)\|, \quad t \geqslant 0,\ n \in \mathbb{N}. \tag{139}$$

Let $n \in \mathbb{N}$ be fixed and let $M_0 := \max_{t\in[0,\alpha_n]} \|f(t)\|$. Then (139) inductively implies that for arbitrary $T \in (0, \alpha_n]$ and for arbitrary $k \in \mathbb{N}$ we have

$$\left\|f(T+k\alpha_n)\right\| \leqslant \sum_{j=0}^{k-1} |c|^{\alpha_n j/\omega} + |c|^{k\alpha_n/\omega} M_0 \leqslant \sum_{j=0}^{\infty} |c|^{j/\omega} + M_0.$$

Therefore, $\|f(t)\| \leqslant \sum_{j=0}^{\infty} |c|^{j/\omega} + M_0$, $t \geqslant 0$, as claimed. The remainder of proof is simple; since the function $f(\cdot)$ is bounded, then we have the existence of an integer $n_1 \in \mathbb{N}$ such that

$$\left\|f(t+\alpha_n)\right\| \leqslant |c|^{\alpha_n/\omega}\|f\|_\infty + (\varepsilon/2) < \varepsilon, \quad t \geqslant 0,\ n \geqslant n_1,$$

and therefore $f \in C_0([0,\infty) : X)$. □

Now we will prove the following result:

PROPOSITION 2.8.22. *Suppose that* $|c| < 1$ *and* $\omega > 0$. *Then* $f \in AP_{\omega,c,2}([0,\infty) : X)$ *if and only if the function* $f_{\omega,c}(\cdot)$ *is bounded continuous and for each* $\varepsilon > 0$ *and* $N > 0$ *the set of all positive real numbers* $\tau > 0$ *such that*

$$\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leqslant \varepsilon, \quad t \in [0, N] \tag{140}$$

is relatively dense in $[0,\infty)$.

PROOF. Suppose first that $f \in AP_{\omega,c,2}([0,\infty) : X)$. Due to Proposition 2.8.16, the function $f_{\omega,c}(\cdot)$ is bounded. Let $\varepsilon > 0$ and $N > 0$ be fixed, and let $\varepsilon_0 > 0$ be such that $\varepsilon_0 |c|^{-N/\omega} \leqslant \varepsilon$. By our assumption, the set of all positive reals $\tau > 0$ such that $\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leqslant \varepsilon_0 |c|^{-t/\omega}$, $t \geqslant 0$ is relatively dense in $[0,\infty)$. If τ belongs to this set, then we have $\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leqslant \varepsilon_0 |c|^{-t/\omega} \leqslant \varepsilon$, $t \in [0, N]$. For the converse, it suffices to assume $f_{\omega,c} \neq 0$. Fix a number $\varepsilon > 0$. In this case, we can find a number $N > 0$ such that $|c|^{t/\omega} \leqslant \varepsilon/(2(1 + \|f_{\omega,c}\|_\infty))$ for all $t \geqslant N$. For this $\varepsilon > 0$ and $N > 0$ we can find a relatively dense set of positive reals τ satisfying (140). If τ belongs to this set, then there exist two possibilities: $t \geqslant N$ or $t \in [0, N)$. In the first case, we have $\|c^{t/\omega}[f_{\omega,c}(t+\tau) - f_{\omega,c}(t)]\| \leqslant \varepsilon |c|^{t/\omega} \leqslant \varepsilon$; in the second case, we have $\|c^{t/\omega}[f_{\omega,c}(t+\tau) - f_{\omega,c}(t)]\| \leqslant (2\varepsilon \|f_{\omega,c}\|_\infty)/(2(1 + \|f_{\omega,c}\|_\infty)) < \varepsilon$. Hence, we have $\|f_{\omega,c}(t+\tau) - f_{\omega,c}(t)\| \leqslant \varepsilon_0 |c|^{-t/\omega}$, $t \geqslant 0$ and the proof of the proposition is thereby complete. $\square$

REMARK 2.8.23.
(i) Let us recall that any Levitan N-almost periodic function $f_{\omega,c} : [0,\infty) \to X$ satisfies that for each $\varepsilon > 0$ and $N > 0$ the set of all positive reals $\tau > 0$ such that (140) holds is relatively dense in $[0,\infty)$ (cf. [**265**, Definition 2, p. 53]). In particular, the restriction of any almost automorphic function $f_{\omega,c} : \mathbb{R} \to X$ satisfies this condition. Denote by $AA_{[0,\infty)}(X)$ the vector space consisting of such functions; thus, $c^{\cdot/\omega} AA_{[0,\infty)}(X) \subseteq AP_{\omega,c,2}([0,\infty) : X)$. Recall also that the function $t \mapsto 1/(2 + \cos t + \cos(\sqrt{2}t))$, $t \geqslant 0$ is Levitan N-almost periodic and unbounded.

(ii) According to [**265**, Definition 2, p. 80], a continuous function $f : I \to X$ is called recurrent if and only if for each $\varepsilon > 0$ and $N > 0$ the set of all positive reals $\tau > 0$ such that (140) holds is relatively dense in $[0,\infty)$ (the case $I = \mathbb{R}$ has been considered in [**265**], only). The Stepanov generalizations of recurrent functions can be also introduced but then it is not clear how one can consider the invariance of recurrence under the action of infinite convolution product given by (55) since the methods proposed in the proof of [**234**, Proposition 2.6.11] and related results do not work in this framework. Note also that we can extend the notion of (ω, c)-almost automorphicity by requiring that the function $f_{\omega,c}(\cdot)$ is recurrent.

(iii) Due to Corollary 2.8.18, $AP_{\omega,c,1}([0,\infty) : X)$ is the vector space together with the usual operations. This is not longer true for the space $AP_{\omega,c,2}([0,\infty) : X)$, which can be deduced from Proposition 2.8.22 and a counterexample constructed by W. A. Veech (see e.g., [**48**, Example 2.8], and the corresponding example given in the pioneering paper [**262**] by B. Ya. Levin, as well as the articles [**?**] by J. Egawa, [**285**] by A. Michalowicz, S. Stoínski and [**97**] by D. N. Cheban). In particular, the space $AP_{\omega,c,2}([0,\infty) : X) \subseteq UR_{\omega,c,2}([0,\infty) : X)$ strictly contains $c^{\cdot/\omega}AA_{[0,\infty)}(X)$. On the other hand, the compactly almost automorphic function constructed by A. M. Fink in [**171**] is not asymptotically uniformly recurrent, as shown earlier. This implies that there exists a function $f \in c^{\cdot/\omega}AA_{[0,\infty)}(X)$ such that $f_{\omega,c}(\cdot)$ is not uniformly recurrent; in particular, $UR_{\omega,c,2}([0,\infty) : X)$ strictly contains $UR_{\omega,c}([0,\infty) : X)$.

(iv) As already seen, there exist two bounded, even, uniformly continuous, uniformly recurrent functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that its sum is not uniformly recurrent. Furthermore, we can choose $f(\cdot)$ and $g(\cdot)$ such that $f(0) = g(0) = 1$ and $|f(t) + g(t)| \leqslant 1$ for $|t| \geqslant 1$. Denote the restrictions of such functions to the non-negative real axis by the same symbols, and define after that $F(t) := 2^{-t}f(t)$, $t \geqslant 0$ and $G(t) := 2^{-t}g(t)$, $t \geqslant 0$. Then $F,\ G \in UR_{1,1/2}([0,\infty) : \mathbb{C}) \subseteq UR_{1,1/2,2}([0,\infty) : \mathbb{C})$ but $F+G \notin UR_{1,1/2,2}([0,\infty) : \mathbb{C})$. If we suppose the contrary, then we would have the existence of a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\geqslant 0}\Big|2^{-t}\big[f(t+\alpha_n)+g(t+\alpha_n)\big]-2^{-t}[f(t)+g(t)]\Big| = 0,$$

which is impossible because for each $n \in \mathbb{N}$ such that $\alpha_n \geqslant 1$ we have

$$\begin{aligned}\sup_{t\geqslant 0}&\Big|2^{-t}\big[f(t+\alpha_n)+g(t+\alpha_n)\big]-2^{-t}[f(t)+g(t)]\Big| \\ &\geqslant \Big|f(0)+g(0)-\big[f(\alpha_n)+g(\alpha_n)\big]\Big| = \Big|2-[f(\alpha_n)+g(\alpha_n)]\Big| \geqslant 1.\end{aligned}$$

In particular, this example can be used to show that the set $UR_{\omega,c,2}([0,\infty) : \mathbb{C})$ does not form a vector space together with the usual operations.

(v) Using Proposition 2.8.22, as well as the arguments contained in the proofs of Proposition 2.8.10 and [**62**, Theorem 8°, pp. 3-4], it follows that $AP_{\omega,c,2}([0,\infty) : X)$ is a complete metric space equipped with the distance $d(\cdot,\cdot) := \|\cdot - \cdot\|_{\omega,c}$.

Keeping in mind the proved results, we will consider the following notion for Stepanov classes, only:

Definition 2.8.24. Let $p \in \mathcal{P}([0,1])$, $c \in \mathbb{C}\backslash\{0\}$, $|c| \leqslant 1$ and $\omega > 0$. Then it is said that a function $f \in L^{p(x)}_{loc}([0,\infty) : X)$ is Stepanov $(p(x),\omega,c)$-uniformly recurrent of type 2, resp. Stepanov $(p(x),\omega,c)$-almost periodic of type 2 if and only if

$$\lim_{n\to+\infty} \sup_{t\geqslant 0}\Big\|c^{\frac{-\alpha_n}{\omega}}f(s+t+\alpha_n)-f(s+t)\Big\|_{L^{p(s)}[0,1]} = 0,$$

resp. for each $\varepsilon > 0$ the set

$$\left\{\tau > 0 : \sup_{t\geqslant 0}\left\|c^{\frac{-\alpha_n}{\omega}} f(s+t+\alpha_n) - f(s+t)\right\|_{L^{p(s)}[0,1]} < \varepsilon\right\}$$

is relatively dense in $[0, \infty)$.

By $S^{p(x)}UR_{\omega,c,2}([0,\infty) : X)$ and $S^{p(x)}AP_{\omega,c,2}([0,\infty) : X)$ we denote the space of all Stepanov $(p(x), \omega, c)$-uniformly recurrent functions of type 2 and the space of all Stepanov $(p(x), \omega, c)$-almost periodic functions of type 2, respectively. If $p(x) \equiv p \in [1, \infty)$, then the above classes are also denoted by $S^pUR_{\omega,c,2}([0,\infty) : X)$ and $S^pAP_{\omega,c,2}([0,\infty) : X)$, respectively.

If $1 \leqslant p(x) \leqslant q(x) < \infty$ and $f \in S^{q(x)}UR_{\omega,c,2}([0,\infty) : X)$, resp. $f \in S^{q(x)}AP_{\omega,c,2}([0,\infty) : X)$, then $f \in S^{p(x)}UR_{\omega,c,2}([0,\infty) : X)$, resp. $f \in S^{p(x)}AP_{\omega,c,2}([0,\infty) : X)$; furthermore, the space $S^{p(x)}UR_{\omega,c,2}([0,\infty) : X)$, resp. $S^{p(x)}AP_{\omega,c,2}([0,\infty) : X)$, contains the space $UR_{\omega,c,2}([0,\infty) : X)$, resp. $AP_{\omega,c,2}([0,\infty) : X)$. It is simply verified that the space $S^{p(x)}UR_{\omega,c,2}([0,\infty) : X)$, resp. $S^{p(x)}AP_{\omega,c,2}([0,\infty) : X)$, consists of those locally $p(x)$-integrable functions $f : I \to X$ for which $\hat{f}(\cdot)$ belongs to the space $UR_{\omega,c,2}([0,\infty) : L^{p(x)}([0,1] : X))$, resp. $AP_{\omega,c,2}([0,\infty) : L^{p(x)}([0,1] : X))$. Keeping in mind this observation, it is straightforward to transfer the previously proved results and the points [4'.-7'.] for the introduced Stepanov classes; details can be omitted. Note, finally, that $S^{p(x)}AP_{\omega,c,2}([0,\infty) : X)$ is a complete metric spaces equipped with the distance $d(\cdot,\cdot) := \|\cdot - \cdot\|_{p,\omega,c}$.

2.8.2. Composition principles for (ω, c)-almost periodic type functions. The methods established in [**250**] enable one to formulate a great number of composition principles for (ω, c)-almost periodic type functions. We will explain this fact only in the case of consideration of [**250**, Theorem 2.9] for Stepanov uniformly recurrent functions. So, let us assume that the function $F : I \times Y \to X$ is continuous and the function $f_{\omega,c}(\cdot)$ is Stepanov p-uniformly recurrent, i.e., the function $f(\cdot)$ is Stepanov (p, ω, c)-almost periodic ($p > 1$, $\omega > 0$, $c \in \mathbb{C} \smallsetminus \{0\}$). Define the function $G : I \times Y \to X$ by

$$G(t, y) := c_1^{-\frac{t}{\omega_1}} F\big(t, c^{t/\omega} y\big), \quad t \in I,\ y \in Y,$$

where $c_1 \in \mathbb{C} \smallsetminus \{0\}$ and $\omega_1 > 0$. If the requirements of the above-mentioned theorem hold with the functions $f(\cdot)$ and $F(\cdot,\cdot)$ replaced respectively with the functions $f_{\omega,c}(\cdot)$ and $G(\cdot,\cdot)$, then the resulting function

$$t \mapsto G\big(t, f_{\omega,c}(t)\big) = c_1^{-t_1/\omega_1} F(t, f(t)), \quad t \in I$$

is Stepanov q-uniformly recurrent so that the function $t \mapsto F(t, f(t))$, $t \in I$ is Stepanov (q, ω_1, c_1)-uniformly recurrent. More precisely, we have:

THEOREM 2.8.25. *Let $I = \mathbb{R}$ or $I = [0, \infty)$, and let $p \in \mathcal{P}([0, 1])$. Suppose that the following conditions hold:*

(i) *The function* $G : I \times Y \to X$ *is Stepanov* $p(x)$*-uniformly recurrent and there exist a function* $r(x) \geqslant \max(p(x), p(x)/(p(x)-1))$ *and a function* $L_G \in L_S^{r(x)}(I)$ *such that* (25) *holds with the functions* $F(\cdot,\cdot)$ *and* $L_F(\cdot)$ *replaced therein with the functions* $G(\cdot,\cdot)$ *and* $L_G(\cdot)$*, respectively.*

(ii) *The function* $f_{\omega,c} : I \to Y$ *is Stepanov* $p(x)$*-uniformly recurrent and there exists a set* $\mathrm{E} \subseteq I$ *with* $m(\mathrm{E}) = 0$ *such that* $K := \{f_{\omega,c}(t) : t \in I \setminus \mathrm{E}\}$ *is relatively compact in* Y*.*

(iii) *For every compact set* $K \subseteq Y$*, there exists a strictly increasing sequence* (α_n) *of positive real numbers tending to plus infinity such that* (61) *holds with the function* $F(\cdot,\cdot)$ *replaced therein with the function* $G(\cdot,\cdot)$*, and* (19) *holds with the function* $f_{\omega,c}(\cdot)$ *and the norm* $\|\cdot\|$ *replaced respectively by the function* $\hat{f_{\omega,c}}(\cdot)$ *and the norm* $\|\cdot\|_{L^{p(x)}([0,1]:Y)}$ *therein.*

Set $q(x) := p(x)r(x)/(p(x)+r(x)) \in [1, p(x))$ *provided* $x \in [0,1]$ *and* $r(x) < +\infty$ *and* $q(x) := p(x)$ *for* $x \in [0,1]$ *and* $r(x) = +\infty$*. Then* $F(\cdot, f(\cdot))$ *is Stepanov* $(q(x), \omega_1, c_1)$*-uniformly recurrent.*

In the remainder of this subsection, we will state and prove some composition principles for (ω, c)-uniformly recurrent functions of type 2; see also Corollary 2.8.18 and Proposition 2.8.21 (we can simply reformulate these results for (ω, c)-almost periodic functions of type 2). Hence, in the continuation of this subsection, we will assume that $|c| \leqslant 1$, $I = [0, \infty)$ and $i = 2$.

Suppose that $F : I \times Y \to X$ is a continuous function and there exists a finite constant $L > 0$ such that (60) holds. Define $\mathcal{F}(t) := F(t, f(t))$, $t \in I$. We will use the following estimate ($\tau \geqslant 0$, $\omega > 0$, $c \in \mathbb{C} \setminus \{0\}$, $t \in I$):

$$\begin{aligned}
&\Big\|c^{(-\tau)/\omega}F(t+\tau, f(t+\tau)) - F(t, f(t))\Big\| \\
&\leqslant \Big\|c^{(-\tau)/\omega}F(t+\tau, f(t+\tau)) - F\Big(t, c^{(-\tau)/\omega}f(t+\tau)\Big)\Big\| \\
&+ \Big\|F\Big(t, c^{(-\tau)/\omega}f(t+\tau)\Big) - F(t, f(t))\Big\| \\
&\leqslant \Big\|c^{(-\tau)/\omega}F(t+\tau, f(t+\tau)) - F\Big(t, c^{(-\tau)/\omega}f(t+\tau)\Big)\Big\| + L\Big\|c^{(-\tau)/\omega}f(t+\tau) - f(t)\Big\|.
\end{aligned} \tag{141}$$

REMARK 2.8.26. Albeit we will not employ this estimate henceforth, it should be noticed that we also have

$$\begin{aligned}
&\Big\|F(t+\tau, f(t+\tau)) - c^{\tau/\omega}F(t, f(t))\Big\| \\
&\leqslant \Big\|F(t+\tau, f(t+\tau)) - F\Big(t+\tau, c^{\tau/\omega}f(t)\Big)\Big\| + \Big\|F\Big(t+\tau, c^{\tau/\omega}f(t)\Big) - c^{\tau/\omega}F(t, f(t))\Big\| \\
&\leqslant L\Big\|f(t+\tau) - c^{\tau/\omega}f(t)\Big\| + \Big\|F\Big(t+\tau, c^{\tau/\omega}f(t)\Big) - c^{\tau/\omega}F(t, f(t))\Big\|.
\end{aligned}$$

Using the proof of [**234**, Theorem 3.29] and (141), we can simply deduce the following result:

THEOREM 2.8.27. *Suppose that* $F : I \times Y \to X$ *is a continuous function and there exists a finite constant* $L > 0$ *such that* (60) *holds.*

(i) *Suppose that $f : I \to Y$ is (ω, c)-uniformly recurrent of type 2. If there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\in I} \Big\| c^{\frac{-\alpha_n}{\omega}} f(t+\alpha_n) - f(t) \Big\| = 0$$

and

$$\lim_{n\to+\infty} \sup_{t\in I} \Big\| c^{(-\alpha_n)/\omega} F\big(t+\alpha_n, f(t+\alpha_n)\big) - F\Big(t, c^{(-\alpha_n)/\omega} f(t+\alpha_n)\Big) \Big\| = 0,$$

then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is (ω, c)-uniformly recurrent of type 2.

(ii) *Suppose that $f : I \to Y$ is (ω, c)-almost periodic of type 2. If for each $\varepsilon > 0$ the set of all positive real numbers $\tau > 0$ such that*

$$\sup_{t\in I} \Big\| c^{\frac{-\tau}{\omega}} f(t+\tau) - f(t) \Big\| < \varepsilon,$$

and

$$\sup_{t\in I} \Big\| c^{(-\tau)/\omega} F(t+\tau, f(t+\tau)) - F\Big(t, c^{(-\tau)/\omega} f(t+\tau)\Big) \Big\| < \varepsilon$$

is relatively dense in $[0, \infty)$, then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is (ω, c)-almost periodic of type 2.

We can similarly reformulate the statements of [**234**, Theorem 3.30, Theorem 3.31] in our context (cf. also [**17**, Theorem 2.11] and [**170**, Theorem 2.11]).

Now we will provide two results for Stepanov classes of (ω, c)-uniformly recurrent functions of type 2. We will first state the following:

THEOREM 2.8.28. *Let $I = [0, \infty)$, $|c| \leqslant 1$, $\omega > 0$, $p(x)$, $q(x) \in [1, \infty)$, $r(x) \in [1, \infty]$, $1/p(x) = 1/q(x) + 1/r(x)$ and the following conditions hold:*

(i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent and there exists a function $L_F \in L_S^{r(x)}(I)$ such that (25) holds.*

(ii) *There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

(142)
$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \Bigg\| \sup_{u\in R(f)} \Big\| c^{-\alpha_n/\omega} F\big(s+t+\alpha_n, u\big) - F\big(s+t, c^{\alpha_n/\omega} u\big) \Big\| \Bigg\|_{L^{p(s)}[0,1]} = 0$$

and

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \Big\| c^{\frac{-\alpha_n}{\omega}} f(s+t+\alpha_n) - f(s+t) \Big\|_{L^{p(s)}[0,1]} = 0.$$

Then the function $F(\cdot, f(\cdot))$ is Stepanov $(p(x), \omega, c)$-uniformly recurrent of type 2. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $p(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $p(x)$-bounded, as well.

PROOF. We will only provide the main details of proof since it is very similar to the proof of [**276**, Theorem 2.2]. Using the arguments contained for proving the estimate (141), we get that ($t \geqslant 0$, $n \in \mathbb{N}$):

$$\begin{aligned}
&\Big\| c^{-\alpha_n/\omega} F\big(t+\alpha_n, f(t+\alpha_n)\big) - F(t, f(t))\Big\| \\
&\leqslant \Big\| c^{(-\alpha_n)/\omega} F(t+\alpha_n, f(t+\alpha_n)) - F\Big(t, c^{(-\alpha_n)/\omega} f(t+\alpha_n)\Big)\Big\| \\
&+ L_F(t)\Big\| c^{(-\alpha_n)/\omega} f(t+\alpha_n) - f(t)\Big\|. \qquad (143)
\end{aligned}$$

Keeping in mind (143), we can repeat almost verbatim the arguments given in the proof of [**276**, Theorem 2.2] so as to conclude that there exists a finite constant $c_p > 0$ such that ($n \in \mathbb{N}$):

$$\begin{aligned}
&\sup_{t\geqslant 0}\Big\| c^{-\alpha_n/\omega} F\big(s+t+\alpha_n, f(s+t+\alpha_n)\big) - F(s, f(s))\Big\|_{L^{p(s)}[0,1]} \\
&\leqslant c_p \Big\| L_F(\cdot)\Big\|^p_{S^{r(x)}} \cdot \sup_{t\geqslant 0}\Bigg\| \Big\| c^{-\alpha_n/\omega} f\big(s+t+\alpha_n\big) - f(s+t)\Big\|^q \, ds\Bigg\|^{p/q}_{L^{p(s)}[0,1]} \\
&+ c_p \sup_{t\geqslant 0}\Bigg\| \sup_{u\in R(f)}\Big\| c^{-\alpha_n/\omega} F\big(s+t+\alpha_n, u\big) - F(s+t, u)\Big\|\Bigg\|_{L^{p(s)}[0,1]}.
\end{aligned}$$

By (179), this yields that the function $F(\cdot, f(\cdot))$ is Stepanov $(p(x), \omega, c)$-uniformly recurrent of type 2. If the function $F(\cdot, 0)$ is Stepanov $p(x)$-bounded, then the arguments given on [**276**, p. 6, l. -1-l. -5] enable one to see that the function $F(\cdot, f(\cdot))$ is Stepanov $p(x)$-bounded, as claimed. $\square$

We can simply formulate a consequence of this result with the usual Lipshitzian condition used. Similarly, we can prove the following result:

THEOREM 2.8.29. *Let $I = [0, \infty)$, $|c| \leqslant 1$, $\omega > 0$, $p \in \mathcal{P}([0, 1])$, and the following conditions hold:*

- (i) *The function $F : I \times Y \to X$ is Stepanov $p(x)$-uniformly recurrent and there exist a function $r(x) \geqslant \max(p(x), p(x)/(p(x)-1))$ and a function $L_F \in L_S^{r(x)}(I)$ such that (25) holds.*
- (ii) *There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n\to+\infty}\sup_{t\geqslant 0}\Bigg\| \sup_{u\in R(f)}\Big\| c^{-\alpha_n/\omega} F\big(s+t+\alpha_n, u\big) - F(s+t, c^{\alpha_n/\omega} u)\Big\|\Bigg\|_{L^{p(s)}[0,1]} = 0$$

and

$$\lim_{n\to+\infty}\sup_{t\geqslant 0}\Big\| c^{\frac{-\alpha_n}{\omega}} f(s+t+\alpha_n) - f(s+t)\Big\|_{L^{p(s)}[0,1]} = 0.$$

Then $q(x) := p(x)r(x)/(p(x)+r(x))$ for $x \in [0,1]$ and $r(x) < +\infty$ and $q(x) := p(x)$ for $x \in [0,1]$ and $r(x) = +\infty$. Then the function $F(\cdot, f(\cdot))$ is Stepanov $(q(x), \omega, c)$-uniformly recurrent of type 2. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov $q(x)$-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov $q(x)$-bounded, as well.

REMARK 2.8.30. Concerning Theorem 2.8.28 and Theorem 2.8.29, it should be noticed that we do not require that there exists a set $\mathrm{E} \subseteq I$ with $m(\mathrm{E}) = 0$ such that the set $K := \{f(t) : t \in I \setminus \mathrm{E}\}$ is relatively compact. For Stepanov (p, ω, c)-uniformly recurrent functions of type 2, we cannot assume, in (179), a slightly weaker condition (see [**276**, Lemma 2.1]):

$$\lim_{n\to+\infty} \sup_{t\geqslant 0} \sup_{u\in R(f)} \Bigl\| c^{-\alpha_n/\omega} F\bigl(s+t+\alpha_n, u\bigr) - F(s+t, c^{\alpha_n/\omega}u) \Bigr\|_{L^{p(s)}[0,1]} = 0.$$

2.8.3. (ω, c)-Almost periodic properties of convolution products and applications to integro-differential equations. In the first part of this subsection, we will examine the invariance of (ω, c)-almost periodic properties of the infinite convolution product (55), where a strongly continuous operator family $(R(t))_{t>0} \subseteq L(X, Y)$ satisfies certain assumptions. As already mentioned, the consideration is simple for the (ω, c)-uniformly recurrent functions, (ω, c)-almost periodic functions and (compactly) (ω, c)-almost automorphic functions because we then need to examine when the function $t \mapsto c^{-(t/\omega)}F(t)$, $t \in \mathbb{R}$ is uniformly recurrent, almost periodic or (compactly) almost automorphic, respectively. But, we have

$$c^{-\frac{t}{\omega}} F(t) = \int_{-\infty}^{t} \Bigl[c^{-\frac{t-s}{\omega}} R(t-s)\Bigr] \Bigl[c^{-\frac{s}{\omega}} f(s)\Bigr]\, ds, \quad t \in \mathbb{R},$$

so that the statements of [**248**, Proposition 3.1, 3.2] (uniform recurrence), [**234**, Proposition 2.6.11] (almost periodicity) and [**234**, Proposition 3.5.3] (almost automorphicity), for instance, can be simply reformulated in our context by replacing respectively the operator family $(R(t))_{t>0}$ and the function $f(\cdot)$ by the operator family $(c^{-\frac{t}{\omega}} R(t))_{t>0}$ and the function $c^{-\frac{\cdot}{\omega}} f(\cdot)$. We will do this only in the case of the last mentioned result (see [**143**] for the notion):

PROPOSITION 2.8.31. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying that*

$$M := \sum_{k=0}^{\infty} \Bigl\| c^{-\frac{\cdot+k}{\omega}} R(\cdot + k) \Bigr\|_{L^{q(x)}[0,1]} < \infty.$$

If $c^{-\frac{\check{\cdot}}{\omega}} f : \mathbb{R} \to X$ is $S^{p(x)}$-almost automorphic, then the function $F : \mathbb{R} \to Y$, given by (55), is well defined and (ω, c)-almost automorphic.

It is worth noting that this result can be applied in both cases $|c| > 1$ and $|c| < 1$, under suitable conditions. It is straightforward to incorporate the above results in the study of the existence and uniqueness of (ω, c)-almost periodic type solutions for the various classes of abstract inhomogeneous integro-differential equations. Keeping in mind Theorem 2.8.12, we will skip all related details concerning the invariance of (ω, c)-uniform recurrence of type 1 (type 2) (ω, c)-almost periodicity of type 1 (type 2) under the actions of infinite convolution products.

Due to the fact that

$$c^{-\frac{t}{\omega}} \int_0^t R(t-s)f(s)\,ds = \int_0^t \Big[c^{-\frac{t-s}{\omega}} R(t-s)\Big]\Big[c^{-\frac{s}{\omega}} f(s)\Big]\,ds, \quad t \geqslant 0, \tag{144}$$

we can similarly analyze the invariance of asymptotical (ω, c)-uniform recurrence, asymptotical (ω, c)-almost periodicity and asymptotical (compact) (ω, c)-almost automorphicity under the actions of finite convolution products; the interested reader may try to reformulate the statements of [**234**, Proposition 2.6.13, Theorem 2.9.5, Theorem 2.9.7, Theorem 2.9.15] in our new context.

If $|c| < 1$ and $\omega > 0$, then it is worth noting that the (ω, c)-uniform recurrence of type 2 and the (ω, c)-almost periodicity of type 2 cannot be so simply retained after the actions of finite convolution products. The situation is much simpler for the classes $AP_{\omega,c,1}([0,\infty) : X)$ and $UR_{\omega,c,1}([0,\infty) : X)$ ($S^{p(x)}AP_{\omega,c,1}([0,\infty) : X)$ and $S^{p(x)}UR_{\omega,c,1}([0,\infty) : X)$) because in this case we can apply Corollary 2.8.18, Proposition 2.8.21 and (144).

In the remainder of this subsection, we will provide a few applications to the abstract integro-differential equations and inclusions in Banach spaces.

1. In the concrete situation of [**359**, Example 4], we know that the unique solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial condition $u(x,0) = f(x)$, is given by

$$u(x,t) := \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-s)^2}{4t}} f(s)\,ds, \quad x \in \mathbb{R},\ t \geqslant 0. \tag{145}$$

Let the number $t_0 > 0$ be fixed, let $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$ and let the function $c^{-\cdot/\omega} f(\cdot)$ be bounded uniformly recurrent (almost periodic, (compactly) almost automorphic). Since $c^{-\frac{\cdot}{\omega}} e^{-\cdot^2/4t_0} \in L^1(\mathbb{R})$, we can apply Proposition 2.8.6 in order to see that the solution $x \mapsto u(x,t_0)$, $x \in \mathbb{R}$ is (ω, c)-uniformly recurrent ((ω, c)-almost periodic/(compactly) (ω, c)-almost automorphic). See also [**17**, Example 2.9].

2. It is worth noting that the notion from Definition 2.8.11 and Definition 2.8.24 can be introduced with the intervals $I = [-a, \infty)$, where $a > 0$ is an arbitrary real number. To explain the importance of this observation, we will reexamine [**359**, Example 5]. It is well known that the unique regular solution of the wave equation $u_{xx}(x,t) = u_{tt}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial conditions $u(x,0) = f(x)$, $x \in \mathbb{R}$, $u_t(x,0) = g(x)$, $x \in \mathbb{R}$, is given by the d'Alembert formula

$$u(x,t) := \frac{1}{2}\big[f(x+t) + f(x-t)\big] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds, \quad x \in \mathbb{R},\ t \geqslant 0.$$

Here we would like to note the following fact about the term

$$H_{t_0}(x) := \frac{1}{2}\int_{x-t_0}^{x+t_0} g(s)\,ds, \quad x \in \mathbb{R},$$

where $t_0 > 0$ is a fixed real number. Suppose that the function $g : [-t_0, \infty) \to \mathbb{C}$ is (ω, c)-uniformly recurrent of type 2, for example (the same comment applies to all other classes of functions introduced in Definition 2.8.11). Then there exists a strictly increasing sequence (α_n) of positive real numbers such that

$$\lim_{n\to+\infty} \sup_{t\geqslant -t_0} \bigl|c^{-\alpha_n/\omega} g(t+\alpha_n) - g(t)\bigr| = 0.$$

If $\varepsilon > 0$ is given, this implies the existence of an integer $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$,

$$\Bigl|c^{-\alpha_n/\omega} H_{t_0}(x+\alpha_n) - H_{t_0}(x)\Bigr| \leqslant \int_{-t_0}^{t_0} \Bigl|c^{-\alpha_n/\omega} g(x+s+\alpha_n) - g(x+s)\Bigr|\, ds \leqslant 2t_0\varepsilon, \quad x \geqslant 0.$$

Hence, the function $H_{t_0} : [0, \infty) \to \mathbb{C}$ is (ω, c)-uniformly recurrent of type 2.

It would be very enticing to provide certain applications of composition principles established in Subsection 2.8.2 in the qualitative analysis of solutions to the abstract semilinear Cauchy inclusions which belongs to the classes $AP_{\omega,c,2}([0, \infty))$ and $UR_{\omega,c,2}([0, \infty))$.

The case $|c| \neq 1$ is still unexplored in the theory of almost periodic functions and we feel it is our duty to say that the classes of (ω, c)-almost periodic type functions with $|c| \neq 1$ have some very unusual and unpleasant features.

2.8.4. (ω, c)-Pseudo almost periodic functions, (ω, c)-pseudo almost automorphic functions and applications. In this subsection, we deal with the interval $I = \mathbb{R}$, only. Let us recall the (ω, c)-mean of a function $h : \mathbb{R} \to X$ is introduced in [**16**] by

$$\mathcal{M}_{\omega,c}(h) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c^{-\sigma/\omega} h(\sigma)\, d\sigma,$$

whenever the limit exists. For example, for $h_1(t) = c^{t/\omega}$ and $h_2(t) = c^{t/\omega} e^{it}$, we have that $\mathcal{M}_{\omega,c}(h_1) = 1$ and $\mathcal{M}_{\omega,c}(h_2) = 0$. Furthermore, $\mathcal{M}_{\omega,c}$ is a linear and continuous operator. Indeed, if $c^{-t/\omega} h_n(t) \to c^{-t/\omega} h(t)$ uniformly as $n \to \infty$, then $\mathcal{M}_{\omega,c}(h_n) \to \mathcal{M}_{\omega,c}(h)$ as $n \to \infty$.

REMARK 2.8.32. If $h(\cdot)$ is (ω, c)-almost periodic, then the mean $\mathcal{M}_{\omega,c}(h)$ always exists, because the function $c^{-(\cdot/\omega)} f(\cdot)$ is almost periodic and the usual mean value of any almost periodic function exists.

We will use the space

$$PAP_{0;\omega,c}(\mathbb{R} : X) := \bigl\{h \in C(\mathbb{R} : X)\ ;\ c^{-\cdot/\omega} h(\cdot) \in PAP_0(\mathbb{R} : X)\bigr\}.$$

A function $h(\cdot)$ is said to be c-ergodic if and only if belongs to this space.

Furthermore, we will use the following two types of (ω, c)-pseudo ergodic components:

DEFINITION 2.8.33. Let $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$.

(i) A function $f \in C(\mathbb{R} \times Y : X)$ is said to be $(\omega, c, 1)$-pseudo ergodic vanishing if and only if $c^{-t/\omega} f(t, \cdot) \in PAP_0(\mathbb{R} \times Y : X)$. The space of all such functions will be denoted by $PAP_{0;\omega,c,1}(\mathbb{R} \times Y : X)$.

(ii) A function $f \in C(\mathbb{R} \times Y : X)$ is said to be $(\omega, c, 2)$-pseudo ergodic vanishing if and only if $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in PAP_0(\mathbb{R} \times Y : X)$. The space of all such functions will be denoted by $PAP_{0;\omega,c,2}(\mathbb{R} \times Y : X)$.

Similarly, we will use two different types of (ω, c)-almost periodic functions, resp. (ω, c)-almost automorphic functions, depending on two variables. Now we would like to introduce the following definitions:

DEFINITION 2.8.34. Let $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$ and $i = 1, 2$.

(i) A function $f \in C(\mathbb{R} \times Y : X)$ is said to be $(\omega, c, 1)$-almost periodic, resp. $(\omega, c, 1)$-almost automorphic, if and only if $c^{-t/\omega} f(t, \cdot) \in AP(\mathbb{R} \times Y : X)$, resp. $c^{-t/\omega} f(t, \cdot) \in AA(\mathbb{R} \times Y : X)$. The space of all such functions will be denoted by $AP_{\omega,c,1}(\mathbb{R} \times Y : X)$, resp. $AA_{\omega,c,1}(\mathbb{R} \times Y : X)$.

(ii) A function $f \in C(\mathbb{R} \times Y : X)$ is said to be $(\omega, c, 2)$-almost periodic, resp. $(\omega, c, 2)$-almost automorphic, if and only if $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in AP(\mathbb{R} \times Y : X)$, resp. $c^{-t/\omega} f(t, c^{t/\omega} \cdot) \in AA(\mathbb{R} \times Y : X)$. The space of all such functions will be denoted by $AP_{\omega,c,2}(\mathbb{R} \times Y : X)$, resp. $AA_{\omega,c,2}(\mathbb{R} \times Y : X)$.

DEFINITION 2.8.35. Let $c \in \mathbb{C} \setminus \{0\}$, $\omega > 0$ and $i = 1, 2$.

(i) A function $f \in C(\mathbb{R} : X)$ is said to be (ω, c)-pseudo almost periodic, resp. (ω, c)-pseudo almost automorphic, if and only if it admits a decomposition $f(t) = g(t) + h(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is (ω, c)-almost periodic, resp. (ω, c)-almost automorphic, and $h \in PAP_{0;\omega,c}(\mathbb{R} : X)$. The space of all such functions will be denoted by $PAP_{\omega,c}(\mathbb{R} : X)$, resp. $PAA_{\omega,c}(\mathbb{R} : X)$.

(ii) A function $f(\cdot, \cdot) \in C(\mathbb{R} \times Y : X)$ is said to be (ω, c, i)-pseudo almost periodic, resp. (ω, c, i)-pseudo almost automorphic, if and only if it admits a decomposition $f(t, x) = g(t, x) + h(t, x)$, $t \in \mathbb{R}$, $x \in X$, where $g(\cdot, \cdot)$ is (ω, c, i)-almost periodic, resp. (ω, c, i)-almost automorphic, and $h(\cdot, \cdot) \in PAP_{0;\omega,i}(\mathbb{R} \times Y : X)$. The space of all such functions will be denoted by $PAP_{\omega,c,i}(\mathbb{R} \times Y : X)$, resp. $PAA_{\omega,c,i}(\mathbb{R} \times Y : X)$.

THEOREM 2.8.36. *Let $f \in C(\mathbb{R} : X)$. Then $f(\cdot)$ is (ω, c)-pseudo almost periodic, resp. (ω, c)-pseudo almost automorphic, if and only if:*

$$f(t) \equiv c^{\wedge}(t) u(t), \quad \text{with } c^{\wedge}(t) \equiv c^{t/\omega}, \ u \in PAP(\mathbb{R} : X), \tag{146}$$

resp.

$$f(t) \equiv c^{\wedge}(t) u(t), \quad \text{with } c^{\wedge}(t) \equiv c^{t/\omega}, \ u \in PAA(\mathbb{R} : X).$$

PROOF. We will consider only (ω, c)-pseudo almost periodic functions for simplicity. It is clear that if $f(\cdot)$ satisfies (146), then $f(\cdot)$ is an (ω, c)-pseudo almost periodic function. In order to show the converse statement, let $f \in PAP_{\omega,c}(\mathbb{R} : X)$. Then there exist $g \in AP_{\omega,c}(\mathbb{R} : X)$ and $PAP_{0;\omega,c}(\mathbb{R} : X)$ such that $f = g + h$. Therefore,

$$u(t) = c^{-t/\omega} g(t) + c^{-t/\omega} h(t) = F_1(t) + F_2(t), \quad t \in \mathbb{R}.$$

So, $u(t)$ is written as a sum of $F_1(\cdot)$ which is almost periodic and $F_2(\cdot)$ which belongs to $PAP_{0;\omega,c}(\mathbb{R} : X)$. $\square$

REMARK 2.8.37. Let us note that the decompositions given in Definition 2.4 are unique; see also [**16**, Remark 2.9]. The proof of this fact can be left to the interested readers.

It can be simply shown that:

(i) We have $f + g \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f + g \in PAA_{\omega,c}(\mathbb{R} : X)$, and $\alpha h \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $\alpha h \in PAA_{\omega,c}(\mathbb{R} : X)$, provided $f,\ g,\ h \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f,\ g,\ h \in PAA_{\omega,c}(\mathbb{R} : X)$, and $\alpha \in \mathbb{C}$.
(ii) If $\tau \in \mathbb{R}$ and $f \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f \in PAA_{\omega,c}(\mathbb{R} : X)$, then $f_\tau(\cdot) \equiv f(\cdot + \tau) \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f_\tau(\cdot) \in PAA_{\omega,c}(\mathbb{R} : X)$.

Now we would like to endow the introduced space of (ω, c)-pseudo almost periodic functions, resp. (ω, c)-pseudo almost automorphic functions, with a certain norm.

PROPOSITION 2.8.38. *The space $PAP_{\omega,c}(\mathbb{R} : X)$, resp. $PAA_{\omega,c}(\mathbb{R} : X)$, equipped with the norm $\|\cdot\|_{\omega,c}$ is a Banach space.*

PROOF. We will consider the space $PAP_{\omega,c}(\mathbb{R} : X)$, only. Let (f_n) be a Cauchy sequence in $PAP_{\omega,c}(\mathbb{R} : X)$. Then, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m,\ n \geqslant N$, we have

$$\|f_n - f_m\|_{\omega,c} < \varepsilon.$$

Since $f_m,\ f_n \in PAP_{\omega,c}(\mathbb{R} : X)$, there exist $u_m,\ u_n \in PAP(\mathbb{R} : X)$ such that $f_m(t) \equiv c^\wedge(t)u_m(t)$ and $f_n(t) \equiv c^\wedge(t)u_n(t)$ for all $t \in \mathbb{R}$. Now, for $m,\ n \geqslant N$ we have $\|u_m - u_n\|_\infty \leqslant \|f_n - f_m\|_{\omega,c} < \varepsilon$. It follows that (u_n) is a Cauchy sequence in $PAP(\mathbb{R} : X)$. Since $PAP(\mathbb{R} : X)$ is complete, there exists $u \in PAP(\mathbb{R} : X)$ such that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Let us define $f(t) := c^\wedge(t)u(t)$, $t \in \mathbb{R}$. We claim that $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Indeed, $\|f_n - f\|_{\omega,c} = \sup_{t\in\mathbb{R}} \|u_n(t) - u(t)\| \to 0 \quad (n \to \infty)$. Hence, $PAP_{\omega,c}(\mathbb{R} : X)$ is a Banach space with the norm $\|\cdot\|_{\omega,c}$. $\square$

LEMMA 2.8.39. *([**16**]) Assume that $k^\sim(\cdot) := c^\wedge(-\cdot)k(\cdot) \in L^1(\mathbb{R})$. Then $h \in PAP_{0;\omega,c}(\mathbb{R} : X)$ implies that $k * h \in PAP_{0;\omega,c}(\mathbb{R} : X)$.*

THEOREM 2.8.40. *Let $f \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $f \in PAA_{\omega,c}(\mathbb{R} : X)$, with $f(\cdot) = c^\wedge(\cdot)p(\cdot)$, $p \in PAP(\mathbb{R} : X)$, resp. $p \in PAA(\mathbb{R} : X)$. If for some $k(\cdot)$ we have that $k^\sim(\cdot) := c^\wedge(-\cdot)k(\cdot) \in L^1(\mathbb{R})$, then*

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t - s)f(s)\, ds = c^\wedge(t)\left(k^\sim * p\right)(t), \quad t \in \mathbb{R}.$$

In particular, $k * f \in PAP_{\omega,c}(\mathbb{R} : X)$, resp. $k * f \in PAA_{\omega,c}(\mathbb{R} : X)$.

PROOF. As before, we will consider the space $PAP_{\omega,c}(\mathbb{R} : X)$ only, because the proof is quite analogous for the space $PAA_{\omega,c}(\mathbb{R} : X)$. Since $p \in PAP(\mathbb{R} : X)$, we have that there exist $p_1 \in AP(\mathbb{R} : X)$ and $p_2 \in PAP_0(\mathbb{R} : X)$ such that $p = p_1 + p_2$. Then $f = f_1 + f_2$, where $f_1(\cdot) = c^\wedge(\cdot)p_1(\cdot) \in AP_{\omega,c}(\mathbb{R} : X)$ and

$f_2(\cdot) = c^{\wedge}(\cdot)p_1(\cdot) \in PAP_{0;\omega,c}(\mathbb{R} : X)$. For every $t \in \mathbb{R}$, we have

$$(k * f)(t) = \int_{-\infty}^{\infty} k(t-s)f(s)\,ds$$
$$= \int_{-\infty}^{\infty} k(t-s)f_1(s)\,ds + \int_{-\infty}^{\infty} k(t-s)f_2(s)\,ds$$
$$= (k * f_1)(t) + (k * f_2)(t) =: I_1(t) + I_2(t).$$

We have that $I_1 \in AP_{\omega,c}(\mathbb{R} : X)$. We have that $I_2 \in PAP_{0;\omega,c}(\mathbb{R} : X)$. Moreover, by definition of $f(\cdot)$, we have $(k*f)(\cdot) = c^{\wedge}(\cdot)\,(k^{\sim} * p)\,(\cdot)$ so that $k*f \in PAP_{\omega,c}(\mathbb{R} : X)$. $\square$

EXAMPLE 2.8.41. Let us consider the heat equation $u_t(x,t) = u_{xx}(x,t)$, $t > 0$, $x \in \mathbb{R}$, with the initial value condition $u(x,0) = f(x)$. Let $u(x,t)$ be a regular solution of this equation; see (145). Fix $t_0 > 0$ and assume that $f(\cdot)$ is an (ω,c)-pseudo almost periodic function. Then, by Theorem 2.2, the solution $u(x,t_0)$ is (ω,c)-pseudo almost periodic with respect to x.

To formulate related composition principles, we will use two lemmae:

LEMMA 2.8.42. *(see [**234**, Lemma 2.12.2]) Let $f \in PAP(\mathbb{R} \times Y : X)$ and $u \in PAP(\mathbb{R} : Y)$. Then the mapping $t \mapsto f(t,u(t))$, $t \in \mathbb{R}$ belongs to the space $PAP(\mathbb{R} : X)$ provided that the following conditions hold:*

(i) *The set $\{f(t,y) : t \in \mathbb{R},\ y \in B\}$ is bounded for every bounded subset $B \subseteq Y$.*
(ii) *$f(t,y)$ is uniformly continuous in each bounded subset of Y uniformly in $t \in \mathbb{R}$. That is, for any $\varepsilon > 0$ and $B \subseteq X$ bounded, there exists $\delta > 0$ such that $x,\ y \in B$ and $\|x-y\| \leqslant \delta$ imply $\|f(t,x) - f(t,y)\| \leqslant \varepsilon$ for all $t \in \mathbb{R}$.*

LEMMA 2.8.43. *(see [**234**, Theorem 3.2.4]) Suppose that $f = g + \phi \in PAA(\mathbb{R} \times Y : X)$ with $g \in AA(\mathbb{R} \times Y : X)$, $\phi \in PAP_0(\mathbb{R} \times Y : X)$ and the following holds:*

(i) *the mapping $(t,y) \mapsto g(t,y)$ is uniformly continuous in any bounded subset $B \subseteq Y$ uniformly for $t \in \mathbb{R}$;*
(ii) *the mapping $(t,y) \mapsto \phi(t,y)$ is uniformly continuous in any bounded subset $B \subseteq Y$ uniformly for $t \in \mathbb{R}$.*

Then for each $u \in PAA(\mathbb{R} : Y)$ one has $f(\cdot,u(\cdot)) \in PAA(\mathbb{R} : X)$.

For simplicity, we will not consider Stepanov p-almost periodic functions and Stepanov p-almost automorphic functions depending on two variables here.

Suppose now that a continuous function $g : \mathbb{R} \times Y \to X$ satisfies $g(t+\omega,y) = cg(t,y)$ for all $t \in \mathbb{R}$ and $y \in Y$, resp. $g(t+\omega,cy) = cg(t,y)$ for all $t \in \mathbb{R}$ and $y \in Y$. Define the functions

$$G_1(t,y) := c^{-\frac{t}{\omega}} g(t,y), \quad t \in \mathbb{R},\ y \in Y \tag{147}$$

and

$$G_2(t,y) := c^{-\frac{t}{\omega}} g\big(t, c^{t/\omega}y\big), \quad t \in \mathbb{R},\ y \in Y. \tag{148}$$

Then, for every $t \in \mathbb{R}$ and $y \in Y$, we have

$$G_1(t+\omega, y) = c^{-\frac{t+\omega}{\omega}} g(t+\omega, y) = c^{-\frac{t+\omega}{\omega}} cg(t+\omega, y) = c^{-\frac{t}{\omega}} g(t, y) = G_1(t, y)$$

and

$$G_2(t+\omega, y) = c^{-\frac{t+\omega}{\omega}} g\Big(t+\omega, c^{\frac{t+\omega}{\omega}} y\Big) = c^{-\frac{t+\omega}{\omega}} cg\big(t, c^{t/\omega} y\big)$$
$$= c^{-t/\omega} g\big(t, c^{t/\omega} y\big) = G_2(t, y).$$

In both cases, the function $G_i(\cdot, \cdot)$ is ω-periodic in time variable ($i = 1, 2$). Furthermore, if the requirements of [**16**, Theorem 2.24] hold (case $i = 2$), then condition (i) of Lemma 2.8.43 holds with the function $g(\cdot, \cdot)$ replaced therein with the function $G_2(\cdot, \cdot)$, and condition (ii) of Lemma 2.8.43 holds with the function $\phi(\cdot, \cdot)$ replaced therein with the function $h_2(t, \cdot) \equiv c^{-t/\omega} h(t, c^{t/\omega} \cdot)$, $t \in \mathbb{R}$. Furthermore, $G_2 \in AA(\mathbb{R} \times Y : X)$ and $h_2 \in PAP_0(\mathbb{R} \times Y : X)$ so that repeating verbatim the arguments used in the proof of [**270**, Theorem 2.4] with appealing to [**17**, Theorem 2.11] in place of [**270**, Lemma 2.2] immediately yields a much simpler proof of [**16**, Theorem 2.24]. Furthermore, the statement of [**17**, Theorem 2.11] can be formulated for continuous functions which maps the space $\mathbb{R} \times Y$ into X; in other words, we can use two different pivot spaces X and Y. Keeping in mind this observation, we can immediately clarify an extension of [**16**, Theorem 2.24] in this context (the interested reader may try to reexamine [**16**, Theorem 2.25] for (ω, c)-pseudo almost periodic functions and (ω, c)-pseudo almost automorphic functions). Furthermore, using Lemma 3.2 we can immediately clarify the following result:

PROPOSITION 2.8.44. (i) *Suppose that $f = g + \phi$ with $g \in AA_{\omega,c,1}(\mathbb{R} \times Y : X)$, $\phi \in PAP_{0;\omega,c,1}(\mathbb{R} \times Y : X)$ and the following holds:*

(a) *the mapping $(t, y) \mapsto G_1(t, y)$ given by (147) is uniformly continuous in any bounded subset $B \subseteq Y$ uniformly for $t \in \mathbb{R}$;*

(b) *the mapping $(t, y) \mapsto \phi_1(t, y)$ given by (147) with the function $g(\cdot, \cdot)$ replaced therein with the function $\phi(\cdot, \cdot)$, is uniformly continuous in any bounded subset $B \subseteq Y$ uniformly for $t \in \mathbb{R}$.*

Then for each $u \in PAA(\mathbb{R} : Y)$ one has $f(\cdot, u(\cdot)) \in PAA_{\omega,c}(\mathbb{R} : X)$.

(ii) *Suppose that $f = g + \phi$ with $g \in AA_{\omega,c,2}(\mathbb{R} \times Y : X)$, $\phi \in PAP_{0;\omega,c,2}(\mathbb{R} \times Y : X)$ and the following holds:*

(c) *the mapping $(t, y) \mapsto G_2(t, y)$ given by (147) is uniformly continuous in any bounded subset $B \subseteq Y$ uniformly for $t \in \mathbb{R}$;*

(d) *the mapping $(t, y) \mapsto \phi_2(t, y)$ given by (147), with the function $g(\cdot, \cdot)$ replaced therein with the function $\phi(\cdot, \cdot)$, is uniformly continuous in any bounded subset $B \subseteq Y$ uniformly for $t \in \mathbb{R}$.*

Then for each $u \in PAA_{\omega,c}(\mathbb{R} : Y)$ one has $f(\cdot, u(\cdot)) \in PAA_{\omega,c}(\mathbb{R} : X)$.

We can also clarify the following result:

PROPOSITION 2.8.45. (i) *Let $f \in PAP_{\omega,c,1}(\mathbb{R} \times Y : X)$ and $u \in PAP(\mathbb{R} : Y)$. Then the mapping $t \mapsto f(t, u(t))$, $t \in \mathbb{R}$ belongs to the space $PAP_{\omega,c}(\mathbb{R} : X)$ provided that the following conditions hold:*

(a) *The set $\{c^{-t/\omega} f(t, y) : t \in \mathbb{R},\ y \in B\}$ is bounded for every bounded subset $B \subseteq Y$.*

(b) $c^{-t/\omega} f(t,y)$ *is uniformly continuous in each bounded subset of* Y *uniformly in* $t \in \mathbb{R}$.

(ii) *Let* $f \in PAP_{\omega,c,2}(\mathbb{R} \times Y : X)$ *and* $u \in PAP_{\omega,c}(\mathbb{R} : Y)$. *Then the mapping* $t \mapsto f(t, u(t))$, $t \in \mathbb{R}$ *belongs to the space* $PAP_{\omega,c}(\mathbb{R} : X)$ *provided that the following conditions hold:*

(a) *The set* $\{c^{-t/\omega} f(t, c^{t/\omega} y) : t \in \mathbb{R}, \ y \in B\}$ *is bounded for every bounded subset* $B \subseteq Y$.

(b) $c^{-t/\omega} f(t, c^{t/\omega} y)$ *is uniformly continuous in each bounded subset of* Y *uniformly in* $t \in \mathbb{R}$.

Consider the semilinear fractional Cauchy inclusion

$$D^{\gamma}_{t,+} u(t) \in \mathcal{A} u(t) + f(t, u(t)), \ t \in \mathbb{R}, \tag{149}$$

where $D^{\gamma}_{t,+}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1]$, $f : \mathbb{R} \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator in X satisfying condition (P). Then there exists a finite constant $M_0 > 0$ such that the degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathcal{A}$ satisfies the estimate $\|T(t)\| \leqslant M_0 e^{-at} t^{\beta-1}$, $t > 0$. By a mild solution of problem (149), we mean any continuous function $t \mapsto u(t)$, $t \in \mathbb{R}$ satisfying

$$u(t) = \int_{-\infty}^{t} T(t-s) f(s, u(s)) \, ds, \quad t \in \mathbb{R}.$$

We will use the following auxiliary result:

LEMMA 2.8.46. *(see the proof of* [**234**, Lemma 2.12.3]*) Suppose that* $f : \mathbb{R} \to X$ *is pseudo-almost periodic (pseudo-almost automorphic) and* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $\|R(t)\| \leqslant M e^{-bt} t^{\beta-1}$, $t > 0$ *for some finite numbers* $M \geqslant 1$, $b > 0$ *and* $\beta \in (0,1]$. *Then the function* $F(t) := \int_{-\infty}^{t} R(t-s) f(s) \, ds$, $t \in \mathbb{R}$ *is well-defined and pseudo-almost periodic (pseudo-almost automorphic).*

Suppose now that

$$0 < M_0 / (a + (\ln |c| / \omega)) < 1 \tag{150}$$

and define the mapping

$$Pu : PAP_{\omega,c}(\mathbb{R} : X) \to PAP_{\omega,c}(\mathbb{R} : X), \text{ resp. } Pu : PAA_{\omega,c}(\mathbb{R} : X) \to PAA_{\omega,c}(\mathbb{R} : X),$$

by

$$(Pu)(t) := \int_{-\infty}^{t} T(t-s) f(s, u(s)) \, ds, \quad t \in \mathbb{R}.$$

Under certain assumptions, the mapping $f(\cdot, u(\cdot))$ belongs to the class $PAP_{\omega,c}(\mathbb{R} : X)$, resp. $PAA_{\omega,c}(\mathbb{R} : X)$. Using the decomposition

$$\int_{-\infty}^{t} T(t-s) f(s, u(s)) \, ds = \int_{-\infty}^{t} \Bigl[c^{-\frac{t-s}{\omega}} T(t-s) \Bigr] \Bigl[c^{-\frac{s}{\omega}} f(s, u(s)) \Bigr] \, ds, \quad t \in \mathbb{R},$$

the estimate (150) yields that the mapping $t \mapsto \int_{-\infty}^{t} T(t-s)f(s,u(s))\,ds,\ t \in \mathbb{R}$ belongs to the class $PAP_{\omega,c}(\mathbb{R}:X)$, resp. $PAA_{\omega,c}(\mathbb{R}:X)$. Hence, the mapping $P(\cdot)$ is well defined. Using a simple calculation, we get that:

$$\|Pu\|_{\omega,c} \leqslant \frac{M_0}{a+(\ln|c|/\omega)}\|Pu\|_{\omega,c}, \quad u \in PAP_{\omega,c}(\mathbb{R}:X) \ \ \big[u \in PAA_{\omega,c}(\mathbb{R}:X)\big].$$

Applying the Banach contraction principle, we get that the mapping $P(\cdot)$ has a unique fixed point, so that there exists a unique solution of the abstract semilinear Cauchy inclusion (149) which belongs to the class $PAP_{\omega,c}(\mathbb{R}:X)$, resp. $PAA_{\omega,c}(\mathbb{R}:X)$.

2.8.5. (ω, c)**-Almost periodic distributions.** Almost periodic distributions extending the classical Bohr and Stepanov almost periodic functions are introduced by L. Schwartz, see [**329**]. Asymptotical almost periodicity of Schwartz distributions was introduced by I. Cioransescu [**111**] (see also [**79**]-[**82**]).

This subsection introduces and investigates $(\omega, c)-$almost periodicity (resp. asymptotic $(w,c)-$almost periodicity) in the setting of Schwartz-Sobolev distributions. For simplicity, we will consider only scalar-valued distributions because the extensions to the vector-valued case are straightforward. For more details about (asymptotically) almost periodic distributions and ultradistributions, see [**234**], the papers by B. Stanković [**335**]-[**336**] and the list of references therein.

By $\mathcal{D} = C_0^{\infty}(\mathbb{R})$, $\mathcal{E} = C^{\infty}(\mathbb{R})$ and $\mathcal{S} = \mathcal{S}(\mathbb{R})$ we denote the Schwartz spaces of test functions, endowed with the usual topologies. If $\emptyset \neq \Omega \subseteq \mathbb{R}$, then by $\mathcal{D}_{\Omega}$ we denote the subspace of $\mathcal{D}$ consisting of those functions $\varphi \in \mathcal{D}$ for which $\operatorname{supp}(\varphi) \subseteq \Omega$; $\mathcal{D}_0 \equiv \mathcal{D}_{[0,\infty)}$ and $\mathcal{D}' := L(\mathcal{D}, \mathbb{C})$ stands for the space consisting of all scalar-valued distributions.

We will first introduce the space of smooth $(w,c)-$almost periodic functions and investigate some of their basic properties. We will use the following notations: (151)

$$\varphi_{w,c}(\cdot) = c^{-\frac{(\cdot)}{w}}\varphi(\cdot), \quad \varphi \in \mathcal{C}^{\infty} \text{ or } L^p, 1 \leqslant p \leqslant +\infty \text{ and } T_{w,c} = c^{-\frac{(\cdot)}{w}}T, \quad T \in \mathcal{D}',$$

where the equality is taken in the usual (resp. Lebesgue, distributional) sense.

To construct the $(w,c)-$smooth almost periodic functions, we need to introduce some new functional spaces. Let $p \in [1,+\infty]$ and $f(\cdot)$ a complex valued measurable function on $\mathbb{R}$.

We say that $f(\cdot)$ is a $(w,c)-$Lebesgue function with exponent p, if

$$\left(\int_{\mathbb{R}} |f_{w,c}(t)|^p\,dt\right)^{\frac{1}{p}} < \infty, \text{ for } 1 \leqslant p < +\infty,$$

and

$$\sup_{t\in\mathbb{R}} |f_{w,c}(t)| < \infty, \text{ for } p = +\infty.$$

We denote by $L^p_{w,c}$ the set of $(w,c)-$Lebesgue functions with exponent p, i.e.,

$$L^p_{w,c} := \{f : \mathbb{R} \longrightarrow \mathbb{C} \text{ measurable };\ f_{w,c} \in L^p\}.$$

When $c = 1$, $L^p_{w,c} := L^p$ is the classical Lebesgue space over $\mathbb{R}$.

PROPOSITION 2.8.47. *The space $L^p_{w,c}$ endowed with the (w,c) −norm*

$$\|f\|_{L^p_{w,c}} := \|f_{w,c}\|_{L^p}\,,\ \textit{for } 1 \leqslant p < +\infty,$$

and

$$\|f\|_{L^\infty_{w,c}} := \|f\|_{w,c}\,,\ \textit{for } p = +\infty,$$

is a Banach space.

PROPOSITION 2.8.48. *$\mathcal{D}$ is dense in $L^p_{w,c}; 1 \leqslant p < \infty$.*

PROOF. Since $\mathcal{D}$ is dense in the space $\mathcal{C}_c$ of continuous functions with compact support, it suffices to show that $\mathcal{C}_c$ is dense in $L^p_{w,c}$ for $1 \leqslant p < \infty$.

Let S be the set of all simple measurable functions s, with complex values, defined on $\mathbb{R}$ and such that

$$m\big(\{t : s\,(t) \neq 0\}\big) < \infty.$$

First, it is clear that S is dense in $L^p_{w,c}$ for $1 \leqslant p < \infty$. Indeed, as $c^{-\frac{t}{w}} s \in L^p$, then $S \subseteq L^p_{w,c}$. Suppose $f \in L^p_{w,c}$ is positive and define the sequence $(s_n)_n$ such that

$$0 \leqslant s_1 \leqslant s_2 \leqslant ... \leqslant f,\ \text{ and for each } t \in \mathbb{R},\ s_n\,(t) \longrightarrow f\,(t)\ \text{ when } n \longrightarrow +\infty.$$

Then $(f - s_n)_{w,c} = c^{-\frac{t}{w}}\,(f - s_n) \in L^p$, hence $s_n \in S$. Furthermore, since

$$\left|c^{-\frac{t}{w}}\,(f - s_n)\right|^p \leqslant f^p,$$

Lebesgue's dominated convergence theorem shows that

$$\left\|(f - s_n)_{w,c}\right\|_{L^p} = \left\|c^{-\frac{t}{w}}\,(f - s_n)\right\|_{L^p} \longrightarrow 0$$

when $n \longrightarrow +\infty$. Hence, $\|f - s_n\|_{L^p_{w,c}} \longrightarrow 0$ when $n \longrightarrow +\infty$. On the other hand, by Lusin's theorem, for $s \in S$ and $\varepsilon > 0$, there exists $g \in \mathcal{C}_c$ such that $g\,(t) = s\,(t)$, except on a set of measure less than ε, and $|g| \leqslant \|s\|_\infty$, and since s takes only a finite number of values, there exists a constant $C > 0$ which depends on c and w such that

$$\left\|(g - s)_{w,c}\right\|_{L^p} = \left(\int_{\mathbb{R}} \left|c^{-\frac{t}{w}}\,(g\,(t) - s\,(t))\right|^p dt\right)^{\frac{1}{p}} \leqslant 2C\varepsilon^{\frac{1}{p}}\,\|s\|_\infty\,.$$

The density of S in $L^p_{w,c}$ completes the proof. $\square$

We define

$$\mathcal{D}_{L^p_{w,c}} := \left\{\varphi \in \mathcal{C}^\infty : \varphi^{(j)}_{w,c} \in \mathcal{D}_{L^p},\ j \in \mathbb{N}\right\}.$$

When $c = 1$, we get $\mathcal{D}_{L^p_{w,c}} := \mathcal{D}_{L^p}$. Moreover, it is easy to show that the space $\mathcal{D}_{L^p_{w,c}}, 1 \leqslant p \leqslant \infty$, endowed with the topology defined by the following countable family of norms

$$|\varphi|_{k,p;w,c} := \sum_{j \leqslant k} \left\|(\varphi_{w,c})^{(j)}\right\|_{L^p},\ k \in \mathbb{N},$$

is a Fréchet subspace of $\mathcal{C}^\infty$.

PROPOSITION 2.8.49. *Let* $1 \leqslant p \leqslant \infty$. *If* $\varphi,\ \psi \in \mathcal{D}_{L^p_{2w,c}}$, *then* $\varphi\psi \in \mathcal{D}_{L^p_{w,c}}$.

PROOF. Let $\varphi,\ \psi \in \mathcal{D}_{L^p_{2w,c}}$, then $\varphi_{2w,c} \in \mathcal{D}_{L^p}$ and $\psi_{2w,c} \in \mathcal{D}_{L^p},\ j \in \mathbb{N}$. So $\varphi^{(j)}_{2w,c} \in L^p$ and $\psi^{(j)}_{2w,c} \in L^p$. By Leibniz's rule, we obtain
(152)

$$\left((\varphi\psi)_{w,c}\right)^{(j)} = \left(c^{-\frac{t}{2w}}\varphi c^{-\frac{t}{2w}}\psi\right)^{(j)} = (\varphi_{2w,c}\psi_{2w,c})^{(j)} = \sum_{i=1}^{j}\left(\begin{array}{c} i \\ j \end{array}\right)\varphi^{(i)}_{2w,c}\psi^{(j-i)}_{2w,c} \in L^p.$$

This shows that

$$(\varphi\psi)_{w,c} \in \mathcal{D}_{L^p}.$$

Hence, $\varphi\psi \in \mathcal{D}_{L^p_{w,c}}$. $\square$

The following result shows that the family of norms $|\cdot|_{k,p;w,c}$ is submultiplicative.

PROPOSITION 2.8.50. *Let* $1 \leqslant p \leqslant \infty$, *if* $\varphi,\ \psi \in \mathcal{D}_{L^p_{2w,c}}$, *then for all* $k \in \mathbb{N}$, *there exists* $C_k > 0$ *such that*

$$|\varphi\psi|_{k,p;w,c} \leqslant C_k\,|\varphi|_{k,p;2w,c}\,.\,|\psi|_{k,p;2w,c}\,.$$

PROOF. Let $\varphi,\ \psi \in \mathcal{D}_{L^p_{2w,c}}$. We have

$$\begin{aligned}
\sum_{j\leqslant k}\left\|\left((\varphi\psi)_{w,c}\right)^{(j)}\right\|_{L^p} &= \sum_{j\leqslant k}\left\|\sum_{i=1}^{j}\left(\begin{array}{c} i \\ j \end{array}\right)(\varphi_{2w,c})^{(i)}\,(\psi_{2w,c})^{(j-i)}\right\|_{L^p} \\
&\leqslant \sum_{j\leqslant k}\sum_{i=1}^{j}\left(\begin{array}{c} i \\ j \end{array}\right)\left\|(\varphi_{2w,c})^{(i)}\,(\psi_{2w,c})^{(j-i)}\right\|_{L^p} \\
&\leqslant \sum_{j\leqslant k}\sum_{i=1}^{j}\left(\begin{array}{c} i \\ j \end{array}\right)\left\|(\varphi_{2w,c})^{(i)}\right\|_{L^p}\sum_{j\leqslant k}\sum_{i=1}^{j}\left(\begin{array}{c} i \\ j \end{array}\right)\left\|(\psi_{2w,c})^{(j-i)}\right\|_{L^p}.
\end{aligned}$$

So, there exists

$$C_k = \left(\sum_{j\leqslant k}\sum_{i=1}^{j}\left(\begin{array}{c} i \\ j \end{array}\right)\right)^2 > 0$$

such that

$$|\varphi\psi|_{k,p;w,c} \leqslant C_k\,|\varphi|_{k,p;2w,c}\,.\,|\psi|_{k,p;2w,c}\,.$$

$\square$

For $1 \leqslant p < \infty$, we have $\mathcal{D} \subseteq \mathcal{D}_{L^p_{w,c}} \subseteq \mathcal{D}_{L^\infty_{w,c}}$. Moreover, we have the following result.

PROPOSITION 2.8.51. *For* $1 \leqslant p < \infty$, *the space* $\mathcal{D}$ *is dense in* $\mathcal{D}_{L^p_{w,c}}$.

PROOF. It follows from the fact that $\mathcal{D}_{L^p_{w,c}} \subseteq L^p_{w,c}$ and the density of $\mathcal{D}$ in $L^p_{w,c}$, see Proposition 2.8.48. $\square$

The space $\mathcal{D}$ is not dense in $\mathcal{D}_{L^\infty_{w,c}}$. We then define $\dot{\mathcal{D}}_{L^\infty_{w,c}}$ as the subspace of all functions in $\mathcal{D}_{L^\infty_{w,c}}$ which vanish at infinity with all their derivatives. This space is the closure of the space $\mathcal{D}_{L^\infty_{w,c}}$ in $\mathcal{D}$. It is clear that $\dot{\mathcal{D}}_{L^\infty_{w,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{w,c}}$, hence it is a Fréchet space. Moreover, it is easy to check the following properties on the structure of $\mathcal{D}_{L^p_{w,c}}$.

Proposition 2.8.52. *For* $1 \leqslant p < \infty$, *we have*

$$\mathcal{D}_{L^p_{w,c}} \hookrightarrow \dot{\mathcal{D}}_{L^\infty_{w,c}} \hookrightarrow \mathcal{D}_{L^\infty_{w,c}},$$

with continuous embedding.

Recall also the following space of smooth almost periodic functions introduced by L. Schwartz

$$\mathcal{B}_{ap} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in AP, \quad j \in \mathbb{N} \right\}.$$

We have the following properties of $\mathcal{B}_{ap}$.

Proposition 2.8.53. (i) $\mathcal{B}_{ap} = AP \cap \mathcal{D}_{L^\infty}$.
(ii) $\mathcal{B}_{ap}$ *is a closed differential subalgebra of* $\mathcal{D}_{L^\infty}$.
(iii) *If* $f \in L^1$ *and* $\varphi \in \mathcal{B}_{ap}$, *then* $f * \varphi \in \mathcal{B}_{ap}$.

Proof. See [**329**]. □

Now, we can introduce the space of smooth (w, c) −almost periodic functions.

Definition 2.8.54. The space of smooth (w, c) −almost periodic functions on $\mathbb{R}$, is defined by

$$\mathcal{B}_{AP_{w,c}} := \left\{ \varphi \in \mathcal{D}_{L^\infty_{w,c}} : \varphi^{(j)}_{w,c} \in \mathcal{B}_{ap}, \quad j \in \mathbb{N} \right\}.$$

We endow $\mathcal{B}_{AP_{w,c}}$ with the topology induced by $\mathcal{D}_{L^\infty_{w,c}}$. Some properties of $\mathcal{B}_{AP_{w,c}}$ are given in the following

Proposition 2.8.55. (i) $\mathcal{B}_{AP_{w,c}} = AP_{w,c} \cap \mathcal{D}_{L^\infty_{w,c}}$.
(ii) $\mathcal{B}_{AP_{w,c}}$ *is a closed subspace of* $\mathcal{D}_{L^\infty_{w,c}}$.
(iii) *If* $f \in L^1_{w,c}$ *and* $\varphi \in \mathcal{B}_{AP_{w,c}}$, *then* $c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c} \right) \in \mathcal{B}_{AP_{w,c}}$.

Proof. (i): Obvious.
(ii): It follows from (i) and the completeness of $\left(AP, \|\cdot\|_\infty \right)$.
(iii): If $f \in L^1_{w,c}$ and $\varphi \in \mathcal{B}_{AP_{w,c}}$, then $f_{w,c} \in L^1$ and $\varphi_{w,c} \in \mathcal{B}_{ap}$. From Proposition 2.8.53, we have $f_{w,c} * \varphi_{w,c} \in \mathcal{B}_{ap}$; hence

$$c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c} \right) \right) \in \mathcal{B}_{ap},$$

which shows that $c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c} \right) \in \mathcal{B}_{AP_{w,c}}$. □

Corollary 2.8.56. *If* $f \in \mathcal{D}_{L^\infty_{w,c}}$ *and* $c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c} \right) \in AP_{w,c}$, $\forall \varphi \in \mathcal{D}$, *then* $f \in \mathcal{B}_{AP_{w,c}}$.

REMARK 2.8.57. It is clear that $\mathcal{B}_{AP_{w,c}} \subseteq AP_{w,c} \cap \mathcal{C}^{\infty}$, whereas the converse inclusion is not true. Indeed, the function

$$f(t) = 2^{-t}\sqrt{2 + \cos t + \cos\sqrt{2}t}, \quad t \in \mathbb{R}$$

is an element of $AP_{w,c} \cap \mathcal{C}^{\infty}$ with $c = 2$ and $w = 1$. However

$$f'(t) = 2^{-t}\left(\frac{-\sin t - \sqrt{2}\sin\sqrt{2}t}{2\sqrt{2 + \cos t + \cos\sqrt{2}t}} - \ln 2\sqrt{2 + \cos t + \cos\sqrt{2}t}\right), \quad t \in \mathbb{R}$$

is not bounded, because $\inf_{t\in\mathbb{R}}\left(2 + \cos t + \cos\sqrt{2}t\right) = 0$ and therefore

$$\frac{-\sin t - \sqrt{2}\sin\sqrt{2}t}{2\sqrt{2 + \cos t + \cos\sqrt{2}t}} \notin AP.$$

Hence, $f \notin \mathcal{B}_{AP_{w,c}}$.

Now we would like to introduce the concept of (w, c) −almost periodicity in the setting of Sobolev-Schwartz distributions. For this we need to introduce the so-called space of $L^p_{w,c}$−distributions, $1 \leqslant p \leqslant \infty$. We first recall the space of L^p−distributions, $1 \leqslant p \leqslant \infty$, which has been introduced for the first time by L. Schwartz in [**329**]. L. Schwartz has introduced the space $\mathcal{D}'_{L^p}$ as topological dual of $\mathcal{D}_{L^q}$, $\frac{1}{p} + \frac{1}{q} = 1$. These spaces is related to Sobolev spaces; for more details, see [**42**] and [**329**].

DEFINITION 2.8.58. Let $1 < p \leqslant \infty$, the space $\mathcal{D}'_{L^p}$ is the topological dual of $\mathcal{D}_{L^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$. An element of $\mathcal{D}'_{L^\infty}$ is called a bounded distribution.

THEOREM 2.8.59. *Let $T \in \mathcal{D}'$. Then the following statements are equivalent:*

(i) $T \in \mathcal{D}'_{L^p}$.
(ii) $T * \varphi \in L^p$, $\varphi \in \mathcal{D}$.
(iii) $\exists k \in \mathbb{N}, \exists\left(f_j\right)_{0\leqslant j\leqslant k} \subseteq L^p : T = \sum_{j=0}^{k} f_j^{(j)}$.

PROOF. See [**42**] or [**329**]. □

Thanks to the density of the space $\mathcal{D}$ in $\mathcal{D}_{L^p_{w,c}}$, $1 \leqslant p < \infty$, (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$), we have that the space $\mathcal{D}_{L^p_{w,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$) is a normal space of distributions, i.e., the elements of topological dual of $\mathcal{D}_{L^p_{w,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$) can be identified with continuous linear forms on $\mathcal{D}$.

DEFINITION 2.8.60. For $1 < p \leqslant \infty$, we denote by $\mathcal{D}'_{L^p_{w,c}}$ the topological dual of $\mathcal{D}_{L^q_{w,c}}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

The following spaces of $L^p_{w,c}$−distributions are needed to define and study the (w, c)−almost periodicity of distributions.

DEFINITION 2.8.61. (i) The topological dual of $\mathcal{D}_{L^1_{w,c}}$, denoted by $\mathcal{B}'_{w,c}$, is called the space of (w, c) −bounded distributions.

(ii) The topological dual of $\dot{\mathcal{D}}_{L^\infty_{w,c}}$, denoted by $\mathcal{D}'_{L^1_{w,c}}$, is called the space of (w, c) −integrable distributions.

By applying Theorem 2.8.59, we can easily show the following characterizations of $L^p_{w,c}$−distributions:

THEOREM 2.8.62. *Let $T \in \mathcal{D}'$. Then the following statements are equivalent:*

(i) $T \in \mathcal{D}'_{L^p_{w,c}}$.

(ii) $c^{\frac{t}{w}} (T_{w,c} * \varphi) \in L^p_{w,c}$, $\varphi \in \mathcal{D}$.

(iii) $\exists k \in \mathbb{N}, \exists (f_j)_{0 \leqslant j \leqslant k} \subseteq L^p_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^{k} (f_{w,c})_j^{(j)}$, *where*

$$\left((f_{w,c})_j\right)_{0 \leqslant j \leqslant k} = \left(c^{-\frac{t}{w}} f_j\right)_{0 \leqslant j \leqslant k}.$$

REMARK 2.8.63. As a consequence of Theorem 2.8.62, we have that $T \in \mathcal{D}'_{L^p_{w,c}}$ if and only if $T_{w,c} \in \mathcal{D}'_{L^p}$.

Returning to the notation (151), we recall that a distribution $T \in \mathcal{D}'$ is zero on an open subset V of $\mathbb{R}$ if

$$\langle T, \varphi \rangle = 0, \quad \forall \varphi \in \mathcal{D}(V),$$

and that two distributions $T,\ S \in \mathcal{D}'$ coincide on V if $T - S = 0$ on V.

LEMMA 2.8.64. *Let $f \in \mathcal{C}^\infty$ and $T \in \mathcal{D}'$. If $fT = 0$, then $T = 0$ on the set* $G = \{x \in \mathbb{R} : f(x) \neq 0\}$.

PROOF. Let $\varphi \in \mathcal{D}$ with $\mathrm{supp}(\varphi) \subseteq G$. Then we have

$$\langle T, \varphi \rangle = \left\langle T, f\frac{\varphi}{f} \right\rangle = \left\langle fT, \frac{\varphi}{f} \right\rangle = 0,$$

because $\frac{\varphi}{f} \in \mathcal{D}$ and by hypothesis $fT = 0$. □

PROPOSITION 2.8.65. *Let $T \in \mathcal{D}'$. Then $T \in \mathcal{D}'_{L^p_{w,c}}, 1 \leqslant p \leqslant \infty$, if and only if, there exists $S \in \mathcal{D}'_{L^p}, 1 \leqslant p \leqslant \infty$, such that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}'$.*

PROOF. $(\rightarrow)$: If $T \in \mathcal{D}'_{L^p_{w,c}}$, then we have (see Remark 2.8.63) $T_{w,c} = c^{-\frac{t}{w}} T \in \mathcal{D}'_{L^p}$, so there exists $S \in \mathcal{D}'_{L^p}$ such that $c^{-\frac{t}{w}} T - S = 0$ in $\mathcal{D}'_{L^p}$, i.e., $c^{-\frac{t}{w}} \left(T - c^{\frac{t}{w}} S\right) = 0$ in $\mathcal{D}'_{L^p}$. By applying Lemma 2.8.64, it follows that

$$T = c^{\frac{t}{w}} S \text{ in } \mathcal{D}'.$$

$(\leftarrow)$: Suppose that $T \in \mathcal{D}'$ and there exists $S \in \mathcal{D}'_{L^p}, 1 \leqslant p \leqslant \infty$, such that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}' >$ Then $c^{-\frac{t}{w}} T = S \in \mathcal{D}'_{L^p}$ and hence $T \in \mathcal{D}'_{L^p_{w,c}}$. □

Recall that the space $\mathcal{B}'_{ap}$ of almost periodic distributions which was introduced and studied by L. Schwartz is based on the topological definition of Bochner's almost periodic functions. Let $h \in \mathbb{R}$ and $T \in \mathcal{D}'$, the translated of T by h, denoted by $\tau_h T$, is defined by

$$\langle \tau_h T, \varphi \rangle := \langle T, \tau_{-h}\varphi \rangle, \ \varphi \in \mathcal{D},$$

where $\tau_{-h}\varphi(x) := \varphi(x+h)$.

The following result gives the basic characterizations of Schwartz almost periodic distributions.

THEOREM 2.8.66. *For any bounded distribution $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent:*

(i) *The set $\{\tau_h T : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$.*
(ii) $T * \varphi \in AP$, $\varphi \in \mathcal{D}$.
(iii) $\exists k \in \mathbb{N}, \ \exists (f_j)_{0 \leqslant j \leqslant k} \subseteq AP : T = \sum_{j=0}^{k} f_j^{(j)}$.

PROOF. See [**329**]. □

The following proposition summarizes the main properties of $\mathcal{B}'_{ap}$.

PROPOSITION 2.8.67. (i) *If $T \in \mathcal{B}'_{ap}$, then $T^{(j)} \in \mathcal{B}'_{ap}$, $j \in \mathbb{N}$.*
(ii) $\mathcal{B}_{ap} \times \mathcal{B}'_{ap} \subseteq \mathcal{B}'_{ap}$.
(iii) $\mathcal{B}'_{ap} * \mathcal{D}'_{L^1} \subseteq \mathcal{B}'_{ap}$.

PROOF. See [**329**]. □

Now we will introduce the following concept:

DEFINITION 2.8.68. A distribution $T \in \mathcal{B}'_{w,c}$ is said to be (w, c) −almost periodic, if and only if, $T_{w,c} \in \mathcal{B}'_{ap}$, i.e., the set $\{\tau_h T_{w,c} : h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$. The set of (w, c) −almost periodic distributions is denoted by $\mathcal{B}'_{AP_{w,c}}$.

EXAMPLE 2.8.69. (i) The associated distribution of a (w, c) −almost periodic function (resp. Stepanov (p, w, c) −almost periodic function) is an (w, c) −almost periodic distribution, i.e.

$$AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}} \text{ (resp. } S^p AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}}).$$

(ii) When $c = 1$ it follows that $\mathcal{B}'_{AP_{w,c}} := \mathcal{B}'_{ap}$.

The main characterizations of (w, c) −almost periodic distributions are given in the following

THEOREM 2.8.70. *Let $T \in \mathcal{B}'_{w,c}$. Then the following statements are equivalent:*

(i) $T \in \mathcal{B}'_{AP_{w,c}}$.
(ii) $c^{\frac{t}{w}} (T_{w,c} * \varphi) \in AP_{w,c}$, $\varphi \in \mathcal{D}$.

(iii) $\exists k \in \mathbb{N},\ \exists\, (f_j)_{0\leqslant j\leqslant k} \subseteq AP_{w,c} : T = c^{\frac{t}{w}} \sum\limits_{j=0}^{k} (f_{w,c})_j^{(j)}$, *where*

$$\Big((f_{w,c})_j\Big)_{0\leqslant j\leqslant k} = \Big(c^{-\frac{t}{w}} f_j\Big)_{0\leqslant j\leqslant k}.$$

PROOF. Since for every $T \in \mathcal{B}'_{AP_{w,c}}$, we have $T_{w,c} \in \mathcal{B}'_{ap}$; hence, the result follows immediately from Theorem 2.8.66. □

The main propreties of $\mathcal{B}'_{AP_{w,c}}$ are given in the following proposition.

PROPOSITION 2.8.71. (i) *If* $T \in \mathcal{B}'_{AP_{w,c}}$, *then* $c^{\frac{t}{w}}\,(T_{w,c})^{(j)} \in \mathcal{B}'_{AP_{w,c}}$, $j \in \mathbb{N}$.

(ii) *If* $\varphi \in \mathcal{B}_{AP_{w,c}}$ *and* $T \in \mathcal{B}'_{AP_{w,c}}$, *then* $\varphi_{w,c}T \in \mathcal{B}'_{AP_{w,c}}$.

(iii) *If* $T \in \mathcal{B}'_{AP_{w,c}}$ *and* $S \in \mathcal{D}'_{L^1_{w,c}}$, *then* $c^{\frac{t}{w}}\,(T_{w,c} * S_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$.

PROOF. (i) Obvious.

(ii) If $\varphi \in \mathcal{B}_{AP_{w,c}}$ and $T \in \mathcal{B}'_{AP_{w,c}}$, then $\varphi_{w,c} \in \mathcal{B}_{ap}$ and $T_{w,c} \in \mathcal{B}'_{ap}$. From Proposition 2.8.67(ii), we get $\varphi_{w,c}T_{w,c} \in \mathcal{B}'_{ap}$ and therefore

$$c^{-\frac{t}{w}}\left(c^{\frac{t}{w}}\,(\varphi_{w,c}T_{w,c})\right) \in \mathcal{B}'_{ap},$$

which gives

$$c^{\frac{t}{w}}\,(\varphi_{w,c}T_{w,c}) \in \mathcal{B}'_{AP_{w,c}}.$$

Hence, $\varphi_{w,c}T \in \mathcal{B}'_{AP_{w,c}}$.

(iii) Let $T \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \mathcal{D}'_{L^1_{w,c}}$. Then $T_{w,c} \in \mathcal{B}'_{ap}$ and $S_{w,c} \in \mathcal{D}'_{L^1}$. According to Proposition 2.8.67(iii), we have $T_{w,c} * S_{w,c} \in \mathcal{B}'_{ap}$, and

$$c^{-\frac{t}{w}}\left(c^{\frac{t}{w}}\,(T_{w,c} * S_{w,c})\right) \in \mathcal{B}'_{ap}.$$

Hence, $c^{\frac{t}{w}}\,(T_{w,c} * S_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$. □

The following result shows that $\mathcal{B}_{AP_{w,c}}$ is dense in $\mathcal{B}'_{AP_{w,c}}$.

PROPOSITION 2.8.72. *Let* $T \in \mathcal{B}'_{w,c}$. *Then* $T \in \mathcal{B}'_{AP_{w,c}}$, *if and only if, there exists* $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}_{AP_{w,c}}$ *such that* $\lim\limits_{n\to+\infty} \varphi_n = T$ *in* $\mathcal{B}'_{w,c}$.

PROOF. If $T \in \mathcal{B}'_{AP_{w,c}}$, then $T_{w,c} \in \mathcal{B}'_{ap}$ and from the density of $\mathcal{B}_{ap}$ in $\mathcal{B}'_{ap}$ there exists $(\psi_n)_{n\in\mathbb{Z}_+} \subseteq \mathcal{B}_{ap}$ such that

$$\lim_{n\to+\infty} \psi_n = T_{w,c} \text{ in } \mathcal{D}'_{L^\infty};$$

this is equivalent to

$$c^{\frac{t}{w}} \lim_{n\to+\infty} \psi_n = \lim_{n\to+\infty}\left(c^{\frac{t}{w}}\psi_n\right) = c^{\frac{t}{w}}T_{w,c} = T \text{ in } \mathcal{B}'_{w,c}.$$

Hence, there exists $(\varphi_n)_{n\in\mathbb{Z}_+} = \left(c^{\frac{t}{w}}\psi_n\right)_{n\in\mathbb{N}} \subseteq \mathcal{B}_{AP_{w,c}}$ such that

$$\lim_{n\to+\infty} \varphi_n = T \text{ in } \mathcal{B}'_{w,c}.$$

$\square$

Now we will introduce the concept of asymptotic (w, c) −almost periodicity of distributions. Asymptotically almost periodic Schwartz distributions have been introduced and studied by I. Cioranescu in [**111**]. We recall the definition and some properties of asymptotically almost periodic Schwartz distributions ($\mathbb{R}_+ \equiv [0, \infty)$).

DEFINITION 2.8.73. A distribution $T \in \mathcal{D}'_{L^\infty}$ is called vanishing at infinity if

$$\forall \varphi \in \mathcal{D}, \quad \lim_{h \to +\infty} \langle \tau_{-h} T, \varphi \rangle = 0 \text{ in } \mathbb{C}.$$

Denote by $\mathcal{B}'_{0+}$ the space of bounded distributions vanishing at infinity.

DEFINITION 2.8.74. A distribution $T \in \mathcal{D}'_{L^\infty}$ is called asymptotically almost periodic if there exist $R \in \mathcal{B}'_{ap}$ and $S \in \mathcal{B}'_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$. The space of asymptotically almost periodic Schwartz distributions is denoted by $\mathcal{B}'_{aap}(\mathbb{R}_+)$.

PROPOSITION 2.8.75. *If $T \in \mathcal{B}'_{aap}(\mathbb{R}_+)$, the decomposition $T = R + S$ on $\mathbb{R}_+$, is unique in $\mathcal{D}'_{L^\infty}$.*

PROOF. See [**111**]. $\square$

Set $\mathcal{D}_+ := \{\varphi \in \mathcal{D} : \operatorname{supp}(\varphi) \subseteq \mathbb{R}_+\}$. Then we have the following characterization of space $\mathcal{B}'_{aap}(\mathbb{R}_+)$.

THEOREM 2.8.76. *Let $T \in \mathcal{D}'_{L^\infty}$. Then the following assertions are equivalent:*

(i) $T \in \mathcal{B}'_{aap}(\mathbb{R}_+)$.

(ii) $T * \check{\varphi} \in AAP(\mathbb{R}_+)$, $\varphi \in \mathcal{D}_+$, *where* $\check{\varphi}(x) := \varphi(-x)$.

(iii) $\exists k \in \mathbb{N}, \exists (f_j)_{0 \leqslant j \leqslant k} \subseteq AAP(\mathbb{R}_+) : T = \sum_{j=0}^{k} f_j^{(j)}$ *on* $\mathbb{R}_+$.

PROOF. See [**111**]. $\square$

Asymptotic (w, c) −almost periodicity of distributions is introduced in the following definition:

DEFINITION 2.8.77. Let $c \in \mathbb{C}$, $|c| \geqslant 1$ and $w > 0$. Then a distribution $T \in \mathcal{B}'_{w,c}$ is said to be asymptotically (w, c) −almost periodic, if and only if, $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$. The space of asymptotically (w, c) −almost periodic distributions is denoted by $\mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$.

REMARK 2.8.78. (i) When $c = 1$ it follows that $\mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+) := \mathcal{B}'_{aap}(\mathbb{R}_+)$.

(ii) The associated distribution of an asymptotically (w, c) −almost periodic function (resp. asymptotically Stepanov (p, w, c) −almost periodic function) is asymptotically (w, c) −almost periodic distribution.

Let us define now the space $\left(\mathcal{B}'_{w,c}\right)_{0+}$ of (w, c) −bounded distributions vanishing at infinity as follows:

DEFINITION 2.8.79. Let $c \in \mathbb{C}$, $|c| \geqslant 1$ and $w > 0$. A distribution $T \in \mathcal{B}'_{w,c}$ is said to be (w, c) −bounded distribution vanishing at infinity, if and only if, $T_{w,c} \in \mathcal{B}'_{0+}$.

We have the following result.

THEOREM 2.8.80. *Let* $c \in \mathbb{C}$, $|c| \geqslant 1$, $w > 0$ *and* $T \in \mathcal{B}'_{w,c}$. *Then* $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$, *if and only if, there exist* $R \in \mathcal{B}'_{AP_{w,c}}$ *and* $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ *such that*

$$T = R + S \text{ on } \mathbb{R}_+. \tag{153}$$

PROOF. $(\Longrightarrow)$: Let $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$. Then $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$ and by definition 2.8.74, there exist $P \in \mathcal{B}'_{ap}$ and $Q \in \mathcal{B}'_{0+}$ such that $T_{w,c} = P + Q$ on $\mathbb{R}_+$. On the other hand, we have

$$\begin{aligned} T_{w,c} &= c^{-\frac{t}{w}} T = P + Q \Longrightarrow \left\langle c^{-\frac{t}{w}} T, \varphi \right\rangle = \langle P, \varphi \rangle + \langle Q, \varphi \rangle, \ \forall \varphi \in \mathcal{D} \\ &\Longrightarrow \ \langle T, \psi \rangle = \left\langle c^{\frac{t}{w}} P, \psi \right\rangle + \left\langle c^{\frac{t}{w}} Q, \psi \right\rangle, \ \forall \psi = c^{-\frac{t}{w}} \varphi \in \mathcal{D}. \end{aligned}$$

Thus there exist $R = c^{\frac{t}{w}} P \in \mathcal{B}'_{AP_{w,c}}$ and $S = c^{\frac{t}{w}} Q \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$.

$(\Longleftarrow)$: If there exist $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$ such that $T = R + S$ on $\mathbb{R}_+$, then $c^{-\frac{t}{w}} T = c^{-\frac{t}{w}} R + c^{-\frac{t}{w}} S$ on $\mathbb{R}_+$, i.e. $T_{w,c} = R_{w,c} + S_{w,c}$ on $\mathbb{R}_+$, where $R_{w,c} \in \mathcal{B}'_{ap}$ and $S_{w,c} \in \mathcal{B}'_{0+}$; hence $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$, which shows that $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$. □

PROPOSITION 2.8.81. *The decomposition* (153) *is unique in* $\mathcal{B}'_{w,c}$.

PROOF. Suppose that $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$ is such that $T = R + S$ on $\mathbb{R}_+$, where $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \left(\mathcal{B}'_{w,c}\right)_{0+}$. Then the result follows from the proof of the implication $(\Longleftarrow)$ of Theorem 2.8.80 and the uniqueness of the decomposition of asymptotically almost periodic distributions. □

Some characterizations of asymptotically (w, c) $-$almost periodic distributions are given in the following result:

THEOREM 2.8.82. *Let* $c \in \mathbb{C}$, $|c| \geqslant 1$, $w > 0$ *and* $T \in \mathcal{B}'_{w,c}$. *The following assertions are equivalent:*

(i) $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$.

(i) $c^{\frac{t}{w}} \left(T_{w,c} * \check{\varphi} \right) \in AAP_{w,c}(\mathbb{R}_+)$, $\varphi \in \mathcal{D}_+$, *where* $\check{\varphi}(x) := \varphi(-x)$.

(iii) $\exists k \in \mathbb{N}$, $\exists (f_j)_{0 \leqslant j \leqslant k} \subseteq AAP_{w,c}(\mathbb{R}_+) : T = c^{\frac{t}{w}} \sum_{j=0}^{k} (f_{w,c})_j^{(j)}$ *on* $\mathbb{R}_+$, *where* $\left((f_{w,c})_j\right)_{0 \leqslant j \leqslant k} = \left(c^{-\frac{t}{w}} f_j\right)_{0 \leqslant j \leqslant k}$.

PROOF. It is clear that if $T \in \mathcal{B}'_{AAP_{w,c}}(\mathbb{R}_+)$ then $T_{w,c} \in \mathcal{B}'_{aap}(\mathbb{R}_+)$. Applying Theorem 2.8.76, we obtain the result. □

2.8.6. Linear differential equations in $\mathcal{B}'_{AP_{w,c}}$. In this subsection, we will study the existence of distributional (w, c) $-$almost periodic solutions of the following system of linear ordinary differential equations

$$T' = AT + S, \tag{154}$$

where $A = (a_{ij})_{1 \leqslant i,j \leqslant k}$, $k \in \mathbb{N}$, is a given square matrix of complex numbers, $S = (S_i)_{1 \leqslant i \leqslant k} \in (\mathcal{D}')^k$ is a vector distribution and $T = (T_i)_{1 \leqslant i \leqslant k}$ is the unknown vector distribution.

First, consider the system (154) with $S \in (AP)^k$ and let us recall the following result.

THEOREM 2.8.83. *If the matrix A has no eigenvalues with real part zero, then for any $S \in (AP)^k$, there exists a unique solution $T \in (AP)^k$ of the system* (154).

PROOF. See [**111**]. $\square$

Let I_k be the unit matrix of order k. The following result gives the (w, c) −almost periodicity of the solution (if it exists) of the system (154).

THEOREM 2.8.84. *Let $S \in \left(\mathcal{B}'_{AP_{w,c}}\right)^k$. If the matrix $A - \frac{\log c}{w} I_k$ has no eigenvalues with real part zero, then the system* (154) *admits a unique solution $T \in \left(\mathcal{D}'_{L^\infty_{w,c}}\right)^k$ which is an (w, c) −almost periodic vector distribution.*

PROOF. Let $\varphi \in \mathcal{D}$. We have

$$c^{-\frac{t}{w}} T' * \varphi = \left(c^{-\frac{t}{w}} T * \varphi\right)' + \frac{\log c}{w} c^{-\frac{t}{w}} T * \varphi. \tag{155}$$

On the other hand, if $T \in \left(\mathcal{D}'_{L^\infty_{w,c}}\right)^k$ satisfies system (154), then

$$c^{-\frac{t}{w}} T' * \varphi = A c^{-\frac{t}{w}} T * \varphi + c^{-\frac{t}{w}} S * \varphi.$$

So from (155), we have

$$\left(c^{-\frac{t}{w}} T * \varphi\right)' = \left(A - \frac{\log c}{w} I_k\right) c^{-\frac{t}{w}} T * \varphi + c^{-\frac{t}{w}} S * \varphi,$$

i.e.

$$(T_{w,c} * \varphi)' = \left(A - \frac{\log c}{w} I_k\right) (T_{w,c} * \varphi) + S_{w,c} * \varphi, \tag{156}$$

where

$$T_{w,c} * \varphi = \left((T_{w,c})_i * \varphi\right)_{1 \leqslant i \leqslant k} = \left(\left(c^{-\frac{t}{w}} T_i\right) * \varphi\right)_{1 \leqslant i \leqslant k},$$

and

$$S_{w,c} * \varphi = \left((S_{w,c})_i * \varphi\right)_{1 \leqslant i \leqslant k} = \left(\left(c^{-\frac{t}{w}} S_i\right) * \varphi\right)_{1 \leqslant i \leqslant k}.$$

Then the system (156) is equivalent in $(\mathcal{C}^\infty)^k$ to the following system of differential equations

$$P' = BP + Q,$$

with $B = A - \frac{\log c}{w} I_k$, $P = T_{w,c} * \varphi \in (\mathcal{C}^\infty)^k$ and $Q = S_{w,c} * \varphi \in (AP)^k$. According to Theorem 2.8.83, it follows that there exists a unique bounded solution P which is almost periodic; therefore $(T_{w,c})_i * \varphi \in AP$, $1 \leqslant i \leqslant k$, $\varphi \in \mathcal{D}$; hence

$c^{\frac{t}{w}}\left((T_{w,c})_i * \varphi\right) \in AP_{w,c}$, $1 \leqslant i \leqslant k$, $\varphi \in \mathcal{D}$. Thus, according to Theorem 2.8.70, we get $(T_i)_{1\leqslant i\leqslant k} \in \left(\mathcal{B}'_{AP_{w,c}}\right)^k$. $\square$

2.8.7. Asymptotically (ω, c)-almost periodic type solutions of abstract degenerate non-scalar Volterra equations. There are by now only a few relevant references concerning abstract non-scalar Volterra equations, degenerate or non-degenerate in time variable. Concerning non-degenerate abstract Volterra equations of non-scalar type, mention should be made of the research monograph [**319**] by J. Prüss, the article [**220**] by M. Jung and the article [**237**] by M. Kostić. In [**238**], we have explained how the methods proposed in [**319**] and [**237**] can be helpful in the analysis of abstract degenerate Volterra equations of non-scalar type. In this subsection, we initate the study of the existence and uniqueness of asymptotically almost periodic type solutions of the abstract degenerate non-scalar Volterra equations. In actual fact, we investigate asymptotically (ω, c)-almost periodic type solutions of the abstract degenerate non-scalar Volterra equations in Banach spaces (we can similarly analyze (ω, c)-asymptotically periodic solutions; the Stepanov, Weyl and Besicovitch generalizations of asymptotically (ω, c)-almost periodic functions will not be considered, as well).

We will first recall the various notions of (A, k, B)-regularized C-pseudoresolvent families introduced in [**238**]; after that, we will analyze the existence and uniqueness of asymptotically (ω, c)-almost periodic type solutions of the abstract degenerate Cauchy problem

$$Bu(t) = f(t) + \int_0^t A(t-s)u(s)\,ds,\ t \in [0, \tau). \tag{157}$$

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)_Y$ be two non-trivial complex Banach spaces such that Y is continuously embedded in X. Let the operator $C \in L(X)$ be injective, and let $\tau \in (0, \infty]$. The norm in X, resp. Y, will be denoted by $||\cdot||_X$, resp. $||\cdot||_Y$. We use the symbol B to denote a closed linear operator with domain and range contained in X; by $\|\cdot\|_{[D(B)]} := \|\cdot\| + \|B\cdot\|$ we denote the corresponding graph norm and by $[D(B)] = (D(B), \|\cdot\|_{[D(B)]})$ we denote the corresponding Banach space. If Z is a general topological space and $Z_0 \subseteq Z$, then by $\overline{Z_0}^Z$ we denote the adherence of Z_0 in Z. We will basically follow the notation employed in the monograph of J. Prüss [**319**] and our paper [**238**].

We start by recalling the following notion introduced in [**238**] (see also [**236**, Section 2.9]):

Definition 2.8.85. Let $k \in C([0, \tau))$ and $k \neq 0$, let $\tau \in (0, \infty]$, $f \in C([0, \tau) : X)$, and let $A \in L^1_{loc}([0, \tau) : L(Y, X))$. Then we say that a function $u \in C([0, \tau) : [D(B)])$ is:

(i) a strong solution of (157) if and only if $u \in L^\infty_{loc}([0, \tau) : Y)$ and (157) holds on $[0, \tau)$,

(ii) a mild solution of (157) if and only if there exist a sequence (f_n) in $C([0,\tau) : X)$ and a sequence (u_n) in $C([0,\tau) : [D(B)])$ such that $u_n(t)$ is a strong solution of (157) with $f(t)$ replaced by $f_n(t)$ and that $\lim_{n\to\infty} f_n(t) = f(t)$ as well as $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on compact subsets of $[0,\tau)$.

The following definition will be invaluably important in our further work ([**238**]):

DEFINITION 2.8.86. Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{loc}([0,\tau) : L(Y,X))$. A family $(S(t))_{t\in[0,\tau)}$ in $L(X,[D(B)])$ is called an (A,k,B)-regularized C-pseudoresolvent family if and only if the following holds:

(S1) The mappings $t \mapsto S(t)x$, $t \in [0,\tau)$ and $t \mapsto BS(t)x$, $t \in [0,\tau)$ are continuous in X for every fixed $x \in X$, $BS(0) = k(0)C$ and $S(t)C = CS(t)$, $t \in [0,\tau)$.

(S2) Put $U(t)x := \int_0^t S(s)x\,ds$, $x \in X$, $t \in [0,\tau)$. Then (S2) means $U(t)Y \subseteq Y$, $U(t)_{|Y} \in L(Y)$, $t \in [0,\tau)$ and $(U(t)_{|Y})_{t\in[0,\tau)}$ is locally Lipschitz continuous in $L(Y)$.

(S3) The resolvent equations

$$BS(t)y = k(t)Cy + \int_0^t A(t-s)\,dU(s)y,\ t \in [0,\tau),\ y \in Y, \tag{158}$$

$$BS(t)y = k(t)Cy + \int_0^t S(t-s)A(s)y\,ds,\ t \in [0,\tau),\ y \in Y, \tag{159}$$

hold; (158), resp. (159), is called the first resolvent equation, resp. the second resolvent equation.

An (A,k,B)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be an (A,k,B)-regularized C-resolvent family if additionally:

(S4) For every $y \in Y$, we have $S(\cdot)y \in L^\infty_{loc}([0,\tau) : Y)$.

An operator family $(S(t))_{t\in[0,\tau)}$ in $L(X,[D(B)])$ is called a weak (A,k,B)-regularized C-pseudoresolvent family if and only if (S1) and (159) hold. Finally, a weak (A,k,B)-regularized C-pseudoresolvent family $(S(t))_{t\in[0,\tau)}$ is said to be a-regular ($a \in L^1_{loc}([0,\tau))$) if and only if $a * S(\cdot)x \in C([0,\tau) : Y)$, $x \in \overline{Y}^X$.

As is well known, condition (S3) can be rewritten in the following equivalent form:

(S3)'

$$BU(t)y = \Theta(t)Cy + \int_0^t A(t-s)U(s)y\,ds,\ t \in [0,\tau),\ y \in Y,$$

$$BU(t)y = \Theta(t)Cy + \int_0^t U(t-s)A(s)y\,ds,\ t \in [0,\tau),\ y \in Y.$$

We also need the following definition from [**238**]:

Definition 2.8.87. Let $k \in C([0,\infty))$, $k \neq 0$, $A \in L^1_{loc}([0,\infty) : L(Y,X))$, $\alpha \in (0,\pi]$, and let $(S(t))_{t\geqslant 0} \subseteq L(X,[D(B)])$ be a (weak) (A,k,B)-regularized C-(pseudo)resolvent family. Then it is said that $(S(t))_{t\geqslant 0}$ is an analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family of angle α, if there exists an analytic function $\mathbf{S} : \Sigma_\alpha \to L(X,[D(B)])$ satisfying $\mathbf{S}(t) = S(t)$, $t > 0$, $\lim_{z\to 0, z\in\Sigma_\gamma} \mathbf{S}(z)x = S(0)x$ and $\lim_{z\to 0, z\in\Sigma_\gamma} B\mathbf{S}(z)x = BS(0)x$ for all $\gamma \in (0,\alpha)$ and $x \in X$. We say that $(S(t))_{t\geqslant 0}$ is an exponentially bounded, analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family, resp. bounded analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family, of angle α, if $(S(t))_{t\geqslant 0}$ is an analytic (weak) (A,k,B)-regularized C-(pseudo)resolvent family of angle α and for each $\gamma \in (0,\alpha)$ there exist $M_\gamma > 0$ and $\omega_\gamma \geqslant 0$, resp. $\omega_\gamma = 0$, such that $||\mathbf{S}(z)||_{L(X)} + ||B\mathbf{S}(z)||_{L(X)} \leqslant M_\gamma e^{\omega_\gamma |z|}$, $z \in \Sigma_\gamma$. Since no confusion seems likely, we shall identify $S(\cdot)$ and $\mathbf{S}(\cdot)$ in the sequel.

In [**238**], we have also introduced the notion of an (A,k,B)-regularized C-uniqueness family with a view to analyze the uniqueness of solutions of the abstract Cauchy problem (157):

Definition 2.8.88. Let $\tau \in (0,\infty]$, $k \in C([0,\tau))$, $k \neq 0$ and $A \in L^1_{loc}([0,\tau) : L(Y,X))$. A strongly continuous operator family $(V(t))_{t\in[0,\tau)} \subseteq L(X)$ is said to be an (A,k,B)-regularized C-uniqueness family if and only if

$$V(t)By = k(t)Cy + \int_0^t V(t-s)A(s)y\,ds,\ t \in [0,\tau),\ y \in Y \cap D(B).$$

We will use the following statements proved in [**238**, Proposition 2]:

[P]: Assume that $(V(t))_{t\in[0,\tau)}$ is an (A,k,B)-regularized C-uniqueness family, $f \in C([0,\tau) : X)$ and $u(t)$ is a mild solution of (157). Then we have $(kC * u)(t) = (V * f)(t)$, $t \in [0,\tau)$.

[Q]: Assume that $(S(t))_{t\in[0,\tau)}$ is an $(A,1,B)$-regularized C-pseudoresolvent family, $C^{-1}f \in C([0,\tau) : X)$ and $f(0) = 0$. Then we know that the following statements hold:

(a) Let $C^{-1}f \in AC_{loc}([0,\tau) : Y)$ and $(C^{-1}f)' \in L^1_{loc}([0,\tau) : Y)$. Then the function $t \mapsto u(t)$, $t \in [0,\tau)$ given by

$$u(t) = \int\limits_0^t S(t-s)(C^{-1}f)'(s)\,ds = \int\limits_0^t dU(s)(C^{-1}f)'(t-s)$$

is a strong solution of (157). Moreover, $u \in C([0,\tau) : Y)$.

(b) Let $(C^{-1}f)' \in L^1_{loc}([0,\tau) : X)$ and $\overline{Y}^X = X$. Then the function $u(t) = \int_0^t S(t-s)(C^{-1}f)'(s)\,ds$, $t \in [0,\tau)$ is a mild solution of (157).

(c) Let $C^{-1}g \in W^{1,1}_{loc}([0,\tau) : \overline{Y}^X)$, $a \in L^1_{loc}([0,\tau))$, $f(t) = (a * g)(t)$, $t \in [0,\tau)$ and let $(S(t))_{t\in[0,\tau)}$ be a-regular. Then the function $u(t) = \int_0^t S(t-s)(a * (C^{-1}g)')(s)\,ds$, $t \in [0,\tau)$ is a strong solution of (157).

The uniqueness of solutions in (a), (b) or (c) holds provided that for each $y \in Y \cap D(B)$ we have $S(t)By = BS(t)y$, $t \in [0,\tau)$.

Even in case that $B = C = I$ and $k(t) \equiv 1$, there exist examples of global not exponentially bounded (A, k, B)-regularized C-pseudoresolvent families (see e.g., [**319**, Example 6.2, pp. 165-166]). For our purposes, it will be crucial to examine whether the operator family $(S(t))_{t\geqslant 0}$ is exponentially decaying as the time variable goes to plus infinity. The existence of a number $\varepsilon_0 \geqslant 0$ such that

$$\int_0^\infty e^{-\varepsilon t}\|A(t)\|_{L(Y,X)}\,dt < \infty,\ \varepsilon > \varepsilon_0, \tag{160}$$

which has been used in [**319**] and [**237**]-[**238**], is not sufficient to ensure the exponential decaying of $(S(t))_{t\geqslant 0}$ as $t \to +\infty$. Therefore, we must impose some extra conditions ensuring this property of $(S(t))_{t\geqslant 0}$, which will be extremely important for us.

Now we will state two simple results concerning this problematic. The both of them are basically deduced in [**238**]:

THEOREM 2.8.89. *Assume* $\varepsilon_0 \geqslant 0$, $k(t)$ *satisfies* (P1), $\omega \geqslant \max(\operatorname{abs}(k), \varepsilon_0)$, (160) *holds,* $\alpha \in (0, \pi/2]$, *there exists an analytic mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ *such that*

(i) $BH(\lambda)y - H(\lambda)\tilde{A}(\lambda)y = \tilde{k}(\lambda)Cy,\ y \in Y,\ \operatorname{Re}\lambda > \omega,\ \tilde{k}(\lambda) \neq 0;\ H(\lambda)C = CH(\lambda),\ \operatorname{Re}\lambda > \omega,$

(ii) $\sup_{\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\gamma}}\Big[\|(\lambda-\omega)H(\lambda)\|_{L(X)} + \|(\lambda-\omega)BH(\lambda)\|_{L(X)}\Big] < \infty$ *for all* $\gamma \in (0, \alpha)$,

(iii) *there exists an operator* $F \in L(X, [D(B)])$ *such that* $BFx = k(0)Cx$, $x \in X$ *and* $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0}\lambda H(\lambda)x = Fx,\ x \in X,$ *and*

(iv) $\lim_{\lambda\to+\infty,\tilde{k}(\lambda)\neq 0}\lambda BH(\lambda)x = k(0)Cx,\ x \in X,$ *provided that* $\overline{Y}^X \neq X$.

If there exists a real number $\omega_0 < 0$ *such that the mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ *can be analytically extended to the sector* $\omega_0 + \Sigma_{\frac{\pi}{2}+\alpha}$ *and condition* (ii) *holds with the number* ω *replaced by the number* ω_0 *therein, then there exists a weak analytic* (A, k, B)*-regularized* C*-pseudoresolvent family* $(S(t))_{t\geqslant 0}$ *of angle* α *such that*

$$\sup_{z\in\Sigma_\gamma}\Big[\|e^{-\omega_0 z}S(z)\|_{L(X)} + \|e^{-\omega_0 z}BS(z)\|_{L(X)}\Big] < \infty \text{ for all } \gamma \in (0, \alpha). \tag{161}$$

PROOF. By [**238**, Theorem 3], we know that there exists a weak analytic (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t\geqslant 0}$ of angle α, satisfying that the estimate (161) holds with the number ω_0 replaced by the number ω. The final statement follows easily from this fact, [**30**, Theorem 2.6.1], the uniqueness theorem for the Laplace transform and the assumption we have made after the formulation of conditions (i)-(iv). $\square$

We can similarly deduce the validity of the following result which corresponds to [**238**, Theorem 4]:

THEOREM 2.8.90. *Assume* $\alpha \in (0, \pi/2]$, $\varepsilon_0 \geqslant 0$, $k(t)$ *satisfies* (P1) *and* (160) *holds. Let* $\omega \geqslant \max(\mathrm{abs}(k), \varepsilon_0)$, *and let there exist an analytic mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ *such that* $H_{|Y} : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(Y)$ *is an analytic mapping, as well as that:*

(i) *One has*

$$\sup_{\lambda \in \omega + \Sigma_{\frac{\pi}{2}+\gamma}} \Big[\big\|(\lambda-\omega)H(\lambda)\big\|_{L(X)} + \big\|(\lambda-\omega)BH(\lambda)\big\|_{L(X)} + \big\|(\lambda-\omega)H(\lambda)\big\|_{L(Y)} \Big] < \infty$$

for all $\gamma \in (0, \alpha)$,

(ii) $BH(\lambda)y - H(\lambda)\tilde{A}(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$; $BH(\lambda)y - \tilde{A}(\lambda)H(\lambda)y = \tilde{k}(\lambda)Cy$, $y \in Y$, $\operatorname{Re}\lambda > \omega$, $\tilde{k}(\lambda) \neq 0$; $H(\lambda)C = CH(\lambda)$, $\operatorname{Re}\lambda > \omega_0$,

(iii) *there exists an operator* $F \in L(X, [D(B)])$ *such that* $BFx = k(0)Cx$, $x \in X$, $\lim_{\lambda \to +\infty, \tilde{k}(\lambda) \neq 0} \lambda H(\lambda)x = Fx$, $x \in X$, *and*

(iv) $\lim_{\lambda \to +\infty, \tilde{k}(\lambda) \neq 0} \lambda BH(\lambda)x = k(0)Cx$, $x \in X$, *provided that* $\overline{Y}^X \neq X$.

If there exists a real number $\omega_0 < 0$ *such that the mapping* $H : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(X, [D(B)])$ *can be analytically extended to the sector* $\omega_0 + \Sigma_{\frac{\pi}{2}+\alpha}$, *the mapping* $H_{|Y} : \omega + \Sigma_{\frac{\pi}{2}+\alpha} \to L(Y)$ *can be analytically extended to the sector* $\omega_0 + \Sigma_{\frac{\pi}{2}+\alpha}$, *and condition* (i) *holds with the number* ω *replaced by the number* ω_0 *therein, then there exists an analytic* (A, k, B)*-regularized* C*-resolvent family* $(S(t))_{t \geqslant 0}$ *of angle* α *such that*

$$\sup_{z \in \Sigma_\gamma} \left[\big\|e^{-\omega_0 z}S(z)\big\|_{L(X)} + \big\|e^{-\omega_0 z}BS(z)\big\|_{L(X)} + \sup_{z \in \Sigma_\gamma} \big\|e^{-\omega_0 z}S(z)\big\|_{L(Y)} \right] < \infty$$

and the mapping $t \mapsto U(t) \in L(Y)$, $t > 0$ *can be analytically extended to the sector* Σ_α.

REMARK 2.8.91. Concerning Theorem 2.8.89, it should be noted that we can also impose condition that there exist a negative real number $\omega < 0$, a real number $\beta \in (0, 1]$ and a number $\alpha_0 \in (0, \pi/2)$ such that $H(\cdot)$ is analytic on the region $\Omega \equiv \omega_0 + \Sigma_{(\pi/2)+\alpha}$, continuous on $\overline{\Omega}$ and satisfies the estimate

$$\sup_{\lambda \in \overline{\Omega}} \left[\big\|(1+|\lambda|)^{-\beta}H(\lambda)\big\|_{L(X)} + \big\|(1+|\lambda|)^{-\beta}BH(\lambda)\big\|_{L(X)} \right] < \infty.$$

Then the integral computation carried out in the proof of [**30**, Theorem 2.6.1] shows that there exists a weak analytic (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t \geqslant 0}$ of angle α such that

$$\sup_{z \in \Sigma_\gamma} \left[\big\|e^{-\omega_0 z}|z|^{\beta-1}S(z)\big\|_{L(X)} + \big\|e^{-\omega_0 z}|z|^{\beta-1}BS(z)\big\|_{L(X)} \right] < \infty \text{ for all } \gamma \in (0, \alpha).$$

A similar comment can be given in the case of consideration of Theorem 2.8.90.

Clearly, it is not trivial to practically verify the requirements of Theorem 2.8.89-Theorem 2.8.90 as well as that these theorems are not suitable for applications to the abstract fractional differential equations of non-scalar type. But, in many concrete situations, the requirements of these theorems can be very simply verified:

EXAMPLE 2.8.92. Suppose that $X = Y$, $B = C = I$, $k(t) \equiv 1$, $\omega_0 < 0$, $0 < \alpha \leqslant \pi/2$ and D is a closed linear operator in X such that for each number $\gamma \in (0, \alpha)$ there exists a finite real number $M_\gamma > 0$ such that

$$\sup_{\lambda \in \omega_0 + \Sigma_{(\pi/2)+\gamma}} \Bigl\| \lambda(\lambda - D)^{-1} \Bigr\| \times \Bigl\| \bigl(\lambda - \omega_0\bigr)(\lambda - D)^{-1} \Bigr\| < \infty.$$

Define $A(\cdot)$ through $\tilde{A}(\lambda) := (2D)/(\lambda) - (D^2)/(\lambda^2)$, $\lambda \neq 0$. Then the assumptions of Theorem 2.8.90 hold true because for each $\gamma \in (0, \pi/2)$ we have

$$\begin{aligned}
\sup_{\lambda \in \omega_0 + \Sigma_{(\pi/2)+\gamma}} & \left| \frac{\lambda - \omega_0}{\lambda} \right| \times \left\| \Bigl(I - \tilde{A}(\lambda) \Bigr)^{-1} \right\| \\
&= \sup_{\lambda \in \omega_0 + \Sigma_{(\pi/2)+\gamma}} \left| \frac{\lambda - \omega_0}{\lambda} \right| \times \left\| \left(I - \frac{2D}{\lambda} + \frac{D^2}{\lambda^2} \right)^{-1} \right\| \\
&= \sup_{\lambda \in \omega_0 + \Sigma_{(\pi/2)+\gamma}} \Bigl\| \lambda(\lambda - D)^{-1} \Bigr\| \times \Bigl\| \bigl(\lambda - \omega_0\bigr)(\lambda - D)^{-1} \Bigr\| < \infty.
\end{aligned}$$

Further possibilities to apply Theorem 2.8.89-Theorem 2.8.90 will be considered somewhere else. In [**237**, Theorem 3] and [**238**, Theorem 2], we have considered the hyperbolic perturbation results for the abstract non-scalar Volterra equations. Before proceeding further, we want also to observe that it is very difficult to say whether the perturbed resolvent solution family will be exponentially decaying if the initial resolvent solution family is exponentially decaying as time marches to plus infinity.

Concerning the exponential decaying rate at infinity of an (A, k, B)-regularized C-pseudoresolvent family $(S(t))_{t \geqslant 0}$, we would like to stress that, in [**234**, Remark 2.6.15], we have presented a simple idea which can be also applied in the qualitative analysis of asymptotically almost periodic type solutions of the abstract degenerate non-scalar Volterra integral equations. This will be the starting point for our investigations carried out in the remainder of paper. First of all, we will clarify the following result which can be also formulated for analytic (A, k, B)-regularized C-pseudoresolvent families:

PROPOSITION 2.8.93. *Suppose that $z \in \mathbb{C}$, $a \in L^1_{loc}([0, \tau))$, $k \neq 0$, $A \in L^1_{loc}([0, \tau) : L(Y, X))$ and $(S(t))_{t \in [0,\tau)}$ is an (A, k, B)-regularized C-pseudoresolvent family (weak (A, k, B)-regularized C-pseudoresolvent family). Define*

$$k_z(t) := e^{-zt} k(t), \quad A_z(t) := e^{-zt} A(t), \quad \textit{and} \quad S_z(t) := e^{-zt} S(t), \quad t \in [0, \tau).$$

Then $(S_z(t))_{t \in [0,\tau)}$ is an (A_z, k_z, B)-regularized C-pseudoresolvent family (weak (A_z, k_z, B)-regularized C-pseudoresolvent family). Furthermore, $(S(t))_{t \in [0,\tau)}$ is a-regular if and only if $(S_z(t))_{t \in [0,\tau)}$ is a_z-regular, where $a_z(t) := e^{-zt} a(t)$, $t \in [0, \tau)$, and $(S_z(t))_{t \in [0,\tau)}$ is an (A_z, k_z, B)-regularized C-resolvent family if $(S(t))_{t \in [0,\tau)}$ is an (A, k, B)-regularized C-resolvent family and $\operatorname{Re} z \leqslant 0$.

PROOF. We will provide the main details of proof for (A, k, B)-regularized C-pseudoresolvent families, only. It is clear that condition (S1) holds true. In order to show (S2), define $U_z(t) := \int_0^t S_z(s)x\,ds$, $x \in X$, $t \in [0, \tau)$ and observe that the partial integration implies

$$U_z(t)x = e^{-zt}U(t)x + z\int_0^t e^{-zs}U(s)x\,ds, \quad x \in X, \ t \in [0, \tau). \tag{162}$$

This simply yields that $U_z(t)Y \subseteq Y$, $U_z(t)_{|Y} \in L(Y)$, $t \in [0, \tau)$ and $(U_z(t)_{|Y})_{t\in[0,\tau)}$ is locally Lipschitz continuous in $L(Y)$. We will prove only the first resolvent equation in (S3)' because the second resolvent equation in (S3)' [or (S3)] can be deduced almost trivially. So, let $y \in Y$ and $t \in [0, \tau)$ be fixed. Applying (162) twice and using the first resolvent equation in (S3)' for $(S(t))_{t\in[0,\tau)}$, we get

$$\begin{aligned}
BU_z(t)y &= e^{-zt}\left[\int_0^t e^{-zs}k(s)Cy\,ds + \int_0^t A(t-s)U(s)y\,ds\right] \\
&\quad + z\int_0^t e^{-zs}\left[\int_0^s e^{-zr}k(r)Cy\,dr + \int_0^s A(s-r)U(r)y\,dr\right]ds \\
&= e^{-zt}\left[\int_0^t e^{-zs}k(s)Cy\,ds + ze^{zt}\int_0^t e^{-zs}\int_0^s e^{-zr}k(r)Cy\,dr\,ds\right] \\
&\quad + \int_0^t A(t-s)e^{-zt}U(s)y\,ds + z\left[e^{-z\cdot}A(\cdot) * 1 * e^{-z\cdot}U(\cdot)y\right](t).
\end{aligned}$$

The use of partial integration yields that

$$e^{-zt}\left[\int_0^t e^{-zs}k(s)Cy\,ds + ze^{zt}\int_0^t e^{-zs}\int_0^s e^{-zr}k(r)Cy\,dr\,ds\right] = \int_0^t e^{-zs}k(s)Cy\,ds$$

and the required statement simply follows because the above equality yields

$$\begin{aligned}
BU_z(t)y &= \int_0^t e^{-zs}k(s)Cy\,ds \\
&\quad + \int_0^t e^{-z(t-s)}A(t-s)\left[e^{-zs}U(s)y + z\int_0^s e^{-zr}U(r)y\,dr\right]ds.
\end{aligned}$$

In order to see that $(S(t))_{t\in[0,\tau)}$ is a-regular if and only if $(S_z(t))_{t\in[0,\tau)}$ is a_z-regular, it suffices to observe that

$$\big(a_z * S_z(\cdot)x\big)(t) = e^{-zt}(a * S(\cdot)x)(t), \quad t \in [0, \tau), \ x \in \overline{Y}^X.$$

The remainder of proof for (A, k, B)-regularized C-resolvent families is trivial. $\square$

Now we will analyze the existence and uniqueness of asymptotically (ω, c)-almost periodic type solutions of the abstract Cauchy problem (157). First of all, we will state the following lemma whose proof is very simple and therefore omitted:

LEMMA 2.8.94. *Let $k \in C([0,\tau))$ and $k \neq 0$, let $\tau \in (0,\infty]$, $z \in \mathbb{C}$, $f \in C([0,\tau) : X)$, and let $A \in L^1_{loc}([0,\tau) : L(Y,X))$. Suppose that $(V(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (A,k,B)-regularized C-uniqueness family. Define $f_z(t) := e^{-zt}f(t)$, $V_z(t) := e^{-zt}V(t)$ and $A_z(t) := e^{-zt}A(t)$ for all $t \in [0,\tau)$. Then we have:*

- (i) *If $u(\cdot)$ is a strong (mild) solution of problem* (157)*, then $u_z(\cdot) \equiv e^{-z\cdot}u(\cdot)$ is a strong (mild) solution of problem obtained by replacing respectively $f(\cdot)$ and $A(\cdot)$ in* (157) *by $f_z(\cdot)$ and $A_z(\cdot)$.*
- (ii) *$(V_z(t))_{t\geqslant 0} \subseteq L(X)$ is an (A_z,k_z,B)-regularized C-uniqueness family.*

Now we will prove the following proposition:

PROPOSITION 2.8.95. *Let $k \in C([0,\infty))$, $k \neq 0$, $\omega_0 \geqslant 0$, $\omega > 0$, $1 > \omega\omega_0$, $A \in L^1_{loc}([0,\infty) : L(Y,X))$ and $(V(t))_{t\geqslant 0} \subseteq L(X)$ is an (A,k,B)-regularized C-uniqueness family such that $\|V(t)\| \leqslant Me^{\omega_0 t}$, $t \geqslant 0$. If $u(\cdot)$ is a mild solution of* (157) *and $f(\cdot)$ is asymptotically (ω,e)-almost periodic (asymptotically (ω,e)-almost automorphic, asymptotically compactly (ω,e)-almost automorphic), then the function $(kC*u)(\cdot)$ is likewise asymptotically (ω,e)-almost periodic (asymptotically (ω,e)-almost automorphic, asymptotically compactly (ω,e)-almost automorphic).*

PROOF. Let $z = 1/\omega$. Due to our assumptions, we have that the operator family $(V_z(t) \equiv e^{-zt}V(t))_{t\geqslant 0}$ is exponentially decaying. By Lemma 2.8.94(i), $u_z(\cdot)$ is a strong (mild) solution of problem obtained by replacing respectively $f(\cdot)$ and $A(\cdot)$ in (157) by $f_z(\cdot)$ and $A_z(\cdot)$. Due to Lemma 2.8.94(ii), we have that $(V_z(t))_{t\geqslant 0} \subseteq L(X)$ is an (A_z,k_z,B)-regularized C-uniqueness family. Applying now [P], we get that

$$\big(k_zC * u_z\big)(t) = \big(V_z * f_z\big)(t), \quad t \geqslant 0,$$

i.e.,

$$e^{-z\cdot}\big(kC * u\big)(t) = \big(V_z * f_z\big)(t), \quad t \geqslant 0.$$

We have that $f_z(\cdot)$ is asymptotically almost periodic (asymptotically almost automorphic, asymptotically compactly almost automorphic), so that the function $t \mapsto (V_z * f_z)(t)$, $t \geqslant 0$ has the same property ([**234**]). This implies the required statement. □

It is clear that, if $(S(t))_{t\in[0,\tau)} \subseteq L(X,[D(B)])$ is a weak (A,k,B)-regularized C-pseudoresolvent family and $BS(t)y = S(t)By$, $t \in [0,\tau)$, $y \in Y \cap D(B)$, then $(S(t))_{t\in[0,\tau)} \subseteq L(X)$ is an (A,k,B)-regularized C-uniqueness family. Using this observation, [P]-[Q] and Proposition 2.8.95, we may deduce the following:

PROPOSITION 2.8.96. *Suppose that $(S(t))_{t\geqslant 0} \subseteq L(X,[D(B)])$ is an $(A,1,B)$-regularized C-pseudoresolvent family, $BS(t)y = S(t)By$, $t \in [0,\tau)$, $y \in Y \cap D(B)$, $\omega_0 \geqslant 0$, $\omega > 0$, $1 > \omega\omega_0$, $\|S(t)\| \leqslant Me^{\omega_0 t}$, $t \geqslant 0$ and $f(\cdot)$ is asymptotically (ω,e)-almost periodic (asymptotically (ω,e)-almost automorphic, asymptotically compactly (ω,e)-almost automorphic). Then we have the following:*

- (i) *Let $C^{-1}f \in AC_{loc}([0,\infty) : Y)$, $(C^{-1}f)' \in L^1_{loc}([0,\infty) : Y)$ and $f(0) = 0$. Then there exists a unique strong solution $u(\cdot)$ of* (157)*; moreover, $u \in C([0,\tau) : Y)$ and the mapping $t \mapsto C\int_0^t u(s)\,ds$, $t \geqslant 0$ is asymptotically*

(ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).

(ii) *Let $(C^{-1}f)' \in L^1_{loc}([0,\infty) : X)$ and $\overline{Y}^X = X$. Then there exists a unique mild solution $u(\cdot)$ of (157); moreover, the mapping $t \mapsto C\int_0^t u(s)\,ds$, $t \geqslant 0$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).*

(iii) *Let $C^{-1}g \in W^{1,1}_{loc}([0,\infty) : \overline{Y}^X)$, $a \in L^1_{loc}([0,\infty))$, $f(t) = (a * g)(t)$, $t \geqslant 0$ and let $(S(t))_{t\geqslant 0}$ be a-regular. Then there exists a unique strong solution $u(\cdot)$ of (157); moreover, the mapping $t \mapsto C\int_0^t u(s)\,ds$, $t \geqslant 0$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).*

It is worth noting that Proposition 2.8.96 can be deduced directly, as well as that some sufficient conditions ensuring the above features of mapping $t \mapsto u(t)$, $t \geqslant 0$ can be also achieved. We will explain this only in the case of consideration of part (i). So, let us assume that $(S(t))_{t\geqslant 0} \subseteq L(X, [D(B)])$ is an $(A, 1, B)$-regularized C-pseudoresolvent family as well as that $C^{-1}f \in AC_{loc}([0,\infty) : Y)$, $(C^{-1}f)' \in L^1_{loc}([0,\infty) : Y)$ and $f(0) = 0$. Then the function $t \mapsto u(t)$, $t \geqslant 0$ given by $u(t) = \int_0^t S(t-s)(C^{-1}f)'(s)\,ds$ is a strong solution of (157). Let $\omega_0 \geqslant 0$, $\omega > 0$, $1 > \omega\omega_0$, let $\|S(t)\| \leqslant Me^{\omega_0 t}$, $t \geqslant 0$, and let the mapping $(C^{-1}f)'(\cdot)$ be asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic). Then we have

$$\begin{aligned} e^{-t/\omega}u(t) &= e^{-t/\omega}\int_0^t S(t-s)(C^{-1}f)'(s)\,ds \\ &= \int_0^t \Big[e^{-(t-s)/\omega}S(t-s)\Big]\Big[e^{-s/\omega}(C^{-1}f)'(s)\Big]\,ds, \quad t \geqslant 0. \end{aligned}$$

Since the operator family $(e^{-t/\omega}S(t))_{t\geqslant 0}$ is exponentially decaying, it follows that the function $t \mapsto e^{-t/\omega}u(t)$, $t \geqslant 0$ is asymptotically almost periodic (asymptotically almost automorphic, asymptotically compactly almost automorphic). Hence, the mapping $t \mapsto u(t)$, $t \geqslant 0$ is asymptotically (ω, e)-almost periodic (asymptotically (ω, e)-almost automorphic, asymptotically compactly (ω, e)-almost automorphic).

Concerning the abstract non-degenerate Volterra equations of non-scalar type, it is clear that the above results can be applied to numerous problems in linear (thermo-)viscoelasticity and electrodynamics with memory (cf. [**319**, Chapter 9, Chapter 13] for more details); for example, in the analysis of viscoelastic Timoshenko beam in case of non-synchronous materials. In both cases, degenerate and non-degenerate, we can make many applications of our results with the regularizing operator $C \neq I$; see e.g., [**237**, Corollary 1, Example 1, Example 2] and the paragraph following [**238**, Theorem 2].

Finally, we would like to say a few words about the following special class of the abstract non-degenerate Volterra equations of non-scalar type:

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s) + f(t), \quad t \geqslant 0; \quad x(0) = x_0, \tag{163}$$

where A generates a strongly continuous semigroup on X and $(B(t))_{t\geqslant 0}$ is a family of linear operators on X such that, for almost every $t \geqslant 0$, the operator $B(t)$ maps continuously the space $Y = [D(A)]$ into X and there exists a locally integrable function $b : [0,\infty) \rightarrow [0,\infty)$ such that $\|B(t)y\|_{L(Y,X)} \leqslant b(t)\|y\|_Y$ for all $y \in Y$ and $t \geqslant 0$; see e.g., [**98**]-[**99**], [**188**]-[**189**] and [**286**] for more details about the subject. By a solution of (163), we mean any function $x \in C([0,\infty) : Y) \cap C^1([0,\infty) : X)$ satisfying the initial condition $x(0) = x_0$ and the first equality in (163) identically for $t \geqslant 0$. In the analysis of (163), the following notion of resolvent family (which is a very special case of the notion introduced in Definition 2.8.86) plays an important role:

DEFINITION 2.8.97. (W. Desch, R. Grimmer, W. Schappacher [**131**, Definition, pp. 220-221]) A strongly continuous operator family $(R(t))_{t\geqslant 0} \subseteq L(X)$ is said to be a resolvent family for (163) if and only if $R(0) = I$, the mapping $y \mapsto R(t)y \in Y$, $t \geqslant 0$ belongs to the class $C([0,\infty) : Y) \cap C^1([0,\infty) : X)$ and the following resolvent equations hold:

$$R'(t)y = AR(t)y + \int_0^t B(t-s)R(s)y\,ds, \quad t \geqslant 0$$

and

$$R'(t)y = R(t)Ay + \int_0^t R(t-s)B(s)y\,ds, \quad t \geqslant 0.$$

In [**131**, Proposition 2(c)], it has been proved that any solution of (163) has the form

$$x(t) = R(t)x_0 + \int_0^t R(t-s)f(s)\,ds, \quad t \geqslant 0.$$

The notion of a resolvent family for (163) has been extended by R. Grimmer in [**187**], where the author has analyzed the wellposedness of the following abstract differential first-oder equation of non-convolution type:

$$x'(t) = A(t)x(t) + \int_0^t B(t,s)x(s) + f(t), \quad x(0) = x_0;$$

here, $A(t)$ and $B(t,s)$ are closed linear operators with fixed domain and the function $f : [0,\infty) \rightarrow X$ is continuous. In this paper, some particular results are given for the convolution case $B(t,s) \equiv B(t-s)$ and the usually considered autonomous case $A(t) \equiv A$, which turns the above equation in (163). We would like to especially emphasize that the author has shown, in [**187**, Theorem 4.1], that there exists an exponentially decaying resolvent family $(R(t))_{t\geqslant 0} \subseteq L(X)$ for (163) which decays exponentially in time. Hence, we can simply apply many structural results obtained so far in the analysis of the existence and uniqueness of asymptotically almost periodic type solutions of (163). As an application, we can consider the existence and uniqueness of asymptotically almost periodic type solutions of the following equation

$$c\Delta_t\theta(x,t) + \beta(0)\frac{\partial}{\partial t}\theta(x,t)$$

$$= \alpha_0 \Delta_x \theta(x,t) - \int_{-\infty}^{t} \beta'(t-s) \frac{\partial}{\partial s} \theta(x,s)\, ds + \int_{-\infty}^{t} \alpha'(t-s) \Delta_x \theta(x,s)\, ds + \frac{\partial}{\partial t} r(x,t),$$

which arises in the study of heat conduction in materials with memory; see [**187**] for further information.

We close this section by recalling that the following special class of second-order abstract Volterra equations of non-scalar type

$$u''(t) = Au(t) + \int_0^t B(t-s)u(s)\, ds + f(t), \quad t \geqslant 0,\ u(0) = x,\ u'(0) = y, \tag{164}$$

where A generates a strongly continuous cosine function and $B \in BV_{loc}([0,\infty) : L([D(A)], X))$, has been systematically investigated starting from the 1970s; see e.g., [**85**]-[**86**], [**122**]-[**123**], [**130**], and references cited therein for more details on the subject. Almost periodic solutions of the abstract second order differential equations of (164) and their generalizations with the added delay or nonlinear dissipative terms have been investigated by [**15**], [**35**], [**207**], [**173**], [**300**], [**312**], [**347**]; see also the reference list of [**234**]. We want also to mention an interesting article [**33**] by M. Arienmughare and T. Diagana, where the authors have employed the Drazin inverses to investigate the existence of almost periodic solutions to some singular systems of first-and second-order differential equations with complex coefficients (see also [**34**] and [**169**]).

2.9. c-Uniformly recurrent functions, c-almost periodic functions and semi-c-periodic functions

Besides the notion depending on two parameters ω and c, it is meaningful to consider the following notion depending only on the parameter c. The main aim of this section is to introduce and analyze the classes of c-almost periodic functions, c-uniformly recurrent functions, semi-c-periodic functions and their Stepanov generalizations, where $c \in \mathbb{C} \setminus \{0\}$. We also introduce and investigate the corresponding classes of c-almost periodic type functions depending on two variables; several composition principles for c-almost periodic type functions are established in this direction. We provide some illustrative examples and applications to the abstract fractional semilinear integro-differential inclusions [before proceeding further, we would like to note that it is not clear how we can introduce and analyze the notion of (compact) c-almost automorphicity in a satisfactory way].

We will use the following auxiliary result, whose proof follows from the argumentation used in the proof that every orbit under an irrational rotation is dense in $S_1 \equiv \{z \in \mathbb{C} : |z| = 1\}$; see e.g. the solution given by C. Blatter in [**67**]:

Lemma 2.9.1. *Suppose that $c = e^{i\pi\varphi}$, where $\varphi \in (-\pi, \pi] \setminus \{0\}$ is not rational. Then for each $c' \in S_1$ there exists a strictly increasing sequence (l_k) of positive integers such that $\sup_{k \in \mathbb{N}}(l_{k+1} - l_k) < \infty$ and $|c^{l_k} - c'| < \varepsilon$.*

Unless stated otherwise, we will always assume here that $c \in \mathbb{C}$ and $|c| = 1$. Let $f : I \to X$ be a continuous function and let a number $\varepsilon > 0$ be given. We call a number $\tau > 0$ an (ε, c)-period for $f(\cdot)$ if $\|f(t+\tau) - cf(t)\| \leqslant \varepsilon$ for all $t \in I$. By $\vartheta_c(f, \varepsilon)$ we denote the set consisting of all (ε, c)-periods for $f(\cdot)$.

We are concerned with the following notion:

Definition 2.9.2. It is said that $f(\cdot)$ is c-almost periodic if and only if for each $\varepsilon > 0$ the set $\vartheta_c(f, \varepsilon)$ is relatively dense in $[0, \infty)$. The space consisting of all c-almost periodic functions from the interval I into X will be denoted by $AP_c(I : X)$.

If $c = -1$, then we recover the notion of almost anti-periodicity ([**254**]).

In general case, it is very simple to prove that the following holds (see e.g., the proof of [**62**, Theorem 4°, p. 2]):

Proposition 2.9.3. *Suppose that $f : I \to X$ is c-almost periodic. Then $f(\cdot)$ is bounded.*

The following generalization of c-almost periodicity is meaningful, as well:

Definition 2.9.4. Let $c \in \mathbb{C} \setminus \{0\}$. Then a continuous function $f : I \to X$ is said to be c-uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to+\infty} \big\| f(\cdot + \alpha_n) - cf(\cdot) \big\|_\infty = 0. \tag{165}$$

If $c = -1$, then we also say that the function $f(\cdot)$ is uniformly anti-recurrent. The space consisting of all c-uniformly recurrent functions from the interval I into X will be denoted by $UR_c(I : X)$.

Define now $\mathbb{S} := \mathbb{N}$ if $I = [0, \infty)$, and $\mathbb{S} := \mathbb{Z}$ if $I = \mathbb{R}$. We will also consider the following notion:

Definition 2.9.5. Let $f \in C(I : X)$. It is said that $f(\cdot)$ is semi-c-periodic if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \big\| f(x + mp) - c^m f(x) \big\| \leqslant \varepsilon.$$

The space of all semi-c-periodic functions will be denoted by $\mathcal{SP}_c(I : X)$.

Suppose that $I = \mathbb{R}$, $f \in C(\mathbb{R} : X)$, $p > 0$ and $m \in \mathbb{N}$. Then we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \big\| f(x + mp) - c^m f(x) \big\| &= \sup_{x\in\mathbb{R}} \big\| f(x) - c^m f(x - mp) \big\| \\
&= \sup_{x\in\mathbb{R}} \big\| c^m \big[c^{-m} f(x) - f(x - mp) \big] \big\| = |c|^m \sup_{x\in\mathbb{R}} \big\| f(x - mp) - c^{-m} f(x) \big\| \\
&= \sup_{x\in\mathbb{R}} \big\| f(x - mp) - c^{-m} f(x) \big\| \in [0, \infty].
\end{aligned}$$

Therefore, we have the following:

Proposition 2.9.6. *Suppose that $f \in C(I : X)$. Then $f(\cdot)$ is semi-c-periodic if and only if*

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{N} \quad \forall x \in I \quad \big\| f(x + mp) - c^m f(x) \big\| \leqslant \varepsilon.$$

Furthermore, if $I = \mathbb{R}$, then the above is also equivalent with

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in -\mathbb{N} \quad \forall x \in I \quad \big\| f(x + mp) - c^m f(x) \big\| \leqslant \varepsilon.$$

It can be very simply shown that any semi-c-periodic function is bounded. Keeping in mind Proposition 2.9.6 and this observation, we may conclude that the notion introduced in Definition 2.9.5 is equivalent and extends the notion of semi-periodicity for case $c = 1$, introduced by J. Andres and D. Pennequin in [**25**], and the notion of semi-anti-periodicity for case $c = -1$, introduced by B. Chaouchi et al in [**94**].

We continue by providing several illustrative examples:

EXAMPLE 2.9.7. Let $f \equiv c \neq 0$. Due to (18), $f \notin ANP(\mathbb{R} : X)$ and clearly $f(\cdot)$ is not semi-anti-periodic. On the other hand, $f(\cdot)$ is periodic and therefore semi-periodic.

EXAMPLE 2.9.8. It can be simply verified that the function $f(x) := \sin x + \sin \pi x\sqrt{2}$, $x \in \mathbb{R}$ is almost anti-periodic but not semi-periodic (see, e.g., [**25**, Remark 3] and [**254**, Example 2.1]).

EXAMPLE 2.9.9. (a slight modification of [**25**, Example 1]) The function

$$f(x) := \sum_{n=1}^{\infty} \frac{e^{ix/(2n+1)}}{n^2}, \quad x \in \mathbb{R}$$

is semi-anti-periodic because it is a uniform limit of $[\pi \cdot (2n+1)!!]$-anti-periodic functions

$$f_N(x) := \sum_{n=1}^{\infty} \frac{e^{ix/(2n+1)}}{n^2}, \quad x \in \mathbb{R} \quad (N \in \mathbb{N}).$$

On the other hand, the function $f(\cdot)$ cannot be periodic.

EXAMPLE 2.9.10. Set $\mathbb{Q}_n := \{(2n+1)/(2m+1) : m,\ n \in \mathbb{Z}\}$. If $\theta > 0$ and $\sum_{\lambda \in \theta \cdot \mathbb{Q}_n} \|a_\lambda(f)\| < \infty$, then the function

$$f(t) := \sum_{\lambda \in \theta \cdot \mathbb{Q}_n} a_\lambda(f) e^{i\lambda t}, \quad t \in \mathbb{R}$$

is semi-anti-periodic. This can be inspected as in the proof of [**25**, Proposition 2] since the function $f_N(\cdot)$ used therein is anti-periodic with the anti-period $\pi q_1 \cdots q_N/\theta$.

The following important result holds true:

PROPOSITION 2.9.11. *Suppose that $f \in UR_c(I : X)$ and $c \in \mathbb{C} \setminus \{0\}$ satisfies $|c| \neq 1$. Then $f \equiv 0$.*

PROOF. Without loss of generality, we may assume that $I = [0, \infty)$. Suppose to the contrary that there exists $t_0 \geqslant 0$ such that $f(t_0) \neq 0$. Inductively, (165) implies

(166)

$$|c|^k m - \frac{|c|^k - 1}{n(|c|-1)} \leqslant \|f(t)\| \leqslant |c|^k M - \frac{|c|^k - 1}{n(|c|-1)}, \quad k \in \mathbb{N},\ t \in \big[k\alpha_n, (k+1)\alpha_n\big].$$

Consider now case $|c| < 1$. Let $0 < \varepsilon < c\|f(t_0)\|$. Then (166) yields that there exist integers $k_0 \in \mathbb{N}$ and $n \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ with $k \geqslant k_0$ we have $\|f(t)\| \leqslant \varepsilon/2$, $t \in [k\alpha_n, (k+1)\alpha_n]$. Then the contradiction is obvious because for each $m \in \mathbb{N}$ with $m > n$ there exists $k \in \mathbb{N}$ such that $t_0 + \alpha_m \in [k\alpha_n, (k+1)\alpha_n]$ and therefore $\|f(t_0 + \alpha_m)\| \geqslant c\|f(t_0)\| - (1/m) \to c\|f(t_0)\| > \varepsilon$, $m \to +\infty$. Consider now case $|c| > 1$; let $n \in \mathbb{N}$ be such that $\|f(t_0)\| > 1/(n(|c|-1))$ and $M := \max_{t\in[0,2\alpha_n]} \|f(t)\| > 0$. Then for each $m \in \mathbb{N}$ with $m > n$ there exists $k \in \mathbb{N}$ such that $\alpha_m \in [(k-1)\alpha_n, k\alpha_n]$ and therefore $\|f(t+\alpha_m)\| \leqslant 1+|c|M$, $t \in [0, 2\alpha_n]$. On the other hand, we obtain inductively from (165) that

$$\|f(t_0 + k\alpha_n)\| \geqslant |c|^k \left[\|f(t_0)\| - \frac{1}{n(|c|-1)}\right] + \frac{1}{n(|c|-1)} \to +\infty \quad \text{as } k \in \mathbb{N},$$

which immediately yields a contradiction. □

In accordance with the established result, it is reasonable to assume $|c| = 1$. This will be our standing assumption till the end of Subsection 2.9.2.

PROPOSITION 2.9.12. *Suppose that $I = \mathbb{R}$ and $f : \mathbb{R} \to X$. Then the function $f(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic) if and only if the function $\check{f}(\cdot)$ is $1/c$-almost periodic ($1/c$-uniformly recurrent, semi-$1/c$-periodic).*

Since for each numbers $t,\ \tau \in I$ and $m \in \mathbb{N}$ we have

$$\Big|\|f(t+\tau)\| - \|f(t)\|\Big| = \Big|\|f(t+\tau)\| - \|c^m f(t)\|\Big| \leqslant \|f(t+\tau) - c^m f(t)\|,$$

the following result simply follows:

PROPOSITION 2.9.13. *Suppose that $f : I \to X$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic). Then $\|f\| : I \to [0, \infty)$ is almost periodic (uniformly recurrent, semi-periodic).*

Further on, we have ($x \in I$, $\tau > 0$, $l \in \mathbb{N}$):

$$f\big(x + l\tau\big) - c^l f(x) = \sum_{j=0}^{l-1} c^j \Big[f\big(x + (l-j)\tau\big) - cf\big(x + (l-j-1)\tau\big)\Big]. \tag{167}$$

Hence,

$$\Big\|f\big(\cdot + l\tau\big) - c^l f(\cdot)\Big\|_\infty \leqslant l\Big\|f\big(\cdot + \tau\big) - cf(\cdot)\Big\|_\infty.$$

The above estimate can be used to prove the following:

PROPOSITION 2.9.14. *Let $f : I \to X$ be a c-almost periodic function (c-uniformly recurrent function, semi-c-periodic), and let $l \in \mathbb{N}$. Then $f(\cdot)$ is c^l-almost periodic (c^l-uniformly recurrent, semi-c^l-periodic).*

Consider now the following condition:

$$p \in \mathbb{Z} \setminus \{0\},\ q \in \mathbb{N},\ (p, q) = 1 \text{ and } \arg(c) = \pi p/q. \tag{168}$$

The next corollary of Proposition 2.9.14 follows immediately by plugging $l = q$:

COROLLARY 2.9.15. *Let $f : I \to X$ be a continuous function, and let* (195) *hold.*

(i) *If p is even and $f(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic), then $f(\cdot)$ is almost periodic (uniformly recurrent, semi-periodic).*
(ii) *If p is odd and $f(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic), then $f(\cdot)$ is almost anti-periodic (uniformly anti-recurrent, semi-anti-periodic).*

Therefore, if $\arg(c)/\pi \in \mathbb{Q}$, then the class of c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) is always contained in the class of almost periodic functions (uniformly recurrent functions, semi-periodic functions); in particular, we have that any almost anti-periodic function (uniformly anti-recurrent function, semi-anti-periodic function) is almost periodic (uniformly recurrent, semi-periodic).

Now we will prove the following:

PROPOSITION 2.9.16. *Let $f : I \to X$ be a continuous function, and let* $\arg(c)/\pi \notin \mathbb{Q}$.

(i) *If $f(\cdot)$ is c-almost periodic, then $f(\cdot)$ is c'-almost periodic for all $c' \in S_1$.*
(ii) *If $f(\cdot)$ is bounded and c-uniformly recurrent, then $f(\cdot)$ is c'-uniformly recurrent for all $c' \in S_1$.*

PROOF. We will prove only (i). Clearly, it suffices to consider the case in which the function $f(\cdot)$ is not identical to zero. Let $c' \in S_1$ and $\varepsilon > 0$ be fixed; then the prescribed assumption implies that the set $\{c^l : l \in \mathbb{N}\}$ is dense in S_1 and therefore there exists an increasing sequence (l_k) of positive integers such that $\lim_{k\to+\infty} c^{l_k} = c'$. By Proposition 2.9.3, the function $f(\cdot)$ is bounded; let $k \in \mathbb{N}$ be such that $|c^{l_k} - c'| < \varepsilon/(2\|f\|_\infty)$, and let $\tau > 0$ be any $(\varepsilon/2, c^{l_k})$-period for $f(\cdot)$. Then we have

$$\big\|f(x+\tau) - c'f(x)\big\| \leqslant \big\|f(x+\tau) - c^{l_k}f(x)\big\| + \big|c^{l_k} - c'\big| \cdot \|f\|_\infty < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for any $x \in I$. This simply completes the proof. $\square$

PROPOSITION 2.9.17. *Let $f : I \to X$ be a continuous function. Then we have the following:*

(i) *If $f(\cdot)$ is semi-c-periodic and* $\arg(c)/\pi \in \mathbb{Q}$, *then $f(\cdot)$ is c'-almost periodic for all $c' \in \{c^l : l \in \mathbb{N}\}$.*
(ii) *If $f(\cdot)$ semi-c-periodic and* $\arg(c)/\pi \notin \mathbb{Q}$, *then $f(\cdot)$ is c'-almost periodic for all $c' \in S_1$.*

PROOF. Let $\varepsilon > 0$ be fixed. To prove (i), it suffices to show that $f(\cdot)$ is c-almost periodic (see Proposition 2.9.14). Since $\arg(c)/\pi \in \mathbb{Q}$ and (195) holds, then we have $c^{1+2lq} = c$ for all $l \in \mathbb{N}$. Then there exists $p > 0$ such that, for every $m \in \mathbb{N}$ and $x \in I$, we have $\|f(x+mp) - c^m f(x)\| \leqslant \varepsilon$. With $m = 1+2lq$, we have $\|f(x+(1+2lq)p) - c^{1+2lq}f(x)\| = \|f(x+(1+2lq)p) - cf(x)\| \leqslant \varepsilon$ so that the conclusion follows from the fact that the set $\{(1+2lq)p : l \in \mathbb{N}\}$ is relatively dense in $[0,\infty)$. Assume now that $\arg(c)/\pi \notin \mathbb{Q}$. To prove (ii), it suffices to consider case $f \neq 0$. Observe first that Lemma 2.9.1 yields that there exists a strictly increasing sequence (l_k) of positive integers such that $\sup_{k\in\mathbb{N}}(l_{k+1}-l_k)<\infty$ and $|c^{l_k}-c'|<\varepsilon/\|f\|_\infty$ for all $k\in\mathbb{N}$. With this sequence and the number $p>0$ chosen as above, we have:

$$\begin{aligned}\big\|f(x+pl_k)-c'f(x)\big\| &\leqslant \big\|f(x+pl_k)-c^{l_k}f(x)\big\|+\big|c^{l_k}-c'\big|\|f\|_\infty\\ &\leqslant \varepsilon+\varepsilon\|f\|_\infty/\|f\|_\infty=2\varepsilon,\quad x\in I,\ k\in\mathbb{N}.\end{aligned}$$

Since the set $\{pl_k : k\in\mathbb{N}\}$ is relatively dense in $[0,\infty)$, the proof is completed. $\square$

In connection with Proposition 2.9.17(ii), it is natural to ask whether the assumptions that the function $f(\cdot)$ is semi-c-periodic and $\arg(c)/\pi\notin\mathbb{Q}$ imply that $f(\cdot)$ is semi-c'-almost periodic for all $c'\in S_1$?

We continue by providing the following extension of [**254**, Theorem 2.2] (see also [**62**, pp. 3-4]):

THEOREM 2.9.18. *Let $f : I\to X$ be c-almost periodic (c-uniformly recurrent, semi-c-periodic), and let $\alpha\in\mathbb{C}$. Then we have:*

(i) *$\alpha f(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic).*
(ii) *If $X=\mathbb{C}$ and $\inf_{x\in\mathbb{R}}|f(x)|=m>0$, then $1/f(\cdot)$ is $1/c$-almost periodic ($1/c$-uniformly recurrent, semi-$1/c$-periodic).*
(iii) *If $(g_n : I\to X)_{n\in\mathbb{N}}$ is a sequence of c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a function $g : I\to X$, then $g(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic).*
(iv) *If $a\in I$ and $b\in I\setminus\{0\}$, then the functions $f(\cdot+a)$ and $f(b\,\cdot)$ are likewise c-almost periodic (c-uniformly recurrent, semi-c-periodic).*

Let us recall that a continuous function $f : I\to X$ is called (p,c)-periodic if and only if $f(x+p)=cf(x)$, $x\in I$. We say that a function $f : I\to X$ is c-periodic if and only if there exists $p>0$ such that the function $f(\cdot)$ is (p,c)-periodic.

Keeping in mind Theorem 2.9.75(iii) and the proofs of [**25**, Lemma 1, Theorem 1], we can clarify the following extension of [**94**, Proposition 3]:

THEOREM 2.9.19. *Let $f\in C_b(I : X)$. Then $f(\cdot)$ is semi-c-periodic if and only if there exists a sequence (f_n) of c-periodic functions in $C_b(I : X)$ such that $\lim_{n\to\infty}f_n(x)=f(x)$ uniformly in I.*

We continue by providing two illustrative examples:

EXAMPLE 2.9.20. (see also [**254**, Example 2.2]) The function $f : \mathbb{R}\to\mathbb{R}$ given by $f(t):=\cos t$, $t\in\mathbb{R}$ is c-almost periodic (c-uniformly recurrent) if and only if $c=\pm1$, while $f(\cdot)$ is semi-c-periodic if and only if $c=1$; the function $f_\varphi : \mathbb{R}\to\mathbb{R}$ given by $f_\varphi(t):=e^{it\varphi}$, $t\in\mathbb{R}$ ($\varphi\in(-\pi,\pi]\setminus\{0\}$) is c-almost periodic (semi-c-periodic) for any $c\in S_1$, while the function $f_0(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic) if and only if $c=1$. Consider now the function $g : \mathbb{R}\to\mathbb{R}$ given by $g(t):=2^{-1}\cos 4t+2\cos 2t$, $t\in\mathbb{R}$. Then we know that the function $g(\cdot)$ is (almost) periodic and not almost anti-periodic. Now we will prove that $g(\cdot)$ is c-almost periodic (c-uniformly recurrent, semi-c-periodic) if and only if $c=1$.

Suppose that (α_n) is a strictly increasing sequence tending to plus infinity such that ($c = e^{i\alpha}$, $\alpha \in (-\pi, \pi]$):

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} \Bigl| 2^{-1} \cos\bigl(4t + \alpha_n\bigr) 2\cos\bigl(2t + \alpha_n\bigr) - e^{i\alpha} \bigl[2^{-1} \cos 4t + 2\cos 2t\bigr] \Bigr| = 0.$$

With $t = \pi$, the above implies

$$\lim_{n\to+\infty} [\cos 4\alpha_n + 4\cos 2\alpha_n - 5\cos\alpha] = 0 \quad \text{and} \quad \lim_{n\to+\infty} 5\sin\alpha = 0, \tag{169}$$

which immediately yields $\alpha = 0$ or $\alpha = \pi$. In the second case, the contradiction is obvious since the first limit equation in (169) cannot be fulfilled, while the case $\alpha = 0$ is possible and equivalent with the usual almost periodicity of $g(\cdot)$.

EXAMPLE 2.9.21. (see also [**25**, Example 1] and [**94**, Example 4, Example 5]) Let p and q be odd natural numbers such that $p-1 \equiv 0$ $(mod\ q)$, and let $c = e^{i\pi p/q}$. The function

$$f(x) := \sum_{n=1}^{\infty} \frac{e^{ix/(2nq+1)}}{n^2}, \quad x \in \mathbb{R}$$

is semi-c-periodic because it is a uniform limit of $[\pi \cdot (1+2q) \cdots (1+2Nq)]$-periodic functions

$$f_N(x) := \sum_{n=1}^{N} \frac{e^{ix/(2nq+1)}}{n^2}, \quad x \in \mathbb{R} \ \ (N \in \mathbb{N}).$$

Now we will state and prove the following

PROPOSITION 2.9.22. *Suppose that $f : I \to \mathbb{R}$ is c-uniformly recurrent (semi-c-periodic) and $f \neq 0$. Then $c = \pm 1$ and moreover, if $f(t) \geqslant 0$ for all $t \in I$, then $c = 1$.*

PROOF. We will consider the class of c-uniformly recurrent functions, only, when we may assume without loss of generality that $I = [0, \infty)$. Then $f \notin C_0([0, \infty) : \mathbb{R})$; namely, if we suppose the contrary, then there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and (165) holds. In particular, for every fixed number $t_0 \geqslant 0$ we have $\lim_{n\to+\infty} |f(t_0+\alpha_n) - cf(t_0)| = 0$. This automatically yields $f(t_0) = 0$ and, since $t_0 \geqslant 0$ was arbitrary, we get $f = 0$ identically, which is a contradiction. Therefore, there exist a strictly increasing sequence $(t_l)_{l\in\mathbb{N}}$ of positive real numbers tending to plus infinity and a positive real number $a \geqslant \limsup_{t\to+\infty} |f(t)| > 0$ such that $|f(t_l)| \geqslant a/2$ for all $l \in \mathbb{N}$. Let $\varepsilon > 0$ be fixed. Then there exist two real numbers $t_0 > 0$ and $n_0 \in \mathbb{N}$ such that $|f(t + \alpha_n) - f(t)| \leqslant \varepsilon$ for all $t \geqslant t_0$ and $n \geqslant n_0$. If $\arg(c) = \varphi \in (-\pi, \pi]$, then we particularly get that for each $t \geqslant t_0$ and $n \geqslant n_0$ we have:

$$\bigl|f(t + \alpha_n) - \cos\varphi \cdot f(t)\bigr| \leqslant \varepsilon \quad \text{and} \quad \bigl|\sin\varphi \cdot f(t)\bigr| \leqslant \varepsilon.$$

Plugging in the second estimate $t = t_l$ for a sufficiently large $l \in \mathbb{N}$ we get that $|\sin\varphi| \leqslant 2\varepsilon/a$. Since $\varepsilon > 0$ was arbitrary, we get $\sin\varphi = 0$ and $c = \pm 1$. Suppose, finally, that $f(t) \geqslant 0$ for all $t \geqslant 0$ and $c = -1$. Then we have $f(t + \alpha_n) + f(t) \leqslant 2\varepsilon$ for all $t \geqslant t_0$ and $n \geqslant n_0$. Plugging again $t = t_l$ for a sufficiently large $l \in \mathbb{N}$ we get that $a \leqslant \varepsilon$ for all $\varepsilon > 0$ and therefore $a = 0$, which is a contradiction. $\square$

By the proof of Proposition 2.9.22, we have:

PROPOSITION 2.9.23. *Suppose that $f : I \to X$ is c-uniformly recurrent (semi-c-periodic) and $f \neq 0$. Then $f \notin C_0(I : X)$.*

We continue by providing some illustrative applications of Proposition 2.9.22:

EXAMPLE 2.9.24. (i) The function $f : \mathbb{R} \to \mathbb{R}$, given by (33), is unbounded, uniformly continuous and uniformly recurrent. By the foregoing, $f(\cdot)$ is c-uniformly recurrent if and only if $c = 1$.

(ii) The function $g : \mathbb{R} \to \mathbb{R}$, given by

$$g(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\Bigl(\frac{t}{3^n}\Bigr)\,dt, \quad t \in \mathbb{R},$$

is unbounded, Lipschitz continuous and uniformly recurrent; furthermore, we have the existence of a positive integer $k_0 \in \mathbb{N}$ such that

$$\frac{1}{3^k\pi}\int_0^{3^k\pi} g(s)\,ds \geqslant \frac{1}{2}(\ln k - 1), \quad k \geqslant k_0 \tag{170}$$

and

$$\sup_{t\in\mathbb{R}}\bigl|g(t+3^n\pi) - g(t)\bigr| \leqslant \frac{\pi}{n+1}\sum_{j=1}^{\infty} 3^{-j}, \quad n \in \mathbb{N}. \tag{171}$$

This can be proved in exactly the same way as in the proof of [**202**, Theorem 1.1]. Define now $f(t) := \sin t \cdot g(t)$, $t \in \mathbb{R}$. Then (171) easily implies

$$\sup_{t\in\mathbb{R}}\bigl|f(t+3^n\pi) + f(t)\bigr| \leqslant \frac{\pi}{n+1}\sum_{j=1}^{\infty} 3^{-j}, \quad n \in \mathbb{N}.$$

Therefore, $f(\cdot)$ is uniformly anti-recurrent and Proposition 2.9.22 yields that the function $f(\cdot)$ is c-uniformly recurrent if and only if $c = \pm 1$. To prove that $f(\cdot)$ is Stepanov unbounded, observe that (170) implies the existence of a sequence $(t_k)_{k\in\mathbb{N}}$ of positive real numbers such that $g(t_k) \geqslant (1/2)(\ln k - 1)$ for all $k \geqslant k_0$. If we denote by $L \geqslant 1$ the Lipschitzian constant of mapping $g(\cdot)$, then the above implies

$$g(x) \geqslant (1/2)(\ln k - 1) - 8L\pi, \quad x \in \bigl[t_k, t_k + 8\pi\bigr],\ k \geqslant k_0. \tag{172}$$

The existence of a constant $M > 0$ such that $\int_t^{t+1} |\sin s| \cdot g(s)\,ds < M$ for all $t \in \mathbb{R}$ would imply by (172) the existence of a sequence (a_k) of positive integers such that $[2a_k\pi + (\pi/2), 2a_k\pi + (\pi/2) + 1] \subseteq \bigl[t_k, t_k + 8\pi\bigr]$ and therefore (take $t = 2a_k\pi + (\pi/2)$)

$$\sin((\pi/2)+1) \cdot \Bigl[(1/2)(\ln k - 1) - 8L\pi\Bigr] \leqslant M, \quad k \geqslant k_0,$$

which is a contradiction.

In connection with Proposition 2.9.22 and Proposition 2.9.23, we would like to present an illustrative example with the complex-valued functions:

EXAMPLE 2.9.25. Let $h : I \to \mathbb{R}$, $q : I \to \mathbb{R}$ and $f(t) := h(t) + iq(t)$, $t \in I$. Suppose that $f : I \to \mathbb{C}$ is c-uniformly recurrent, where $c = e^{i\varphi}$ and $\sin\varphi \neq 0$. Then $h \in C_0(I : \mathbb{R})$ or $q \in C_0(I : \mathbb{R})$ implies $f \equiv 0$. To show this, observe that the c-uniform recurrence of $f(\cdot)$ implies the existence of a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that

$$\lim_{n\to+\infty} \sup_{t\in I}\bigl|h(t+\alpha_n) - \cos\varphi \cdot h(t) + \sin\varphi \cdot q(t)\bigr| = 0, \text{ and}$$

$$\lim_{n\to+\infty} \sup_{t\in I}\bigl|q(t+\alpha_n) - \cos\varphi \cdot q(t) - \sin\varphi \cdot h(t)\bigr| = 0.$$

Since we have assumed that $\sin\varphi \neq 0$, the assumption $h \in C_0(I : \mathbb{R})$ ($q \in C_0(I : \mathbb{R})$) implies by the first (second) of the above equalities that $q \in C_0(I : \mathbb{R})$ ($h \in C_0(I : \mathbb{R})$). Hence, $f \in C_0(I : \mathbb{C})$ and the claimed statement follows by Proposition 2.9.23.

The space consisting of all almost periodic functions ($c = 1$) is the only function space from those introduced in Definition 2.9.2, Definition 2.9.4 and Definition 2.9.5 which has a linear vector structure:

EXAMPLE 2.9.26.
- (i) Suppose that $c = 1$. Then the set of all c-almost periodic functions is a vector space together with the usual operations, while the set of c-uniformly recurrent functions and the set of semi-c-periodic functions are not vector spaces together with the usual operations.
- (ii) Suppose that $c = -1$. Then the set of all c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) is not a vector space together with the usual operations ([**254**]).
- (iii) Suppose that $c \neq \pm 1$. Then the set of all c-almost periodic functions (c-uniformly recurrent functions, semi-c-periodic functions) is not a vector space together with the usual operations. Speaking-matter-of-factly, the functions $f_{\varphi,\pm} : \mathbb{R} \to \mathbb{R}$ given by $f_{\varphi,\pm}(t) := e^{\pm it\varphi}$, $t \in \mathbb{R}$ ($\varphi \in (-\pi, \pi] \setminus \{0\}$) are c-almost periodic (semi-c-periodic); see Example 2.9.20. Its sum $f_{\varphi,+}(\cdot) + f_{\varphi,-}(\cdot) = 2\cos\varphi \cdot$ is not c-uniformly recurrent due to Proposition 2.9.22.

Similarly, we have:

EXAMPLE 2.9.27. Let $f : I \to \mathbb{C}$ and $g : I \to X$.
- (i) Suppose that $c = 1$. If $f \in AP(I : \mathbb{C})$ and $g \in AP(I : X)$, then $f \cdot g \in AP(I : X)$; furthermore, there exist $f \in UR(I : \mathbb{C})$ and $g \in UR(I : X)$ such that $f \cdot g \notin UR(I : X)$ ([**248**]). It can be simply proved that the pointwise product of anti-periodic functions $f(t) := \cos t$, $t \in \mathbb{R}$ and $g(t) := \cos\sqrt{2}t$, $t \in \mathbb{R}$ is not a semi-periodic function (see e.g., [**25**, Lemma 2]).
- (ii) Suppose that $c = -1$. Then there exist an anti-periodic function $f(\cdot)$ and an anti-periodic function $g(\cdot)$ such that $f \cdot g(\cdot)$ is not anti-uniformly recurrent. We can simply take $X = \mathbb{C}$ and $f(t) := g(t) := \cos t$, $t \in I$.
- (iii) Suppose that $c \neq \pm 1$. Then there exist a semi-c-periodic function $f(\cdot)$ and a semi-c-periodic function $g(\cdot)$ such that $f \cdot g(\cdot)$ is not c-uniformly recurrent. Consider again the functions $f_{\varphi,\pm} : \mathbb{R} \to \mathbb{R}$ given by $f_{\varphi,\pm}(t) :=$

$e^{\pm it\varphi}$, $t \in \mathbb{R}$ ($\varphi \in (-\pi, \pi] \setminus \{0\}$). They are semi-$c$-periodic but their pointwise product $f_{\varphi,+}(\cdot) \cdot f_{\varphi,-}(\cdot) = 1$ is not c-uniformly recurrent due to Proposition 2.9.22.

Let us recall that $ANP_0(I : X)$ and $ANP(I : X)$ stand for the linear span of almost anti-periodic functions $I \mapsto E$ and its closure in $AP(I : X)$, respectively; by (18), we have $ANP(I : X) = AP_{\mathbb{R}\setminus\{0\}}(I : X)$. Now we will prove the following extension of this equality:

THEOREM 2.9.28. *Denote by* $AP_{c,0}(I : X)$ *and* $\mathbf{AP}_{c,0}(I : X)$ *the linear span of* c*-almost periodic functions* $f : I \to X$ *and its closure in* $AP(I : X)$*, respectively. Then the following holds:*

(i) *Let* $\arg(c) \in \pi \cdot \mathbb{Q}$. *Then we have* $\mathbf{AP}_{c,0}(I : X) = AP_{\mathbb{R}\setminus\{0\}}(I : X)$.
(ii) *Let* $\arg(c) \notin \pi \cdot \mathbb{Q}$. *Then we have* $\mathbf{AP}_{c,0}(I : X) \supseteq AP_{\mathbb{R}\setminus\{0\}}(I : X)$.

PROOF. Assume first that $f \in AP_{\mathbb{R}\setminus\{0\}}(I : X)$. By spectral synthesis, we have

$$f \in \overline{span\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\}},$$

where the closure is taken in the space $C_b(I : X)$. Since $\sigma(f) \subseteq \mathbb{R} \setminus \{0\}$ and the function $t \mapsto e^{i\mu t}$, $t \in I$ ($\mu \in \mathbb{R} \setminus \{0\}$) is c-almost periodic for all $c \in S_1$, we have that $span\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\} \subseteq AP_{c,0}(I : X)$. Hence, $f \in \mathbf{AP}_{c,0}(I : X)$. To complete the proof, it remains to consider case $\arg(c) \in \pi \cdot \mathbb{Q}$ and show that any function $f \in \mathbf{AP}_{c,0}(I : X)$ belongs to the space $AP_{\mathbb{R}\setminus\{0\}}(I : X)$. Furthermore, it suffices to consider case in which (195) holds with the number p even because otherwise we can apply Corollary 2.9.73(ii) and Proposition 2.9.74(i) to see that $AP_{c,0}(I : X) \subseteq ANP_0(I : X)$ and therefore $\mathbf{AP}_{c,0}(I : X) \subseteq ANP(I : X)$, so that the statement directly follows from [**254**, Theorem 2.3]. We will prove that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(s)\, ds = 0; \tag{173}$$

clearly, by almost periodicity of $f(\cdot)$, the limit in (173) exists. Let $\varepsilon > 0$ be fixed, and let $l > 0$ satisfy that every interval of $[0, \infty)$ of length l contains a point τ such that $\|f(t+\tau) - cf(t)\| \leqslant \varepsilon$, $t \geqslant 0$. We have $c^q = 1$ and therefore $1 + c + \cdots + c^{q-1} = 0$; using this equality and decomposition ($s \geqslant 0$, $n \in \mathbb{N}$)

$$\begin{aligned}
&\big\|f(s + (n-1)\tau) + f(s + (n-2)\tau) + \cdots + f(s)\big\| \\
&\quad \leqslant \varepsilon + \big\|(1 + c)f(s + (n-2)\tau) + f(s + (n-3)\tau) + \cdots + f(s)\big\| \\
&\quad \leqslant \varepsilon + \big\|(1 + c)f(s + (n-2)\tau) - (1 + c)cf(s + (n-3)\tau)\big\| \\
&\quad + \big\|[1 + (1 + c)c]f(s + (n-3)\tau) + f(s + (n-4)\tau) \cdots + f(s)\big\| \\
&\quad \leqslant \varepsilon + |1 + c|\varepsilon + \Big\|[1 + c + c^2]f(s + (n-3)\tau) + f(s + (n-4)\tau) + \cdots + f(s)\Big\| \\
&\quad \leqslant \varepsilon + |1 + c|\varepsilon + \big|1 + c + c^2\big|\varepsilon + ... \\
&\quad \leqslant \varepsilon + |1 + c|\varepsilon + \big|1 + c + c^2\big|\varepsilon + ... + \big|1 + c + c^2 + \cdots + c^{q-2}\big|\varepsilon \\
&\quad + \big\|f(s) + f(s + \tau) + \cdots + f(s + (n - 1 - q)\tau)\big\|,
\end{aligned}$$

we immediately get that there exists a finite constant $A \geqslant 1$ such that, for every $s \geqslant 0$ and $n \in \mathbb{N}$,

$$\big\| f(s+(n-1)\tau) + f(s+(n-2)\tau) + \cdots + f(s) \big\| \leqslant A\varepsilon \lceil n/q \rceil + A\|f\|_\infty .$$

Integrating this estimate over the segment $[0, n\tau]$, we get that, for every $s \geqslant 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \int_0^{n\tau} f(s)\, ds \right\| &= \left\| \int_0^{\tau} \big[f(s+(n-1)\tau) + f(s+(n-2)\tau) + \cdots + f(s) \big]\, ds \right\| \\ &\leqslant A\tau\varepsilon \lceil n/q \rceil + A\tau \|f\|_\infty . \end{aligned}$$

Dividing the both sides of the above inequality with $n\tau$, we get that

$$\lim_{n \to +\infty} \left\| \frac{1}{n\tau} \int_0^{n\tau} f(s)\, ds \right\| \leqslant A\varepsilon / q.$$

Since $\varepsilon > 0$ was arbitrary, this immediately yields (173). $\square$

Now we will state and prove the following result:

Proposition 2.9.29. *Suppose that $f : [0, \infty) \to X$ is c-almost periodic (semi-c-periodic). Then $\mathbb{E}f : \mathbb{R} \to X$ is a unique c-almost periodic extension (semi-c-periodic extension) of $f(\cdot)$ to the whole real axis.*

Proof. The proof for the class of c-almost periodic functions is very similar to the proof of [**254**, Proposition 2.2] and therefore omitted. For the class of semi-c-periodic functions, the proof can be deduced as follows. Due to Proposition 2.9.17, we have that the function $f : [0, \infty) \to X$ is almost periodic, so that the function $\mathbb{E}f : \mathbb{R} \to X$ is a unique almost periodic extension of $f(\cdot)$ to the whole real axis. Therefore, it remains to be proved that $\mathbb{E}f(\cdot)$ is semi-c-periodic. Let $\varepsilon > 0$ be fixed. Then there exists $p > 0$ such that for all $m \in \mathbb{N}$ and $x \geqslant 0$ we have $\|f(x + mp) - c^m f(x)\| \leqslant \varepsilon$. For every fixed number $m \in \mathbb{N}$, the function $\mathbb{E}f(\cdot + mp) - c^m \mathbb{E}f(\cdot)$ is almost periodic so that the supremum formula implies

$$\begin{aligned} \sup_{x \in \mathbb{R}} \big\| \mathbb{E}f(x+mp) - c^m \mathbb{E}f(x) \big\| &= \sup_{x \geqslant 0} \big\| \mathbb{E}f(x+mp) - c^m \mathbb{E}f(x) \big\| \\ &= \sup_{x \geqslant 0} \big\| f(x+mp) - c^m f(x) \big\| \leqslant \varepsilon. \end{aligned}$$

This completes the proof. $\square$

We continue by introducing the following notion:

Definition 2.9.30. A continuous function $f : I \to X$ is called asymptotically c-uniformly recurrent (asymptotically c-almost periodic, asymptotically semi-c-periodic) if and only if there are a c-uniformly recurrent (c-almost periodic, semi-c-periodic) function $g : \mathbb{R} \to X$ and a function $h \in C_0(I : X)$ such that $f(x) = g(x) + h(x)$, $x \in I$.

Definition 2.9.31. Let $p \in \mathcal{P}([0,1])$, and let $f \in L^{p(x)}_{loc}(I : X)$.

(i) It is said that $f(\cdot)$ is Stepanov $(p(x), c)$-uniformly recurrent (Stepanov $(p(x), c)$-almost periodic, Stepanov semi-$(p(x), c)$-periodic) if and only if the function $\hat{f} : I \to L^{p(x)}([0, 1] : X)$, defined by (21), is c-uniformly recurrent (c-almost periodic, semi-c-periodic).

(ii) It is said that $f(\cdot)$ is asymptotically Stepanov $(p(x), c)$-uniformly recurrent (asymptotically Stepanov $(p(x), c)$-almost periodic, asymptotically Stepanov semi-$(p(x), c)$-periodic) if and only if there are a Stepanov $(p(x), c)$-uniformly recurrent (Stepanov $(p(x), c)$-almost periodic, Stepanov semi-$(p(x), c)$-periodic) function $h(\cdot)$ and $q \in C_0(I : L^{p(x)}([0, 1] : X))$ such that $f(t) = h(t) + q(t)$ for a.e. $t \in I$.

If $p(x) \equiv p \in [1, \infty)$, then we also say that the function $f(\cdot)$ is Stepanov (p, c)-uniformly recurrent (Stepanov (p, c)-almost periodic, Stepanov semi-(p, c)-periodic) and so on.

In case $c = 1$, resp. $c = -1$, we also say that an (asymptotically) Stepanov $(p(x), c)$-uniformly recurrent ((asymptotically) Stepanov $(p(x), c)$-almost periodic/ (asymptotically) Stepanov semi-$(p(x), c)$-periodic) function is (asymptotically) Stepanov $p(x)$-uniformly recurrent, resp. (asymptotically) Stepanov $p(x)$-uniformly anti-recurrent ((asymptotically) Stepanov $p(x)$-almost periodic, resp. (asymptotically) Stepanov $p(x)$-almost anti-periodic/(asymptotically) Stepanov semi-$p(x)$-periodic, resp. (asymptotically) Stepanov semi-$p(x)$-anti-periodic).

QUESTION 2.9.32. Assume that $\alpha,\ \beta \in \mathbb{R}$ and $\alpha\beta^{-1}$ is a well-defined irrational number. We would like to raise the question whether the functions $f(\cdot)$ and $g(\cdot)$, given by (22) and (23) respectively, are Stepanov q-semi-periodic for any $1 \leqslant q < \infty$?

EXAMPLE 2.9.33. Let us consider the function $f(x) := \sin x + \sin \pi x\sqrt{2}$, $x \in \mathbb{R}$. Then a simple analysis involving the identity $f(x) = 2 \sin x \frac{1+\pi\sqrt{2}}{2} \cos x \frac{\pi\sqrt{2}-1}{2}$, $x \in \mathbb{R}$ shows that the function $\operatorname{sign}(f(\cdot))$ is identically equal to a function $F(\cdot)$ of the following, much more general form: Let $(a_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers satisfying $\lim_{n\to+\infty}(a_{n+1} - a_n) = +\infty$, $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to-\infty} a_n = -\infty$. Suppose that $(b_n)_{n\in\mathbb{Z}}$ is a sequence of non-zero real numbers satisfying that the sets $\{n \in \mathbb{Z} : b_n < 0\}$ and $\{n \in \mathbb{Z} : b_n > 0\}$ are infinite, as well as that there exists a finite positive constant $c > 0$ such that $c \leqslant |b_n - b_l|$ for any $n,\ l \in \mathbb{Z}$ such that $b_n b_l < 0$. Define the function $F : \mathbb{R} \to \mathbb{R}$ by $F(x) := b_n$ if $x \in [a_n, a_{n+1})$, for any $n \in \mathbb{Z}$. Then $F(\cdot)$ cannot be Stepanov q-semi-periodic for any finite real number $q \geqslant 1$. Otherwise, for $\varepsilon \in (0, c^q)$ we would be able to find a number $p > 0$ such that for each $m \in \mathbb{Z}$ and $x \in \mathbb{R}$ one has:

$$\int_0^1 \Big|F(x + mp + s) - F(x + s)\Big|^q\, ds < (1/2)^q.$$

Let $n \in \mathbb{Z}$ be such that $[x, x + 1] \subseteq [a_n, a_{n+1})$ and $b_n < 0$, say. Without loss of generality, we may assume that the set $\{n \in \mathbb{N} : b_n > 0\}$ is infinite. Then the contradiction is obvious because, for every sufficiently large numbers $l \in \mathbb{N}$ with

$b_l > 0$, we can find $m \in \mathbb{N}$ such that $[x+mp, x+mp+1] \subseteq [a_l, a_{l+1})$ so that

$$\int_0^1 \Bigl|F(x+mp+s) - F(x+s)\Bigr|^q \, ds \geqslant \bigl|b_n - b_l\bigr|^q \geqslant c^q.$$

In the remainder of this subsection, we will present two statements concerning the invariance of c-almost periodicity, c-uniform recurrence and semi-c-periodicity under the actions of infinite convolution products. We first state the following slight generalization of [**254**, Proposition 3.1], which can be deduced by using almost the same arguments as in the proof of Proposition 2.4.39 (similarly we can generalize [**254**, Proposition 3.2] for asymptotical c-almost type periodicity):

PROPOSITION 2.9.34. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$ and $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(x)}[0,1]} < \infty$. If $\check{f} : \mathbb{R} \to X$ is is Stepanov $(p(x),c)$-almost periodic (Stepanov $p(x)$-bounded and Stepanov $(p(x),c)$-uniformly recurrent/Stepanov $p(x)$-bounded and Stepanov semi-$(p(x),c)$-periodic), then the function $F(\cdot)$, given by* (55)*, is well-defined and c-almost periodic (bounded c-uniformly recurrent/bounded and semi-c-periodic).*

We can also consider the situation in which the forcing term $f(\cdot)$ is not Stepanov $p(x)$-bounded (see Propostion 2.4.41):

PROPOSITION 2.9.35. *Suppose that $p,\ q \in \mathcal{P}([0,1])$, $1/p(x) + 1/q(x) = 1$, $\check{f} : \mathbb{R} \to X$ is Stepanov $(p(x),c)$-almost periodic (Stepanov $(p(x),c)$-uniformly recurrent/Stepanov semi-$(p(x),c)$-periodic), $(R(t))_{t>0} \subseteq L(X,Y)$ is a strongly continuous operator family and there exists a continuous function $P : \mathbb{R} \to [1,\infty)$ such that* (56)-(57) *hold. If the function $\hat{f} : \mathbb{R} \to L^{p(x)}([0,1] : X)$ is uniformly continuous, then the function $F : \mathbb{R} \to Y$, given by* (55)*, is well-defined and c-almost periodic (c-uniformly recurrent/semi-c-periodic).*

2.9.1. Composition principles for c-almost periodic type functions. In this subsection, we will clarify and prove several composition principles for c-almost periodic functions and c-uniformly recurrent functions.

Suppose that $F : I \times Y \to X$ is a continuous function and there exists a finite constant $L > 0$ such that (60) holds. Define $\mathcal{F}(t) := F(t, f(t))$, $t \in I$. We need the following estimates ($\tau \geqslant 0$, $c \in \mathbb{C} \smallsetminus \{0\}$, $t \in I$):

$$\begin{aligned} &\Bigl\|F(t+\tau, f(t+\tau)) - cF(t, f(t))\Bigr\| \\ &\leqslant \Bigl\|F(t+\tau, f(t+\tau)) - F\Bigl(t+\tau, cf(t)\Bigr)\Bigr\| + \Bigl\|F\Bigl(t+\tau, cf(t)\Bigr) - cF(t, f(t))\Bigr\| \\ &\leqslant L\Bigl\|f(t+\tau) - cf(t)\Bigr\| + \Bigl\|F\Bigl(t+\tau, cf(t)\Bigr) - cF(t, f(t))\Bigr\|. \end{aligned} \tag{174}$$

Using (174), we can simply deduce the following result:

THEOREM 2.9.36. *Suppose that $F : I \times Y \to X$ is a continuous function and there exists a finite constant $L > 0$ such that* (60) *holds.*

(i) *Suppose that $f : I \to Y$ is c-uniformly recurrent. If there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that*

$$\lim_{n\to+\infty} \sup_{t\in I} \bigl\| f(t+\alpha_n) - cf(t) \bigr\| = 0 \tag{175}$$

and

$$\lim_{n\to+\infty} \sup_{t\in I} \Bigl\| F\Bigl(t+\alpha_n, cf(t)\Bigr) - cF(t, f(t)) \Bigr\| = 0, \tag{176}$$

then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is c-uniformly recurrent.

(ii) *Suppose that $f : I \to Y$ is c-almost periodic. If for each $\varepsilon > 0$ the set of all positive real numbers $\tau > 0$ such that*

$$\sup_{t\in I} \bigl\| f(t+\tau) - cf(t) \bigr\| < \varepsilon \tag{177}$$

and

$$\sup_{t\in I} \Bigl\| F\Bigl(t+\tau, cf(t)\Bigr) - cF(t, f(t)) \Bigr\| < \varepsilon, \tag{178}$$

is relatively dense in $[0, \infty)$, then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is c-almost periodic.

For the class of asymptotically c-almost periodic functions, the following result simply follows from the previous theorem and the argumentation used in the proof of [**135**, Theorem 3.49]:

THEOREM 2.9.37. *Suppose that $F : I \times Y \to X$ and $Q : I \times Y \to X$ are continuous functions and there exists a finite constant $L > 0$ such that* (60) *holds as well as that* (60) *holds with the function $F(\cdot, \cdot)$ replaced therein with the function $Q(\cdot, \cdot)$.*

(i) *Suppose that $g : I \to E$ is a c-uniformly recurrent function, $h \in C_0(I : Y)$ and $f(x) = g(x) + h(x)$, $x \in I$. If there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that* (175) *and* (176) *hold with the function $f(\cdot)$ replaced therein with the function $g(\cdot)$, $\lim_{|t|\to+\infty} Q(t, y) = 0$ uniformly for $y \in R(f)$, then the mapping $\mathcal{H}(t) := (F + Q)(t, f(t))$, $t \in I$ is asymptotically c-uniformly recurrent.*

(ii) *Suppose that $g : I \to Y$ is a c-almost periodic function, $h \in C_0(I : Y)$ and $f(x) = g(x) + h(x)$, $x \in I$. If for each $\varepsilon > 0$ the set of all positive real numbers $\tau > 0$ such that* (177) *and* (178) *hold with the function $f(\cdot)$ replaced therein with the function $g(\cdot)$, $\lim_{|t|\to+\infty} Q(t, y) = 0$ uniformly for $y \in R(f)$, then the mapping $\mathcal{H}(t) := (F + Q)(t, f(t))$, $t \in I$ is asymptotically c-almost periodic.*

For the Stepanov classes, we can also clarify certain results:

THEOREM 2.9.38. *Let $p(x)$, $q(x) \in [1, \infty)$, $r(x) \in [1, \infty]$, $1/p(x) = 1/q(x) + 1/r(x)$ and the following conditions hold:*

(i) *Let $F : I \times Y \to X$ and let there exist a function $L_F \in L_S^{r(x)}(I)$ such that* (25) *holds.*

(ii) *There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n\to+\infty}\sup_{t\in I}\sup_{u\in R(f)}\Bigl\|F\bigl(s+t+\alpha_n,cu\bigr)-cF(s+t,u)\Bigr\|_{L^{p(s)}[0,1]}=0 \tag{179}$$

and

$$\lim_{n\to+\infty}\sup_{t\in I}\Bigl\|f(s+t+\alpha_n)-cf(s+t)\Bigr\|_{L^{q(s)}[0,1]}=0. \tag{180}$$

Then the function $F(\cdot,f(\cdot))$ is Stepanov $(p(x),c)$-uniformly recurrent. Furthermore, the assumption that $F(\cdot,0)$ is Stepanov $p(x)$-bounded implies that the function $F(\cdot,f(\cdot))$ is Stepanov $p(x)$-bounded, as well.

Similarly, we can prove the following

THEOREM 2.9.39. *Suppose that $p\in\mathcal{P}([0,1])$ and the following conditions hold:*

(i) *Let $F:I\times Y\to X$ and there exist a function $r(x)\geqslant\max(p(x),p(x)/(p(x)-1))$ and a function $L_F\in L_S^{r(x)}(I)$ such that (25) holds.*

(ii) *There exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that (179) holds and (180) holds with the function $q(\cdot)$ replaced by the function $p(\cdot)$ therein.*

Then $q(x):=p(x)r(x)/(p(x)+r(x))$ for $x\in[0,1]$ and $r(x)<+\infty$ and $q(x):=p(x)$ for $x\in[0,1]$ and $r(x)=+\infty$. Then the function $F(\cdot,f(\cdot))$ is Stepanov $(q(x),c)$-uniformly recurrent. Furthermore, the assumption that $F(\cdot,0)$ is Stepanov $q(x)$-bounded implies that the function $F(\cdot,f(\cdot))$ is Stepanov $q(x)$-bounded, as well.

The above results can be simply reformulated for the class of Stepanov $(p(x),c)$-almost periodic functions. For the classes of asymptotically Stepanov $(p(x),c)$-uniformly recurrent (asymptotically Stepanov $(p(x),c)$-almost periodic) functions, we can simply extend the assertions of [**234**, Proposition 2.7.3, Proposition 2.7.4]. Details can be left to the interested readers.

2.9.2. Applications to the abstract Volterra integro-differential inclusions. In this subsection, we will present some illustrative applications of our abstract results in the analysis of the existence and uniqueness of c-almost periodic type solutions to the abstract (semilinear) Volterra integro-differential inclusions.

Concerning semilinear problems, we can apply our results in the study of the existence and uniqueness of c-almost periodic solutions and c-uniformly recurrent solutions of the fractional semilinear Cauchy inclusion (149), where $D_{t,+}^{\gamma}$ denotes the Riemann-Liouville fractional derivative of order $\gamma\in(0,1)$, $F:\mathbb{R}\times Y\to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator satisfying condition [**234**, (P)]. To explain this in more detail, fix a strictly increasing sequence (α_n) of positive reals tending to plus infinity and define

$$BUR_{(\alpha_n);c}(\mathbb{R}:X):=\Bigl\{f\in UR_c(\mathbb{R}:X)\,;\,f(\cdot)\text{ is bounded and}$$
$$\lim_{n\to+\infty}\sup_{t\in\mathbb{R}}\bigl\|f(t+\alpha_n)-cf(t)\bigr\|_{\infty}=0\Bigr\}.$$

Equipped with the metric $d(\cdot,\cdot) := \|\cdot-\cdot\|_{\infty}$, $BUR_{(\alpha_n);c}(\mathbb{R} : X)$ becomes a complete metric space. Let $(R_\gamma(t))_{t>0}$ be the operator family considered in [**234**]. It is said that a continuous function $u : \mathbb{R} \to X$ is a mild solution of (149) if and only if

$$u(t) = \int_{-\infty}^{t} R_\gamma(t-s)F\bigl(s,u(s)\bigr)\,ds, \quad t \in \mathbb{R}.$$

Now we are able to state the following result, which is very similar to [**234**, Theorem 3.1] (for simplicity, we will consider the constant coefficient $p(x) \equiv p > 1$ here):

THEOREM 2.9.40. *Suppose that the function $F : \mathbb{R} \times X \to X$ satisfies that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\in\mathbb{R}} \sup_{y\in B} \|F(t,y)\| \leqslant M_B$. Suppose, further, that the function $F : \mathbb{R} \times X \to X$ is Stepanov (p,c)-uniformly recurrent with $p > 1$, and there exist a number $r \geqslant \max(p, p/(p-1))$ and a function $L_F \in L_S^r(I)$ such that $q := pr/(p+r) > 1$ and* (25) *holds with $I = \mathbb{R}$. If* (68) *holds and there exists an integer $n \in \mathbb{N}$ such that $M_n < 1$, where*

$$\begin{aligned} M_n := \sup_{t\geqslant 0} \int_{-\infty}^{t} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \Bigl\| R_\gamma(t-x_n) \Bigr\| \\ \times \prod_{i=2}^{n} \Bigl\| R_\gamma(x_i - x_{i-1}) \Bigr\| \prod_{i=1}^{n} L_F(x_i)\,dx_1\,dx_2 \cdots\, dx_n, \end{aligned}$$

and (179) *holds with the set $R(f)$ replaced therein with an arbitrary bounded set $B \subseteq X$, then the abstract semilinear fractional Cauchy inclusion* (149) *has a unique bounded uniformly recurrent solution which belongs to the space $BUR_{(\alpha_n);c}(\mathbb{R} : X)$.*

PROOF. Define $\Upsilon : BUR_{(\alpha_n);c}(\mathbb{R} : X) \to BUR_{(\alpha_n);c}(\mathbb{R} : X)$ by

$$(\Upsilon u)(t) := \int_{-\infty}^{t} R_\gamma(t-s)F(s,u(s))\,ds, \quad t \in \mathbb{R}.$$

Suppose that $u \in BUR_{(\alpha_n);c}(\mathbb{R} : X)$. Then $R(u) = B$ is a bounded set and the mapping $t \mapsto F(t,u(t))$, $t \in \mathbb{R}$ is bounded due to the prescribed assumption. Applying Theorem 2.9.39, we have that the function $F(\cdot,u(\cdot))$ is Stepanov (q,c)-uniformly recurrent. Define $q' := q/(q-1)$. By (66) and (68), we have $\|R_\gamma(\cdot)\| \in L^{q'}[0,1]$ and $\sum_{k=0}^{\infty} \|R_\gamma(\cdot)\|_{L^{q'}[k,k+1]} < \infty$. Applying Proposition 2.9.34, we get that the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t-s)F(s,u(s))\,ds$, $t \in \mathbb{R}$ is bounded and c-uniformly recurrent, implying that $\Upsilon u \in BUR_{(\alpha_n);c}(\mathbb{R} : X)$, as claimed. Furthermore, a simple calculation shows that

$$\Bigl\| \bigl(\Upsilon^n u_1\bigr) - \bigl(\Upsilon^n u_2\bigr) \Bigr\|_{\infty} \leqslant M_n \bigl\| u_1 - u_2 \bigr\|_{\infty}, \quad u_1,\ u_2 \in BUR_{(\alpha_n);c}(\mathbb{R} : X),\ n \in \mathbb{N}.$$

Since there exists an integer $n \in \mathbb{N}$ such that $M_n < 1$, the well known extension of the Banach contraction principle shows that the mapping $\Upsilon(\cdot)$ has a unique fixed point, finishing the proof of the theorem. $\square$

Similarly we can analyze the existence and uniqueness of asymptotically Stepanov (p, c)-almost periodic solutions and Stepanov (p, c)-uniformly recurrent solutions of the fractional semilinear Cauchy inclusion $(DFP)_{f,\gamma,s}$.

As mentioned earlier, the unique regular solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial condition $u(x,0) = f(x)$, is given by (145). Let the number $t_0 > 0$ be fixed, and let the function $f(\cdot)$ be bounded c-uniformly recurrent (c-almost periodic, semi-c-periodic). Since $e^{-\cdot^2/4t_0} \in L^1(\mathbb{R})$, we can use the fact that the space of bounded c-uniformly recurrent functions (c-almost periodic functions, semi-c-periodic functions) is convolution invariant in order to see that the solution $x \mapsto u(x, t_0)$, $x \in \mathbb{R}$ is bounded and c-uniformly recurrent (c-almost periodic, semi-c-periodic).

2.9.3. Semi-c-periodic functions. Let us recall that $\mathbb{S} := \mathbb{N}$ if $I = [0, \infty)$, and $\mathbb{S} := \mathbb{Z}$ if $I = \mathbb{R}$. In this subsection, we will first extend the notion introduced in Definition 2.9.5 with general parameter $c \in \mathbb{C} \setminus \{0\}$:

DEFINITION 2.9.41. Let $f \in C(I : X)$.

(i) It is said that $f(\cdot)$ is semi-c-periodic of type 1 if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \big\| f(x + mp) - c^m f(x) \big\| \leqslant \varepsilon. \tag{181}$$

(ii) It is said that $f(\cdot)$ is semi-c-periodic of type 2 if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \big\| c^{-m} f(x + mp) - f(x) \big\| \leqslant \varepsilon. \tag{182}$$

The space of all semi-c-periodic functions of type i will be denoted by $\mathcal{SP}_{c,i}(I : X)$, $i = 1, 2$.

DEFINITION 2.9.42. Let $f \in C(I : X)$.

(i) It is said that $f(\cdot)$ is semi-c-periodic of type 1_+ if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{N} \quad \forall x \in I \quad \big\| f(x + mp) - c^m f(x) \big\| \leqslant \varepsilon. \tag{183}$$

(ii) It is said that $f(\cdot)$ is semi-c-periodic of type 2_+ if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{N} \quad \forall x \in I \quad \big\| c^{-m} f(x + mp) - f(x) \big\| \leqslant \varepsilon. \tag{184}$$

The space of all semi-c-periodic functions of type i_+ will be denoted by $\mathcal{SP}_{c,i,+}(I : X)$, $i = 1, 2$.

We have already seen that the notion of a semi-c-periodicity of type i (i_+), where $i = 1, 2$, is equivalent with the notion of semi-c-periodicity introduced in Definition 2.9.5, provided that $|c| = 1$.

Now we will focus our attention to the general case $c \in \mathbb{C} \setminus \{0\}$. We will first state the following:

LEMMA 2.9.43. (i) *If $|c| \geqslant 1$ and $f : I \to X$ is semi-c-periodic of type 1_+, then $f(\cdot)$ is semi-c-periodic of type 2_+.*

(ii) *If $|c| \leqslant 1$ and $f : I \to X$ is semi-c-periodic of type 2_+, then $f(\cdot)$ is semi-c-periodic of type 1_+.*

PROOF. If $x \in I$, $p > 0$, $m \in \mathbb{N}$ and $|c| \geqslant 1$, then we have

$$\left\|f(x+mp) - c^m f(x)\right\| \leqslant \varepsilon \Rightarrow \left\|c^{-m} f(x+mp) - f(x)\right\| \leqslant \varepsilon,$$

which implies (i); the proof of (ii) is similar. □

Using the proofs of [**25**, Lemma 1, Theorem 1], we can clarify the following important lemma:

LEMMA 2.9.44. *Suppose that* $|c| \leqslant 1$, *resp.* $|c| \geqslant 1$, *and* $f : [0,\infty) \to X$ *is semi-c-periodic of type* 1_+, *resp.* 2_+. *Then there exists a sequence* $(f_n : [0,\infty) \to X)_{n\in\mathbb{N}}$ *of c-periodic functions which converges uniformly to* $f(\cdot)$.

Now we are able to prove the following result:

THEOREM 2.9.45. *Let* $|c| \neq 1$, $i = 1, 2$ *and* $f : I \to X$. *Then* $f(\cdot)$ *is c-periodic if and only if* $f(\cdot)$ *is semi-c-periodic of type* i (i_+).

PROOF. Suppose that the function $f(\cdot)$ is (p, c)-periodic. Then we have $f(x + mp) = c^m f(x)$, $x \in I$, $m \in \mathbb{S}$, so that $f(\cdot)$ is automatically semi-c-periodic of type i (i_+). To prove the converse statement, let us observe that any semi-c-periodic function of type i is clearly semi-c-periodic of type i_+. Suppose first that $|c| > 1$. Due to Lemma 2.9.43(i), it suffices to show that, if $f(\cdot)$ is semi-c-periodic of type 2_+, then $f(\cdot)$ is c-periodic. Assume first $I = [0,\infty)$. Using Lemma 2.9.44, we get the existence of a sequence $(f_n : [0,\infty) \to X)_{n\in\mathbb{N}}$ of c-periodic functions which converges uniformly to $f(\cdot)$. Let $f_n(t + p_n) = cf_n(t)$, $t \geqslant 0$ for some sequence (p_n) of positive real numbers. Consider first case that (p_n) is bounded. Then there exist a strictly increasing sequence (n_k) of positive integers and a number $p \geqslant 0$ such that $\lim_{k\to+\infty} p_{n_k} = p$. Let $\varepsilon > 0$ be given. Then there exists an integer $k_0 \in \mathbb{N}$ such that $\|f(t) - f_{n_k}(t)\| \leqslant \varepsilon/(2+2|c|^{-1})$ for all real numbers $t \geqslant 0$ and all integers $k \geqslant k_0$. Furthermore, we have

$$\begin{aligned}
&\left\|c^{-1} f(t + p_{n_k}) - f(t)\right\| \\
&\leqslant \left\|c^{-1} f(t + p_{n_k}) - c^{-1} f_{n_k}(t + p_{n_k})\right\| \\
&+ \left\|c^{-1} f_{n_k}(t + p_{n_k}) - f_{n_k}(t)\right\| + \left\|f_{n_k}(t) - f(t)\right\| \\
&= \left\|c^{-1} f(t + p_{n_k}) - c^{-1} f_{n_k}(t + p_{nk})\right\| + \left\|f_{n_k}(t) - f(t)\right\| \\
&\leqslant 2(1 + |c|^{-1})\varepsilon/(2 + 2|c|^{-1}) = \varepsilon,
\end{aligned}$$

for all real numbers $t \geqslant 0$ and all integers $k \geqslant k_0$. Letting $k \to +\infty$ we get $f(t + p) = cf(t)$ for all $t \geqslant 0$. If $p > 0$ the above yields that $f(\cdot)$ is (p, c)-periodic, while the assumption $p = 0$ yields $f \equiv 0$ or $c = 1$, i.e., $f(\cdot) \equiv 0$; in any case, $f(\cdot)$ is (p, c)-periodic. Suppose now that (p_n) is unbounded. Then, with the same notation as above, we may assume that $\lim_{k\to+\infty} p_{n_k} = +\infty$. Using the same computation, it follows that $\lim_{k\to+\infty}\left\|c^{-1} f(\cdot + p_{n_k}) - f(\cdot)\right\|_\infty = 0$, so that $f \in UR_c([0,\infty) : X)$. Due to Proposition 2.9.11, we get $f(\cdot) \equiv 0$. Assume now $I = \mathbb{R}$. By the foregoing arguments, we know that there exists $p > 0$ such that $f(x+p) = cf(x)$ for all $x \geqslant 0$. Let $x < 0$ and $\varepsilon > 0$ be fixed. Since $f(\cdot)$ is semi-c-periodic, there exists $p_\varepsilon > 0$ such that $\|c^{-m} f(x + p + mp_\varepsilon) - f(x + p)\| \leqslant \varepsilon$ and $\|c^{1-m} f(x + mp_\varepsilon) - cf(x)\| \leqslant \varepsilon$

for all $m \in \mathbb{N}$. For all sufficiently large integers $m \in \mathbb{N}$ we have $x + mp_\varepsilon > 0$ so that $c^{-m} f(x+p+mp_\varepsilon) = c^{1-m} f(x+mp_\varepsilon)$ and therefore $\|f(x+p) - cf(x)\| \leqslant 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $f(x+p) = cf(x)$, which completes the proof in case $|c| > 1$. Suppose now that $|c| < 1$. Due to Lemma 2.9.43(ii), it suffices to show that, if $f(\cdot)$ is semi-c-periodic of type 1_+, then $f(\cdot)$ is c-periodic. But, then we can apply Lemma 2.9.44 again and the similar arguments as above to complete the whole proof. □

COROLLARY 2.9.46. *Let $c \in \mathbb{C} \smallsetminus \{0\}$, let $i = 1, 2$, and let $f(\cdot)$ be semi-c-periodic of type i (i_+). Then there exist two finite real constants $M > 0$ and $p > 0$ such that $\|f(t)\| \leqslant M|c|^{t/p}$, $t \in I$.*

Using Theorem 2.9.19 and the proof of Theorem 2.9.45, we may deduce the following corollaries:

COROLLARY 2.9.47. *Let $f \in C(I : X)$ and $c \in \mathbb{C} \smallsetminus \{0\}$. Then $f(\cdot)$ is semi-c-periodic if and only if there exists a sequence (f_n) of c-periodic functions in $C(I : X)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in I.*

COROLLARY 2.9.48. *Let $f \in C(I : X)$ and $|c| \neq 1$. If (f_n) is a sequence of c-periodic functions and $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in I, then $f(\cdot)$ is c-periodic.*

For the Stepanov classes, we will use the following notion:

DEFINITION 2.9.49. Let $p \in \mathcal{P}([0,1])$, and let $f \in L^{p(x)}_{loc}(I : X)$.

(i) It is said that $f(\cdot)$ is Stepanov semi-$(p(x), c)$-periodic of type i (i_+) if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (21), is semi-c-periodic of type i (i_+).

(ii) It is said that $f(\cdot)$ is asymptotically Stepanov semi-$(p(x), c)$-periodic of type i (i_+) if and only if there are a Stepanov semi-$(p(x), c)$-periodic function of type i (i_+) $h(\cdot)$ and $q \in C_0(I : L^{p(x)}([0,1] : X))$ such that $f(t) = h(t) + q(t)$ for a.e. $t \in I$.

If $p(x) \equiv [1, \infty)$, then we also say that the function $f(\cdot)$ is Stepanov semi-(p, c)-periodic of type i (i_+) and so on.

REMARK 2.9.50. Let us observe that we can also analyze the following notion in case that the parameter c is not given in advance (compare with (181)):

$$\forall \varepsilon > 0 \quad \exists c > 0 \ \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \big\|f(x+mp) - c^m f(x)\big\| \leqslant \varepsilon. \tag{185}$$

Fairly complete analysis of class consisting of all continuous functions $f : I \to X$ satisfying (185) and corresponding Stepanov class is without scope of this paper.

Semi-periodic functions depending on parameter have been introduced in [**25**, Definition 4], where the authors have considered case in which $I = \mathbb{R}$, $E = \mathbb{R}^k$ and $c = 1$. We will not introduce the related notion in case $|c| = 1$, which will be standing till the end of subsection.

The composition theorems for semi-c-periodic functions have not been considered elsewhere even in case $c = 1$. In order to formulate the first result in this direction, suppose that $t \in I$, $p > 0$, $m \in \mathbb{S}$ and $c \in \mathbb{C} \smallsetminus \{0\}$. Let $F : I \times Y \to X$ be a continuous function. If there exists a finite constant $L \geqslant 1$ such that (60) holds, then we have

$$\begin{aligned}
&\bigl\|F(t+mp, f(t+mp)) - c^m F(t, f(t))\bigr\| \\
&\leqslant \bigl\|F(t+mp, f(t+mp)) - F\bigl(t+mp, c^m f(t)\bigr)\bigr\| \\
&+ \bigl\|F\bigl(t+mp, c^m f(t)\bigr) - c^m F(t, f(t))\bigr\| \\
&\leqslant L\bigl\|f(t+mp) - c^m f(t)\bigr\| + \bigl\|F\bigl(t+mp, c^m f(t)\bigr) - c^m F(t, f(t))\bigr\|.
\end{aligned} \tag{186}$$

Therefore, it is natural to consider the following condition:

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall t \in I \quad \bigl\|F(t+mp, c^m f(t)) - c^m F(t, f(t))\bigr\| \leqslant \varepsilon. \tag{187}$$

Using these estimates, we can immediately clarify the following result which can be simply formulated for semi-c-periodic functions:

THEOREM 2.9.51. *Suppose that $F : I \times Y \to X$ is a continuous function satisfying that there exists a finite real constant $L > 0$ such that* (60) *holds, $f : I \to Y$ is a continuous function and for each $\varepsilon > 0$ there exists $p > 0$ such that* (181) *and* (187) *hold. Then the function $t \mapsto F(t, f(t))$, $t \in I$ is semi-c-periodic.*

In the following result, we reconsider [**135**, Theorem 3.31] for semi-c-periodic functions:

THEOREM 2.9.52. *Suppose that $F : I \times Y \to X$ is a continuous function, $f : I \to Y$ is a continuous function and $F(\cdot, \cdot)$ is uniformly continuous on set $\{\eta f(t) : \eta \in \mathbb{C},\ t \in I\}$, uniformly in $t \in I$ (that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f(t, x) - f(t, y)\| \leqslant \varepsilon$ for all $t \in I$ and $x,\ y \in \{\eta f(t) : \eta \in \mathbb{C},\ t \in I\}$). Suppose that for each $\varepsilon > 0$ there exists $p > 0$ such that* (181) *and* (187) *hold. Then the function $t \mapsto F(t, f(t))$, $t \in I$ is semi-c-periodic.*

PROOF. Since (187) holds, the statement easily follows from the estimate (186) and the prescribed assumptions. $\square$

For the Stepanov classes, we will first clarify the following result:

THEOREM 2.9.53. *Suppose that $p_1 \in \mathcal{P}([0, 1])$, $r(x) \geqslant \max(p_1(x)/(p_1(x) - 1))$, and there exists a function $L_F \in L_S^{r(x)}(I)$ such that* (25) *holds. Suppose, further, that for each $\varepsilon > 0$ there exists $p > 0$ such that*

$$\forall m \in \mathbb{S} \quad \forall t \in I \quad \bigl\|F(s+t+mp, c^m f(s+t)) - c^m F(s+t, f(s+t))\bigr\|_{L^{p_1(s)}[0,1]} \leqslant \varepsilon \tag{188}$$

holds, as well as (181) *holds, with the function $f(\cdot)$ and the space Y replaced therein with the function $\hat{f}(\cdot)$ and the space $L^{p_1(x)}([0, 1] : Y)$. Then the function $F(\cdot, f(\cdot))$ is Stepanov semi-$(q(x), c)$-periodic with $q(x) := p(x)r(x)/(p(x)+r(x))$ for $x \in [0, 1]$ and $r(x) < \infty$ and $q(x) := p(x)$ for $x \in [0, 1]$ and $r(x) = +\infty$.*

PROOF. We will prove the thorem with the constant coefficient $p_1(x) \equiv p_1 \in [1, \infty)$. Let $\varepsilon > 0$ be given and let the number $p > 0$ satisfy the above requirements. Fix numbers $t \in I$ and $m \in \mathbb{Z}$. Arguing as in the proof of estimate (186), we get:

$$\bigl\|F(t+mp, f(t+mp)) - c^m F(t, f(t))\bigr\|$$

$$\leqslant L_F(t)\big\|f(t+mp)-c^m f(t)\big\| + \big\|F\big(t+mp, c^m f(t)\big) - c^m F(t, f(t))\big\|.$$

Using the Hölder inequality and the inequality $q < p_1$, we get:

$$\Biggl(\int_t^{t+1}\big\|F(s+mp, f(s+mp)) - c^m F(s, f(s))\big\|^q\, ds\Biggr)^{1/q}$$
$$\leqslant 2^{(q-1)/q}\Bigl[\big\|L_F(\cdot)\big\|_{L^r[t,t+1]}\big\|f(\cdot+mp)-c^m f(\cdot)\big\|_{L^{p_1}[t,t+1]}$$
$$+\big\|F\big(\cdot+mp, c^m f(\cdot)\big) - c^m F(\cdot, f(\cdot))\big\|_{L^{p_1}[t,t+1]}\Bigr].$$

This completes the proof of the theorem in a routine manner. □

Remark 2.9.54. We will not reconsider the statement of [**276**, Lemma 2.1] here.

We can similarly prove the result which naturally corresponds to [**242**, Theorem 2.1] and the consequence for the usual Lipschitz condition used. Finally, we will clarify an interesting result concerning the existence and uniqueness of semi-c-periodic solutions of the following abstract semilinear fractional Cauchy problem

$$D^\alpha u(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)\, ds + F(t, u(t)), \quad t \in \mathbb{R}, \tag{189}$$

where $D^\alpha u(t)$ denotes the Weyl-Liouville fractional derivative of order $\alpha > 0$, $a \in L^1_{loc}([0,\infty))$ is a scalar-valued kernel, the function $F(\cdot,\cdot)$ enjoys some properties and A generates a non-degenerate α-resolvent operator family $(S_\alpha(t))_{t\geqslant 0}$ on X satisfying that $\int_0^\infty \|S_\alpha(t)\|\, dt < \infty$ (see R. Ponce [**318**] for more details; equations of this kind arise in the study of heat flow in materials with memory as well as in certain equations of population dynamics). By a mild solution of (189), we mean any continuous function $u : \mathbb{R} \to X$ such that

$$u(t) = \int_{-\infty}^t S_\alpha(t-s)F(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

Now we are able to formulate the following theorem:

Theorem 2.9.55. *Suppose that $F : \mathbb{R} \times X \to X$ is a continuous function satisfying that there exists a finite real constant $L > 0$ such that* (60) *holds. If $L\int_0^\infty \|S_\alpha(t)\|\, dt < 1$, then the abstract fractional semilinear Cauchy inclusion* (189) *has a unique semi-c-periodic solution.*

Proof. It can be easily shown that the set $\mathcal{SP}_{c,1}(\mathbb{R} : X)$, equipped with the distance $d(u,v) := \sup_{t\in\mathbb{R}} \|u(t)-v(t)\|$, $u,\ v \in \mathcal{SP}_{c,1}(\mathbb{R} : X)$, is a complete metric space. Define the mapping

$$(\Lambda u)(t) := \int_{-\infty}^t S_\alpha(t-s)F(s, u(s))\, ds, \quad t \in \mathbb{R} \ \Bigl(u \in \mathcal{SP}_{c,1}(\mathbb{R} : X)\Bigr).$$

Applying Theorem 2.9.51 and the foregoing arguments, we get that the mapping $\Lambda(\cdot)$ is well defined. Moreover, our assumption $L\int_0^\infty \|S_\alpha(t)\|\, dt < 1$ easily implies that $\Lambda(\cdot)$ is a contraction. The proof completes an application of the Banach contraction principle. $\square$

2.9.4. Semi-Bloch k-periodicity. In this subsection, we will always assume that $I = [0, \infty)$ or $I = \mathbb{R}$; as before, we set $\mathbb{S} := \mathbb{N}$ if $I = [0, \infty)$, and $\mathbb{S} := \mathbb{Z}$ if $I = \mathbb{R}$. For the convenience of the reader, we recall that a bounded continuous function $f : I \to X$ is said to be Bloch (p, k)-periodic, or Bloch periodic with period p and Bloch wave vector or Floquet exponent k if and only if $f(x + p) = e^{ikp} f(x)$, $x \in I$, with $p > 0$ and $k \in \mathbb{R}$. The space of all functions $f : I \to X$ that are Bloch (p, k)-periodic will be denoted by $\mathcal{B}_{p,k}(I : X)$. If $f \in \mathcal{B}_{p,k}(I : X)$, then we have

$$f(x + mp) = e^{ikmp} f(x), \quad x \in I, \ m \in \mathbb{S}.$$

Given $k \in \mathbb{R}$, we set $\mathcal{B}_k(I : X) := \bigcup_{p>0} \mathcal{B}_{p,k}(I : X)$. Observing that $f \in P_c(I : X)$ satisfies $f(x + p) = f(x)$ for all $x \in I$ and some $p > 0$ if and only if the function $F(x) := e^{ikx} f(x)$, $x \in I$ satisfies $F(x+p) = e^{ikp} F(x)$, $x \in I$, we may conclude that

$$\mathcal{B}_k(I : X) := \{e^{ik\cdot} f(\cdot) : f \in P_c(I : X)\}. \tag{190}$$

For more details on the Bloch (p, k)-periodic functions, see [**204**] and references cited therein.

Let us define the notion of a semi-Bloch k-periodic function as follows:

DEFINITION 2.9.56. Let $k \in \mathbb{R}$. A function $f \in C_b(I : X)$ is said to be semi-Bloch k-periodic if and only if

$$\forall \varepsilon > 0 \quad \exists p > 0 \quad \forall m \in \mathbb{S} \quad \forall x \in I \quad \|f(x + mp) - e^{ikmp} f(x)\| \leqslant \varepsilon. \tag{191}$$

The space of all semi-Bloch k-periodic functions will be denoted by $\mathcal{S}B_k(I : X)$.

It is clear that Definition 2.9.56 provides a generalization of [**25**, Definition 2 and Definition 3], given only in the case that $I = \mathbb{R}$. Speaking-matter-of-factly, a function $f : \mathbb{R} \to X$ is semi-periodic in the sense of above-mentioned (equivalent) definitions if and only if $f : \mathbb{R} \to X$ is semi-Bloch 0-periodic. Further on, it can be easily seen that for each $k \in \mathbb{R}$ any constant function $f \equiv c$ belongs to the space $\mathcal{S}B_k(I : X)$; for this, it is only worth noticing that for each $\varepsilon > 0$ and $k \neq 0$ we can take $p = 2\pi/k$ and (191) will be satisfied.

REMARK 2.9.57. It is not so easy to introdude the concept of almost Bloch k-periodicity, where $k \in \mathbb{R}$. In order to explain this in more detail, assume that a function $f \in C_b(I : X)$ and a number $\varepsilon > 0$ are given. Let us say that a real number $p > 0$ is an (ε, k)-Bloch period for $f(\cdot)$ if and only if

$$\left\| f(x + p) - e^{ikp} f(x) \right\| \leqslant \varepsilon, \quad x \in I, \tag{192}$$

and $f(\cdot)$ is almost Bloch k-periodic if and only if for each $\varepsilon > 0$ the set constituted of all (ε, k)-Bloch periods for $f(\cdot)$ is relatively dense in $[0, \infty)$. But, then we have that $f(\cdot)$ is almost Bloch k-periodic if and only if $f(\cdot)$ is almost periodic. To see this, it suffices to observe that (192) is equivalent with

$$\left\| e^{-ik(x+p)} f(x + p) - e^{-ikx} f(x) \right\| \leqslant \varepsilon, \quad x \in I,$$

so that, actually, the function $f(\cdot)$ is almost Bloch k-periodic if and only if the function $e^{-ik\cdot}f(\cdot)$ is almost periodic, which is equivalent to say that the function $f(\cdot)$ is almost periodic. Further on, let $f(\cdot) \in \mathcal{S}B_k(I : X)$. Then for each number $\varepsilon > 0$ we have that the set constituted of all (ε, k)-Bloch periods for $f(\cdot)$ is relatively dense in $[0, \infty)$ since it contains the set $\{mp : m \in \mathbb{N}\}$, where $p > 0$ is determined by (191). In view of our previous conclusions, we get that $f(\cdot)$ is almost periodic. In particular, any Bloch (p, k)-periodic function needs to be almost periodic, which has not been observed in the researches of Bloch periodic functions carried out so far (see e.g., [**150**] and [**204**]).

Now we will prove the following result:

PROPOSITION 2.9.58. *Let $k \in \mathbb{R}$ and $f \in C_b(I : X)$. Then the following holds:*

(i) *$f(\cdot)$ is semi-Bloch k-periodic if and only if $e^{-ik\cdot}f(\cdot)$ is semi-periodic.*
(ii) *$f(\cdot)$ is semi-Bloch k-periodic if and only if there exists a sequence (f_n) in $P_c(I : X)$ such that $\lim_{n\to\infty} e^{ikx}f_n(x) = f(x)$ uniformly in I.*
(iii) *$f(\cdot)$ is semi-Bloch k-periodic if and only if there exists a sequence (f_n) in $\mathcal{B}_k(I : X)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in I.*

PROOF. The proof of (i) follows similarly as above. Since [**25**, Lemma 1 and Theorem 1] hold for the functions defined on the interval $I = [0, \infty)$, we have that (i) implies that $f(\cdot)$ is semi-Bloch k-periodic if and only if there exists a sequence (f_n) in $P_c(I : X)$ such that $\lim_{n\to\infty} e^{ikx}f_n(x) = f(x)$ uniformly in I. This proves (ii). For the proof of (iii), it suffices to apply (ii), (190) and the conclusion preceding it. $\square$

Let $k \in \mathbb{R}$. Using Proposition 2.9.58 and [**25**, Proposition 2], we may construct a substantially large class of semi-Bloch k-periodic functions, which do not form a vector space due to a simple example in the second part of [**25**, Remark 3]; [**25**, Lemma 2] can be straightforwardly reformulated for semi-Bloch k-periodic functions, while the function given in [**25**, Example 1] can be simply used to provide an example of a scalar-valued semi-Bloch k-periodic function which is not contained in the space $\mathcal{B}_k(I : \mathbb{C})$. If we define Bloch k-quasi periodic function

$$\mathcal{B}_{k;q}(I : X) := \{e^{ik\cdot}f(\cdot) : f \in QP^0(I : X)\},$$

where $QP^0(I : X)$ denotes the space of all quasi-periodic functions from I into X (see [**25**], [**71**] and references cited therein for the notion), then [**25**, Theorem 2] can be also reformulated in our context; this also holds for [**25**, Example 2, Example 3].

By the foregoing, we have:

$$\mathcal{B}_k(I : X) \subseteq \mathcal{S}B_k(I : X) \subseteq AP(I : X) \subseteq BUC(I : X), \quad k \in \mathbb{R}.$$

EXAMPLE 2.9.59. The function $f(x) := \cos x$, $x \in \mathbb{R}$ is anti-periodic. Now we will prove that $f \in \mathcal{S}B_k(I : X)$ if and only if $k \in \mathbb{Q}$. For $k \in \mathbb{Q}$, this is clear because we can take p in (191) as a certain multiple of 2π. Let us assume now that $k \notin \mathbb{Q}$. Then it suffices to show that the function $e^{-ik\cdot}f(\cdot)$ is not semi-periodic. Towards see this, let us observe that $\sigma(e^{-ik\cdot}f(\cdot)) = \{1 - k, -1 - k\}$ so that there does not exist a positive real number $\theta > 0$ such that $\sigma(e^{-ik\cdot}f(\cdot)) \subseteq \theta \cdot \mathbb{Q}$, which can be simply approved and which contradicts [**25**, Lemma 2].

REMARK 2.9.60. Let $a \in AP(I : \mathbb{C})$. Then we can introduce and analyze the following notion: A function $f \in C_b(I : X)$ is said to be semi $_a$-periodic if and only if there exists a sequence (f_n) in $P_c(I : X)$ such that $\lim_{n\to\infty} a(x)f_n(x) = f(x)$ uniformly in I. Any such function needs to be almost periodic. We will analyze this notion somewhere else.

EXAMPLE 2.9.61. Roughly speaking, it is well known that the unique solution of the heat equation $u_t(x,t) = u_{xx}(x,t)$, $x \in \mathbb{R}$, $t \geqslant 0$, accompanied with the initial condition $u(x,0) = f(x)$, is given by (145). By the conclusion from [**204**, Example 2.1], we know that, if the function $f(\cdot)$ is Bloch (p,k)-periodic, then the solution $u(x,\cdot)$ is likewise Bloch (p,k)-periodic ($p > 0$, $k \in \mathbb{R}$). Using this fact, the dominated convergence theorem and Proposition 2.9.58, we get that, if $f(\cdot)$ is semi-Bloch k-periodic, then the solution $u(x,\cdot)$ will be likewise semi-Bloch k-periodic.

PROPOSITION 2.9.62. *Let $k \in \mathbb{R}$, let $p > 0$, and let a function $f \in C_b([0,\infty) : X)$ be given. If $f(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic), then the function $\mathbb{E}f(\cdot)$ is likewise Bloch (p,k)-periodic (semi-Bloch k-periodic).*

PROOF. Suppose first that $f(\cdot)$ is Bloch (p,k)-periodic. Then $f(x+p) = e^{ikp}f(x)$, $x \geqslant 0$; we need to show that $(\mathbb{E}f)(x+p) = e^{ikp}(\mathbb{E}f)(x)$, $x \in \mathbb{R}$, i.e., $[W(x+p)f](0) = e^{ikp}[W(x)f](0)$, $x \in \mathbb{R}$. Since $W(x+p) = W(x)W(p)$, $x \in \mathbb{R}$, it suffices to show that $[W(x)f(\cdot+p)](0) = e^{ikp}[W(x)f](0)$, $x \in \mathbb{R}$, i.e., $[W(x)e^{ikp}f(\cdot)](0) = e^{ikp}[W(x)f](0)$, $x \in \mathbb{R}$, which is true. If $f(\cdot)$ is semi-Bloch k-periodic, then Proposition 2.9.58(iii) yields that there exists a sequence (f_n) in $\mathcal{B}_k([0,\infty) : X)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly in $[0,\infty)$. Due to the supremum formula, we have that $\lim_{n\to\infty}(\mathbb{E}f_n)(x) = (\mathbb{E}f)(x)$ uniformly in $\mathbb{R}$. By the first part of proof, we know that for each $n \in \mathbb{N}$ the function $(\mathbb{E}f_n)(\cdot)$ belongs to the space $\mathcal{B}_k(\mathbb{R} : X)$. Applying again Proposition 2.9.58(iii), we get that $\mathbb{E}f(\cdot)$ is likewise semi-Bloch k-periodic. $\square$

The proof of following simple proposition is left to the interested reader:

PROPOSITION 2.9.63. *Let $k \in \mathbb{R}$, let $p > 0$, and let $f : I \to X$. Then we have:*

(i) *If $f(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic), then $cf(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic) for any $c \in \mathbb{C}$.*
(ii) *If $X = \mathbb{C}$, $\inf_{x\in\mathbb{R}} |f(x)| = m > 0$ and $f(\cdot)$ is Bloch (p,k)-periodic (semi-Bloch k-periodic), then $1/f(\cdot)$ is Bloch $(p,-k)$-periodic (semi-Bloch $(-k)$-periodic).*

Now we will introduce the following definition.

DEFINITION 2.9.64. Let $f \in C_b(I : X)$ and $k \in \mathbb{R}$. Then we say that $f(\cdot)$ is asymptotically semi Bloch k-periodic if and only if there exist a function $\phi \in C_0(I : X)$ and a semi Bloch k-periodic function $g : \mathbb{R} \to X$ such that $f(t) = g(t) + \phi(t)$ for all $t \geqslant 0$.

As already mentioned, the notion of Stepanov semi-periodicity has not been analyzed in [**25**]. We will use the following definitions:

DEFINITION 2.9.65. Let $k \in \mathbb{R}$ and $p \in \mathcal{P}([0,1])$. Then we say that a function $f \in L_S^{p(x)}(I : X)$ is Stepanov $p(x)$-semi-Bloch k-periodic if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (21), is semi-Bloch k-periodic.

If $p(x) \equiv p \in [1,\infty)$, then we also say that the function $f(\cdot)$ is Stepanov p-semi-Bloch k-periodic.

DEFINITION 2.9.66. Let $k \in \mathbb{R}$ and $p \in \mathcal{P}([0,1])$. Then we say that a function $f \in L_S^{p(x)}(I : X)$ is asymptotically Stepanov $p(x)$-semi-Bloch k-periodic if and only if the function $\hat{f} : I \to L^{p(x)}([0,1] : X)$, defined by (21), is asymptotically semi-Bloch k-periodic.

If $p(x) \equiv p \in [1,\infty)$, then we also say that the function $f(\cdot)$ is asymptotically Stepanov p-semi-Bloch k-periodic.

Let $p > 0$ and $k \in \mathbb{R}$. It should be noted that, if $f : I \to X$ is Bloch (p,k)-periodic, then $\hat{f} : I \to L^q([0,1] : X)$ is likewise Bloch (p,k)-periodic. Further on, it immediately follows from the corresponding definitions that, if $f : I \to X$ is semi-Bloch k-periodic, then $f(\cdot)$ is Stepanov q-semi-Bloch k-periodic for every number $q \in [1,\infty)$; a large class of non-continuous periodic or Bloch (p,k)-periodic functions can be used to provide that the converse statement does not hold in general. If $1 \leqslant q < q' < \infty$ and $f : I \to X$ is (asymptotically) Stepanov q'-semi-Bloch k-periodic, then $f(\cdot)$ is (asymptotically) Stepanov q-semi-Bloch k-periodic. To see that the converse statement does not hold in general, we will provide only one illustrative example:

EXAMPLE 2.9.67. Suppose that $1 < q < \infty$. Let us revisit the example of H. Bohr and E. Følner once more; they have constructed an example of a Stepanov 1-almost periodic function $F : \mathbb{R} \to \mathbb{R}$ that is not Stepanov q-almost periodic (see [**77**, p. 70]). Moreover, for each $n \in \mathbb{N}$ there exists a bounded periodic function $F_n : \mathbb{R} \to \mathbb{R}$ with at most countable points of discontinuity such that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \int_t^{t+1} \big|F_n(s) - F(s)\big|\, ds = 0. \tag{193}$$

Therefore, $\hat{F_n} : \mathbb{R} \to L^1([0,1] : \mathbb{R})$ is a bounded periodic function and, in addition to the above, $\hat{F_n}(\cdot)$ is continuous ($n \in \mathbb{N}$). Due to (193), we have that $\lim_{n\to\infty} \hat{F_n}(t) = \hat{F}(t)$ uniformly in $t \in \mathbb{R}$. This implies that the function $F(\cdot)$ is Stepanov 1-semi-periodic but not Stepanov q-semi-periodic because it is not Stepanov q-almost periodic.

The above conclusions can be clarifed for Stepanov $p(x)$-semi-Bloch k-periodic functions, as well. Concerning the invariance of semi-Bloch k-periodicity under the actions of infinite convolution products, we have the following result:

PROPOSITION 2.9.68. *Suppose that* $k \in \mathbb{R}$, $p,\ q \in \mathcal{P}([0,1])$, $1/p(x)+1/q(x) = 1$ *and* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that*

$M := \sum_{k=0}^{\infty} \|R(\cdot + k)\|_{L^{q(\cdot)}[0,1]} < \infty$. *If* $\check{f} : \mathbb{R} \to X$ *is Stepanov* $p(x)$*-semi-Bloch* $(-k)$*-periodic, then the function* $F(\cdot)$*, given by* (55)*, is well-defined and semi-Bloch* k*-periodic.*

Proof. Using the same arguments as in the proof of Propostion 2.5.17, we have that $F(\cdot)$ is well defined and continuous. It remains to be proved that $F(\cdot)$ is semi-Bloch k-periodic. Let a number $\varepsilon > 0$ be given in advance. Then we can find a finite number $p > 0$ such that, for every $m \in \mathbb{Z}$ and $t \in \mathbb{R}$, we have

$$\big\|\check{f}(t+mp) - e^{-ikmp}\check{f}(t)\big\|_{L^{p(x)}[0,1]} \leqslant \varepsilon, \quad t \in \mathbb{R}.$$

Applying Hölder inequality and this estimate, we get that

$$\begin{aligned}
&\big\|F(t+mp) - e^{ikmp}F(t)\big\| \\
&\quad \leqslant \int_0^{\infty} \|R(r)\| \cdot \big\|f(t+mp-r) - e^{ikmp}f(t-r)\big\|\, dr \\
&\quad = \sum_{k=0}^{\infty} \int_0^1 \|R(r+k)\| \cdot \big\|f(t+k+mp-r) - e^{ikmp}f(t+k-r)\big\|\, dr \\
&\quad \leqslant 2\sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(\cdot)}[0,1]} \big\|e^{-ikmp}\check{f}(r-t-mp-k) - \check{f}(r-t-k)\big\|_{L^{p(r)}[0,1]} \\
&\quad \leqslant 2\sum_{k=0}^{\infty} \|R(\cdot+k)\|_{L^{q(\cdot)}[0,1]}\varepsilon = 2M\varepsilon, \quad t \in \mathbb{R},
\end{aligned}$$

which clearly implies the required. □

The above result can be simply applied in the study of existence and uniqueness of semi-Bloch k-periodic solutions of the fractional Cauchy inclusion (58). We can also analyze the invariance of asymptotical semi-Bloch k-periodicity under the actions of finite convolution products, applying the obtained results in the qualitative analysis of asymptotically (Stepanov) semi-Bloch k-periodic solutions of the abstract fractional Cauchy inclusion $(\mathrm{DFP})_{f,\gamma}$.

Let $p > 0$ and $k \in \mathbb{R}$. If $f : \mathbb{R} \to X$ is Bloch (p, k)-periodic and $a \in L^1(\mathbb{R})$, then the function $a * f(\cdot)$ is likewise Bloch (p, k)-periodic. Using the Young inequality and our previous results, it can be simply shown that the space of semi-Bloch k-periodic functions is convolution invariant.

Finally, let B be a subset of $\mathbb{R}^s$ and $f : \mathbb{R} \times B \to \mathbb{R}^s$. Then we say that the function $f(\cdot)$ is uniformly semi-Bloch k-periodic function if and only if for any compact subset K of B, we have

$$\forall \varepsilon > 0\ \exists p \geqslant 0\ \forall m \in \mathbb{Z}\ \forall x \in \mathbb{R}\ \forall \alpha \in K\ \|f(x+mp,\alpha) - e^{ikmp}f(x,\alpha)\|_{\mathbb{R}^s} \leqslant \varepsilon.$$

We close the subsection with the observation that we can simply reformulate [**25**, Proposition 3] for uniformly semi-Bloch k-periodic functions and provide certain applications to the matrix differential equations, as it has been done in [**25**, Theorem 4] for semi-periodic functions.

2.9.5. Weyl-(p, c)-almost periodic type functions. The material of the next three subsections is taken from [**225**], outr joint paper with Prof. M. T. Khalladi, A. Rahmani, M. Pinto and D.Velinov.

In this subsection, we first introduce the notion of an (equi-)Weyl-(p, c)-almost periodic function as follows:

DEFINITION 2.9.69. Let $1 \leqslant p < \infty$ and $f \in L^p_{loc}(I : X)$.

(i) We say that the function $f(\cdot)$ is equi-Weyl-(p, c)-almost periodic, $f \in e - W^p_{ap;c}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \left\| f(t+\tau) - cf(t) \right\|^p dt \right]^{1/p} \leqslant \varepsilon. \tag{194}$$

(ii) We say that the function $f(\cdot)$ is Weyl-(p, c)-almost periodic, $f \in W^p_{ap;c}(I : X)$ for short, if and only if for each $\varepsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l \to +\infty} \sup_{x \in I} \left[\frac{1}{l} \int_x^{x+l} \left\| f(t+\tau) - cf(t) \right\|^p dt \right]^{1/p} \leqslant \varepsilon.$$

If $c = 1$, resp. $c = -1$, then we also say that $f(\cdot)$ is (equi-)Weyl-p-almost periodic, resp. (equi-)Weyl-p-almost anti-periodic.

It is clear that any equi-Weyl-(p, c)-almost periodic function is Weyl-(p, c)-almost periodic. The proofs of following results are trivial and therefore omitted:

PROPOSITION 2.9.70. *Suppose that $f : I \to X$ is (equi-)Weyl-(p, c)-almost periodic. Then $\|f\| : I \to [0, \infty)$ is (equi-)Weyl-p-almost periodic.*

PROPOSITION 2.9.71. *Let $1 \leqslant p < \infty$ and $f \in L^p_{loc}(I : X)$. If the function $f(\cdot)$ is (equi-)Weyl-(p, c)-almost periodic and $I = \mathbb{R}$, then the function $\check{f} : \mathbb{R} \to X$ is (equi-)Weyl-$(p, 1/c)$-almost periodic.*

We will include the proof of following proposition for the sake of completeness:

PROPOSITION 2.9.72. *Let $1 \leqslant p < \infty$ and $f \in L^p_{loc}(I : X)$. If the function $f(\cdot)$ is (equi-)Weyl-(p, c)-almost periodic and $m \in \mathbb{N}$, then the function $f(\cdot)$ is (equi-)Weyl-(p, c^m)-almost periodic.*

PROOF. We will give the proof for the class of equi-Weyl-(p, c)-almost periodic functions. Let $\varepsilon > 0$ be fixed; then we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (194) holds true. Clearly, integrating the estimate (167) (with the number l replaced by the number m therein) over the segment $[x, x + l]$, where $x \in I$, we obtain the existence of a finite constant $c_p > 0$ such that

$$\left[\frac{1}{l} \int_x^{x+l} \left\| f(t+m\tau) - c^m f(t) \right\|^p dt \right]^{1/p}$$

$$\leqslant c_p\left[\sum_{j=0}^{m-1}\frac{|c|^{jp}}{l}\int_x^{x+l}\Big\|f\big(t+(m-j)\tau\big)-cf\big(t+(m-j-1)\tau\big)\Big\|^p\,dt\right]^{1/p}$$
$$\leqslant c_p\left[\sum_{j=0}^{m-1}\frac{|c|^{jp}}{l}\int_{x+(m-j-1)\tau}^{x+l+(m-j-1)\tau}\Big\|f(t+\tau)-cf(t)\Big\|^p\,dt\right]^{1/p}$$
$$\leqslant c_p\varepsilon\left[\sum_{j=0}^{m-1}|c|^{jp}\right]^{1/p}.$$

Therefore, for this number $\varepsilon > 0$, we can take the numbers $l > 0$ and $mL > 0$ in definition of equi-Weyl-(p, c)-almost periodicity. This completes the proof. □

Consider now the following condition:

$$m \in \mathbb{Z} \smallsetminus \{0\},\ n \in \mathbb{N},\ (m,n)=1,\ |c|=1 \text{ and } \arg(c)=\pi m/n. \tag{195}$$

The next corollary of Proposition 2.9.72 follows immediately:

COROLLARY 2.9.73. *Let $1 \leqslant p < \infty$, $f \in L^p_{loc}(I : X)$, and let (195) hold.*

(i) *If m is even and $f(\cdot)$ is an (equi-)Weyl-(p, c)-almost periodic function, then $f(\cdot)$ is (equi-)Weyl-p-almost periodic.*
(ii) *If m is odd and $f(\cdot)$ is an (equi-)Weyl-(p, c)-almost periodic function, then $f(\cdot)$ is (equi-)Weyl-p-almost anti-periodic.*

PROPOSITION 2.9.74. *Let $1 \leqslant p < \infty$, $f \in L^p_{loc}(I : X)$, and let $|c| = 1$, $\arg(c)/\pi \notin \mathbb{Q}$. If $f(\cdot)$ is (equi-)Weyl-(p, c)-almost periodic and Stepanov p-bounded, then $f(\cdot)$ is (equi-)Weyl-(p, c')-almost periodic for all $c' \in S_1$.*

PROOF. It suffices to consider case in which the function $f(\cdot)$ is not almost everywhere equal to zero. Let the numbers $c' \in S_1$ and $\varepsilon > 0$ be fixed; then the set $\{c^l : l \in \mathbb{N}\}$ is dense in S_1 and therefore there exists an increasing sequence (l_k) of positive integers such that $\lim_{k\to+\infty} c^{l_k} = c'$. Let $k \in \mathbb{N}$ be such that $|c^{l_k} - c'| < \varepsilon/(2\|f\|_{S^p})$, and let $\varepsilon > 0$ be given. Then we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that (194) holds. Then we have

$$\big\|f(x+\tau)-c'f(x)\big\| \leqslant \big\|f(x+\tau)-c^{l_k}f(x)\big\| + \big|c^{l_k}-c'\big|\cdot\|f(x)\|,$$

for any $x \in I$. Then the conclusion follows from Proposition 2.9.72, after integrating the above estimate over the segment $[x, x + l]$ and using the estimate

$$\frac{1}{l}\int_x^{x+l}\|f(t)\|^p\,dt \leqslant \frac{1}{l}(1+\lfloor l\rfloor)\|f\|^p_{S^p}.$$

□

The main structural properties of (equi-)Weyl-(p, c)-almost periodic functions are collected in the following theorem (see also [**234**, Proposition 2.3.5]):

THEOREM 2.9.75. *Let $f : I \to X$ be (equi-)Weyl-(p, c)-almost periodic, and let $\alpha \in \mathbb{C}$. Then we have:*

(i) $\alpha f(\cdot)$ *is (equi-)Weyl-(p,c)-almost periodic.*
(ii) *If $X = \mathbb{C}$ and $ess\inf_{x\in\mathbb{R}} |f(x)| = m > 0$, then $1/f(\cdot)$ is (equi-)Weyl-$(p,1/c)$-almost periodic).*
(iii) *If $(g_n : I \to X)_{n\in\mathbb{N}}$ is a sequence of bounded, continuous, (equi-)Weyl-(p,c)-almost periodic functions and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a function $g : I \to X$, then $g(\cdot)$ is (equi-)Weyl-(p,c)-almost periodic.*
(iv) *If $a \in I$ and $b \in I \setminus \{0\}$, then the functions $f(\cdot + a)$ and $f(b\,\cdot)$ are likewise (equi-)Weyl-(p,c)-almost periodic.*

Now we will provide two illustrative examples:

EXAMPLE 2.9.76. Set $f(t) := \chi_{[0,1/2]}(t)$, $t \in \mathbb{R}$. Then for each number $l > 0$ we have
$$\frac{1}{l}\int_x^{x+l} |f(t+\tau) - cf(t)|^p \, dt \leqslant \frac{1}{2l}(1+|c|)^p, \quad x \in \mathbb{R}.$$
This implies that $f(\cdot)$ is equi-Weyl-(p,c)-almost periodic for each complex number $c \in \mathbb{C} \setminus \{0\}$ and for each finite exponent $p \geqslant 1$.

EXAMPLE 2.9.77. Set $f(t) := \chi_{[0,\infty)}(t)$, $t \in \mathbb{R}$. Then for each number $l > 0$ we have
$$\sup_{x\in\mathbb{R}} \frac{1}{l}\int_x^{x+l} |f(t+\tau) - cf(t)|^p \, dt \geqslant |1-c|^p,$$
so that $f(\cdot)$ cannot be Weyl-(p,c)-almost periodic for $c \neq 1$. On the other hand, it is well known that $f(\cdot)$ is Weyl-$(p,1)$-almost periodic for any finite exponent $p \geqslant 1$.

Concerning the invariance of (equi-)Weyl-(p,c)-almost periodicity under the actions of convolution products, we will only note that the statements of [**234**, Proposition 2.11.1, Theorem 2.11.4, Proposition 2.11.6] can be simply reformulated in our framework. The interested reader can try to slightly generalize the notions and results of this subsection for variable exponents $p(x)$.

2.9.6. S-asymptotically (ω,c)-periodic functions. We start this subsection by introducing the following notion:

DEFINITION 2.9.78. Let $\omega \in I$. Then we say that a continuous function $f : I \to X$ is S-asymptotically (ω,c)-periodic if and only if $\lim_{|t|\to\infty} \|f(t+\omega) - cf(t)\| = 0$; a continuous function $f : I \to X$ is said to be S_c-asymptotically periodic if and only if there exists $\omega > 0$ such that $f(\cdot)$ is S-asymptotically (ω,c)-periodic. By $SAP_{\omega;c}(I : X)$ and $SAP_c(I : X)$ we denote the spaces consisting of all such functions; if $c = -1$, then we also say that the function $f(\cdot)$ is S-asymptotically ω-anti-periodic, resp. S-asymptotically anti-periodic.

This definition extends the well known definition of an S-asymptotically ω-periodic function, introduced by H. Henríquez et al. [**209**] for case $I = \mathbb{R}$ and M. Kostić [**247**] for case $I = [0,\infty)$.

DEFINITION 2.9.79. Let $p \in \mathcal{P}([0,1])$. A $p(x)$-locally integrable function $f(\cdot)$ is said to be Stepanov $p(x)$-asymptotically (ω,c)-periodic if and only if
$$\lim_{|t|\to\infty} \big\| f(s+t+\omega) - cf(s+t) \big\|_{L^{p(s)}[0,1]} = 0;$$

a $p(x)$-locally integrable function $f : I \to X$ is said to be Stepanov $p_c(x)$-asymptotically periodic if and only if there exists $\omega > 0$ such that $f(\cdot)$ is Stepanov $p(x)$-asymptotically (ω, c)-periodic.

By $S^{p(x)}SAP_{\omega;c}(I : X)$ and $S^{p(x)}SAP_c(I : X)$ we denote the spaces consisting of all such functions; if $c = -1$, then we also say that the function $f(\cdot)$ is Stepanov $p(x)$-asymptotically ω-anti-periodic, resp. Stepanov $p(x)$-asymptotically anti-periodic.

If $p(x) \equiv p \in [1, \infty)$, then by $S^pSAP_{\omega;c}(I : X)$ and $S^pSAP_c(I : X)$ we denote the spaces consisting of all such functions; if $c = -1$, then we also say that the function $f(\cdot)$ is Stepanov p-asymptotically ω-anti-periodic, resp. Stepanov p-asymptotically anti-periodic.

Now we will introduce the class of quasi-asymptotically c-almost periodic functions:

Definition 2.9.80. It is said that a continuous function $f : I \to X$ is quasi-asymptotically c-almost periodic if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\|f(t + \tau) - cf(t)\| \leqslant \varepsilon, \text{ provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

Denote by $Q - AAP_c(I : X)$ the set consisting of all quasi-asymptotically c-almost periodic functions from I into X; if $c = -1$, then we also say that the function $f(\cdot)$ is quasi-asymptotically almost anti-periodic.

Now we will introduce the following notion of Stepanov (p, c)-quasi-asymptotical almost periodicity:

Definition 2.9.81. Let $p \in \mathcal{P}([0, 1])$. A $p(x)$-locally integrable function $f(\cdot)$ is said to be Stepanov $(p(x), c)$-quasi-asymptotically almost periodic if and only if for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon, \tau) > 0$ such that

$$\|f(s + t + \tau) - cf(s + t)\|_{L^{p(s)}[0,1]} \leqslant \varepsilon^p, \text{ provided } t \in I \text{ and } |t| \geqslant M(\varepsilon, \tau).$$

By $S^{p(x)}Q - AAP_c(I : X)$ we denote the set consisting of all Stepanov $p(x)$-quasi-asymptotically c-almost periodic functions from I into X; if $c = -1$, then we also say that the function $f(\cdot)$ is Stepanov $p(x)$-quasi-asymptotically almost anti-periodic.

If $p(x) \equiv p \in [1, \infty)$, then we accept the usual terminology and then we denote the above space by $S^pQ - AAP_c(I : X)$.

Remark 2.9.82. A px)-locally integrable function $f(\cdot)$ is Stepanov $(p(x0, c)$-quasi-asymptotically almost periodic if and only if the function $f : I \to L^{p(x)}([0, 1] : X)$ is quasi-asymptotically c-almost periodic. Similar statements hold for the class of Stepanov $p(x)$-asymptotically (ω, c)-periodic functions. This observation enables one to see that many results clarified below, like Proposition 2.9.83, Corollary 2.9.84 and Theorem 2.9.86, continue to hold for the corresponding Stepanov classes of functions under our consideration.

It is very simple to prove that any asymptotically c-almost periodic function is quasi-asymptotically c-almost periodic. Furthermore, (167) easily implies:

PROPOSITION 2.9.83. *Let $\omega > 0$, $f : I \to X$ be an S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), and let $m \in \mathbb{N}$. Then $f(\cdot)$ is S-asymptotically $(m\omega, c^m)$-periodic (S_{c^m}-asymptotically periodic, quasi-asymptotically c^m-almost periodic).*

The next corollary of Proposition 2.9.83 follows immediately:

COROLLARY 2.9.84. *Let $f : I \to X$ be a continuous function, and let* (195) *hold.*

(i) *If m is even and $f(\cdot)$ is S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), then $f(\cdot)$ is S-asymptotically ω-anti-periodic (S-asymptotically anti-periodic, quasi-asymptotically almost anti-periodic).*
(ii) *If m is odd and $f(\cdot)$ is S-asymptotically (ω, c)-periodic (S_c-asymptotically periodic, quasi-asymptotically c-almost periodic), then $f(\cdot)$ is S-asymptotically ω-periodic (S-asymptotically periodic, quasi-asymptotically almost periodic).*

Therefore, if $\arg(c)/\pi \in \mathbb{Q}$, then the class of S-asymptotically (ω, c)-periodic functions (S_c-asymptotically periodic functions, quasi-asymptotically c-almost periodic functions) is always contained in the class of S-asymptotically ω-periodic functions (S-asymptotically periodic functions, quasi-asymptotically almost periodic functions).

The following result holds true:

COROLLARY 2.9.85. *Let $|c| = 1$ and $\arg(c)/\pi \notin \mathbb{Q}$. If $f(\cdot)$ is bounded S-asymptotically (ω, c)-periodic (bounded S_c-asymptotically periodic, bounded quasi-asymptotically c-almost periodic), then $f(\cdot)$ is S-asymptotically ω-periodic (S-asymptotically periodic, quasi-asymptotically almost periodic).*

Further on, a slight modification of the proof of [**247**, Theorem 2.5] shows that the following statement holds:

THEOREM 2.9.86. *Let $F(I : X)$ be any space consisting of continuous functions $h : I \to X$ such that $\sup_{t\in I} \|h(t + \tau) - ch(t)\| = \sup_{t\geqslant a} \|h(t + \tau) - ch(t)\|$, $a \in I$. Then the following holds:*

(i) $AAA_c(I : X) \cap Q - AAP_c(I : X) = AAP_c(I : X)$.
(ii) $AA_c(\mathbb{R} : X) \cap Q - AAP_c(\mathbb{R} : X) = AP_c(\mathbb{R} : X)$.

We will include the proof of the following proposition for the sake of completeness (see also the proof of [**247**, Proposition 2.7]):

PROPOSITION 2.9.87. *Let $|c| \leqslant 1$. Then $SAP_{\omega;c}(I : X) \subseteq Q - AAP_c(I : X)$.*

PROOF. Let $\varepsilon > 0$ be given. Then we can take $L(\varepsilon) = 2\omega$ in definition of space $Q - AAP_c(I : X)$. Then any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains a number

$\tau = n\omega$ for some $n \in \mathbb{N}$. For this n and ε, there exists a finite number $M(\varepsilon, n) > 0$ such that $\|f(t+\omega) - cf(t)\| \leqslant \varepsilon/n\omega$ for $|t| \geqslant M(\varepsilon, n)$. Then we have

$$\|f(t+n\omega) - cf(t)\| \leqslant \sum_{k=0}^{n-1} |c|^{n-k-1} \|f(t+(k+1)\omega) - cf(t+k\omega)\|$$
$$\leqslant \sum_{k=0}^{n-1} \|f(t+(k+1)\omega) - cf(t+k\omega)\| \leqslant \sum_{k=0}^{n-1} \frac{\varepsilon}{n\omega} = \varepsilon/\omega,$$

provided $|t| \geqslant M(\varepsilon, n) + n\omega$. This completes the proof. □

The following proposition can be deduced from the argumentation contained in the proof of [**247**, Proposition 2.12]:

PROPOSITION 2.9.88. *We have* $S^pQ - AAP_c(I : X) \subseteq W^p_{ap;c}(I : X)$.

The structural properties of quasi-asymptotically almost periodic functions clarified in [**247**, Theorem 2.13] can be slightly generalized in the following manner:

THEOREM 2.9.89. *Let* $f : I \to X$ *be a quasi-asymptotically* c*-almost periodic function (Stepanov* (p, c)*-quasi-asymptotically almost periodic function). Then we have:*

(i) $\alpha f(\cdot)$ *is quasi-asymptotically* c*-almost periodic (Stepanov* (p, c)*-quasi-asymptotically almost periodic) for any* $\alpha \in \mathbb{C}$.
(ii) *If* $X = \mathbb{C}$ *and* $\inf_{x\in I} |f(x)| = m > 0$ *(ess*$\inf_{x\in I} |f(x)| = m > 0$*), then* $1/f(\cdot)$ *is quasi-asymptotically* $1/c$*-almost periodic (Stepanov* $(p, 1/c)$*-quasi-asymptotically almost periodic).*
(iii) *If* $(g_n : I \to X)_{n\in\mathbb{N}}$ *is a sequence of quasi-asymptotically* c*-almost periodic functions and* $(g_n)_{n\in\mathbb{N}}$ *converges uniformly to a function* $g : I \to X$*, then* $g(\cdot)$ *is quasi-asymptotically* c*-almost periodic.*
(iv) *If* $(g_n : I \to X)_{n\in\mathbb{N}}$ *is a sequence of Stepanov* (p, c)*-quasi-asymptotically almost periodic functions and* $(g_n)_{n\in\mathbb{N}}$ *converges to a function* $g : I \to X$ *in the space* $L^p_S(I : X)$*, then* $g(\cdot)$ *is Stepanov* (p, c)*-quasi-asymptotically almost periodic.*
(v) *The functions* $f(\cdot + a)$ *and* $f(b\,\cdot)$ *are likewise quasi-asymptotically* c*-almost periodic (Stepanov* (p, c)*-quasi-asymptotically almost periodic), where* $a \in I$ *and* $b \in I \setminus \{0\}$.

The space of quasi-asymptotically c-almost periodic functions is not closed under pointwise addition and multiplication (see also [**247**, Proposition 2.15, Example 2.16-Example 2.18]).

Concerning the invariance of quasi-asymptotical c-almost periodicity under the actions of convolution products, the structural results clarified in [**247**, Section 3] continue to hold for (Stepanov p-) bounded forcing terms $f(\cdot)$:

PROPOSITION 2.9.90. (i) *Suppose that* $(R(t))_{t>0} \subseteq L(X, Y)$ *is a strongly continuous operator family and* $\int_0^\infty \|R(s)\|\, ds < \infty$*. If the function* $f \in Q - AAP_c([0, \infty) : X)$ *is bounded, then the function* $F(\cdot)$*, defined through*

(131), *with the function* $\mathrm{F}(\cdot)$ *replaced therein with the function* $f(\cdot)$, *belongs to the class* $Q-AAP_c([0,\infty):Y)$.

(ii) *Suppose that* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $\int_0^\infty \|R(s)\|\,ds < \infty$. *If* $f \in Q-AAP_c(\mathbb{R}:X)$ *is bounded, then the function* $\mathbf{F}(t)$, *defined through* (55), *belongs to the class* $Q-AAP_c(\mathbb{R}:Y)$.

PROPOSITION 2.9.91. (i) *Suppose that* $1/p+1/q=1$, $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[k,k+1]} < \infty$. *If* $f \in S^pQ-AAP_c([0,\infty):X)$ *is Stepanov* p*-bounded, then the function* $F(\cdot)$, *defined by* (131), *belongs to the class* $Q-AAP_c([0,\infty):Y)$.

(ii) *Suppose that* $1/p+1/q=1$, $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family and* $\sum_{k=0}^{\infty}\|R(\cdot)\|_{L^q[k,k+1]} < \infty$. *If* $f \in S^pQ-AAP_c(\mathbb{R}:X)$ *is Stepanov* p*-bounded, then the function* $\mathbf{F}(\cdot)$, *defined by* (55), *belongs to the class* $Q-AAP_c(\mathbb{R}:Y)$.

Before switch to the next subsection, let us note the obvious fact that the various notions of Stepanov quasi-asymptotically almost periodic functions in Lebesgue spaces with variable exponent, among many other classes of generalized almost periodic functions, can be slightly generalized by the use of difference $f(\cdot+\tau)-cf(\cdot)$. Fairly complete analysis of corresponding classes is without scope of this book.

2.9.7. Composition principles for quasi-asymptotically c-almost periodic functions. The main aim of this subsection is to introduce the class of quasi-asymptotically c-almost periodic functions depending on two parameters, its Stepanov generalization and to formulate several composition principles for quasi-asymptotically c-almost periodic functions. First of all, we will introduce the folowing definition:

DEFINITION 2.9.92. Suppose that $F: I\times Y \to X$ is a continuous function and $\mathcal{F}$ is a non-empty collection of subsets of Y. Then we say that $F(\cdot,\cdot)$ is quasi-asymptotically c-almost periodic, uniformly on $\mathcal{F}$ if and only if for each $\varepsilon>0$ there exists a finite number $L(\varepsilon)>0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number holds with a number $M(\varepsilon,\tau)>0$ such that for each subset $B \in \mathcal{F}$ we have:

$$\big\|F(t+\tau,x)-cF(t,x)\big\|_Y \leqslant \varepsilon, \text{ provided } t\in I,\ x\in B \text{ and } |t| \geqslant M(\varepsilon,\tau).$$

Denote by $Q-AAP_{c;\mathcal{F}}(I\times Y:X)$ the set consisting of all quasi-asymptotically c-almost periodic functions $F: I\times Y\to X$ on $\mathcal{F}$.

Suppose that $F: I\times Y\to X$ is a continuous function and there exists a finite constant $L>0$ such that (60) holds. Define $\mathcal{F}(t):=F(t,f(t))$, $t\in I$. Using (174) and the proofs of [**135**, Theorem 3.30, Theorem 3.31], we may deduce the following composition principles:

THEOREM 2.9.93. *Suppose that* $F \in Q-AAP_c(I\times Y:X)$ *and* $f \in Q-AAP_c(I:Y)$. *If there exists a finite number* $L>0$ *such that* (60) *holds and for each* $\varepsilon>0$ *there exists a finite number* $L(\varepsilon)>0$ *such that any interval* $I' \subseteq I$ *of length* $L(\varepsilon)$

contains at least one number $\tau \in I'$ satisfying that

$$\Big\| F\big(t+\tau, cf(t)\big) - cF(t, f(t))\Big\| \leqslant \varepsilon, \quad t \in I, \tag{196}$$

then the function $t \mapsto F(t, f(t))$, $t \in I$ belongs to the class $Q - AAP_c(I : X)$.

THEOREM 2.9.94. *Suppose that $F \in Q - AAP_c(I \times Y : X)$ and $f \in Q - AAP_c(I : Y)$. If the function $x \mapsto F(t, x)$, $t \in I$ is uniformly continuous on $R(f)$ uniformly for $t \in I$ and for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that* (196) *holds, then the function $t \mapsto F(t, f(t))$, $t \in I$ belongs to the class $Q - AAP_c(I : X)$.*

The notion of a Stepanov (p, c)-quasi-asymptotically almost periodic function depending on two parameters can be also introduced, and [**247**, Theorem 2.23, Theorem 2.24] can be slightly generalized in this framework.

In [**247**, Section 4], we have analyzed the qualitative solutions of the abstract nonautonomous differential equations (135)-(136) and their semilinear analogues. We close the section with the observation that the structural results established in [**247**, Theorem 4.1, Theorem 4.3] can be simply reformulated in our context; for example, in the formulation of [**247**, Theorem 4.1], we can assume that

$$\sum_{k=0}^{\infty} \big\| \Gamma(t+\tau, t+\tau-\cdot) - c\Gamma(t, t-\cdot)\big\|_{L^q[k,k+1]} \leqslant \varepsilon, \quad \text{provided } t \geqslant M(\varepsilon, \tau),$$

in place of condition [**247**, (4.1)]. Then the unique mild solution $u(\cdot)$ of the abstract Cauchy problem (136) will belong to the class $Q - AAP_c([0, \infty) : X) + \mathcal{F}$; see [**247**] for the notation. The structural results established for the abstract nonautonomous semilinear differential equations [**247**, Theorem 4.6, Theorem 4.7] can be slightly generalized in our framework, as well.

2.10. Notes and appendicies

In this section, we will briefly consider several important topics which have not been discussed in the previous part of this monograph.

Recurrent strongly continuous semigroups. The notion of a uniformly recurrent operator is closely connected with the notion of a recurrent operator in a complex Banach space X. Let us recall that a linear operator $T : X \to X$ is called recurrent if and only if for every non-empty open subset U of X there exists some $k \in \mathbb{N}$ such that $U \cap T^{-k}(U) \neq \emptyset$. A vector $x \in X$ is said to be recurrent for T if and only if there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}x \to x$ as $n \to +\infty$; the set consisting of all reccurent vectors of T will be denoted by $Rec(T)$. A much stronger notion than the recurrence is the measure theoretic rigidity, introduced in the ergodic theoretic setting by H. Furstenberg and B. Weiss ([**178**]; see also [**177**]). This concept, in the context of topological dynamical systems, is known as the (uniform) rigidity, which was introduced by S. Glasner and D. Maon ([**182**]). We say that a bounded linear operator $T : X \to X$ is rigid if and only if there exists a strictly increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $T^{k_n}x \to x$ as $n \to +\infty$, for every $x \in X$. A bounded linear operator

$T : X \to X$ is called uniformly rigid if and only if there exists an increasing sequence of positive integers $(k_n)_{n\in\mathbb{N}}$ such that $\|T^{k_n} - I\| = \sup_{\|x\|\leqslant 1} \|T^{k_n}x - x\| \to 0$ as $n \to +\infty$. For more details about recurrent and rigid operators on Banach spaces, see the research articles [**119**] by G. Costakis, I. Parissis and [**118**] by G. Costakis, A. Manoussos, I. Parissis.

For families of bounded linear operators, we will use the following notion:

Definition 2.10.1. Let $I = [0, \infty)$ or $I = \mathbb{R}$. We say that a family $(W(t))_{t\in I}$ of bounded linear operators on X is recurrent if and only if for every open non-empty set $U \subseteq X$ there exists some $t \in I$ such that $U \cap (W(t))^{-1}(U) \neq \emptyset$. A vector $x \in X$ is called a recurrent vector for $(W(t))_{t\in I}$ if and only if there exists an unbounded sequence of numbers (t_k) in I such that $W(t_k)x \to x$ as $k \to +\infty$. By $Rec(W(t))$ we denote the set consisting of all recurrent vectors for $(W(t))_{t\in I}$.

Definition 2.10.2. We say that a family $(W(t))_{t\in I}$ of bounded linear operators on X is rigid if and only if there exists an unbounded sequence of numbers (t_k) in I such that $W(t_k)x \to x$ as $k \to +\infty$, for every $x \in X$, i.e. $W(t_k) \to I$ as $k \to +\infty$ in the strong operator topology, while $(W(t))_{t\in I}$ is called uniformly rigid if and only if there exists an unbounded sequence (t_k) in I such that $\|W(t_k) - I\| \to 0$ as $k \to \infty$.

The following result is fundamental:

Theorem 2.10.3. *Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0, \infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. The following statements are equivalent:*

(i) *$(T(t))_{t\in I}$ is recurrent.*
(ii) *$\overline{Rec(T(t))} = X$.*

If this is the case, the set of recurrent vectors for $(T(t))_{t\in I}$ is a G_δ-subset of X.

Proof. First we will show that (ii) $\Rightarrow$ (i). Let $\overline{Rec(T(t))} = X$ and U be an arbitrary open non-empty subset in X. Let x be a recurrent vector and $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$, where $B(x, \varepsilon) = \{y \in X : \|x - y\| < \varepsilon\}$. Then there exists $t \in I$ such that $\|T(t)x - x\| < \varepsilon$. Thus $x \in U \cap T(t)(U) \neq \emptyset$, so $(T(t))_{t\in I}$ is recurrent. Now, we will show that (i) $\Rightarrow$ (ii). Let $(T(t))_{t\in I}$ be recurrent and let $B = B(x, \varepsilon)$ be an open ball in X, for fixed $x \in X$ and $\varepsilon < 1$. The proof will end if we show that there exists a recurrent vector in B. We use the recurrence property of $(T(t))_{t\in I}$. So, there exists $t_1 \in I$ such that $x_1 \in B \cap T(t_1)^{-1}(B)$, for some $x_1 \in E$. Since $(T(t))_{t\in I}$ is strongly continuous, we have that there exists $\varepsilon_1 < \frac{1}{2}$ such that $B_2 = B(x_1, \varepsilon_1) \subseteq B \cap T(t_1)^{-1}(B)$. Since $(T(t))_{t\in I}$ is recurrent, there exists $t_2 \in I$ with $|t_2| > |t_1|$ and some $x_2 \in E$ such that $x_2 \in B_2 \cap T(t_2)^{-1}(B_2)$. Using the same argument with strong continuity and recurrence of $(T(t))_{t\in I}$, we can inductively construct a sequence (x_n) in X, an unbounded sequence (t_n) in I and a decreasing sequence of positive real numbers (ε_n), such that for every integer $n \in \mathbb{N}$ one has $\varepsilon_n < 2^{-n}$,

$$B(x_n, \varepsilon_n) \subseteq B(x_{n-1}, \varepsilon_{n-1}) \quad \text{and} \quad T(t_n)\big(B(x_n, \varepsilon_n)\big) \subseteq B(x_{n-1}, \varepsilon_{n-1}).$$

By Cantor's theorem we have that

$$\bigcap_{n=1}^{\infty} B(x_n, \varepsilon_n) = \{y\},$$

for some $y \in X$. It is clear that $T(t_n)y \to y$ as $n \to +\infty$. Hence $y \in B$ is a recurrent vector in the open ball B, so the proof of (ii) $\Rightarrow$ (i) is finished. Let us prove that

$$Rec(T(t)) = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \left\{ x \in X \, : \, \left\| T(q_n)x - x \right\| < \frac{1}{k} \right\} =: \mathrm{R}(T(t)), \tag{197}$$

where (q_n) denotes the sequence consisting of all rational numbers which do have the modulus strictly greater than 1. It simply follows that $Rec(T(t))$ is contained in the set $\mathrm{R}(T(t))$. For the opposite inclusion, for each element $x \in \mathrm{R}(T(t))$ and for each integer $k \in \mathbb{N}$ we can pick up a rational number q_k which do have the module strictly greater than 1 and for which $\|T(q_k)x - x\| < 1/k$. If the sequence (q_k) is unbounded, we have done. If not, then there exists a convergent subsequence (q_{n_k}) of (q_n) such that $\lim_{k\to\infty} q_{n_k} = q$ for some real number $q \in I$ such that $|q| \geqslant 1$. In this case, the strong continuity of $(T(t))_{t\in I}$ shows that $x = T(q)x$ so that clearly $x \in Rec(T(t))$ because, in this case, we have $T(nq)x = x$ for all $n \in \mathbb{N}$. Hence, (197) holds and $(T(t))_{t\in I}$ is a G_δ subset of X. $\square$

Using the representation formula (197) and the proof of [**118**, Proposition 2.6], it can be easily seen that the following result holds good:

THEOREM 2.10.4. *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group on X. Then $(T(t))_{t\geqslant 0}$ is recurrent if and only if $(T(-t))_{t\geqslant 0}$ is recurrent.*

We continue by stating the following continuous analogue of [**118**, Proposition 2.3(i)]:

THEOREM 2.10.5. *Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0, \infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. Then, for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we have $Rec(T(t)) = Rec(\lambda T(t))$.*

PROOF. It is enough to show that $Rec(T(t)) \subseteq Rec(\lambda T(t))$. For $x \in Rec(T(t))$, we define the set $L = \big\{ |\mu| = 1 \, : \, \lambda^n T(t_n)x \to \mu x$, for some unbounded sequence (t_n) in $I \big\}$. To finish the proof, we have to prove that $1 \in L$. First of all, let us note that $L \neq \emptyset$. Since $x \in Rec(T(t))$, there exists an unbounded sequence (t_n) in I such that $T(t_n)x \to x$. There exists a subsequence of (t_n), denoted by (t_{n_k}), such that $\lambda^{t_{k_n}} \to \rho$ as $k \to \infty$, for some $|\rho| = 1$. Hence, we have $\lambda^{t_{k_n}} T(t_{n_k})x \to \rho x$ as $k \to \infty$, which means that $\rho \in L$. Let $\mu_1,\ \mu_2 \in L$ and $\varepsilon > 0$ be fixed. Since $\mu_1 \in L$, there exist a positive integer $n_1 \in \mathbb{N}$ and a real number $t_1 \in I$, with modulus sufficiently large, such that

$$\left\| \lambda^{n_1} T(t_1)x - \mu_1 x \right\| < \frac{\varepsilon}{2}.$$

Since $\mu_2 \in L$, there is a positive integer $n_2 \in \mathbb{N}$ and a real number $t_2 \in I$, with module sufficiently large, such that

$$\left\|\lambda^{n_2} T(t_2)x - \mu_2 x\right\| < \frac{\varepsilon}{2\|T(t_1)\|}.$$

Hence,

$$\begin{aligned}\left\|\lambda^{n_1+n_2} T(t_1+t_2)x - \mu_1\mu_2 x\right\| &\leqslant \left\|\lambda^{n_1} T(t_1)(\lambda^{n_2} T(t_2)x - \mu_2 x)\right\| + \left\|\mu_2(\lambda^{n_1} T(t_1)x - \mu_1 x)\right\| \\ &\leqslant \left\|T(t_1)\right\|\left\|(\lambda^{n_2} T(t_2))x - \mu_2 x\right\| + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

so that $\mu_1\mu_2 \in L$. Hence, $\mu^n \in L$ for $\mu \in L$. If μ is a rational rotation, this means that $1 \in L$ and we are done. If μ is an irrational rotation, there is a strictly increasing sequence of positive integers (s_k) such that $\mu^{s_k} \to 1$. Since L is closed, it follows that $1 \in L$. $\square$

THEOREM 2.10.6. *Let $(T(t))_{t\in I}$ be a C_0-semigroup if $I = [0, \infty)$, resp. C_0-group if $I = \mathbb{R}$, of bounded linear operators on X. If $(T(t) \oplus T(t))_{t\in I}$ is recurrent, then $(T(t))_{t\in I}$ is likewise recurrent.*

PROOF. Let $x_1 \oplus x_2$ be a recurrent vector for $(T(t) \oplus T(t))_{t\in I}$. Then it is clear that x_1 and x_2 are recurrent vectors for $(T(t))_{t\in I}$; hence, $(T(t))_{t\in I}$ is recurrent. $\square$

The question whether the direct sum $(T(t) \oplus T(t))_{t\in I}$ of recurrent strongly continuous operator families $(T(t))_{t\in I}$ is recurrent is not simple. The answer is affirmative if $(T(t))_{t\in I}$ possesses some extra properties (see [**118**] for more details about single-valued case).

The following continuous analogue of [**118**, Proposition 2.3(ii)] appears in this monograph for the first time:

THEOREM 2.10.7. *Let $(T(t))_{t\in\mathbb{R}}$ be a C_0-group. Then the following assertions are equivalent:*

(i) *$(T(t))_{t\geqslant 0}$ is recurrent.*
(ii) *For every $t_0 > 0$, the operator $T(t_0)$ is recurrent.*
(iii) *There exists $t_0 > 0$ such that the operator $T(t_0)$ is recurrent.*

If this is the case, then for every $t_0 \in I \smallsetminus \{0\}$, we have $Rec(T(t)) = Rec(T(t_0))$.

PROOF. The only non-trivial part is that (i) implies (ii), with the equality $Rec(T(t)) = Rec(T(t_0))$ for any fixed number $t_0 > 0$. To see this, assume that $(T(t))_{t\geqslant 0}$ is a recurrent C_0-semigroup. Then it is clear that $Rec(T(t)) \supseteq Rec(T(t_0))$ and, owing to Theorem 2.10.3, all that we need to prove is that the preassumption $x \in Rec(T(t))$ implies $x \in Rec(T(t_0))$. Without loss of generality, we can assume that $t_0 = 1$. Indeed, we can consider the semigroup $(\tilde{T}(t))_{t\geqslant 0}$, with $\tilde{T}(t) := T(tt_0)$, for every $t \geqslant 0$. It is clear that x is a recurrent vector for $(\tilde{T}(t))_{t\geqslant 0}$ and $\tilde{T}(1) = T(t_0)$. Denote by $\mathbb{T}$ the unit sphere in $\mathbb{C}$ and define the mapping $\phi : [0, \infty) \to \mathbb{T}$ by $\phi(t) := e^{2\pi i t}$, $t \geqslant 0$. For every $u \in X$, we define the set

$$F_u := \Big\{\lambda \in \mathbb{T} : \ \exists (t_n)_n \in (0, \infty) \text{ s.t. } \lim_{n\to\infty} t_n = \infty, \lim_{n\to\infty} T(t_n)u = u \text{ and } \lim_{n\to\infty} \phi(t_n) = \lambda\Big\}.$$

Note that the set F_u is not empty by its definition and the recurrence property of the semigroup $(T(t))_{t\geqslant 0}$. The set F_u is closed for $u \in X$, as it can be easily checked. Next, we will prove that if $u \in X$ and $\lambda,\ \mu \in F_u$, then $\lambda\mu \in F_u$. Let U be an open balanced neighborhood of zero in X and $\varepsilon > 0$ arbitrary. Then we can find $t_1 > 0$ such that $\|T(t_1)u - \lambda u\| \leqslant \varepsilon/2$ and $|\phi(t_1) - \mu| < \varepsilon/2$. Choose an open balanced neighborhood of zero V in X and number $t_2 > 0$ such that $T(t_1)(V) \subseteq U$, $T(t_2)u - \mu u \in V$ and $|\phi(t_2) - \lambda| < \varepsilon/2$. Hence,

$$\begin{aligned} T(t_1 + t_2)u - \lambda\mu u &= T(t_1)\big(T(t_2)u - \mu u\big) + \mu\big(T(t_1)u - \lambda u\big) \\ &\in T(t_1)(V) + B(0, \varepsilon/2) \subseteq U + B(0, \varepsilon/2), \end{aligned}$$

so that

$$\big|\phi(t_1 + t_2) - \lambda\mu\big| = \big|\phi(t_1)\phi(t_2) - \lambda\mu\big| \leqslant \big|\phi(t_1) - \mu\big| \cdot \big|\phi(t_2)\big| + |\mu| \cdot \big|\phi(t_2) - \lambda\big| < \varepsilon.$$

This simply implies that $\lambda\mu \in F_u$ as claimed. Further on, it is clear that there exists $x \in (-\pi, \pi]$ such that $e^{ix} = \lambda \in F_u$. If x is rational, then using the fact that F_u is closed under multiplication immediately gives $1 \in F_u$. If x is not rational, then F_u is dense in $\mathbb{T}$ since it contains the set $\{e^{inx} : n \in \mathbb{N}\}$ so that $1 \in F_u$ again. Hence, $1 \in F_u$. Suppose now $u \in Rec(T(t))$. Then we have the existence of a sequence $(t_n)_{n \in \mathbb{N}}$ of positive real numbers tending to infinity such that $\lim_{n\to\infty} T(t_n)u = u$ and $\lim_{\to\infty} \phi(t_n) = 1$. Let (k_n) be a sequence of positive integers and $\varepsilon_n \in [-1, 1]$ such that $t_n = k_n + \varepsilon_n$ for all $n \in \mathbb{N}$. Obviously, $\lim_{n\to\infty} \varepsilon_n = 0$. Hence, $\|T(k_n)u - u\| \leqslant \|T(-\varepsilon_n)[T(t_n)u - u] + [T(-\varepsilon_n)u - u]\| \leqslant \sup_{\xi\in[-1,1]} \|T(\xi)\| \cdot \|T(t_n)u - u\| + \|T(-\varepsilon_n)u - u\| \to 0$ as $n \to +\infty$. As a consequence, we have $u \in Rec(T(1))$. $\square$

REMARK 2.10.8. Condition that $(T(t))_{t\geqslant 0}$ can be extended to a C_0-group seems to be slightly redundant. Due to [**306**, Theorem 6.5, p. 24], this is the case provided that there exists a finite number $t_0 > 0$ such that $[T(t_0)]^{-1} \in L(X)$.

Suppose that $\Delta = [0, \infty)$ or $\Delta = \mathbb{R}$. A measurable function $\rho : \Delta \to (0, \infty)$ is said to be an admissible weight function if and only if there exist constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\rho(t) \leqslant Me^{\omega|t'|}\rho(t + t')$ for all $t,\ t' \in \Delta$. Let us introduce the Banach spaces

$$L^p_\rho(\Delta, \mathbb{C}) := \big\{u : \Delta \to \mathbb{C}\ ; u(\cdot) \text{ is measurable and } ||u||_p < \infty\big\},$$

where $p \in [1, \infty)$ and $||u||_p := (\int_\Delta |u(t)|^p \rho(t)\, dt)^{1/p}$, and

$$C_{0,\rho}(\Delta, \mathbb{C}) := \Big\{u : \Delta \to \mathbb{K}\ ; u(\cdot) \text{ is continuous and } \lim_{t\to\infty} u(t)\rho(t) = 0\Big\},$$

with $||u|| := \sup_{t\in\Delta} |u(t)\rho(t)|$. For any function $f : \Delta \to \mathbb{C}$, we define $T(t)f := f(\cdot + t)$, $t \in \Delta$. If $\rho(\cdot)$ is an admissible weight function and $\Delta = [0, \infty)$, resp. $\Delta = \mathbb{R}$, then the translation semigroup, resp. group, $(T(t))_{t\in\Delta}$ is strongly continuous on $L^p_\rho(\Delta, \mathbb{C})$ and $C_{0,\rho}(\Delta, \mathbb{C})$. Recently, Z. Yin and Y. Wei have considered weak recurrence of translation operators on weighted Lebesgue spaces and weighted continuous function spaces ([**355**]). They have shown that the existence of a function $f \in X$, where $X = L^p_\rho([0, \infty), \mathbb{C})$ or $X = C_{0,\rho}([0, \infty), \mathbb{C})$, satisfying that there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that (compare with (19))

$$\lim_{n\to+\infty} \big\|f(\cdot + \alpha_n) - f(\cdot)\big\|_X = 0$$

is equivalent to saying that $\liminf_{t\to+\infty} \rho(t) = 0$ (the hypercyclicity of $(T(t))_{t\geqslant 0}$); see also the preprint [**83**] by W. Brian and J. P. Kelly.

For more details about recurrent sets of operators, we refer the reader to the recent paper [**22**] by M. Amouch and O. Benchiheb.

Lower and upper densities. In Subsection 2.4.1, we have used various notions of lower and upper densities for a subset $A \subseteq [0,\infty)$ which can take, generally speaking, any value in the range $[0,\infty]$. Without any doubt, the most important densities are those ones with values in the range $[0,1]$. As in the discrete case, the minimal conditions which should satisfy any lower or upper density $d : P([0,\infty)) \to [0,1]$ are: $d(\emptyset) = 0$, $d([0,\infty)) = 1$ and $d(A) \leqslant d(B)$, whenever $A,\ B \subseteq [0,\infty)$ and $A \subseteq B$. But, some other axioms are needed for obtaining a good definition of density. For example, following A. R. Freedman and J. J. Sember [**175**] we can consider the upper density $\delta^\star(\cdot) : P([0,\infty)) \to [0,1]$ with the following properties:

(l1) $\delta^\star(A\cup B) \leqslant \delta^\star(A) + \delta^\star(B)$;
(l2) $\delta^\star(A) = \delta^\star(B)$, provided that $A\Delta B$ is bounded;
(l3) $\delta_\star(A) \leqslant \delta^\star(A)$.

It is also worth noting that we can consider the upper density $\nu^\star : P([0,\infty)) \to [0,1]$ with the following properties introduced recently by P. Leonetti and S. Tringali in the discrete case ([**261**]):

(f1) $\nu^\star(A\cup B) \leqslant \nu^\star(A) + \nu^\star(B)$;
(f2) $\nu^\star(\alpha A) = \alpha^{-1}\nu^\star(A)$, provided that $\alpha > 0$;
(f3) $\nu^\star(A+\alpha) = \nu^\star(A)$, provided that $\alpha > 0$.

Besides that, it could be of some importance to analyze many other notions of lower and upper densities in the continuous setting, like the notions of upper logarithmic, upper Buck, upper Pólya or upper analytic densities (see also the classical studies by A. S. Besicovitch [**63, 64, 65**], the monograph [**128**] by C. De Lellis and the doctoral dissertation of N. F. G. Martin [**283**]). For further information, see also [**185**], [**235**] and references cited therein.

Almost periodic functions of complex variables. The theory of almost periodic functions of one complex variable, initiated already by H. Bohr in the third part of [**75**], is still very popular and attracts the attention of numerous mathematicians (see e.g., [**165**], [**217**], [**330**]). Suppose that $-\infty \leqslant \alpha < \beta \leqslant +\infty$ and the function $f : \Omega \equiv \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\} \to X$ is analytic. Then we say that $f(\cdot)$ is almost periodic if and only if for any $\varepsilon > 0$ and every reduced strip $\{z \in \mathbb{C} : \alpha' < \operatorname{Re} z < \beta'\}$, where $\alpha < \alpha' < \beta' < \beta$, there exists a number $l > 0$ such that each subinterval of length l of $\mathbb{R}$ contains a number τ satisfying the inequality

$$\|f(z+i\tau) - f(z)\| \leqslant \varepsilon \text{ for } \alpha' < \operatorname{Re} z < \beta'.$$

In particular, this definition implies that, for any fixed $\sigma \in (\alpha,\beta)$, the function $f_\sigma(t) := f(\sigma+it)$, $t\in\mathbb{R}$ is almost periodic. Moreover, the definition implies that the almost periodicity should be uniform on the various straight lines, with the meaning clear. The Fourier series of these functions can be obtained from a certain exponential series with complex coefficients; the associated series is called the

Dirichlet series of $f(\cdot)$. As for the functions of one real variable, Bohr's notion of almost periodicity of $f(\cdot)$ in a vertical strip Ω is equivalent to the relative compactness of the set of its vertical translates, $\{f(\cdot + ih) : h \in \mathbb{R}\}$, with the topology of the uniform convergence on reduced strips. Mean motions and zeros of generalized almost periodic analytic functions have been analyzed by V. Borchsenius and B. Jessen in [**78**], where the reader can find several important applications to the Riemann zeta function (see also [**292**] and the references there for further information about applications of results from the theory of almost periodic analytic functions to the Riemann zeta function).

We would like to accent that the notions of uniform recurrence and $\odot_g$-almost periodicity for the functions of one real variable can be simply modified and introduced for the functions of one complex variable. For more details about almost periodic analytic functions of several complex variables, we refer the reader to [**166, 322**] and references quoted therein.

$C^{(n)}$-almost periodic functions. The notion of $C^{(n)}$-almost periodicity was introduced by M. Adamczak [**6**] in 1997 and later received a grat attention of many other authors. In this monograph, we will not consider $C^{(n)}$-almost periodic type functions and solutions of integro-differential equations; we shall only say a few words about generalized $C^{(n)}$-almost periodic functions and possibilites for further expansions.

Several different classes of Stepanov-like $C^{(n)}$-pseudo almost automorphic functions have been analyzed by T. Diagana, V. Nelson and G. M. N'Guérékata in [**144**]. For example, let $1 \leqslant p < \infty$, let $n \in \mathbb{N}$, and let $f \in L^p_{loc}(I : X)$.

(i) We say that the function $f(\cdot)$ is Stepanov-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - APS^p(I : X)$ for short, if and only if for each $k = 0, 1, \cdots, n$, we have that $f^{(k)} \in APS^p(I : X)$.

(ii) We say that the function $f \in L^p_{loc}([0,\infty) : X)$ is asymptotically Stepanov-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - AAPS^p([0,\infty) : X)$ for short, if and only if for each $k = 0, 1, \cdots, n$, we have that $f^{(k)} \in AAPS^p([0,\infty) : X)$. The following definitions have been analyzed in [**234**]:

(iii) We say that the function $f(\cdot)$ is equi-Weyl-p-$C^{(n)}$-almost periodic, $f \in e - C^{(n)} - W^p_{ap}(I : X)$ for short, if and only if for each $k = 0, 1, \cdots, n$, we have that $f^{(k)} \in e - W^p_{ap}(I : X)$.

(iv) We say that the function $f(\cdot)$ is Weyl-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - W^p_{ap}(I : X)$ for short, if and only if for each $k = 0, 1, \cdots, n$, we have that $f^{(k)} \in W^p_{ap}(I : X)$.

(v) We say that he function $f(\cdot)$ is Besicovitch-Doss-p-$C^{(n)}$-almost periodic, $f \in C^{(n)} - \mathrm{B}^p(I : X)$ for short, if and only if for each $k = 0, 1, \cdots, n$, we have that $f^{(k)} \in \mathrm{B}^p(I : X)$.

Using the same idea, we can introduce and analyze a great number of $C^{(n)}$-almost automorphic function spaces ([**234**]). For example, the function

$$f(t)=\sum_{n=1}^{\infty}\frac{\sin nt}{n^4},\quad t\in\mathbb{R}$$

is $C^{(2)}$-almost periodic but not $C^{(3)}$-almost automorphic. Furthermore, for any real-valued function $g\in C^{(3)}-AA(\mathbb{R}:\mathbb{C})$ satisfying $\inf_{t\in\mathbb{R}}g'''(t)>0$, we have that the function

$$f(t)=\sum_{n=1}^{\infty}\frac{g(nt)}{n^4},\quad t\in\mathbb{R},$$

belongs to the space $C^{(2)}$-$AAS^1(\mathbb{R}:\mathbb{C})\setminus C^{(3)}$-$AAS^1(\mathbb{R}:\mathbb{C})$; see e.g. [**144**, Example 2.23]. It is clear that we can slightly generalize the notion of all above-mentioned function spaces by using the definitions and results from the theory of $L^{p(x)}$-spaces.

Riemann-Stepanov almost periodicity, Riemann-Weyl almost periodicity and Riemann-Besicovitch almost periodicity. In [**157**], R. Doss has analyzed the classes of Riemann-Stepanov almost periodic functions, Riemann-Weyl almost periodic functions and Riemann-Besicovitch almost periodic functions. All considerations in this paper are carried out for the scalar-valued functions.

Following [**157**, Definition 1], we say that an essentially bounded function $f:I\rightarrow X$ is Riemann-Stepanov almost periodic if and only if for every $\varepsilon>0$ there exist $\delta>0$ and numbers $\pi_1\in I,\cdot\cdot\cdot,\pi_m\in I$ such that

$$\sup_{x\in I}\overline{\int_x^{x+1}}\|f(t+\tau_t)-f(t)\|\,dt<\varepsilon \tag{198}$$

provided that $|\tau_t|<\delta\pmod{\pi_k}$, $k\in\mathbb{N}_m$; here, $\overline{\int}$ denotes the upper Lebesgue integral. If we replace the quantity in (198) with

$$\limsup_{l\rightarrow+\infty}\sup_{x\in I}\frac{1}{l}\overline{\int_x^{x+l}}\|f(t+\tau_t)-f(t)\|\,dt<\varepsilon$$

resp.,

$$\limsup_{l\rightarrow+\infty}\frac{1}{2l}\overline{\int_{-l}^{l}}\|f(t+\tau_t)-f(t)\|\,dt,\ \text{if } I=\mathbb{R},\ \text{resp.}$$

$$\limsup_{l\rightarrow+\infty}\frac{1}{l}\overline{\int_{0}^{l}}\|f(t+\tau_t)-f(t)\|\,dt,\ \text{if } I=[0,\infty),$$

then we say that $f(\cdot)$ is Riemann-Weyl almost periodic, resp. Riemann-Besicovitch almost periodic.

Following A. S. Kovanko [**259**], R. Doss has also introduced the classes of Kovanko-Stepanov almost periodic functions, Kovanko-Weyl almost periodic functions and Kovanko-Besicovitch almost periodic functions (see [**157**, Definition 2]). This definition can be simply introduced in the vector-valued case.

For any measurable set $E \subseteq I$, we introduce the quantities

$$S(E) := \sup_{x\in I} \int_x^{x+1} \chi_E(t)\, dt$$

$$W(E) := \lim_{l\to+\infty} \sup_{x\in I} \frac{1}{l} \int_x^{x+l} \chi_E(t)\, dt$$

and

$$B(E) := \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \chi_E(t)\, dt, \text{ if } I = \mathbb{R}, \text{ resp.}$$

$$B(E) := \limsup_{l\to+\infty} \frac{1}{l} \int_{0}^{l} \chi_E(t)\, dt, \text{ if } I = [0,\infty).$$

In [**157**, Theorem 1], it has been proved that an essentially bounded function $f : I \to X$ is Riemann-Stepanov almost periodic if and only if for every $\varepsilon > 0$ there exist a measurable set $E \subseteq I$ and numbers $\delta > 0$, $\pi_1 \in I, \cdots, \pi_m \in I$ such that $S(I \setminus E) < \varepsilon$ and $|f(x) - f(x')| < \varepsilon$ provided $x \in E$ and $|x - x'| < \delta \pmod{\pi_k}$, $k \in \mathbb{N}_m$. For the Riemann-Weyl almost periodicity and the Riemann-Besicovitch almost periodicity, we have the same statement with the quantity $S(I\setminus E)$ replaced respectively by $W(I \setminus E)$ and $B(I \setminus E)$. We would like to note that the proof of necessity in this theorem works for the vector-valued functions, as it can be simply inspected. But, the proof of sufficiency in this theorem and the statement of [**157**, Theorem 2] are intended solely for the scalar-valued functions. Furthermore, in the scalar-valued case, we have that the concepts Riemann-Weyl almost periodicity and the Riemann-Besicovitch almost periodicity coincide.

Due to [**157**, Theorem 3], we have that an essentially bounded function $f : I \to X$ is Riemann-Stepanov almost periodic if and only if for every $\varepsilon > 0$ there exist a measurable set $E \subseteq I$ and a trigonometric polynomial $q(\cdot)$ such that $S(I \setminus E) < \varepsilon$ and $|f(x) - q(x)| < \varepsilon$ provided $x \in E$. For the Riemann-Weyl almost periodicity and the Riemann-Besicovitch almost periodicity, we have the same statement with the quantity $S(I \setminus E)$ replaced respectively by $W(I \setminus E)$ and $B(I \setminus E)$. We would like to note that the proof of sufficiency in this theorem works for the vector-valued functions.

Nemytskii operators between Stepanov almost periodic function spaces. Let p and q be two real numbers belonging to the interval $[1,\infty)$, and let $T > 0$. It is said that $f : (0,T) \times X \to Y$ is a Carathéodory function if and only if the following holds:

(i) the mapping $t \mapsto f(t,x)$, $t \in (0,T)$ is measurable for any fixed element $x \in X$;

(ii) for a.e. $t \in (0,T)$ the function $f(t,\cdot)$ is continuous from X and Y.

Consider now the Nemytskii operator $\mathcal{N}_f : L^p((0,T) : X) \to L^q((0,T) : Y)$ by

$$\big[\mathcal{N}_f(\omega)\big](t) := f(t,\omega(t)), \quad t \in (0,T),\ \omega \in L^p((0,T) : X).$$

The well known result of R. Lucchetti and F. Patrone [**278**, Theorem 3.1] states that the Nemytskii operator is a well defined between these spaces if and only if there exist $a > 0$ and $b \in L^p((0,T))$ such that for all $x \in X$ and a.e. $t \in (0,T)$ we have

$$\|f(t,x)\| \leqslant a\|x\|^{p/q} + b(t).$$

In this case, the Nemytskii operator is continuous.

Concerning the Nemytskii operator between the spaces of almost periodic functions $AP(\mathbb{R} : X)$ and $AP(\mathbb{R} : Y)$, it should be noted that we have the equivalence of the following statements (see e.g. J. Blot, P. Cieutat, G. M. NGuérékata and D. Pennequin [**69**]) :

(i) The Nemytskii operator $\mathcal{N}_f : AP(\mathbb{R} : X) \to AP(\mathbb{R} : Y)$ is continuous.
(ii) For each compact set $K \subseteq X$ and for each $\varepsilon > 0$ the set

$$\Big\{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} \sup_{x\in K} \|f(t+\tau) - f(t,x)\| \leqslant \varepsilon\Big\}$$

is relatively dense in $\mathbb{R}$.
(iii) For all $x \in X$, $f(\cdot, x) \in AP(\mathbb{R} : Y)$ and for each compact set $K \subseteq X$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $x_1,\ x_2 \in K$ and for each $t \in \mathbb{R}$ we have the implication: $\|x_1 - x_2\| \leqslant \delta \Rightarrow \|f(t,x_1) - f(t,x_2)\| \leqslant \varepsilon$.

A similar statement holds for the continuity of Nemytskii operator between the spaces of almost automorphic functions $AA(\mathbb{R} : X)$ and $AA(\mathbb{R} : Y)$; see e.g., the recent paper [**107**, Theorem 2.3] by P. Cieutat. Several necessary and sufficient conditions clarifying the continuity of Nemytskii operators between almost periodic and almost automorphic spaces in the sense of Stepanov can be found in [**107**, Section 4].

Geometric properties of generalized almost periodic function spaces of Orlicz type. In his fundamental paper [**211**], T. R. Hillmann has investigated the Besicovitch-Orlicz spaces of almost periodic functions. After that, numerous mathematicians working in the field of almost periodic functions have investigated the geometric properties of generalized almost periodic function spaces of Orlicz type.

We will insribe here the results of M. Morsli, M. Smaali established in [**296**] and the results of F. Bedouhene, Y. Djabri, F. Boulahia established in [**55**], only; for more details on the subject, we refer the reader to [**57**], [**100**], [**293**]-[**295**] and the list of references quoted in these papers. Assume that the function $\varphi : \mathbb{R} \times [0,\infty) \to [0,\infty)$ satisfies the following conditions:

(i) For every $t \in \mathbb{R}$, we have $\varphi(t,0) = 0$.
(ii) For every $t \in \mathbb{R}$, the mapping $u \mapsto \varphi(t,u)$, $u \geqslant 0$ is convex.
(iii) $\varphi(t+1,u) = \varphi(t,u)$ for all $t \in \mathbb{R}$ and $u \geqslant 0$.
(iv) For every $u > 0$, we have $\inf_{t\in\mathbb{R}} \varphi(t,u) = \phi(u) > 0$.

If $f : \mathbb{R} \to [0,+\infty]$ is a measurable function, then it is well known that the functional

$$f \mapsto \rho_\varphi(f) := \limsup_{t\to+\infty} \frac{1}{2t} \int_{-t}^{t} \varphi\big(t, |f(t)|\big)\, dt, \quad f \in M(\mathbb{R}),$$

is convex and pseudomodular.

In [**296**], the authors have defined the Besicovitch-Musielak-Orlicz space associated to $\varphi(\cdot,\cdot)$ by

$$B^{\varphi}(\mathbb{R}) := \Big\{ f \in M(\mathbb{R}) : \lim_{\alpha\to 0+} \rho_{\varphi}(\alpha f) = 0 \Big\}.$$

We have

$$B^{\varphi}(\mathbb{R}) = \Big\{ f \in M(\mathbb{R}) : (\exists \alpha > 0) \ \ \rho_{\varphi}(\alpha f) < \infty \Big\}.$$

The space $B^{\varphi}(\mathbb{R})$ is eqipped with the pseudonorm

$$\|f\|_{\varphi} := \Big\{ k > 0 : \rho_{\varphi}(f/k) \leqslant 1 \Big\}.$$

The authors have introduced two different types of Besicovitch-Musielak-Orlicz spaces of almost periodic functions, $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ and $B^{\varphi}_{a.p.}(\mathbb{R})$, as follows: A function $f : \mathbb{R} \to \mathbb{C}$ is said to belong the space $B^{\varphi}_{a.p.}(\mathbb{R})$, resp. $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$, if and only if there exists a sequence (f_n) of trigonometric polynomials such that for every $k > 0$, resp. there exists $k > 0$, such that $\lim_{n\to+\infty} \rho_{\varphi}(k(f_n - f)) = 0$. Then we clearly have

$$B^{\varphi}_{a.p.}(\mathbb{R}) \subseteq \tilde{B}^{\varphi}_{a.p.}(\mathbb{R}) \subseteq B^{\varphi}(\mathbb{R}).$$

If $\varphi(t,|x|) = |x|$, then by $B^1_{a.p.}(\mathbb{R})$, $\tilde{B}^1_{a.p.}(\mathbb{R})$ and $B^1(\mathbb{R})$ we denote the respective spaces.

Let us recall that a function $\varphi : \mathbb{R} \times [0,\infty) \to [0,\infty)$ is strictly convex if and only if $\varphi(t, \lambda u + (1-\lambda)v) < \lambda\varphi(t,u) + (1-\lambda)\varphi(t,v)$ for a.e. $t \in \mathbb{R}$ and for all $\lambda \in (0,1)$, $0 \leqslant u < v < \infty$. On the other hand, a normed linear space $(E, \|\cdot\|)$ is said to be strictly convex if and only if

$$\Big\| \frac{x+y}{2} \Big\| < 1, \text{ provided that } \|x\| = \|y\| = 1 \text{ and } x \neq y.$$

It is said that the function $\varphi(\cdot,\cdot)$ satisfies the Δ_2-condition if and only if there exist a number $k > 1$ and a measurable nonnegative function $h(\cdot)$ such that $\rho_{\varphi}(h) < \infty$ and $\varphi(t,2u) \leqslant k\varphi(t,u)$ for almost all $t \in \mathbb{R}$ and all $u \geqslant h(t)$.

Let $f \in B^{\varphi}_{a.p.}(\mathbb{R})$. Then, due to [**296**, Proposition 1], we have $\varphi(\cdot, |f(\cdot)|) \in B^1_{a.p.}(\mathbb{R})$ so that the limit

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(t, |f(t)|)\, dt$$

always exists and is finite. The main result of paper is [**296**, Theorem 1], which states that the space $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ is strictly convex if and only if $\varphi(\cdot,\cdot)$ is strictly convex and satisfies the Δ_2-condition.

Ergodicity in Stepanov-Orlicz spaces has been investigated in [**55**]. Let us recall that a convex function $\phi : \mathbb{R} \to [0,\infty)$ is said to be an Orlicz function if and only if it is non-decreasing, even and continuous on $\mathbb{R}$ and satisfies $\phi(0) = 0$, $\phi(u) > 0$ for $u > 0$ and $\lim_{u\to+\infty} \phi(u) = +\infty$. In the newly arisen situation, we say that the function $\phi(\cdot)$ satisfies the Δ_2-condition if and only if there exist real numbers $k > 1$ and $u_0 > 0$ such that $\phi(2u) \leqslant k\varphi(u)$ for $|u| \geqslant u_0$. For any Orlicz function

$\phi : \mathbb{R} \to [0, \infty)$, it can be simply proved that $f \in PAP_0(\mathbb{R} : X)$ if and only if $\phi(\|f\|) \in PAP_0(\mathbb{R} : X)$.

For any vector-valued measurable function $f : \mathbb{R} \to X$, we define the positive functional

$$\rho_{S^\phi}(f) := \sup_{x\in\mathbb{R}} \int_x^{x+1} \phi(\|f(s)\|)\, ds.$$

The Stepanov-Orlicz function space generated by ϕ is defined by

$$BS^{\phi}(\mathbb{R}, X) := \Big\{ f \in M(\mathbb{R} : X)\ ;\ (\exists\alpha > 0)\ \ \rho_{S^\phi}(\alpha f) < \infty \Big\}$$

We know that the vector space $BS^{\phi}(\mathbb{R}, X)$ equipped with the Luxemburg norm

$$\|f\|_{S^\phi} := \inf\left\{ k > 0 : \sup_{x\in\mathbb{R}} \int_x^{x+1} \phi(\|f(s)\|/k)\, ds \leqslant 1 \right\}.$$

is a Banach space. It is also worth noting that the Morse-Transue space type

$$\widetilde{BS}^{\phi}(\mathbb{R}, X) := \Big\{ f \in M(\mathbb{R} : X)\ ;\ (\exists\alpha > 0)\ \ \rho_{S^\phi}(\alpha f) < \infty \Big\}$$

equipped with the Luxemburg norm is a closed subspace of $BS^{\phi}(\mathbb{R}, X)$, which is commonly called the Besicovitch-Orlicz class. We know that $BS^{\phi}(\mathbb{R}, X) = \widetilde{BS}^{\phi}(\mathbb{R}, X)$ if and only if $\phi(\cdot)$ satisfies the Δ_2-condition.

Further on, for any $p \in C_+(\mathbb{R})$ we define the Musielak-Orlicz modular type space

$$BS^{p(\cdot)}(\mathbb{R}, X) := \left\{ f \in M(\mathbb{R} : X)\ ;\ (\exists\alpha > 0)\ \ \sup_{x\in\mathbb{R}} \int_x^{x+1} \big(\|f(s)\|/k\big)^{p(s)}\, ds \leqslant 1 \right\}.$$

For any function $f \in BS^{p(\cdot)}(\mathbb{R}, X)$, the notion of $BS^{p(\cdot)}(\mathbb{R}, X)$-ergodicity in norm sense and the notion of $BS^{p(\cdot)}(\mathbb{R}, X)$-ergodicity in modular sense are introduced in [**55**, Definition 3.1] and [**55**, Definition 3.2], respectively. Due to [**55**, Proposition 3.4], these concepts are equivalent.

Let $\phi : \mathbb{R} \to [0, \infty)$ be an Orlicz function. In [**55**, Definition 3.6], the authors introduce the notions of norm ergodicity in Stepanov Orlicz sense, modular ergodicity in Stepanov Orlicz sense and strongly modular ergodicity in Stepanov Orlicz sense for a given function $f \in BS^{\phi}(\mathbb{R}, X)$. After that, the authors further explore this notion in [**55**, Theorem 3.8, Theorem 3.10, Theorem 3.11] and provide several illustrative examples in [**55**, Section 4].

Density theorems for almost periodic functions in Hilbert spaces. In this part, we will inscribe a few relevant results obtained by A. Haraux and V. Komornik in [**201**]; these results have been obtained in their investigation of the oscillatory properties of the wave equation. Denote by X_T the vector space of all square-integrable functions with zero mean

$$X_T := \left\{ f \in L^2_{loc}(\mathbb{R} : \mathbb{C})\,;\, f(t+T) \equiv f(t),\ \int_0^T f(t)\, dt = 0 \right\},$$

where $T > 0$. If the set $A = \{T_1, \cdot\cdot\cdot, T_N\}$ is a given set of positive real numbers, we define

$$X := X_{T_1} + \cdots + X_{T_N}.$$

If V is a certain collection of complex-valued functions and I is an interval in $\mathbb{R}$, then we set $V_I := \{f_I : f \in V\}$. In [**201**, Theorem 1], the authors have proved that there exists a positive real number $T(A)$ such that for any interval $I \subseteq \mathbb{R}$ we have

$$X_I \text{ is dense in } L^2(I) \text{ if and only if } |I| < T(A),$$

where $|I|$ denotes the length of interval I; furthermore, the orthogonal complement of X_I in $L^2(I)$ is finite dimensional if $|I| = T(A)$ and infinite dimensional if $|I| > T(A)$. Suppose that $|I| = T(A)$ and the orthogonal complement of X_I in $L^2(I)$ is p-dimensional for some integer $p \in \mathbb{N}$. If P_{p-1} denotes the vector space consisting of all complex polynomials of degree $\leqslant p-1$ (including also the zero polynomial), then [**201**, Theorem 3(a)] states that Y_I is dense in $L^2(I)$, where $Y := P_{p-1} + X$; furthermore, $Y_I = L^2(I)$ if and only if $p = 1$, which is equivalent to saying that $P_i/P_j \in \mathbb{Q}$ for $1 \leqslant i \leqslant j \leqslant N$. Due to [**201**, Theorem 3(b)], there exists a real-valued function $h \in L^2(I)$ such that the functions $h,\ h', \cdot\cdot\cdot,\ h^{p-1}$ span X_I; furthermore, if we extend the function $h(\cdot)$ by zero to the whole real line and denote the obtained function by $H(\cdot)$, then we know that the function $H(\cdot)$ is a nonzero finite linear combinations of Dirac measures.

Almost periodicity in chaos. In this part, we will only draw the attention of the readers to the results presented in the tenth chapter of the recent research monograph [**18**] by M. Akhmet. In [**18**, Section 10], the author has investigated the dynamical properties of the following system

$$y' = Ay + G(t, y) + H(x(t)), \quad t \in \mathbb{R}, \tag{199}$$

where $G : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous in both variables, almost periodic in variable t uniformly for $y \in \mathbb{R}^n$, the function $H : \mathbb{R}^m \to \mathbb{R}^n$ is continuous, and all eigenvalues of the constant $n \times n$ real matrix A have negative real parts. Roughly speaking, if the perturbation part $H(x(t))$ is chaotic in a certain sense, then the system (199) has the interesting feature of chaos with infinitely many almost periodic motions. The obtained results are well illustrated with several numerical tests involving the coupled Duffing oscillators, for which it is well known that play an important role in modeling of the enhanced signal propagation (see also [**20**] and [**19**]). The most important notion used in [**18**, Section 10] is the notion of Li-Yorke chaotic set with infinitely many almost periodic motions, which is introduced in [**18**, Definition 10.1] for the equicontinuous families of uniformly bounded functions $x : \mathbb{R} \to \Lambda$, where Λ is a non-empty compact subset of $\mathbb{R}^m$. We would like to note here that this notion can be introduced in the infinite-dimensional setting, even for other types of chaos like distributional chaos or mean Li-Yorke chaos ([**235**]).

New classes of weighted pseudo-ergodic components. As is well known, the notion of a weighted pseudo almost-periodic function was introduced by T. Diagana in [**137**] (2006); cf. also [**139**]-[**140**]. This notion was extended by J. Blot, G. M.

Mophou, G. M. N'Guérékata and D. Pennequin in [**70**] (2009) by introducing the notion of a weighted pseudo-almost automorphic function.

Let us recall the following fundamental notion:

a. $\mathbb{U} := \{\rho \in L^1_{loc}(\mathbb{R}) : \rho(t) > 0 \text{ a.e. } t \in \mathbb{R}\}$,

b. $\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \inf_{x\in\mathbb{R}} \rho(x) < \infty$ and $\nu(T,\rho) := \lim_{T\to+\infty} \int_{-T}^{T} \rho(t)\, dt = \infty\}$,

c. $\mathbb{U}_b := L^\infty(\mathbb{R}) \cap \mathbb{U}_\infty$.

Then $\mathbb{U}_b \subseteq \mathbb{U}_\infty \subseteq \mathbb{U}$. It is said that the weights $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are equivalent, $\rho_1 \sim \rho_2$ for short, if and only if $\rho_1/\rho_2 \in \mathbb{U}_b$. By $\mathbb{U}_T$ we denote the space consisting of all weights $\rho \in \mathbb{U}_\infty$ satisfying that ρ is equivalent with all its translations.

The following spaces of double weighted ergodic components have been analyzed by many authors, even their generalizations in the light of measure theory:

Suppose that $\rho_1,\ \rho_2 \in \mathbb{U}_\infty$. Set

$$PAP_0\big(\mathbb{R}, X, \rho_1, \rho_2\big) := \left\{ f \in C_b(\mathbb{R} : X) : \lim_{T\to+\infty} \frac{1}{2\int_{-T}^{T} \rho_1(t)\, dt} \int_{-T}^{T} \|f(t)\|\rho_2(t)\, dt = 0 \right\}.$$

The main aim of this part, which closes the whole monograph, is to simply explain how one can introduce and analyze several new classes of weighted pseudo-ergodic components which generalize the space $PAP_0(\mathbb{R}, X, \rho_1, \rho_2)$. In order to do that, we modify our approach from Subsection 2.5.5.

Suppose that $\phi : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to [0,\infty)$ and $F : (0,\infty) \to (0,\infty)$ are given functions and $p \in \mathcal{P}(\mathbb{R})$. Then we can consider the following generalizations of the space $PAP_0(\mathbb{R}, X, \rho_1, \rho_2)$:

$$PAP_{0,p}(\mathbb{R}, X, F, \phi) := \left\{ f \in M(\mathbb{R} : X)\ ;\ \lim_{T\to+\infty} F(T)\big[\phi(\|f(x)\|)\big]_{L^{p(x)}[-T,T]} = 0 \right\},$$

$$PAP^1_{0,p}(\mathbb{R}, X, F, \phi, \psi) := \left\{ f \in M(\mathbb{R} : X)\ ;\ \lim_{T\to+\infty} F(T)\phi\Big(\big[\psi(\|f(x)\|)\big]_{L^{p(x)}[-T,T]}\Big) = 0 \right\}$$

and

$$PAP^2_{0,p}(\mathbb{R}, X, F, \phi, \psi) := \left\{ f \in M(\mathbb{R} : X)\ ;\ \lim_{T\to+\infty} \phi\Big(F(T)\big[\psi(\|f(x)\|)\big]_{L^{p(x)}[-T,T]}\Big) = 0 \right\}.$$

Concerning two-parameter double weighted ergodic components, the following space is of crucial importance:

$$PAP_0\big(\mathbb{R}\times Y, X, \rho_1, \rho_2\big) := \Bigg\{ f \in C_b(\mathbb{R}\times Y : X)\ ;$$

$$\lim_{T\to+\infty} \frac{1}{2\int_{-T}^{T} \rho_1(t)\, dt} \int_{-T}^{T} \|f(t,y)\|\rho_2(t)\, dt = 0, \text{ uniformly on bounded subsets of } Y \Bigg\}.$$

Following the above ideas, we can similarly introduce and analyze the corresponding spaces of two-parameter double weighted ergodic components which generalize the space $PAP_0(\mathbb{R} \times Y, X, \rho_1, \rho_2)$. The analysis can be also carried out for the functions defined on the non-negative real axis (the set $[0, \infty) \times Y$).

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