Singularity Theorems in General Relativity and Distributional Geometry

Michael Kunzinger

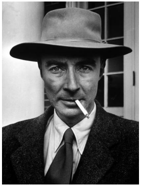
University of Vienna

GF2020

Messier 87 – Supermassive Black Hole

First mathematical derivation of existence of black holes

BER 1, 1939 PHYSICAL REVIEW VOL



On Continued Gravitational Contraction

J. R. OPPENHEIMER AND H. SNYDER
University of California, Berkeley, California

(Received July 10, 1939)

When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. Unless fission due to rotation, the radiation of mass, or the blowing off of mass by radiation, reduce the star's mass to the order of that of the sun, this contraction will continue indefinitely. In the present paper we study the solutions of the gravitational field equations which describe this process. In I, general and qualitative arguments are given on the behavior of the metrical tensor as the contraction progresses: the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles. In II, an analytic solution of the field equations confirming these general arguments is obtained for the case that the pressure within the star can be neglected. The total time of collapse for an observer comoving with the stellar matter is finite, and for this idealized case and typical stellar

No singularity within a black hole?

Investigations in Relativistic Cosmology†

By E. M. Lifshitz and I. M. Khalatnikov

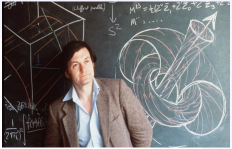
Institute for Physical Problems, Academy of Sciences, Moscow, U.S.S.R.

Abstract (by translator)

A detailed report is given here of the general investigations carried out by the authors in the field of relativistic cosmology during the past years. The paper consists of two parts.

The first part is devoted to a study of the singularities of the cosmological solutions of the gravitational equations. An attempt is made to provide an answer to one of the principal questions of modern cosmology : 'does the general solution of the gravitational equations have a singularity?' The authors give a negative answer to this question. The study carried out leads, in fact, to the general conclusion that the presence of a singularity with respect to time is not a necessary property of cosmological models of the general theory of relativity, and that the general case of an arbitrary distribution of matter and gravitational field does not lead to the appearance of a singularity. This result, however, does not exclude the possibility of the existence of more

Topology to the rescue



GRAVITATIONAL COLLAPSE AND SPACE-TIME SINGULARITIES

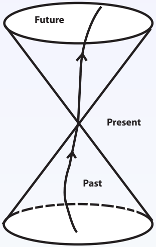
Roger Penrose
Department of Mathematics, Birkbeck College, London, England
(Received 18 December 1964)

The discovery of the quasistellar radio sources has stimulated renewed interest in the question of gravitational collapse. It has been suggested by some authors[1] that the enormous amounts of energy that these objects apparently emit may result from the collapse of a mass of the order of $(10^6\text{-}10^8)M_\odot$ to the neighborhood of its Schwarzschild radius, accompanied by a violent release of energy, possibly in the form of gravitational radiation. The detailed mathematical discussion of such situations is difficult since the full complexity of general relativity is required. Consequently, most exact calculations concerned with the implications of gravitational collapse have employed the simplifying assumption of spherical symmetry. Unfortunately, this precludes any detailed discussion of gravitational radiation—which requires at least a quadripole structure.

The general situation with regard to a spher- measured by local comoving observers, the body passes within its Schwarzschild radius $r = 2m$. (The densities at which this happens need not be enormously high if the total mass is large enough.) To an outside observer the contraction to $r = 2m$ appears to take an infinite time. Nevertheless, the existence of a singularity presents a serious problem for any complete discussion of the physics of the interior region.

The question has been raised as to whether this singularity is, in fact, simply a property of the high symmetry assumed. The matter collapses radially inwards to the single point at the center, so that a resulting space-time catastrophe there is perhaps not surprising. Could not the presence of perturbations which destroy the spherical symmetry alter the situation drastically? The recent rotating solution of Kerr[3] also possesses a physical

General relativity in a nutshell



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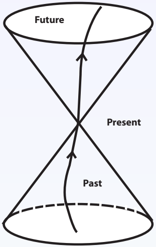

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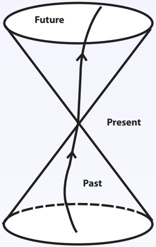

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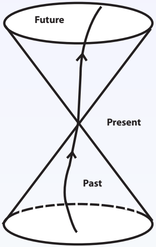

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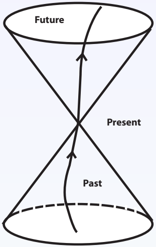

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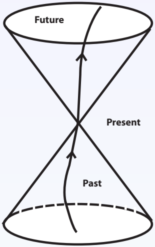

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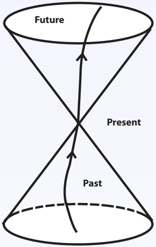

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- Geodesic (in-)completeness of M as a criterion for existence of singularities.

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Blueprint of the generic singularity theorem (J. Senovilla):

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- Below $C^{1,1}$: unbounded curvature, non-unique geodesics: singular.
- Hence $C^{1,1}$ is the natural threshold for singularity theorems.

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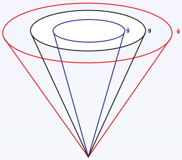
Strategy: Use approximations adapted to the causal structure.

Regularization techniques



Th. g a $\mathcal{C}^{1,1}$-metric, $\mathrm{Ric}(X,X) \geq 0$ for every smooth TL vector field X. Then $\forall K \Subset M\ \forall C > 0\ \forall \delta > 0\ \forall \kappa < 0\ \forall^0 \varepsilon\ \forall X \in TM|_K,\ \check{g}(X,X) \leq 0, \|X\|_h \leq C : \mathrm{Ric}[\check{g}_\varepsilon](X,X) > -\delta.$

Proof uses:

- $\check{g}_\varepsilon - g * \rho_\varepsilon \to 0$ in $\mathcal{C}^2 \rightsquigarrow$ suffices to consider $g_\varepsilon := g * \rho_\varepsilon$.
- $R_{jk} = R^i_{jki} = \partial_{x^i}\Gamma^i_{kj} - \partial_{x^k}\Gamma^i_{ij} + \Gamma^i_{im}\Gamma^m_{kj} - \Gamma^i_{km}\Gamma^m_{ij}$
- Blue terms$|_\varepsilon$ converge uniformly.
- For red terms use variant of Friedrichs-Lemma:

$$(R_{jk}X^jX^k) * \rho_\varepsilon - R_{jk}[g_\varepsilon]X^jX^k \to 0 \text{ uniformly}$$

$$\rho_\varepsilon \geq 0 \Rightarrow (R_{jk}X^jX^k) * \rho_\varepsilon \geq 0.$$

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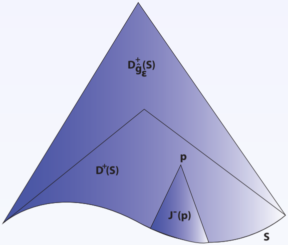

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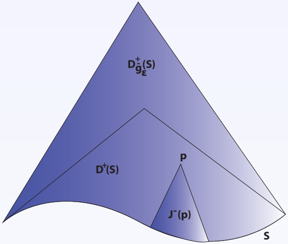

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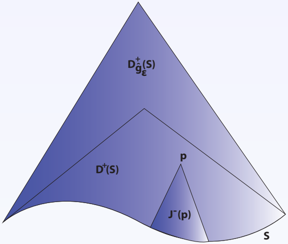

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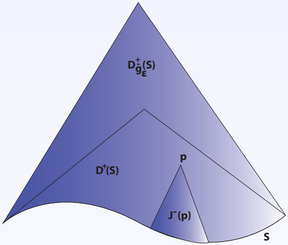

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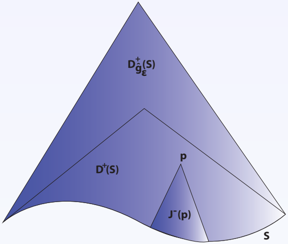

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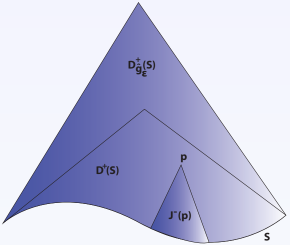

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- Derive a contradiction as in the $\mathcal{C}^\infty$-case.

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However:

- Geodesics still exist $\rightsquigarrow$ notion of geodesic completeness (*all* geodesics inextendable).

From $C^{1,1}$ to C^1: Motivation and new challenges

Motivation:

- Regularity of solutions to the Einstein equations: $\in H^s_{loc}$, $s > 5/2$, $\Gamma^i_{jk} \in L^2_{loc}$ lowest regularity where weak solutions are well defined.
- How low can you go?

New problems:

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- Curvature well-defined as a distribution.

Hawking's singularity theorem for C^1-metrics

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[M. Graf, Comm. Math. Phys., 2020]

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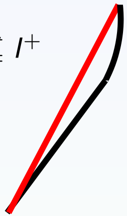
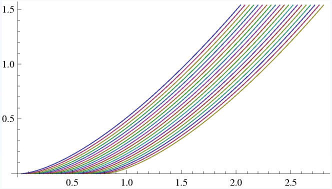
$(J^+ \circ I^+) \not\subseteq I^+$



(2) *light cones bubble up*



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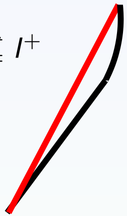

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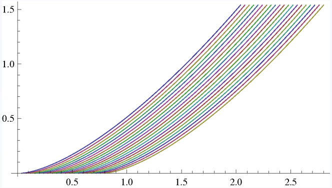

- Failure of convexity in $C^{1,\alpha}$, both IVP and BVP for geodesics problematic

The future is not always open (below $C^{0,1}$)

$$ds^2 = 2\left[-\sin 2\theta(x)\,dt^2 - 2\cos 2\theta(x)\,dx\,dt + \sin 2\theta(x)\,dx^2\right]$$

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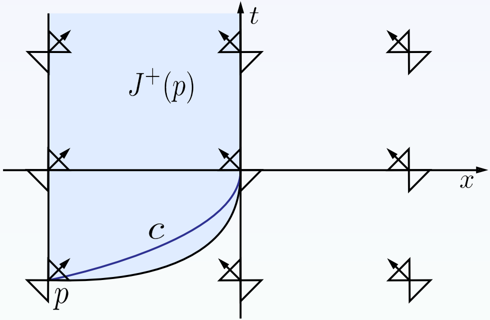

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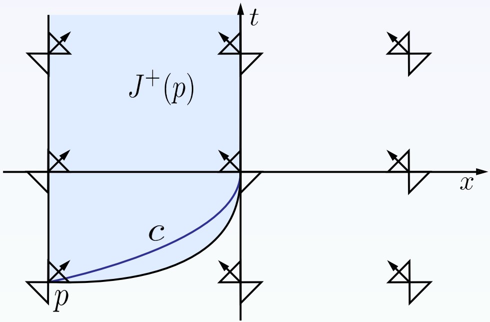
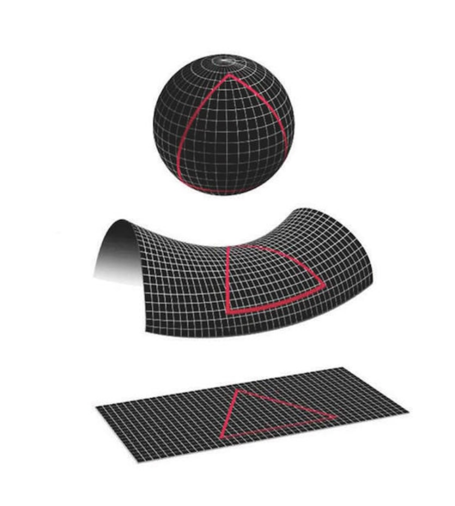


- c is a Lip. timelike curve reaching $\partial J^+(p)$ (C^1 & null at single pt.)

$\Rightarrow$ $I^+(p)$ contains segment of t-axis $\Rightarrow$ $I^+(p)$ not open

[Grant, K, Sämann, Steinbauer, Lett. Math. Phys. '19]

Metric curvature bounds via triangles



Lorentzian pre-length spaces

$(X, \ll, \leq)$ causal space, d metric on X, $\tau\colon X \times X \to [0, \infty]$ lower semicontinuous (with respect to d)

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Examples

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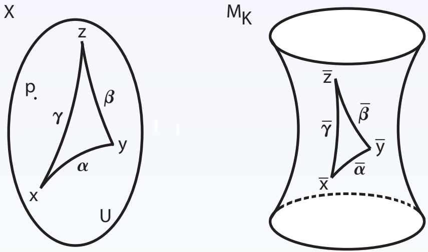
- Finite directed graphs

[K., Sämann, Ann. Glob. Anal. Geom., 2018]

Timelike curvature bounds in Lorentzian Length Spaces



Curvature singularities

Synthetic singularity Theorems in Lorentzian Length Spaces

- Hawking's Theorem in warped products $I \times_f X$ with TL lower curvature bound ([Alexander, Graf, K, Sämann, '19]).
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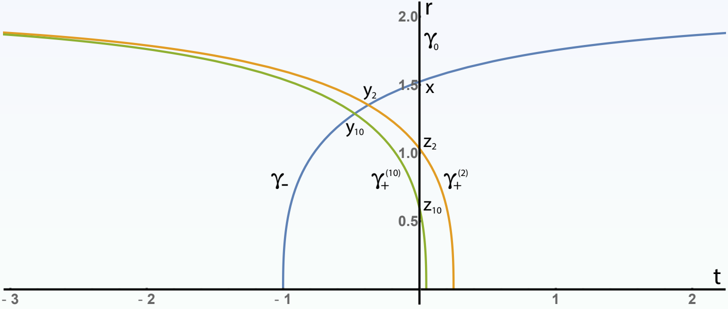

Schwarzschild has timelike curvature unbounded below

Analytic versus synthetic approach

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Geodesics

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- Solution of geodesic equation (classical, distributional, Caratheodory, Filipov, ...) vs longest (shortest) curves. Notions coincide for $g \in C^1$ (Lytchak/Yaman '06, Schinnerl '20), open questions in $C^{0,\alpha}$.

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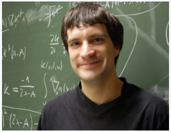
No "vs" yet: Impulsive gravitational waves ($g \in \mathcal{D}' \rightsquigarrow \mathcal{G}$ (Colombeau)): Podolsky, Steinbauer, Sämann. Wave equations in $C^{1,1}$, Green operators, QFT (Hörmann, Sanchez, Spreitzer, Vickers) – $\mathcal{G}$ as a tool

Meet the team

M. Graf

J. Grant

G. Hörmann



C. Sämann

B. Schinnerl

C. Spreitzer

R. Steinbauer

M. Stojković

J. Vickers