TECHNISCHE UNIVERSITÄT WIEN

The stochastic heat equation with singular potential

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Stochastic problem

- We consider stochastic evolution equations with space depending singular potential, random driving force and random initial condition of the form

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \mathcal{L})\, U(t,x,\omega) + q(x) \cdot U(t,x,\omega) &= F(t,x,\omega) \\
U(0,x,\omega) &= U^0(x,\omega)
\end{aligned}
$$

$$
t \in [0, T],\ x \in \mathbb{R}^d,\ \omega \in \Omega
$$

- $-\mathcal{L}$ is the second order elliptic operator
- q is a space depending singular potential
- F and U^0 are Kondratiev-type generalized stochastic processes

Literature review

M. Nedeljkov, S. Pilipović, D. Rajter-Ćirić
Heat equation with singular potential and singular data.
Proceedings of the Royal Society of Edinburgh Section A Mathematics 135 (04), 863–886, 2005

Semilinear parabolic equations with singular potential and singular initial data of the form

$$\frac{\partial}{\partial t} u(t,x) - \triangle u(t,x) + q(x) \cdot u(t,x) = f(t, u(t,x)),$$
$$u(0,x) = u^0(x)$$

are solved in suitable generalized function algebras.

- Potential q and initial data u^0 are singular distributions (e.g. the delta distribution or its powers)
- f satisfies a Lipschitz-type condition.

Literature review

A. Altybay, M. Ruzhansky, M. E. Sebih, N. Tokmagambetov
The heat equation with singular potentials.
Preprint, arXiv:2004.11255v2, 2020

Cauchy problem for the heat equation with strongly singular potential

$$\frac{\partial}{\partial t} u(t,x) - \triangle u(t,x) + q(x) \cdot u(t,x) = 0,$$

$$u(0,x) = u^0(x),$$

$$(t,x) \in [0,T] \times \mathbb{R}^d$$

- the potential q is singular, eather positive or negative

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Cauchy problem for the heat equation with strongly singular potential

$$\frac{\partial}{\partial t}u(t,x) - \triangle u(t,x) + q(x)\cdot u(t,x) = 0,$$
$$u(0,x) = u^0(x),$$
$$(t,x) \in [0,T] \times \mathbb{R}^d$$

- the potential q is singular, eather positive or negative
- The existence and the uniqueness of very weak solution are proved and the consistency of very weak solution with the classical solution is shown

Literature review

F. Russo, M. Oberguggenberger
White noise driven stochastic partial differential equations: triviality and non-triviality.
Chapman and Hall CRC Research Notes in Mathematics 315–334, 1999

Stochastic semilinear heat equation which is driven by a space-time Gaussian white noise

$$(\frac{\partial}{\partial t} - \mathcal{L})U(t,x) = \lambda\, F(U(t,x))\ \chi(x) + \dot{W}(t,x) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d$$
$$U(0,\cdot) = U^0$$

in suitable algebras of generalized functions

- $\mathcal{L}$ is a uniformly elliptic symmetric PDO with C_b^∞-coefficients
- F is a globally Lipschitz real valued function
- $U_0 \in \mathcal{S}'(\mathbb{R}^d)$
- χ is smooth with compact support, $\lambda \in \mathbb{R}$ and
- $\dot{W}(t,x)$ is Gaussian space-time white noise

Literature review

T. Levajković, S. Pilipović, D. Seleši, M. Žigić
Stochastic evolution equations with multiplicative noise.
Electronic Journal of Probability 20 (19), 1–23, 2015.

Stochastic evolution equations with multiplicative noise

$$\frac{\partial}{\partial t} U(t,x,\omega) = \mathcal{A}\, U(t,x,\omega) + \mathcal{B} \lozenge U(t,x,\omega) + F(t,x,\omega)$$
$$U(0,x,\omega) = U^0(x,\omega)$$

- $t \in [0, T]$, $\omega \in \Omega$, $U(t,\cdot,\omega)$ belongs to a Banach space X
- $\mathcal{A}$ is densely defined, generating a C_0-semigroup
- $\mathcal{B}$ is a linear bounded operator which combined with the Wick product $\lozenge$ introduces convolution-type perturbations
- F and U^0 are Kondratiev-type generalized stochastic processes

Our method

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) U(t,x,\omega) + q(x) \cdot U(t,x,\omega) = F(t,x,\omega)$$
$$U(0,x,\omega) = U^0(x,\omega)$$

- We combine the chaos expansion method from white noise analysis with the concept of very weak solutions

White noise probability space

- $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ Gaussian white noise probability space
  - $\mathcal{B}$- Borel σ algebra generated by the weak topology in $\mathcal{S}'(\mathbb{R})$
  - μ- Gaussian white noise measure given by

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle \omega, f\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}}, \quad f \in \mathcal{S}(\mathbb{R}).$$

- $(L)^2 = L^2(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ Hilbert space of random variables
  - $\{H_\alpha\}_{\alpha \in \mathcal{I}}$ - Fourier–Hermite polynomials defined by

$$H_\alpha(\omega) = \prod_{\alpha \in \mathcal{I}} h_{\alpha_k}(\langle \omega, \xi_k \rangle), \qquad \alpha \in \mathcal{I} = (\mathbb{N}_0^{\mathbb{N}})_c$$

    form an orthogonal basis of $(L)^2$.

  - Wiener–Itô chaos expansion theorem
    Each $F \in (L)^2$ can be uniquely represented in the form
    $F(\omega) = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha(\omega)$ with $f_\alpha \in \mathbb{R}$ such that $\|F\|^2_{(L)^2} = \sum_{\alpha \in \mathcal{I}} f_\alpha^2 \, \alpha! < \infty$.

Kondratiev spaces

- $(S)_1$, $(S)_{-1}$ Kondratiev spaces

  Let $G(\omega) = \sum_{\alpha \in \mathcal{I}} g_\alpha H_\alpha(\omega)$. Then,

  - $G \in (S)_1 \Leftrightarrow \sum_{\alpha \in \mathcal{I}} \alpha!^2 \, g_\alpha^2 \, (2\mathbb{N})^{p\alpha} < \infty$ for all $p \in \mathbb{N}_0$
  - $G \in (S)_{-1} \Leftrightarrow \sum_{\alpha \in \mathcal{I}} g_\alpha^2 \, (2\mathbb{N})^{-p\alpha} < \infty$ for some $p \in \mathbb{N}_0$,

  where $(2\mathbb{N})^\alpha = \prod_{k \in \mathbb{N}} (2k)^{\alpha_k}$ for $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{I}$

- $(S)_1 \subseteq (L)^2 \subseteq (S)_{-1}$

- Wick product $F \lozenge G$

  of $F(\omega) = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha(\omega)$ and $G(\omega) = \sum_{\beta \in \mathcal{I}} g_\beta H_\beta(\omega)$ is defined by

$$F \lozenge G(\omega) = \sum_{\gamma \in \mathcal{I}} \left(\sum_{\alpha+\beta=\gamma} f_\alpha g_\beta \right) H_\gamma(\omega), \qquad f_\alpha, g_\beta \in \mathbb{R}, \ \alpha, \beta \in \mathcal{I}.$$

Stochastic processes

- Let X be an arbitrary Banach space
- Classes of stochastic processes

Let $U = \sum_{\alpha \in \mathcal{I}} u_\alpha \ H_\alpha, \quad u_\alpha \in X, \ \alpha \in \mathcal{I}.$

- $U \in X \otimes (L)^2 \quad \Leftrightarrow \quad \sum_{\alpha \in \mathcal{I}} \alpha! \ \|u_\alpha\|_X^2 < \infty$
- $U \in X \otimes (S)_1 \quad \Leftrightarrow \quad \sum_{\alpha \in \mathcal{I}} \alpha!^2 \ \|u_\alpha\|_X^2 \, (2\mathbb{N})^{p\alpha} < \infty$ for all $p \in \mathbb{N}_0$
- $U \in X \otimes (S)_{-1} \quad \Leftrightarrow \quad \sum_{\alpha \in \mathcal{I}} \|u_\alpha\|_X^2 \, (2\mathbb{N})^{-p\alpha} < \infty$ for some $p \in \mathbb{N}_0$

- Example

Singular white noise

$$W_t(\omega) = \sum_{k \in \mathbb{N}} \xi_k(t) \ H_{\varepsilon(k)}(\omega)$$

is an element of $C^\infty(\mathbb{R}) \otimes (S)_{-1}$.

Assumptions

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) U(t,x,\omega) + q(x)\cdot U(t,x,\omega) = F(t,x,\omega)$$
$$U(0,x,\omega) = U^0(x,\omega)$$

Assume:

(a1) $-\mathcal{L}$ be a second order elliptic operator whose action on a process $U(t,x,\omega) = \sum_{\alpha\in\mathcal{I}} u_\alpha(t,x)\, H_\alpha(\omega)$ is given by

$$\mathcal{L}U(t,x,\omega) = \sum_{\gamma\in\mathcal{I}} \mathcal{L}u_\gamma(t,x)\, H_\gamma(\omega),$$

(a2) q is a singular space potential

(a3) $U^0 \in Dom(\mathcal{L})$ having the form $U^0(x,\omega) = \sum_{\gamma\in\mathcal{I}} u^0_\alpha(x)\, H_\gamma(\omega)$ such that $\sum_{\gamma\in\mathcal{I}} \|u^0_\gamma\|^2\, (2\mathbb{N})^{-p_1\gamma} < \infty$ for some $p_1 > 0$

(a4) $F(t,x,\omega) = \sum_{\gamma\in\mathcal{I}} f_\gamma(t,x) H_\gamma(\omega)$ such that $\sum_{\gamma\in\mathcal{I}} \|f_\gamma\|^2\, (2\mathbb{N})^{-p_1\gamma} < \infty$ for some $p_2 > 0$

Moderate nets

- Let $(X, \|\cdot\|_X)$ be a Banach space. A net of elements $(k_\varepsilon)_{\varepsilon\in(0,1]}$ in X is $X-$moderate if there exist $N \in \mathbb{N}_0$ and $C > 0$ such that

$$\|k_\varepsilon\|_X \leq C\varepsilon^{-N}.$$

- Nets of generalized stochastic processes in $X \otimes (S)_{-1}$ whose coefficients are X-moderate nets are also called moderate.

Solution concept

- A net of moderate stochastic processes $(U_\varepsilon)_\varepsilon$ in $X \otimes (S)_{-1}$ of the form

$$U_\varepsilon = \sum_{\gamma \in \mathcal{I}} (u_\gamma)_\varepsilon \, H_\gamma$$

is a very weak solution to the stochastic problem

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \mathcal{L}\right) U(t,x,\omega) + q(x) \cdot U(t,x,\omega) &= F(t,x,\omega) \\ U(0,x,\omega) &= U^0(x,\omega)\end{aligned}$$

if there exist an $L^\infty-$moderate log-type regularisation $(q_\varepsilon)_\varepsilon$ of the singular potential q, such that $[(u_\gamma)_\varepsilon]_\varepsilon$, for $\gamma \in \mathcal{I}$ solves the regularized equations

$$\begin{aligned}\frac{\partial}{\partial t} u_\gamma(t,x) - \mathcal{L} u_\gamma(t,x) + q_\varepsilon(x) \cdot u_\gamma(t,x) &= f_\gamma(x,t) \\ u(0,x) &= u^0(x)\end{aligned}$$

for every $\varepsilon \in (0,1]$.

Existence Theorem

Theorem

Let the assumptions $(a1)$-$(a4)$ hold. Then, the stochastic evolution problem

$$\frac{\partial}{\partial t}U(t,x,\omega) - \mathcal{L}u(t,x,\omega) + q(x)\cdot U(t,x,\omega) = F(x,t,\omega)$$
$$U(0,x,\omega) = U^0(x,\omega).$$

has a very weak solution $(U_\varepsilon)_\varepsilon$.

Steps of the proof

(1) Let the unknown process be of the form $U(t,x,\omega) = \sum_{\gamma\in\mathcal{I}} u_\gamma(t,x)\, H_\alpha(\omega)$.

(2) By applying the method of chaos expansions, the initial problem reduces to a family of deterministic problems with singular potential

$$\begin{aligned} \frac{\partial}{\partial t} u_\gamma(t,x) - \mathcal{L} u_\gamma(t,x) + q(x)\cdot u_\gamma(t,x) &= f_\gamma(x,t) \\ u(0,x) &= u^0(x) \end{aligned}$$

for all $\gamma \in \mathcal{I}$.

Steps of the proof

(3) We regularise the potential q with a Friedrichs-mollifier and obtain net of smooth functions

$$q_\varepsilon(x) = q * \psi_\varepsilon(x), \quad \varepsilon \in (0,1],$$

where $\psi_\varepsilon(x) = \frac{1}{\varepsilon^d}\psi\left(\frac{x}{\varepsilon}\right) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, for $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\psi \geq 0$, $\int \psi = 1$. Additionally, we assume that the regularization is such that q_ε is of log-type, i.e.

$$\|q_\varepsilon\|_{L^\infty} \leq N_q \log \frac{1}{\varepsilon}.$$

Then, we obtain the system of equations with regularized potential

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) u_\gamma(t,x) + q_\varepsilon(x) \cdot u_\gamma(t,x) = f_\gamma(t,x)$$

$$u_\gamma(0,x) = u_\gamma^0(x),$$

which for each $\gamma \in \mathcal{I}$ has a very weak solution $[(u_\gamma)_\varepsilon]_\varepsilon$, i.e.

$$\|(u_\gamma)_\varepsilon(t,\cdot)\|_{L^2} \leq \left(Me^{wt}\|u_\gamma^0(\cdot)\|_{L^2} + \tfrac{M}{2} \cdot D_{2w}(T) + \tfrac{M}{2} \int_0^t \|f_\gamma(s,\cdot)\|_{L^2}^2 \, ds\right) \cdot \varepsilon^{-N}$$

holds for some $N > 0$.

Steps of the proof

(4) For every $\varepsilon \in (0,1]$ a process defined by

$$U_\varepsilon(t,x,\omega) = \sum_{\gamma \in \mathcal{I}} (u_\gamma(t,x))_\varepsilon \, H_\gamma(\omega),$$

where $[(u_\gamma)_\varepsilon]_\varepsilon$, is a net of moderate functions which solves the regularized equations, i.e. $(U_\varepsilon)_\varepsilon$ is a stochastic very weak solution to the initial stochastic problem.

Namely, we need to prove that for some $p > 0$ it holds

$$\sum_{\gamma \in \mathcal{I}} \|(u_\gamma)_\varepsilon\|^2 \, (2\mathbb{N})^{-p\gamma} < \infty.$$

Work in progress: General problem

- We consider stochastic evolution equations with space depending singular random potential, random driving force and random initial condition of the form

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right) U(t,x,\omega) + Q(t,x,\omega) \lozenge U(t,x,\omega) = F(t,x,\omega)$$
$$U(0,x,\omega) = U^0(x,\omega)$$

$$t \in [0,T],\ x \in \mathbb{R}^d,\ \omega \in \Omega$$

  - $-\mathcal{L}$ is the second order elliptic operator
  - Q is a singular random potential
  - F and U^0 Kondratiev-type generalized stochastic processes
  - $\lozenge$ denotes the Wick product

Thank you for your attention!

How did we celebrate in 2010?


Carlsberg
Probably the be

GF 2018

GF 2009

GF 2009