What are continuously differentiable functions on compact sets?

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Joint work with
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A function f, continuous on K, belongs to $C^1(K)$ if there exits a continuous function df on K with values in the linear maps from $\mathbb{R}^d$ to $\mathbb{R}$ such that, for all $x \in K$,

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$\mathcal{J}^1(K) := \{(f; df) \in C(K)^{d+1} \ : \ df \text{ is a continuous derivative of } f \text{ on } K\}$

endowed with the norm

$$\|(f; df)\|_{\mathcal{J}^1(K)} = \|f\|_K + \|df\|_K,$$

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$$C^1(K) = \pi_1(\mathcal{J}^1(K)),$$

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$$\|f\|_{C^1(K)} = \|f\|_K + \inf\{\|df\|_K \ : \ df \text{ is a continuous derivative of } f \text{ on } K\}$$

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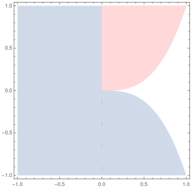

$$K = \{(x,y) \in [-1,1]^2 \ : \ |y| \geq x^3 \text{ for } x \geq 0\}$$

$$f(x,y) = x^2 \text{ on the red part } f(x,y) = 0 \text{ otherwise}$$

$f \in C^1(K)$ but $f \notin C^1(\mathbb{R}^2|K)$ because f is not Lipschitz continuous near the origin.

Theorem (Whitney,1934)

A function $f \in C^1(\mathbb{R}^d|K)$ if and only if $f \in \pi_1(\mathscr{E}^1(K))$

A jet $(f, df) \in \mathscr{E}^1(K)$ if

$$\lim_{\substack{y \to x \\ y \in K}} \frac{f(y) - f(x) - \langle df(x), y - x\rangle}{|y - x|} = 0.$$

uniformly on $x \in K$.

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If K is topologically regular ($\overline{\mathring{K}} = K$)

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$$C^1(K) \subseteq C^1_{\text{int}}(K)$$

Compact set with infinitely many connected components

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2. Extract a convergent subsequence $x_{k(j)} \longrightarrow x_0 \in K$, this limit cannot belong to one of the $K_{k(j)}$ as those subsets are open and pairwise disjoint.
3. $$f_n(x) = \begin{cases} |x_{k(j)} - x_0| & \text{if } x \in K_{k(j)} \text{ and} j \leq n \\ 0 & \text{if } x \in K \setminus \bigcup_{j=1}^{n} K_{k(j)}. \end{cases}$$

f_n is a Cauchy sequence in $(C^1(K), \|\cdot\|_{C^1(K)})$.

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We say that a set $A \subseteq \mathbb{R}^d$ is *(Whitney) regular* if there exists $C > 0$ such that any two points $x, y \in A$ can be joined by a rectifiable path in A of length bounded by $C|x - y|$.

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We say that A is *pointwise (Whitney) regular* if for any $x \in A$ there exists a neighbourhood V_x of x in A and $C_x > 0$ such that any $y \in V_x$ is joined to x by a rectifiable path in A of length bounded by $C_x|x - y|$.

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Generalized F.T.C. (L. Frerick, L.L., J. Wengenroth)

For each $f \in C^1(K)$ with a continuous derivative df and each rectifiable path $\gamma : [a, b] \to K$ we have

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

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1. Take $((f_j; df_j))_{j \in \mathbb{N}}$ a Cauchy sequence in $(\mathcal{J}^1(K), \|\cdot\|_{\mathcal{J}^1(K)})$, from the completeness of $(C(K), \|\cdot\|_K)$ we obtain uniforms limits f and df.
2. Given $x \in K$ and a path γ_y from x to y of length $L(\gamma_y) \leq C_x|x-y|$

$$f_j(y) - f_j(x) - \langle df(x), y - x \rangle = \int_{\gamma_y} (df_j - df(x))$$

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By taking the quotient map

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We fix $x \in K$, for all $y \in K \setminus \{x\}$, we define a linear and continuous functional on $C^1(K)$ by

$$\Phi_y(f) = \frac{f(y) - f(x)}{|y - x|} \quad \forall f \in C^1(K).$$

Conclusion by Uniform boudedness principle.

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$$d_\varepsilon(y) := \inf\{L_\gamma \ : \ \gamma \text{ rectifiable path from } x \text{ to } y \text{ in } K_{2\varepsilon}\}.$$

($K_{2\varepsilon}$ open connected neighbourhood of K)

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$y_0 \in K$ fixed, set $u_\varepsilon(y) = \min\{d_\varepsilon(y), d_\varepsilon(y_0)\}$. If y and y' are closed enough in $K_{2\varepsilon}$, we have

$$|u_\varepsilon(y) - u_\varepsilon(y')| \leq |y - y'|,$$

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$$|(u_\varepsilon * \phi)(x) - (u_\varepsilon * \phi)(y_0)| \leq C_x(d_\varepsilon(y_0) + d)|x - y_0|$$

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$y_0 \in K$ fixed

$\mathrm{supp}\phi \longrightarrow \{0\}$

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For any $y_0 \in B(x, \frac{1}{2C_x}) \cap K$, there exists a rectifiable path from x to y_0 in $K_{2\varepsilon}$ of length bounded by $2C_x d|x - y_0| + \varepsilon$.Using a parametrization by arc length for these paths gives an uniformly equicontinuous and uniformly bounded family of functions.
By Arzelà-Ascoli Theorem, we find a convergent subsequence an the limit is a path of length bounded by $2C_x d|x - y_0|$.

Characterization of the completness of $C^1(K)$ (L. Frerick, L.L., J. Wengenroth)

Let K be a compact set, $(C^1(K), \|\cdot\|_{C^1(K)})$ is a Banach space if and only if K has finitely many components which are pointwise Whitney regular.

We want to show that

$$\overline{C^1(\mathbb{R}^d|K)}^{C^1(K)} = C^1(K).$$

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Consequence of Hahn-Banach theorem

It suffices to check that, if Φ is a bounded linear functional on $C(K)^{d+1}$ such that $\Phi_{|\mathscr{E}^1(K)} = 0$ then $\Phi_{|\mathcal{J}^1(K)} = 0$

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We consider such a Φ.

Thanks to Riesz representation theorem, Φ is represented by $(\mu; \mu_1, \cdots, \mu_d)$: for all $(f, f_1, \cdots, f_d) \in C(K)^{d+1}$,

$$\Phi((f, f_1, \cdots, f_d)) = \int f \, d\mu + \int f_1 \, d\mu_1 + \ldots + \int f_d \, d\mu_d.$$

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As, $\Phi_{|\mathscr{E}^1(K)} = 0$, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\int \varphi \, d\mu = -\int \partial_1 \varphi \, d\mu_1 - \cdots - \int \partial_d \varphi \, d\mu_d$$

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$$\Phi((f, f_1, \cdots, f_d)) = \int f \, d\mu + \int f_1 \, d\mu_1 + \ldots + \int f_d \, d\mu_d.$$

As, $\Phi_{|\mathscr{E}^1(K)} = 0$, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\mu[\varphi] = (\partial_1 \mu_1)[\varphi] + \cdots + (\partial_d \mu_d)[\varphi]$$

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$$\mu = \operatorname{div}(T)$$

where $T = (\mu_1, \cdots, \mu_d)$ is a vector-field of measures (charge).

If $\gamma : [a, b] \to K$ is a (Lipschitz) path and $F = (F_1, \cdots, F_d) \in C(K)^d$,

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If Γ is the set of all Lipschitz paths.

Smirnov (1993)

Every charge T with compact support such that $\operatorname{div}(T)$ is a signed measure can be decomposed into elements of Γ, i.e., there is a positive finite measure ν on Γ such that

$$T = \int_\Gamma T_\gamma d\nu(\gamma) \text{ and } \|T\| = \int_\Gamma \|T_\gamma\| d\nu(\gamma).$$

Theorem (L. Frerick, L.L., J. Wengenroth)

For any compact set K, the space of restrictions to K of continuously differentiable function on $\mathbb{R}^d$ is dense in $(C^1(K), \|\cdot\|_{C^1(K)})$.

Criterium for the equality $C^1(\mathbb{R}^d|K) = C^1(K)$ (L. Frerick, L.L., J. Wengenroth)

$C^1(K) = C^1(\mathbb{R}^d|K)$ with equivalent norms if and only if K has only finitely many components which are all Whitney regular.

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- Two half of a broken heart behave better than the intact heart...
- For $M = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ and $K = M \times [0,1]$ we have $C^1(K) \neq C^1(\mathbb{R}^2|K)$.

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If $F \in C^1(K)$, $dF = (0,0)$ and if we consider the path γ consisting of one horizontal lines crossing K, we have

$$\int_\gamma dF = 0 \quad \text{while} \quad F(\gamma(1)) - F(\gamma(0)) = f(1) - f(0) = 1.$$

Whitney (1934)

Let K be a topologically regular compact set. If $\mathring{K}$ is Whitney regular, then $C^1_{\text{int}}(K) = C^1(\mathbb{R}^d|K)$.

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Whitney conjecture : what about the reverse?

Criterium for the equality $C^1(K) = C^1_{\text{int}}(K)$ (L. Frerick, L.L., J. Wengenroth)

Let K be a topologically regular compact set and assume that, for all $x \in \partial K$, there exist $C_x > 0$ and a neighbourhood V_x of x such that each $y \in V_x$ can be joined from x by a rectifiable path in $\mathring{K} \cup \{x, y\}$ of length bounded by $C_x|x - y|$. Then $C^1_{\text{int}}(K) = C^1(K)$.

Let Ω be the open unit disk in $\mathbb{R}^2$ from which we remove tiny disjoints balls which accumulate at $\{0\} \times (-\frac{1}{2}, \frac{1}{2})$.

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$\overset{\circ}{K}$ is not Whitney regular, because $\{0\} \times (-\frac{1}{2}, \frac{1}{2})$ is not contained in $\overset{\circ}{K}$, but the assumptions of the previous criterium are satisfied and $C^1(K) = C^1_{\mathrm{int}}(K)$.

Equality between $C^1(K)$ and $C^1(\mathbb{R}|K)$

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K compact set of $\mathbb{R}$ with infinitely many connected components, for all $\xi \in K$

$$\sigma(\xi) := \lim_{\varepsilon \to 0^+} \sup \left\{ \frac{\sup\{|y-\xi| \, : \, y \in G\}}{\ell(G)} \; : \; G \text{ gap } \subseteq (\xi-\varepsilon, \xi+\varepsilon) \right\}$$

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Theorem (L. Frerick, L.L., J. Wengenroth)

Let $K \subset \mathbb{R}$ be a compact set with infinitely many connected components. We have $C^1(K) = C^1(\mathbb{R}|K)$ if and only if $\sigma(\xi) < \infty$ for all $\xi \in K$.

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3. $K = \{0\} \cup \bigcup_{n\in\mathbb{N}} [x_n, x_n + r_n]$ For $r_n = e^{-2n}$ we get $\sigma(0) < \infty$, e.g., for $x_n = e^{-n}$ and $\sigma(0) = \infty$ for $x_n = 1/n$

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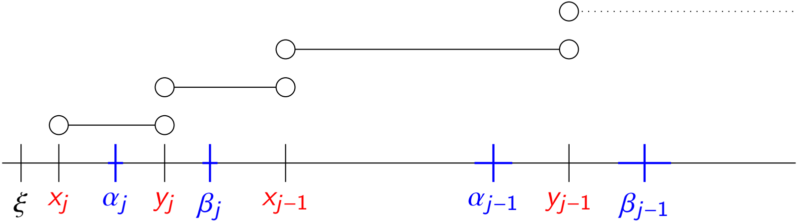

To go further :

Leonhard Frerick, Laurent Loosveldt and Jochen Wengenroth, *Continously differentiable functions on compact sets*, Submitted for publication.

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