Gelfand–Shilov Classification of Semigroups Related to Random Processes

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1. Formulation of the Problem

Semigroups under consideration are:

$$U(t)u(x) = \int_{\mathbb{R}} u(y)P(0, x; t, dy) = \langle u(\cdot),\, p(0, x; t, \cdot)\rangle.$$

Here P is a transition probability and p is corresponding transition density.

Our goal is

- to consider semigroups related to basic processes: shift, Wiener, and Poisson;
- to show that generators are ΨDOs and operators with distribution kernels;
- to extend Gelfand–Shilov classification introduced for differential systems to systems with ΨDOs.

2. Shift Semigroups U_1

(i). Consider the shift semigroup $\{U_1(t), t \geq 0\}$ on $C(\mathbb{R})$:

$$U_1(t)u(x) = u(x,t) = \langle u(\cdot), p_1(0,x;t,\cdot)\rangle = \langle u, \delta_{x+t}\rangle = u(x+t).$$

Here $p_1(0,x;t,y) = \delta_{x+t}(y)$ is the transition probability density.

The semigroup is strongly continuous (c_0-class) on $C(\mathbb{R})$, $C_0(\mathbb{R})$, $C_c(\mathbb{R})$ and $u(x,t)$ is the solution to the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)}{\partial x}, \quad x \in \mathbb{R},\ t \geq 0, \quad u(x,0) = u(x).$$

Hence, the generator is $\frac{\partial}{\partial x}$.

3. Heat Semigroup U_2

(ii). Consider the semigroup on $L_2(\mathbb{R})$ and on $C_0(\mathbb{R})$:

$$U_2(t)u(x) = u(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} u(y) e^{-\frac{(x-y)^2}{2t}} dy.$$

Here $p_2(0,x;t,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$ is the normal density and $u(x,t)$ is the solution to the heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in \mathbb{R},\ t \geq 0, \quad u(x,0) = u(x).$$

The semigroup is strongly continuous on $L_2(\mathbb{R})$ and on $C_0(\mathbb{R})$.

4. Poisson Semigroups U_3

(iii). Consider the Poisson semigroup on $C(\mathbb{R})$:

$$U_3(t)u(x) = \langle u(\cdot), p_3(0, x; t, \cdot)\rangle,$$

Here p_3 is the transition density of Poisson process with jump value q and intensity λ:

$$p_3(0, x; t, y) = \sum_{k=0}^{c_q} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \delta_{x+kq}(y),$$

defined by $P_3(0, x; t, y) = \sum_{k=0}^{c_q} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, where c_q depends on $y - x$: $c_q = \left[\frac{y-x}{q}\right]$ if $\left[\frac{y-x}{q}\right] \neq \frac{y-x}{q}$ and $\left[\frac{y-x}{q} - 1\right]$ if not.

5. The Poisson Semigroup Generator

By the definition of generator:

$$A_3 u(x) = \lim_{t \to 0} \frac{1}{t} \left[U_3(t) - I \right] u(x)$$

$$= \lim_{t \to 0} \frac{1}{t} \left[\langle u(y), \sum_{k=0}^{c_q} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \delta_{x+kq}(y) \rangle - u(x) \right]$$

$$= \lambda(u(x+q) - u(x)) = \lambda \langle u, \delta_{x+q} - \delta_x \rangle.$$

It follows $u(x,t) = U_3(t) u(x)$ is the solution to

$$\frac{\partial u(x,t)}{\partial t} = \lambda(u(x+q,t) - u(x,t)), \quad x \in \mathbb{R},\ t \geq 0, \quad u(x,0) = u(x)$$

and U_3 is strongly continuous on $C(\mathbb{R})$, $C_0(\mathbb{R})$.

6. Compound Poisson (CP) Semigroup U_4

(iv). Let $\{z_k(t), t \geq 0\}$ be iid random processes with general distribution law μ_z, then

$$Z(t) = z_1(t) + \ \ldots \ + z_N(t)$$

is a CP process.

For the characteristic function of Z

$$\Phi_Z(\alpha) := \int_{\mathbb{R}} e^{i(\alpha,y)} \mu_Z(dy) = \mathcal{F}[\mu_Z](-\alpha)$$

(here $\mu_Z(dy) = P(0,0;t,dy)$) we proved the equality:

$$\Phi_Z(\alpha) = e^{t\lambda(\Phi_z(\alpha)-1)}.$$

7. Compound Poisson Semigroup U_4

For invariant w.r.t. shifts (spatially homogeneous) semigroups $p(0, x; t, y) = p(t, y - x)$.

U_4 (and $U_1 - U_3$) are invariant w.r.t. shifts, hence they are related to Levy processes ([1]: Böttcher, Schilling, ...) and

$$U_4(t)u(x) = \langle u(\cdot),\, p_4(t, \cdot - x)\rangle.$$

Due to obtained equality for Φ_Z, for $X(t) := x + Z(t)$ we have

$$\Phi_X(x, \alpha) = \mathcal{F}[p_4](t, x, -\alpha) = e^{ix\alpha}\Phi_Z(\alpha) = e^{ix\alpha}e^{t\lambda(\Phi_z(\alpha)-1)}.$$

8. Generators of $U_1 - U_4$ are ΨDOs

ΨDOs are operators K of the form

$$K\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\alpha} s(x,\alpha)\hat{\varphi}(\alpha)d\alpha,\ x \in \mathbb{R}, \quad \varphi \in \mathcal{S},$$

with symbols s of not more then power growth in α.
Symbols are (generalized) Fourier transforms of operators K.

For a differential operator with coefficients $a_k(x)$, the symbol is $\sum_{k=0}^{n} a_k(x)(i\alpha)^k$. Hence, generators of U_1, U_2, as special cases, are ΨDOs. For A_3 we have

$$A_3\varphi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\alpha} s(x,\alpha)\hat{\varphi}(\alpha)d\alpha,\ \ s(x,\alpha) = \lambda e^{ix\alpha}(e^{iq\alpha} - 1).$$

The U_4 can be considered as a special case related to Levy processes.

9. Generators Related to Levy Processes are ΨDOs

Semigroups $U_1 - U_4$ are related to Levy processes: $X(t) = X(0) + at + bW(t) + \int_{|y|\geq q} yN(t,dy) + \int_{|y|<q} y\widetilde{N}(t,dy),\ q > 0.$

To prove the ΨDO-property we can using the Levy–Khintchine formulae for $\Phi_X = e^{ix\alpha}e^{ts(\alpha)}$:

$$-s(\alpha) = \underbrace{-ib\alpha}_{U_1} + \underbrace{\frac{1}{2}\alpha\, Q\alpha}_{U_2} + \underbrace{\int_{\mathbb{R}^m/\{0\}} \left(1 - e^{i\alpha y} + i\alpha y\chi_{[-1,1]}(y)\right)\nu(dy)}_{U_3, U_4, \ldots}$$

and power estimate (in α) for the integral term.
Here ν is a Levy measure: $\int_{\mathbb{R}^m/\{0\}} \max\{1,y\}\nu(dy) < \infty$.

For Poisson generators the integral term is
$\int_{\mathbb{R}^m/\{0\}} \left(1 - e^{i\alpha y}\right)\nu(dy)$.

10. Generators are Operators with Distribution Kernels

To show this property we use ΨDO-property and the kernel Schwartz theorem:

There exists one-to-one correspondence between operators $K : \mathcal{D}(X) \to \mathcal{D}'(Y)$ and distributions $\mathcal{K} \in \mathcal{D}'(X \times Y)$ called kernels of K.

In our case generator and operators of semigroups are $K : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ and we find $\mathcal{K}$ writing $K\varphi$ on test functions ψ

$$\langle \psi(x), 2\pi K\varphi(x)\rangle = \int_{\mathbb{R}} \psi(x)dx \overline{\int_{\mathbb{R}} s(x,\alpha)d\alpha \int_{\mathbb{R}} e^{i(x-y)\alpha}\varphi(y)dy}$$

$$=: \langle \psi(x)\varphi(y), \mathcal{K}(y,x)\rangle, \ \varphi, \psi \in \mathcal{S}.$$

11. Examples of Kernels

The integral in the definition of $\mathcal{K}$ converges due to estimates for $s(x, \alpha)$ and ψ, φ.

Examples of kernels for semigroups $U_1 - U_3$:

$$\delta_{x+t}(y), \quad e^{-\frac{(x-y)^2}{2t}}, \quad \sum_{k=0}^{c_q} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \delta_{x+kq}(y).$$

12. Gelfand–Shilov Systems

"Classical" Gelfand–Shilov classification is given to the Cauchy problem for the differential system

$$\frac{\partial}{\partial t}u(t,x) = A\left(i\frac{\partial}{\partial x}\right)u(t,x),\ t \geq 0,\ x \in \mathbb{R}^m,\ u(0,x) = f(x),$$

where $A\left(i\frac{\partial}{\partial x}\right)$ is a matrix of linear differential operators of order $\leq l$.

The classification is constructed by $e^{tA(\alpha)}$, the solution operator of the Fourier transformed Cauchy problem and due to estimates

$$e^{t\Lambda(\alpha)} \leq \left\|e^{tA(\alpha)}\right\|_{\mathbb{R}^m} \leq C(1+|\alpha|)^{l(m-1)} \cdot e^{t\Lambda(\alpha)}, \quad t \geq 0,$$

by $\Lambda(\alpha) = \max Re\lambda_k(\alpha)$, where $\lambda_k(\alpha)$ are eigenvalues of $A(\alpha)$.

13. "Classical" Gelfand–Shilov Classification

The system is

- *Petrovsky correct*, if $\Lambda(\alpha) \leq C$, in particular,

parabolic, if $\exists C, h, C_1 > 0 : \Lambda(\alpha) \leq -C|\alpha|^h + C_1$,

hyperbolic, if $\exists C, C_1 : \Lambda(\alpha) \leq C,\ \Lambda(\alpha) \leq C|\alpha| + C_1$;

- *conditionally correct*, if
$\exists C, C_1 > 0, 0 < h < 1 : \Lambda(\alpha) \leq C|\alpha|^h + C_1$,

- *incorrect*, if $\Lambda(\alpha) \leq C|\alpha|^{l_0}$,
where $l_0 \geq 1$ is the reduced order of the system.

Depending on the type of the system, correctness spaces of test and generalized functions are defined for Fourier transformed and original systems ([2]-[4]): Gelfand, Melnikova, ...)

14. Main Result: Extended G-S classification

Using the Levy–Khintchine formulae, we include ΨDOs **A** in the classification.
Let the Fourier transform of **A** be

$$\mathbf{A}(\alpha) = A(\alpha) + \int_{\mathbb{R}/\{0\}} \left(1 - e^{i\alpha y} + i\alpha y \chi_{[-1,1]}(y)\right) \nu(dy) =: A(\alpha) + I(\alpha).$$

For $I(\alpha)$ is known the estimate: $\|I(\alpha)\| \leq C(1+|\alpha|)^2$.
Then by estimates for $e^{t\mathbf{A}(\alpha)} = e^{tA(\alpha)} \cdot e^{tI(\alpha)}$ we obtain

- for all types of the classification for the system with A, except parabolic, the system with **A** is incorrect;
- in the parabolic case for A:
if $h > 2$, the system with **A** is also parabolic, if $2 - h < 1$, it is conditionally correct, and if $2 - h \geq 1$, it is incorrect.

15. Classification for the special case of **A**

If the Fourier transform of $A\left(i\frac{\partial}{\partial x}\right)$ is equal to the first two term in the Levy–Khintchine formulae:

$$A(\alpha) = -ib\alpha + \frac{1}{2}\alpha\, Q\alpha$$

and the integral term contains just the first part, corresponding Poisson processes:

$$I(\alpha) = \int_{\mathbb{R}} \left(1 - e^{i\alpha y}\right) \nu(dy),$$

that is

if $\quad \mathbf{A}(\alpha) = -ib\alpha + \frac{1}{2}\alpha\, Q\alpha + \int_{\mathbb{R}} \left(1 - e^{i\alpha y}\right) \nu(dy),$

then the system with **A** is Petrovsky correct.

16. References

1. Böttcher B., Schilling R., Wang J. Lévy matters III. Lévy-type processes: construction, approximation and sample path properties. Springer, 2013.

2. Gelfand I.M. and Shilov G.E. Generalized functions. Vol. 3, Academic Press. 1967.

3. Melnikova I.V. Stochastic Cauchy Problems in Infinite Dimensions. Regularized and Generalized Solutions. CRC Press, 2016.

4. Melnikova I.V., Alekseeva U.A. Semigroup Classification and Gelfand–Shilov Classification of Systems of Partial Differential Equations. Mathematical Notes, 2018, Vol.104, 886-899.

Thank you for your attention!