Certain results on the existence of the convolution of Roumieu ultradistributions

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Dedicated to Professor Stevan Pilipović
on the occasion of his 70th birthday

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We consider complex-valued $\mathcal{C}^\infty$-functions and Roumieu ultradistributions defined on an open subset Ω of $\mathbb{R}^d$ using the standard multi-dimensional notation in $\mathbb{R}^d$.

The spaces of test functions and Roumieu ultradistributions are defined by a given weight sequence $(M_p) = (M_p)_{p\in\mathbb{N}_0}$ of positive numbers. Usually the following conditions are imposed on the sequence (M_p):

(M.1) $\quad M_p^2 \leqslant M_{p-1}M_{p+1}, \qquad p \in \mathbb{N};$

(M.2) $\quad M_p \leqslant AH^p M_q M_{p-q}, \qquad p,\, q \in \mathbb{N}_0,\ q \leqslant p;$

(M.2') $\quad M_p \leqslant AH^p M_{p-1}, \qquad p \in \mathbb{N},\ q \leqslant p;$

(M.3) $\quad \sum_{p=q+1}^\infty M_{p-1}M_p^{-1} \leqslant AqM_qM_p^{-1}, \quad q \in \mathbb{N};$

(M.3') $\quad \sum_{p=1}^\infty M_{p-1}M_p^{-1} < \infty,$

where the inequality in (M.3) is assumed to be satisfied for a certain constant $A > 0$ and the inequalities in (M.2) and in (M.2') for some constants $A > 0$ and $H > 0$.

In the sequel, we will assume some of the above conditions. Clearly, we can assume and we will assume that the constant H in condition (M.2') satisfies $H \geqslant 1$.

After P-P [4], denote by $\mathfrak{R}$ the class of numerical sequences $(r_p) = (r_p)_{p\in\mathbb{N}_0}$ (with $r_0 = 1$) which monotonously increase to infinity. We call the sequence $(R_p) = (R_p)_{p\in\mathbb{N}_0}$, where $R_p := \prod_{i=0}^{p} r_i$ for $p \in \mathbb{N}_0$ (clearly $R_0 = 1$), the product sequence corresponding to $(r_p) \in \mathfrak{R}$.

Lemma 1 (H. Komatsu 1982)

Let $(a_k)_{k\in\mathbb{N}_0}$ be a sequence of nonnegative numbers. Then

$$\sup_{k\in\mathbb{N}_0} \frac{a_k}{h^k} < \infty \quad \text{for} \quad \text{some } h > 0$$

if and only if

$$\sup_{k\in\mathbb{N}_0} \frac{a_k}{R_k} < \infty \quad \text{for} \quad \text{all } (r_k) \in \mathfrak{R},$$

where (R_k) is the product sequence corresponding to (r_p).

For a given complex-valued function φ on an open set $\Omega \subseteq \mathbb{R}^d$ and a compact set $K \subset \Omega$ denote

$$\|\varphi\|_K := \sup_{x\in K} |\varphi(x)|; \qquad \|\varphi\|_\Omega := \sup_{x\in\Omega} |\varphi(x)|.$$

For a given sequence (M_p), a regular compact set K in $\mathbb{R}^d$ and $h > 0$ the symbol $\mathcal{E}_{K,h}^{\{M_p\}}$ means the l.c.s. of all $\mathcal{C}^\infty$-functions φ on Ω s. t.

$$\|\varphi\|_{K,h} := \sup_{k\in\mathbb{N}_0^d} \frac{\|D^k\varphi\|_K}{h^{|k|}M_k} < \infty, \tag{1}$$

with the topology defined by the semi-norm $\|\cdot\|_{K,h}$ given above. The Banach space of all $\mathcal{C}^\infty$-functions φ satisfying (1) and having supports contained in K, with the topology of the norm $\|\cdot\|_{K,h}$, is denoted by $\mathcal{D}_{K,h}^{\{M_p\}}$.

For a fixed sequence (M_p) and an open set $\Omega \subseteq \mathbb{R}^d$, we consider the following locally convex spaces of ultradifferentiable functions on Ω:

$$\mathcal{D}_K^{\{M_p\}} := \varinjlim_{h\to\infty} \mathcal{D}_{K,h}^{\{M_p\}}; \qquad \mathcal{D}^{\{M_p\}}(\Omega) := \varinjlim_{K\Subset\Omega} \mathcal{D}_K^{\{M_p\}};$$

and

$$\mathcal{E}^{\{M_p\}}(\Omega) := \varprojlim_{K\Subset\Omega} \varinjlim_{h\to\infty} \mathcal{E}_{K,h}^{\{M_p\}}.$$

On the other hand, for a given regular compact set $K \subset \Omega$ and given sequences (M_p) and $(r_p) \in \mathfrak{R}$, we denote by $\mathcal{D}^{\{M_p\}}_{K,(r_p)}$, the Banach space of all $\mathcal{C}^\infty$-functions φ on Ω having supports contained in K such that

$$\|\varphi\|_{K,(r_p)} := \sup_{k \in \mathbb{N}_0^d} \frac{\|D^k \varphi\|_K}{R_{|k|} M_k} < \infty \tag{2}$$

with the norm $\| \cdot \|_{K,(r_p)}$ defined above. Then we have

$$\varprojlim_{(r_p) \in \mathfrak{R}} \mathcal{D}^{\{M_p\}}_{K,(r_p)} = \mathcal{D}^{\{M_p\}}_{K}.$$

For given (M_p) and $(r_p) \in \mathfrak{R}$, we consider the Banach space $\mathcal{D}^{\{M_p\}}_{L^\infty,(r_p)}(\Omega)$ of all $\mathcal{C}^\infty$-functions φ on Ω s. t.

$$\|\varphi\|_{(r_p)} := \sup_{k \in \mathbb{N}_0^d} \frac{\|D^k \varphi\|_\Omega}{R_{|k|} M_k} < \infty, \tag{3}$$

with the norm $\| \cdot \|_{(r_p)}$ and denote

$$\mathcal{D}^{\{M_p\}}_{L^\infty}(\Omega) := \varprojlim_{(r_p) \in \mathfrak{R}} \mathcal{D}^{\{M_p\}}_{L^\infty,(r_p)}(\Omega).$$

The completion of $\mathcal{D}^{\{M_p\}}(\Omega)$ in $\mathcal{D}_{L^\infty}^{\{M_p\}}(\Omega)$ is denoted by $\dot{\mathcal{B}}^{\{M_p\}}(\Omega)$.

Definition 1

The strong dual of $\mathcal{D}^{\{M_p\}}(\Omega)$, denoted by $\mathcal{D}'^{\{M_p\}}(\Omega)$, is called the space of Roumieu ultradistributions.

The strong dual of $\dot{\mathcal{B}}^{\{M_p\}}(\Omega)$, denoted by $\mathcal{D}'^{\{M_p\}}_{L^1}(\Omega)$, is called the space of integrable Roumieu ultradistributions.

For given a weight sequence (M_p) and $(r_p) \in \mathfrak{R}$, we consider the corresponding weight sequence (N_p), given by $N_p := M_p R_p$ for $p \in \mathbb{N}_0$ and define the respective associated functions by the formulas:

$$M(\rho) := \sup\{\log_+ \frac{\rho^p}{M_p} : \ p \in \mathbb{N}_0\}, \quad \rho > 0, \tag{4}$$

$$N(\rho) = N_{(r_p)}(\rho) := \sup\{\log_+ \frac{\rho^p}{N_p} : \ p \in \mathbb{N}_0\}, \quad \rho > 0. \tag{5}$$

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For given (M_p) and $(r_p),(u_p) \in \mathfrak{R}$, we consider the Banach space

$$\mathcal{S}^{M_p}_{(r_p),(u_p),d} := \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^d) : \ \|\varphi\|_{(r_p),(u_p)} < \infty\},$$

where

$$\|\varphi\|_{(r_p),(u_p)} := \sup_{k \in \mathbb{N}_0^d} \frac{\|D^k \varphi \, e^{N_{(u_p)}(|\cdot|)}\|_\infty}{R_{|k|} M_k}.$$

Let (M_p) satisfy conditions (M.1), (M.2') and (M.3'). After C-K-P [1], we define the space of ultradifferentiable functions $\mathcal{S}^{\{M_p\}}_d$ by

$$\mathcal{S}^{\{M_p\}}_d := \varprojlim_{(r_p),(u_p) \in \mathfrak{R}} \mathcal{S}^{M_p}_{(r_p),(u_p),d}. \tag{6}$$

Definition 2

The strong dual $\mathcal{S}'^{\{M_p\}}_d$ of the space $\mathcal{S}^{\{M_p\}}_d$ is called the space of tempered Roumieu ultradistributions.

The space $\mathcal{S}^{\{M_p\}}_d$ is a (DFS)-space and $\mathcal{S}'^{\{M_p\}}_d$ is an (FS)-space. If (M_p) satisfies condition (M.2), then both spaces are nuclear.

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Suppose that (M_p) satisfies (M.1) and J is an arbitrary set of indices.

Theorem 1 (M. Valdivia 2009)

For each j in a set J, let $\{s_{\alpha,j}:\ \alpha \in \mathbb{N}_0^d\}$ be a family of Radon measures on an open $\Omega \subseteq \mathbb{R}^d$. If for every compact set $K \subset \Omega$ there is an $(r_p) \in \mathfrak{R}$ s. t.

$$\sup_{\alpha \in \mathbb{N}_0^d, j \in J} R_{|\alpha|} M_\alpha \|s_{\alpha,j}\|_{\mathcal{C}'(K)} < \infty, \tag{7}$$

then there is a bounded subset $\{S_j:\ j \in J\}$ in $\mathcal{D}'^{\{M_p\}}(\Omega)$ such that

$$\langle \varphi, S_j \rangle = \sum_{|\alpha|=0}^{\infty} \langle D^\alpha \varphi, s_{\alpha,j} \rangle, \qquad \varphi \in \mathcal{D}^{\{M_p\}}(\Omega). \tag{8}$$

Moreover the series in (8) converges absolutely and uniformly as j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_p\}}(\Omega)$.

Conversely, if $\{S_j:\ j \in J\}$ is a bounded set in $\mathcal{D}'^{\{M_p\}}(\Omega)$, then for every $j \in J$ there is a family $(s_{\alpha,j}:\ \alpha \in \mathbb{N}_0^d)$ of Radon measures in Ω satisfying condition (7) for every compact $K \subset \Omega$ and some $(r_p) \in \mathfrak{R}$. Moreover the series (8) converges absolutely and uniformly as j varies in J and φ varies in any given bounded subset of $\mathcal{D}^{\{M_p\}}(\Omega)$.

Proposition 1

Let $S \in \mathcal{D}_d'^{\{M_p\}}$ and $S_n \in \mathcal{D}_d'^{\{M_p\}}$ for $n \in \mathbb{N}$. Suppose that $S_n \to S$ in $\mathcal{D}_d'^{\{M_p\}}$ as $n \to \infty$. Then for each bounded open set $G \in \mathbb{R}^d$ there are Radon measures $s_\alpha, s_{n\alpha} \in \mathcal{C}'(G)$ for $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^d$, a sequence $(r_p) \in \mathfrak{R}$ and a constant $B > 0$ such that

$$S_{|G} = \sum_{|\alpha|=0}^{\infty} D^\alpha s_\alpha, \qquad S_{n|G} = \sum_{|\alpha|=0}^{\infty} D^\alpha s_{n\alpha}, \tag{9}$$

and the equalities

$$\|s_\alpha\|_{\mathcal{C}'(K)} \leqslant B(R_{|\alpha|}M_\alpha)^{-1}, \qquad \|s_{n\alpha}\|_{\mathcal{C}'(K)} \leqslant B(R_{|\alpha|}M_\alpha)^{-1} \tag{10}$$

hold for every compact set $K \subset G$ and arbitrary $\alpha \in \mathbb{N}_0^d$ and $n \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} \|s_{n\alpha} - s_\alpha\|_{\mathcal{C}'(K)} = 0 \quad \text{for every } \alpha \in \mathbb{N}_0^d.$$

Theorem 2 (S. Pilipović 1996)

Let (M_p) satisfy (M.1) and (M.2'). Assume $S \in \mathcal{D}_d'^{\{M_p\}}$. Then the Roumieu ultradistribution S is tempered, i.e. $S \in \mathcal{S}_d'^{\{M_p\}}$, if and only if S is of the following form

$$S = \sum_{|\alpha|,|\beta| \in \mathbb{N}_0} D^\alpha (\langle \cdot \rangle^\beta F_{\alpha,\beta}) \tag{11}$$

with the series convergent in $\mathcal{S}_d'^{\{M_p\}}$, where the symbol $\langle \cdot \rangle^\beta$ means the function defined by

$$\langle x \rangle^\beta := \prod_{j=1}^{d} \left(1 + x_j^2\right)^{\beta_j/2}, \quad x \in \mathbb{R}^d,\ \beta \in \mathbb{N}_0^d$$

and $(F_{\alpha,\beta})_{\alpha,\beta \in \mathbb{N}_0^d}$ is a matrix of elements of L^∞ for which there exists a sequence $(r_p) \in \mathfrak{R}$ such that

$$\sup_{\alpha,\beta \in \mathbb{N}_0^d, x \in \mathbb{R}^d} R_{|\alpha+\beta|}\, M_\alpha M_\beta |F_{\alpha,\beta}(x)| < \infty.$$

Proposition 2

Let $S_n \in \mathcal{S}_d'^{\{M_p\}}$ for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Suppose that $S_n \to S_0$ in $\mathcal{S}_d'^{\{M_p\}}$ as $n \to \infty$. Then for each bounded open set $G \in \mathbb{R}^d$, there exist functions $F_{\alpha,n} \in L^\infty(G)$ for $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$, constants $\lambda, B > 0$ and a sequence $(r_p) \in \mathfrak{R}$ such that

$$S_{n|G} = \sum_{|\alpha|=0}^{\infty} D^\alpha (e_\lambda^M F_{\alpha,n}) \qquad (n \in \mathbb{N}_0),$$

$$\|F_{\alpha,n}\|_\infty \leqslant B(R_{|\alpha|} M_\alpha)^{-1} \text{ on } G \qquad (\alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}_0)$$

and

$$\lim_{n\to\infty} \|F_{\alpha,n} - F_{\alpha,0}\|_\infty = 0 \quad \text{on } G \qquad (\alpha \in \mathbb{N}_0^d),$$

where $e_\lambda^M(x) := e^{M(\lambda|x|)}$ for $x \in \mathbb{R}^d$.

Definition 3

By an $\mathfrak{R}$-approximate unit we mean a sequence (Π_n) of ultradifferentiable functions $\Pi_n \in \mathcal{D}_d^{\{M_p\}}$ which converges to 1 in $\mathcal{E}_d^{\{M_p\}}$ such that, for every sequence $(r_p) \in \mathfrak{R}$, we have

$$\sup_{n\in\mathbb{N}} \|\Pi_n\|_{(r_p)} = \sup_{n\in\mathbb{N}} \sup_{k\in\mathbb{N}_0^d} (R_{|k|}M_k)^{-1} \|D^k \Pi_n\|_\infty < \infty, \tag{12}$$

where (R_p) is the product sequence corresponding to (r_p).

Definition 4

By a special $\mathfrak{R}$-approximate unit we mean an $\mathfrak{R}$-approximate unit (Π_n) such that for every compact set $K \subset \mathbb{R}^d$ there exists an index $n_0 \in \mathbb{N}$ such that $\Pi_n(x) = 1$ for all $n \geqslant n_0$ and $x \in K$.

We denote the class of all $\mathfrak{R}$-approximate units on $\mathbb{R}^d$ by $\mathbb{U}_d^{\{M_p\}}$ and the class of all special $\mathfrak{R}$-approximate units on $\mathbb{R}^d$ by $\overline{\mathbb{U}}_d^{\{M_p\}}$.

Definition 5

We say that $S, T \in \mathcal{D}'^{\{M_p\}}_d$ are convolvable with respect to 1° (V) and 2° $(\overline{\mathrm{V}})$, whenever

1° the sequence $\big(\langle S \otimes T,\ \Pi_n\, \varphi^{\triangle}\rangle_{2d}\big)$ is Cauchy for all $(\Pi_n) \in \mathbb{U}^{\{M_p\}}_{2d}$ for every $\varphi \in \mathcal{D}^{\{M_p\}}_d$ and

2° the sequence $\big(\langle S \otimes T,\ \Pi_n\, \varphi^{\triangle}\rangle_{2d}\big)$ is Cauchy for all $(\Pi_n) \in \overline{\mathbb{U}}^{\{M_p\}}_{2d}$ for every $\varphi \in \mathcal{D}^{\{M_p\}}_d$, respectively.

One can prove that both types of convolvability with respect to (V) and $(\overline{\mathrm{V}})$ defined in 1° and 2° are equivalent.

Definition 6

We define the convolution in $\mathcal{D}'^{\{M_p\}}_d$ of two convolvable Roumieu ultradistributions $S, T \in \mathcal{D}'^{\{M_p\}}_d$ by

$$\langle S * T, \varphi\rangle_d = \lim_{n\to\infty} \langle S \otimes T, \Pi_n\, \varphi^{\triangle}\rangle_{2d}, \qquad \varphi \in \mathcal{D}^{\{M_p\}}_d, \quad (\Pi_n) \in \overline{\mathbb{U}}^{\{M_p\}}_{2d}. \tag{13}$$

Definition 7

We say that $S, T \in \mathcal{S}_d'^{\{M_p\}}$ are t-convolvable with respect to $1^{\circ\circ}$(V); $2^{\circ\circ}$($\overline{\mathrm{V}}$), whenever

$$1^{\circ\circ} \quad \big(\langle S \otimes T,\ \Pi_n\, \varphi^{\triangle}\rangle_{2d}\big) \quad \text{is a C. s. for all}\,(\Pi_n) \in \mathbb{U}_{2d}^{\{M_p\}}, \varphi \in \mathcal{S}_d^{\{M_p\}};$$

$$2^{\circ\circ} \quad \big(\langle S \otimes T,\ \Pi_n\, \varphi^{\triangle}\rangle_{2d}\big) \quad \text{is a C. s. for all}\,(\Pi_n) \in \overline{\mathbb{U}}_{2d}^{\{M_p\}}, \varphi \in \mathcal{S}_d^{\{M_p\}},$$

respectively.

If $S, T \in \mathcal{S}_d'^{\{M_p\}}$ are t-convolvable with respect to $1^{\circ\circ}$ (V_t); $2^{\circ\circ}$ ($\overline{\mathrm{V}}_t$), respectively, then the t-convolution of S and T in $\mathcal{S}_d'^{\{M_p\}}$ is defined by

$$1^{\circ\circ}\ \langle S *_t T, \varphi\rangle_d \ := \ \lim_{n\to\infty} \langle S \otimes T, \Pi_n\, \varphi^{\triangle}\rangle_{2d}, \quad \varphi \in \mathcal{S}_d^{\{M_p\}},\ (\Pi_n) \in \mathbb{U}_{2d}^{\{M_p\}};$$

$$2^{\circ\circ}\ \langle S \overline{*}_t T, \varphi\rangle_d \ := \ \lim_{n\to\infty} \langle S \otimes T, \Pi_n\, \varphi^{\triangle}\rangle_{2d}, \quad \varphi \in \mathcal{S}_d^{\{M_p\}},\ (\Pi_n) \in \overline{\mathbb{U}}_{2d}^{\{M_p\}}$$

respectively.

The following definition of the t-convolution of tempered Roumieu ultradistributions is an analogue of one of the known Schwartz definitions of the convolution of distributions:

Definition 8

If $S, T \in \mathcal{S}_d'^{\{M_p\}}$ satisfy the condition:

$$(\star) \qquad (S \otimes T)\, \varphi^{\triangle} \in \mathcal{D}'^{\{M_p\}}_{L^1,2d}, \qquad \varphi \in \mathcal{S}_d^{\{M_p\}},$$

then we say that tempered Roumieu ultradistributions S, T are convolvable in the sense of $(\star)$ and the convolution $S \star_t T$ is defined by the formula

$$\langle S \star_t T, \varphi\rangle_d := \langle (S \otimes T)\, \varphi^{\triangle}, 1\rangle_{2d}, \qquad \varphi \in \mathcal{S}_d^{\{M_p\}}. \tag{14}$$

It can be proved that both the sequential versions of the t-convolution of two tempered Roumieu ultradistributions S and T given in Definition 7 are equivalent to Definition 8 and, moreover,

$$S *_t T = S \overline{*}_t T = S \star_t T.$$

According to this, we will use the common notation $S *_t T$ for the t-convolution of tempered Roumieu ultradistributions S and T.

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According to this, we will use the common notation $S *_t T$ for the t-convolution of tempered Roumieu ultradistributions S and T.

Proposition 3

Let $X, Y \subseteq \mathbb{R}^d$ be arbitrary sets. The following conditions are equivalent:

$(C1.)$ the set $(X \times Y) \cap K^{\Delta}$ is bounded in $\mathbb{R}^{2d}$ for every K bounded in $\mathbb{R}^d$, where

$$K^{\Delta} := \{(x, y) \in \mathbb{R}^{2d} :\ x + y \in K\};$$

$(C2.)$ for every $R > 0$ the set

$$W_R := \{(x, y) :\ x \in X, y \in Y, |x + y| \leqslant R\}$$

is bounded in $\mathbb{R}^{2d}$;

$(C3.)$ the following implication holds:

$$\lim_{n \to \infty} |x_n| + |y_n| = \infty \quad \Rightarrow \quad \lim_{n \to \infty} |x_n + y_n| = \infty,$$

whenever $x_n \in X$ and $y_n \in Y$ for $n \in \mathbb{N}$.

Definition 9

The sets $X, Y \subseteq \mathbb{R}^d$ are called compatible if any of the three equivalent conditions $(C1)$, $(C2)$, $(C3)$ is satisfied.

Analogously to the case of Beurling tempered ultradistributions, we define the notion of M-compatibility of subsets of $\mathbb{R}^d$, applying the the associated function M defined in (4).

Definition 10

The sets $X, Y \subseteq \mathbb{R}^d$ are called M-compatible if

$$M(|x|) + M(|y|) \leqslant M(b|x+y|) + b, \qquad x \in X,\ y \in Y \tag{15}$$

for some $b \geqslant 1$.

Theorem 3

Suppose that (M_p) satisfies conditions (M.1), (M.2) and (M.3').

If $S, T \in \mathcal{D}_d'^{\{M_p\}}(\mathcal{S}_d'^{\{M_p\}})$ are (tempered) ultradistributions whose supports $\Sigma = \operatorname{supp} S$ and $\Theta = \operatorname{supp} T$ are compatible (M-compatible), then the convolution $S{*}T$ $(S{*_t}T)$ exists in $\mathcal{D}_d'^{\{M_p\}}$ $(\mathcal{S}_d'^{\{M_p\}})$.

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Inverse results

That the conditions of compatibility and M-compatibility cannot be weakened in the above existence theorem the following inverse results show:

Theorem 4

Let $\Sigma, \Theta \subseteq \mathbb{R}^d$. Assume that the convolution $S * T$ exists in $\mathcal{D}_d'^{\{M_p\}}$ for each pair $S, T \in \mathcal{D}_d'^{\{M_p\}}$ of Roumieu ultradistributions with supports contained in Σ and Θ, respectively. Then sets Σ and Θ are compatible.

Theorem 4'

Let $\Sigma, \Theta \subseteq \mathbb{R}^d$. Assume that the convolution $S *_t T$ exists in $\mathcal{S}_d'^{\{M_p\}}$ for each pair $S, T \in \mathcal{S}_d'^{\{M_p\}}$ of Roumieu tempered ultradistributions with supports contained in Σ and Θ, respectively. Then sets Σ and Θ are M-compatible.

Sequential continuity of convolution

Assume that (M_p) satisfies conditions (M.1), (M.2) and (M.3').

Theorem 5

If for ultradistributions $S_n, T_n \in \mathcal{D}_d'^{\{M_p\}}$ with supp $S_n \subset \Sigma$, supp $T_n \subset \Theta$ where Σ and Θ are compatible sets in $\mathbb{R}^d$, the convergence $S_n \to S,\ T_n \to T$ holds in $\mathcal{D}_d'^{\{M_p\}}$, then $S_n * T_n \to S * T$ in $\mathcal{D}_d'^{\{M_p\}}$ as $n \to \infty$.

Theorem 5'

Suppose that supports of tempered ultradistributions $S_n, T_n \in \mathcal{S}_d'^{\{M_p\}}$ satisfy the inclusions supp $S_n \subset \Sigma$ and supp $T_n \subset \Theta$, where the sets Σ and Θ are M-compatible.

If the convergence $S_n \to S,\ T_n \to T$ holds in $\mathcal{S}_d'^{\{M_p\}}$, then $S_n *_t T_n \to S *_t T$ in $\mathcal{S}_d'^{\{M_p\}}$ as $n \to \infty$.

Examples

Using the parametric equations of the logarithmic spiral:

$$x(\theta) = ae^{b\theta}\cos\theta \quad y(\theta) = ae^{b\theta}\sin\theta$$

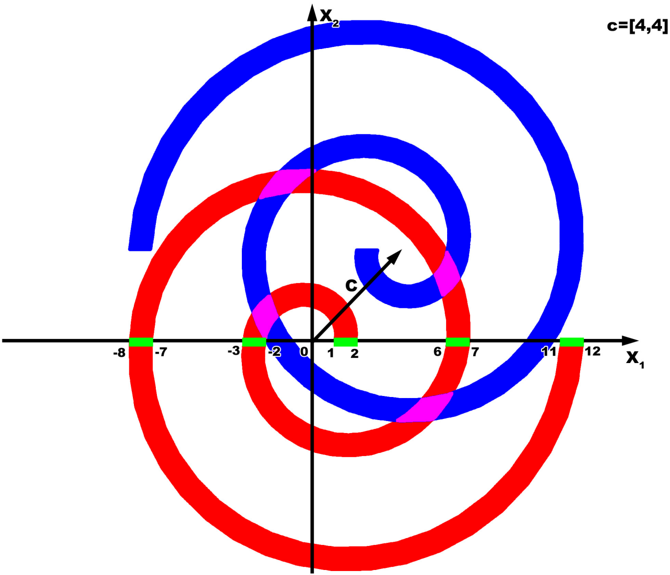
for $a \in \mathbb{R}$ and $b, \theta > 0$, we construct suitable sets $\Sigma, \Theta \subset \mathbb{R}^2$.

One may verify that the sequences $(P_n)_{n\in\mathbb{N}}$ and $(Q_n)_{n\in\mathbb{N}}$ of points $P_n := (x'_n, y'_n) \in \Sigma$ and $Q_n := (x''_n, y''_n) \in \Theta$ satisfy condition $(C3.)$, i.e.

$$\lim_{n\to\infty} \|P_n\|_2 + \|Q_n\|_2 = \infty \;\Rightarrow\; \lim_{n\to\infty} \|P_n + Q_n\|_2 = \infty.$$

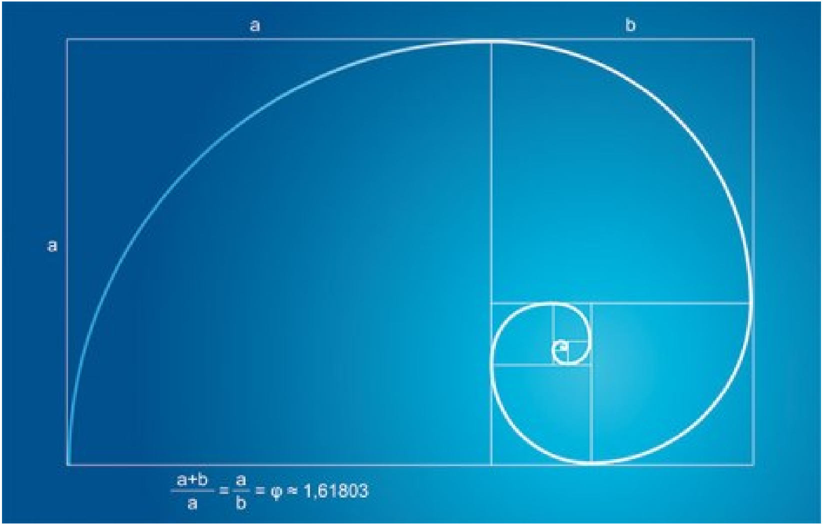
Then every two Roumieu ultradistributions $S, T \in \mathcal{D}_2'^{\{M_p\}}$ with supports supp S and supp T contained in Σ and Θ, respectively, are convolvable.

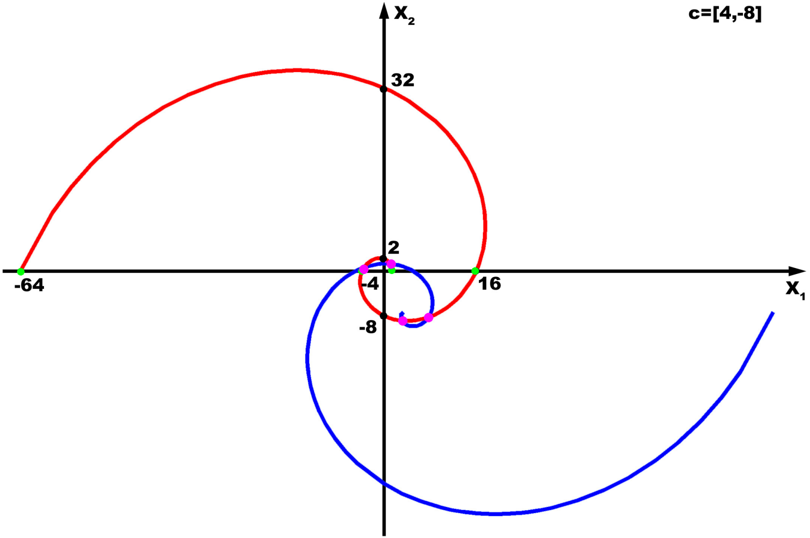
Logarithmic spiral



Fibonacci spiral

Parametric equation $r(\theta) = ae^{b\theta}$ with $a, b > 0$.




x₂
c=[4,-8]
32
2
-64
-4
16
x₁
-8

References

R. D. Carmichael, A. Kamiński, S. Pilipović, Boundary Values and Convolution in Ultradistribution Spaces, World Scientific, (2007).

Komatsu H., Ultradistributions, III: Vector valued ultradistributions and the theory of kernels, J. Fac. Sci. Univ. Tokyo, Sect. I A Math., 29, 653-717, (1982).

Mincheva-Kaminska, S. A sequential approach to the convolution of Roumieu ultradistributions, Adv. Oper. Theory, accepted.

Pilipović, S., Prangoski, B., On the convolution of Roumieu ultradistributions through the ε tensor product, Monatsh. Math., 173, 83-105 (2014).

Pilipović, S., Prangoski, B., Vindas, J., On quasianalytic classes of Gelfand-Shilov type. Parametrix and convolution, J. Math.PuresAppl., 116, 174-210 (2018).

Valdivia M., On the structure of ceratain ultradistributions, Rev. R. Acad. Cien. Serie A Mat., 103(1), 17-48 (2009).

Thanks

THANK YOU FOR YOUR ATTENTION!