A Fourier transform for all generalized functions

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Fourier transform in the Colombeau setting

Definition

Let $u = [u_\varepsilon] \in \mathcal{G}^s(\Omega)$ be a CGF and $K \Subset \Omega$. Then $\int_K u(x)\,\mathrm{d}x := \left[\int_K u_\varepsilon(x)\,\mathrm{d}x\right] \in \widetilde{\mathbb{R}}$. Similarly, for $\int_\Omega u$ if u is compactly supported.

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1. This notion of Fourier transform shares several properties with the classical one, but it lacks e.g. the Fourier inversion theorem, which holds only at the level of equality in the sense of generalized tempered distributions.
2. We can say, the use of the multiplicative damping measure introduces a perturbation of infinitesimal frequencies that inhibit several results.

Hyperfinite Fourier transform (HFT)

Definition

Let $k \in {}^{\rho}\widetilde{\mathbb{R}}$ be a positive infinite number and let $K := [-k, k]^n \subseteq {}^{\rho}\widetilde{\mathbb{R}}^n$ be a functionally compact set. We define the n-dimensional hyperfinite Fourier transform of the GSF $f \in {}^{\rho}\mathcal{GC}^{\infty}(K, \widetilde{\mathbb{C}}^n)$ as follows:

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$$\mathcal{F}_k(f)(\omega) := \int_K f(x)\, e^{-ix\omega}\,\mathrm{d}x = \int_{-k}^{k} \mathrm{d}x_1 \ldots \int_{-k}^{k} f(x_1 \ldots x_n)\, e^{-ix\cdot\omega}\,\mathrm{d}x_n,$$

where the product $x\cdot\omega$ on ${}^{\rho}\widetilde{\mathbb{R}}^n$ denotes the usual dot product $x\cdot\omega = \sum_{j=1}^{n} (x_j\omega_j)$.

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$\forall x \in [-\infty, -k]^n \cup [k, \infty]^n : f(x) = g(x) = 0$. Then $\forall x, y \in {}^{\rho}\widetilde{\mathbb{R}}^n$,
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For this type convolution we have the usual commutative, associative and distributive properties. Moreover, for $f \in {}^{\rho}\mathcal{GC}^{\infty}\left({}^{\rho}\widetilde{\mathbb{R}}, {}^{\rho}\widetilde{\mathbb{R}}\right)$ such that $\forall x \in {}^{\rho}\widetilde{\mathbb{R}}\, \exists r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} \exists c \in {}^{\rho}\widetilde{\mathbb{R}}\, \forall y \in \overline{B^{\mathrm{E}}}_{r}(x)\, \forall j \in \mathbb{N} :\ \left|d^{j} f(y)\right| \leq c$ we have $f * \delta = f$, where δ is the Dirac delta distribution.

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The inverse HFT and the inversion theorem

Definition

Let $f \in {}^{\rho}\mathcal{GC}^{\infty}\left(K, {}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$, we define $\mathcal{F}_{k}^{-1}(f)(x) = \left(\frac{1}{2\pi}\right)^{n} \int_{K} f(\omega)\, e^{ix\cdot\omega}\, \mathrm{d}\omega$ for all $\omega \in {}^{\rho}\widetilde{\mathbb{R}}$. This operation is called inverse hyperfinite Fourier transform.

We can in fact prove that inverse HFT shares the same properties as the HFT does. In the next theorem we prove that one is the inverse operation of the other.

Theorem

Let $f, g, h \in {}^{\rho}\mathcal{GC}^{\infty}\left(K, {}^{\rho}\widetilde{\mathbb{R}}^{n}\right)$ we have that

1. $\int_{K} \mathcal{F}_{k}(f)\, g(x) = \int_{K} f(x)\, \mathcal{F}_{k}(g)$.
2. *Hyperfinite Fourier inversion:*
   $\mathcal{F}_{k}^{-1}\left[\mathcal{F}_{k}(f)\right] = f = \mathcal{F}_{k}\left[\mathcal{F}_{k}^{-1}(f)\right] = \left(\frac{1}{2\pi}\right)^{n} \int_{K} \mathcal{F}_{k}(f)(\omega)\, e^{ix\omega}\, \mathrm{d}\omega$.

Examples

1. If $\delta(x) = \delta(x_1)\,\delta(x_2)\ldots\delta(x_n)$ is n-dimensional Dirac delta distribution with $x = (x_1, x_2, \ldots, x_n) \in {}^{\rho}\widetilde{\mathbb{R}}^n$ and $k \geq \mathrm{d}\rho^{-a}, \forall a \in {}^{\rho}\widetilde{\mathbb{R}}$ then $\mathcal{F}_k(\delta)(\omega) = 1$ if $\forall\omega$ finite and $\mathcal{F}_k(\delta)(\omega) = 0$ if $|\omega| \geq k$.

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2. Let $f(x) = e^{-\frac{|x|^2}{2}} \in {}^{\rho}\mathcal{GC}^{\infty}\left({}^{\rho}\widetilde{\mathbb{R}}^n\right)$, $x \in {}^{\rho}\widetilde{\mathbb{R}}^n$. Then

   $\mathcal{F}_k(f)(\omega) = (2\pi)^{\frac{n}{2}}\, e^{-\frac{|\omega|^2}{2}}$.

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3. If $f(x) = e^{-a|x|}$ (with $x \in {}^{\rho}\widetilde{\mathbb{R}}$) then $\mathcal{F}(f)(\omega) = \frac{1-e^{(a-i\omega)k}}{a-i\omega} - \frac{e^{-(a+i\omega)k}-1}{a+i\omega}$.

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4. If $f(x) = e^x$ where $|x| \leq k$ and $k = -\log(\mathrm{d}\rho)$ then $\mathcal{F}_k(f)(\omega) = \frac{e^{k(1-i\omega)}-e^{-k(1-i\omega)}}{1-i\omega} = \frac{\mathrm{d}\rho^{(i\omega-1)}-\mathrm{d}\rho^{(1-i\omega)}}{1-i\omega}$.

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where $b \geq \mathrm{d}\rho^{-a}$, $a \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ *is a linear embedding that commutes with partial derivatives and* $\forall f \in \mathcal{S}(\mathbb{R}^{n})$, $\mathcal{F}_{k}\left(\iota_{\Omega}^{b}(f)\right) = \iota_{\Omega}^{b}(\mathcal{F}(f))$.

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$$\mathcal{F}_k\left(\iota_\Omega^b(T) \cdot \mathbf{1}\right)(\omega) = \left(\iota_\Omega^b\left(\widehat{T}\right)\right)(\omega), \forall \omega \in \widetilde{\Omega}_c, \mathbf{1} := \mathcal{F}_k(\delta).$$

Examples of HFT applied in ODE

Examples

1. n-th order homogeneous generalized ODE
$a_n y^{(n)} + \ldots a_1 y^{(1)} + a_0 y = 0, y \in {}^{\rho}\mathcal{GC}^{\infty}(K, {}^{\rho}\widetilde{\mathbb{R}}), a_n \in {}^{\rho}\widetilde{\mathbb{R}}^{*}, n \in \mathbb{N}_{\geq 1},$
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3. We finally apply the inversion theorem, at $\omega \in \widetilde{\Omega}_c$, so that we can recover the initial CGF u.

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4. Paley-Wiener theorem using hyperfinite series.

Contact

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Thank you for your attention!