The nuclearity of Gelfand-Shilov spaces and kernel theorems

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International Conference on Generalized Functions
Ghent, Belgium, September 3, 2020

joint work with Andreas Debrouwere and Jasson Vindas

Introduction

The goal of this talk is to characterize when the Gelfand-Shilov space $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]}$ is nuclear.

- Gel'fand and Shilov characterized the nuclearity of the space $\mathcal{S}_{(\mathscr{W}),\infty}$.
- For weight sequences, the nuclearity of the space $\mathcal{S}^{[M]}_{[A]}$ was shown under $(M.1)$ and $(M.2)$ by Pilipović, Prangoski, and Vindas.
- Boiti, Jornet, and Oliaro recently considered the nuclearity of isotropic Beurling-Björk spaces $\mathcal{S}^{(\omega)}_{(\omega)}$ with respect to the weight function ω.
- Our results deal with more general spaces and provide full characterizations of their nuclearity. This talk is based on:

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Nuclear spaces - Definition

Let E and F be locally convex Hausdorff spaces (=lcHs).
A linear map $A : E \to F$ is nuclear if there exists

- an equicontinuous sequence $(a_n)_n$ in E';
- a sequence $(b_n)_n$ contained in a bounded Banach disk of F;
- an absolutely summable complex sequence $(\lambda_n)_n$;

such that

$$A(x) = \sum_{n \in \mathbb{N}} \lambda_n \langle a_n, x \rangle b_n, \qquad \forall x \in E.$$

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Nuclear spaces - Properties

Nuclear spaces enjoy several nice properties.

- Nuclear spaces are stable under taking subspaces, Hausdorff quotients, projective and inductive limits, completion, tensor products, ...
- Nuclear spaces are Schwartz.
  $\hookrightarrow$ Every quasi-complete barrelled nuclear space is Montel (i.e. every bounded set is relatively compact).
  $\hookrightarrow$ An infinite dimensional Banach space is not nuclear.
- For a nuclear complete lcHs E and any complete lcHs F the well-known topologies on $E \otimes F$ coincide. Moreover, the extension of

$$E \otimes F \to \mathcal{L}_b(E'_b, F) : \quad e \otimes f \mapsto (e' \mapsto \langle e', e \rangle f)$$

  gives the canonical isomorphism

$$E \widehat{\otimes} F \cong \mathcal{L}_b(E'_b, F)$$

- For example we have $\mathcal{S}(\mathbb{R}^{d_1+d_2})'_b \cong \mathcal{L}_b(\mathcal{S}(\mathbb{R}^{d_1}), \mathcal{S}(\mathbb{R}^{d_2})'_b)$, i.e. for any continuous linear $L : \mathcal{S}(\mathbb{R}^{d_1}) \to \mathcal{S}(\mathbb{R}^{d_2})'_b$ there exists a kernel $f \in \mathcal{S}(\mathbb{R}^{d_1+d_2})'$ such that

$$\langle L(\varphi_1), \varphi_2 \rangle = \int f(x,y)\varphi_1(x)\varphi_2(y)dxdy, \quad \varphi_j \in \mathcal{S}(\mathbb{R}^{d_j}),\ j = 1,2$$

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Nuclear spaces - Fréchet and (DF)-spaces

Let E be a lcHs and $x = (x_n)_{n\in\mathbb{N}}$ be some sequence in E. Then x is called

- weakly summable if $\sum_{n=0}^{\infty} |\langle x', x_n\rangle| < \infty$ for all $x' \in E'$
- absolutely summable if $\sum_{n=0}^{\infty} p(x_n) < \infty$ for every continuous seminorms p on E

Proposition (Grothendieck, 1955)

Let E be a Fréchet space or a (DF)-space. Then, E is nuclear if and only if every weakly summable sequence in E is absolutely summable.

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The Beurling and Roumieu case

Throughout this presentation, we will simultaneously consider the Beurling and Roumieu cases.

Notation

We use the brackets $(\cdot)$ to denote the Beurling case, while we use $\{\cdot\}$ for the Roumieu case. We employ $[\cdot]$ as a common notation.

We will encounter several conditions which use the indices λ and μ. These should always be preceded by the quantifiers:

- Beurling case: $\forall \lambda \in \mathbb{R}_+ \ \exists \mu \in \mathbb{R}_+$;
- Roumieu case: $\forall \mu \in \mathbb{R}_+ \ \exists \lambda \in \mathbb{R}_+$.

Köthe sequence spaces

A family $A = \{(a_j^\lambda)_{j \in \mathbb{Z}^d} : \lambda \in \mathbb{R}_+\}$ of sequences of positive numbers is called a Köthe set if $a_j^\lambda \leq a_j^\mu$ when $\mu \leq \lambda$.

For any $q \in [1, \infty]$ we consider the Köthe sequence space $\lambda^q[A]$ of all $(c_j)_{j \in \mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d}$ such that

$$\|c_j a_j^\lambda\|_{\ell^q(\mathbb{Z}^d)} < \infty, \quad \forall \lambda \in \mathbb{R}_+ \text{ (resp. } \exists \lambda \in \mathbb{R}_+)$$

A is said to satisfy [N] if $\sum_{j \in \mathbb{Z}^d} a_j^\lambda / a_j^\mu < \infty$

Proposition

The following are equivalent:

(*i*) A satisfies [N]

(*ii*) $\lambda^q[A]$ is nuclear for all (resp. for some) $q \in [1, \infty]$

(*iii*) $\lambda^q[A] = \lambda^r[A]$ for all $q, r \in [1, \infty]$ (resp. for some $q \neq r$)

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Weight function systems

A family $\mathscr{W} = \{w^\lambda : \lambda \in \mathbb{R}_+\}$ of positive continuous functions is called a weight function system if $1 \leq w^\lambda \leq w^\mu$ when $\mu \leq \lambda$.

We consider the conditions

[wM] $\sup_{|y|\leq 1} w^\lambda(x+y) \leq C w^\mu(x)$

[M] $w^\lambda(x+y) \leq C w^\mu(x) w^\mu(y)$

[N] $w^\lambda / w^\mu \in L^1(\mathbb{R}^d)$

To $\mathscr{W}$ we associate the Köthe set $A_{\mathscr{W}} = \{(w^\lambda(j))_{j\in\mathbb{Z}^d}\}$

Proposition

Suppose $\mathscr{W}$ satisfies [wM]. The following are equivalent:

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A weight sequence system $\mathfrak{M} = \{M^\lambda : \lambda \in \mathbb{R}_+\}$ is a family of weight sequences such that $M^\lambda \leq M^\mu$ when $\lambda \leq \mu$.

- We consider the following conditions on $\mathfrak{M}$:

  [L] $\forall L > 0 : L^{|\alpha|} M^\mu_\alpha \leq C M^\lambda_\alpha$;

  $[\mathfrak{M}.2]'$ $\exists H > 0 : M^\mu_{\alpha+e_j} \leq C H^{|\alpha|} M^\lambda_\alpha$.
- $\mathfrak{M}$ is called accelerating if $M^\lambda_{\alpha+e_j}/M^\lambda_\alpha \leq M^\mu_{\alpha+e_j}/M^\mu_\alpha$ when $\lambda \leq \mu$.
- $\mathfrak{M}$ is called isotropically decomposable if, after some permutation of the indices, it can be written as

$$\mathfrak{M} = \mathfrak{M}_1 \otimes \cdots \otimes \mathfrak{M}_k$$

where for each $\mathfrak{M}_j = \{(M^\lambda_{j,\alpha})_{\alpha\in\mathbb{N}^d} : \lambda \in \mathbb{R}_+\}$ it holds that $M^\lambda_{j,\alpha} = M^\lambda_{j,\beta}$ whenever $|\alpha| = |\beta|$.

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A weight sequence $M = (M_\alpha)_\alpha$ is a sequence of positive numbers such that $\lim_{\alpha\to\infty} M_\alpha^{1/|\alpha|} = \infty$ and for which $M_{\alpha+e_j}^2 \leq M_\alpha M_{\alpha+2e_j}$, $\forall\alpha \in \mathbb{N}^d$.

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Weight sequence systems - Definition

A weight sequence $M = (M_\alpha)_\alpha$ is a sequence of positive numbers such that $\lim_{\alpha\to\infty} M_\alpha^{1/|\alpha|} = \infty$ and for which $M_{\alpha+e_j}^2 \leq M_\alpha M_{\alpha+2e_j}$, $\forall \alpha \in \mathbb{N}^d$.

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Weight sequence systems - Properties

For any $\lambda \in \mathbb{R}_+$ the associated function of M^λ is defined as

$$\omega_{M^\lambda}(t) = \sup_{\alpha \in \mathbb{N}^d} \log \frac{|t^\alpha| M_0^\lambda}{M_\alpha^\lambda}$$

Then, $\mathscr{W}_{\mathfrak{M}} = \{\exp\ \omega_{M^\lambda}(\cdot) : \lambda \in \mathbb{R}_+\}$ is a weight function system.

Proposition

Let $\mathfrak{M}$ be an isotropically decomposable weight sequence system satisfying $[L]$.

(a) $\mathscr{W}_{\mathfrak{M}}$ satisfies [M].

(b) Consider the statements:

(i) $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$

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Weight sequence systems - Examples

Weight sequence systems allow us to simultaneously consider different definitions of Gelfand-Shilov spaces such as:

- Via weight sequences. Here we put for a weight sequence M:

$$\mathfrak{M}_M = \{(\lambda^{|\alpha|} M_\alpha)_{\alpha \in \mathbb{N}^d} : \lambda \in \mathbb{R}_+ \}, \qquad \mathscr{W}_M = \{\exp \omega_M(\cdot/\lambda) : \lambda \in \mathbb{R}_+ \}$$

- Via Braun-Meise-Taylor weight functions $\omega : \mathbb{R}^d \to \mathbb{R}_+$ where we set

$$\mathfrak{M}_\omega = \{(\exp(\frac{1}{\lambda}\phi^*(\lambda|\alpha|)))_{\alpha \in \mathbb{N}^d} : \lambda \in \mathbb{R}_+\}, \quad \mathscr{W}_\omega = \{\exp(\frac{1}{\lambda}\omega(\cdot)) : \lambda \in \mathbb{R}_+\}$$

where $\phi^*(y) = \sup_{x \geq 0}(xy - \omega(e^x))$ is the Young conjugate of $\omega(e^x)$.

In this case, we characterized the nuclearity for a larger class of spaces in:

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Gelfand-Shilov spaces

For a weight sequence system $\mathfrak{M}$, weight function system $\mathscr{W}$ and $q \in [1, \infty]$ we define $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ as the space of all $\varphi \in C^\infty(\mathbb{R}^d)$ s.t.

$$\|\varphi\|_{\mathcal{S}^{M^\lambda}_{w^\lambda,q}} = \sup_{\alpha \in \mathbb{N}^d} \frac{1}{M^\lambda_{|\alpha|}} \|\varphi^{(\alpha)} w^\lambda\|_{L^q(\mathbb{R}^d)} < \infty, \quad \forall \lambda \in \mathbb{R}_+ \text{ (resp. } \exists \lambda \in \mathbb{R}_+)$$

Theorem 1 (Debrouwere, N. and Vindas, 2020)

Let $\mathfrak{M}$ satisfy [L] and $[\mathfrak{M}.2]'$, $\mathscr{W}$ satisfy [wM] and suppose $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \neq \{0\}$ for some $q \in [1, \infty]$. Consider the statements

(i) $\mathscr{W}$ satisfies [N]

(ii) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}]} = \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} = \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],r}$ as locally convex spaces for all $q, r \in [1, \infty]$

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Sufficient conditions for nuclearity

Theorem 2 (Debrouwere, N. and Vindas, 2020)

Let $\mathfrak{M}$ satisfy [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfy [wM] and [N].
Then, $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ is nuclear for all $q \in [1, \infty]$.

Proof: By Theorem 1 it suffices to show $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],\infty}$ is nuclear.
Let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],\infty}$ be weakly summable, then one shows

$$\forall \mu \in \mathbb{R}_+ \ (\exists \mu \in \mathbb{R}_+) : \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} \frac{1}{M^\mu_\alpha} \sum_{n=0}^{\infty} |\varphi_n^{(\alpha)}(x)| w^\mu(x) < \infty$$

Then one finds for any $\lambda \in \mathbb{R}_+$ a $\mu \in \mathbb{R}_+$ (for any $\mu \in \mathbb{R}_+$ a $\lambda \in \mathbb{R}_+$) s.t.

$$\sum_{n=0}^{\infty} \|\varphi_n\|_{\mathcal{S}^{M^\lambda}_{w^\lambda,1}} = \sum_{n=0}^{\infty} \sup_{\alpha\in\mathbb{N}^d} \frac{1}{2^{|\alpha|} M^\mu_\alpha} \int_{\mathbb{R}^d} |\varphi_n^{(\alpha)}(x)| w^\lambda(x) dx = O\left(\left\|w^\lambda / w^\mu\right\|_{L^1(\mathbb{R}^d)}\right)$$

Then $(\varphi_n)_{n\in\mathbb{N}}$ is absolutely summable in $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],1} = \mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],\infty}$.

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Necessary conditions for nuclearity

Lemma (Petzsche, 1978)

Let A be a Köthe set. Let E be a lcHs s.t. E is nuclear (E'_b is nuclear). Suppose the following diagram of continuous functions

$$\begin{tikzcd} \lambda^1[A] \arrow[r, "T"] \arrow[dr, "\iota"'] & E \arrow[d, "S"] \\ & \lambda^\infty[A] \end{tikzcd}$$

commutes, then $\lambda^1[A]$ is nuclear.

To get necessary conditions for nuclearity: find continuous embeddings such that the diagram commutes in the case of $A = A_{\mathscr{W}}$ and $A = A_{\mathscr{W}_{\mathfrak{M}}}$.

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Characterization of nuclearity - General

The nuclearity of the Gelfand-Shilov spaces $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ may now be characterized as follows :

Theorem 3 (Debrouwere, N. and Vindas, 2020)

Let $\mathfrak{M}$ be an isotropically decomposable accelerating weight sequence system satisfying [L].
Let $\mathscr{W}$ be a weight function system satisfying [M].
Suppose that $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \neq \{0\}$ for some $q \in [1, \infty]$.

The following are equivalent:

- (i) $\mathfrak{M}$ satisfies $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfies [N]
- (ii) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ is nuclear for some $q \in [1, \infty]$
- (iii) $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q}$ is nuclear for all $q \in [1, \infty]$

Characterization of nuclearity - Fixed $\mathfrak{M}$

If we fix a weight sequence $\mathfrak{M}$, our result becomes:

Theorem 4 (Debrouwere, N. and Vindas, 2020)

Let $\mathfrak{M}$ be a weight sequence system satisfying [L] and $[\mathfrak{M}.2]'$.
Let $\mathscr{W}$ be a weight function system satisfying [M].
Suppose that $\mathcal{S}^{[\mathfrak{M}]}_{[\mathscr{W}],q} \neq \{0\}$ for some $q \in [1, \infty]$.

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Application - Projective description

Let

$$\overline{V}(\mathscr{W}) = \{w : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \text{ is upper semicontinuous such that } \sup_{x \in \mathbb{R}^d} w(x)/w^\lambda(x) < \infty, \forall \lambda \in \mathbb{R}_+\}$$

$$\overline{V}(\mathfrak{M}) = \{M = (M_\alpha)_{\alpha \in \mathbb{N}^d} \in \mathbb{R}_+^{\mathbb{N}^d} \text{ such that } \sup_{\alpha \in \mathbb{N}^d} M_\alpha^\lambda / M_\alpha < \infty, \forall \lambda \in \mathbb{R}_+\}$$

Theorem 5 (Debrouwere, N. and Vindas, 2020)

Let $\mathfrak{M}$ satisfy [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfy [wM] and [N].
Then,

$$\varphi \in \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} \iff \sup_{(\alpha,x) \in \mathbb{N}^d \times \mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)| w(x)}{M_\alpha} < \infty, \quad \forall w \in \overline{V}(\mathscr{W}), M \in \overline{V}(\mathfrak{M})$$

Moreover, the topology of $\mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$ is generated by the latter seminorms.

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Let $\mathfrak{M}$ satisfy [L] and $[\mathfrak{M}.2]'$ and $\mathscr{W}$ satisfy [wM] and [N].
Then,

$$\varphi \in \mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}} \iff \sup_{(\alpha,x) \in \mathbb{N}^d \times \mathbb{R}^d} \frac{|\varphi^{(\alpha)}(x)| w(x)}{M_\alpha} < \infty, \quad \forall w \in \overline{V}(\mathscr{W}), M \in \overline{V}(\mathfrak{M})$$

Moreover, the topology of $\mathcal{S}^{\{\mathfrak{M}\}}_{\{\mathscr{W}\}}$ is generated by the latter seminorms.

Application - Kernel theorems

Theorem 6 (Debrouwere, N. and Vindas, 2020)

Let $\mathfrak{M}_j$ be a weight sequence system on $\mathbb{N}^{d_j}$ satisfying [L] and $[\mathfrak{M}.2]'$. (j=1,2)
Let $\mathscr{W}_j$ be a weight function systems on $\mathbb{R}^{d_j}$ satisfying [wM] and [N]. (j=1,2)

Then,

$$\mathcal{S}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}) \widehat{\otimes} \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2}) \cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1})'_b, \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2}))$$

and

$$\mathcal{S}^{[\mathfrak{M}_1 \otimes \mathfrak{M}_2]}_{[\mathscr{W}_1 \otimes \mathscr{W}_2]}(\mathbb{R}^{d_1+d_2})'_b \cong \mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1})'_b \widehat{\otimes} \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})'_b \cong \mathcal{L}_b(\mathcal{S}^{[\mathfrak{M}_1]}_{[\mathscr{W}_1]}(\mathbb{R}^{d_1}), \mathcal{S}^{[\mathfrak{M}_2]}_{[\mathscr{W}_2]}(\mathbb{R}^{d_2})'_b)$$