Colombeau solutions to stochastic hyperbolic systems

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Stochastic Cauchy problem for linear hyperbolic systems:

$$\begin{aligned}(\partial_t + \lambda(x,t)\partial_x)\,u &= f(x,t)u + g(x,t), \quad (x,t) \in \mathbb{R}^2,\\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}.\end{aligned}$$

Here $u = (u_1, \ldots, u_n)$, $g = (g_1, \ldots, g_n)$; λ and f are $(n \times n)$-matrix valued functions. The matrix λ is real-valued and diagonal.

The coefficient functions λ, f, g, u_0 will be **random fields or stochastic processes** on some probability space (Ω, Σ, P).

Issue: the involved random fields or stochastic processes often have paths that are too irregular to admit classical or weak solutions.

Setting: Colombeau stochastic processes of the type $\mathcal{G}_{L^p}(\Omega, \mathbb{R}^d)$, $p = 0$ or $1 \leq p \leq \infty$.

Plan of Talk

- Classical deterministic theory
- Random classical solutions
- Colombeau random functions
- Existence and uniqueness of Colombeau random solutions
- Applications

Joint work with *Jelena Karakašević* and *Martin Schwarz.*

Based on work with *Snežana Gordić, Stevan Pilipović, Dora Seleši*. (Monatshefte für Mathematik 186, 2018, 609 - 633.)

Characteristic curves:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\gamma_j(x_0,t_0,t) &= \lambda_j\big(\gamma_j(x_0,t_0,t),t\big),\\ \gamma_j(x_0,t_0,t_0) &= x_0.\end{aligned}$$

For global classical existence, we need that $\lambda_j(x,t)$ is continuous in (x,t) and either globally Lipschitz in x or locally Lipschitz and globally bounded in x, uniformly for t in compact intervals.

Integral equations for $u(x,t)$: For $i = 1,\dots,n$,

$$u_i(x,t) = u_{0i}\big(\gamma_i(x,t,0)\big) + \int_0^t \Big(\sum_{j=1}^n f_{ij}\big(\gamma_i(x,t,\tau),\tau\big)u_j\big(\gamma_i(x,t,\tau),\tau\big) + g_i\big(\gamma_i(x,t,\tau),\tau\big)\Big)\,\mathrm{d}\tau$$

Fixed point arguments: For $\mathcal{C}^k$ data ($k \geq 0$), there is a global unique solution in $\mathcal{C}^k$.

Let (Ω, Σ, P) be a probability space and S a subset of $\mathbb{R}^n$. A *random field* A on S is a map

$$S \times \Omega \to \mathbb{R}, (x, \omega) \to A(x, \omega)$$

such that the map $\omega \to A(x, \omega)$ is measurable for every $x \in S$.

Assumption: The entries of λ, f, g are random fields on $\mathbb{R}^2$ with almost surely continuous paths. To apply the fixed point theorem in the random case, we need:

For any $K \times I \subset \mathbb{R}^2$, $L \subset \mathbb{R}^2$, there is a constant c, independent of $\omega \in \Omega$, and $\ell(\omega)$, $C(\omega)$ such that, for almost all ω,

$$\begin{aligned}
&|\lambda_i(x, t, \omega)| \le c \quad \text{for all } x \in \mathbb{R}, t \in I, \\
&|\lambda_i(x, t, \omega) - \lambda_i(y, t, \omega)| \le \ell(\omega)|x - y| \quad \text{for all } x, y \in K, t \in I, \\
&|f(x, t, \omega)| \le C(\omega) \quad \text{for all } (x, t) \in L.
\end{aligned}$$

Proposition. There is an almost surely unique process $u = u(x,t,\omega)$ on $\mathbb{R}^2$ with continuous paths which solves

$$\begin{aligned}(\partial_t + \lambda(x,t,\omega)\partial_x)\,u &= f(x,t,\omega)u + g(x,t,\omega), \quad (x,t) \in \mathbb{R}^2,\\ u(x,0,\omega) &= u_0(x,\omega), \quad x \in \mathbb{R}\end{aligned}$$

for almost all $\omega \in \Omega$. If the paths of u_0, λ, f, g are in $\mathcal{C}^k$, so is u.

Important estimate:

For $p \geq 1$, the p-th moments satisfy (if C is independent of ω)

$$\mathrm{E}\Big(\sup_{(x,t)\in K_T} |u(x,t)|^p\Big) \leq 2^p\,\mathrm{E}\Big(\sup_{x\in K_0} |u_0(x)|^p + T^p \sup_{(x,t)\in K_T} |g(x,t)|^p\Big)\exp(pTC)$$

where K_T is a trapezoidal domain of determinacy with baseline K_0.

Moderate/negligible families $(r_\varepsilon)_{\varepsilon\in(0,1]}$ *of real or complex numbers:*

$\mathcal{E}_M$: $\exists a \geq 0, |r_\varepsilon| = \mathcal{O}(\varepsilon^{-a})$; / $\mathcal{N}$: $\forall b \geq 0, |r_\varepsilon| = \mathcal{O}(\varepsilon^{b}) \quad (\varepsilon \to 0)$.

Families of random fields $(u_\varepsilon)_{\varepsilon\in(0,1]}$ *on* $S \times \Omega$, properties:

- For almost all $\omega \in \Omega$, all $0 < \varepsilon \leq 1$, $x \to u_\varepsilon(x,\omega)$ is in $\mathcal{C}^\infty(S)$.
- For all $x \in S$, $0 < \varepsilon \leq 1$, $\omega \to u_\varepsilon(x,\omega)$ is measurable.

L^0-version:

$\mathcal{E}_{M,L^0}(\Omega, S)$: for almost all $\omega \in \Omega$, $\big(\sup_{x\in K} |\partial^\alpha u_\varepsilon(x,\omega)|\big)_{\varepsilon\in(0,1]} \in \mathcal{E}_M$.

$\mathcal{N}_{L^0}(\Omega, S)$: for almost all $\omega \in \Omega$, $\big(\sup_{x\in K} |\partial^\alpha u_\varepsilon(x,\omega)|\big)_{\varepsilon\in(0,1]} \in \mathcal{N}$.

L^p-version $(1 \leq p \leq \infty)$:

$\mathcal{E}_{M,L^p}(\Omega, S)$: for all $K \Subset S$, $\alpha \in \mathbb{N}_0^n$, $\big(\sup_{x\in K} \|\partial^\alpha u_\varepsilon(x)\|_{L^p(\Omega)}\big)_{\varepsilon\in(0,1]} \in \mathcal{E}_M$.

$\mathcal{N}_{L^p}(\Omega, S)$: for all $K \Subset S$, $\alpha \in \mathbb{N}_0^n$, $\big(\sup_{x\in K} \|\partial^\alpha u_\varepsilon(x)\|_{L^p(\Omega)}\big)_{\varepsilon\in(0,1]} \in \mathcal{N}$.

Colombeau random functions:

$$\mathcal{G}_{L^p}(\Omega, S) = \mathcal{E}_{M,L^p}(\Omega, S)/\mathcal{N}_{L^p}(\Omega, S)$$

Decisive Lemma:

For all $K \Subset S$, $\alpha \in \mathbb{N}_0^n$, $\left(\sup_{x\in K} \|\partial^\alpha u_\varepsilon(x)\|_{L^p(\Omega)}\right)_{\varepsilon\in(0,1]} \in \mathcal{E}_M$,

if and only if

for all $K \Subset S$, $\alpha \in \mathbb{N}_0^n$, $\left(\left\|\sup_{x\in K} |\partial^\alpha u_\varepsilon(x)|\right\|_{L^p(\Omega)}\right)_{\varepsilon\in(0,1]} \in \mathcal{E}_M$.

The same holds for $\mathcal{N}$ in place of $\mathcal{E}_M$.

Proposition. Let $1 \le p' \le \infty$ and $u \in \mathcal{G}_{L^{p'}}(\Omega, S)$. Then $\mathrm{E}(u^p)$ is a well-defined element of $\mathcal{G}(S)$ for every integer $p \le p'$.

Corollary. Let $u \in \mathcal{G}_{L^2}(\Omega, S)$. Then the autocovariance function

$$(x, y) \to C(x, y) = \mathrm{E}\big(u(x)u(y)\big) - \mathrm{E}\big(u(x)\big)\mathrm{E}\big(u(y)\big)$$

is a well-defined element of $\mathcal{G}(S \times S)$.

Special types of Colombeau random functions $v \in \mathcal{G}_{L^p}(\Omega, \mathbb{R}^2)$:

Globally bounded on strips: For all $I \Subset \mathbb{R}$ there is $c > 0$ such that $\sup_{x\in\mathbb{R}, t\in I} |v_\varepsilon(x,t,\omega)| \leq c$, for almost all $\omega \in \Omega$, all $\varepsilon \in (0,1]$.

Locally of logarithmic type: For all $K \Subset \mathbb{R}^2$ there is $c > 0$ such that $\sup_{(x,t)\in K} |v_\varepsilon(x,t,\omega)| \leq c|\log\varepsilon|$, for almost all $\omega \in \Omega$, all $\varepsilon \in (0,1]$.

The Cauchy problem for Colombeau random functions:

$$\begin{aligned} (\partial_t + \lambda(x,t,\omega)\partial_x)\, u &= f(x,t,\omega)u + g(x,t,\omega), \quad (x,t) \in \mathbb{R}^2, \\ u(x,0,\omega) &= u_0(x,\omega), \quad x \in \mathbb{R}. \end{aligned}$$

Theorem, $\mathbf{p = 0}$. Assume that λ, f and g belong to $\mathcal{G}_{L^0}(\Omega, \mathbb{R}^2)$, λ is globally bounded on strips and $\partial_x\lambda$ and f are locally of logarithmic type. Let $u_0 \in \mathcal{G}_{L^0}(\Omega, \mathbb{R})$. Then the Cauchy problem has a unique global solution $u \in \mathcal{G}_{L^0}(\Omega, \mathbb{R}^2)$.

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Theorem, p $\geq$ 1. Let $1 \leq p \leq \infty$. Assume that λ and f belong to $\mathcal{G}_{L^\infty}(\Omega, \mathbb{R}^2)$, λ is globally bounded on strips, $\partial_x \lambda$ and f are locally of logarithmic type, $g \in \mathcal{G}_{L^p}(\Omega, \mathbb{R}^2)$. Let $u_0 \in \mathcal{G}_{L^p}(\Omega, \mathbb{R})$. Then the Cauchy problem has a unique global solution $u \in \mathcal{G}_{L^p}(\Omega, \mathbb{R}^2)$.

Selected Applications (1)

Imbedding of generalized stochastic processes on $S \subset \mathbb{R}^n$:

A *generalized stochastic process* is a weakly measurable map $H : \Omega \to \mathcal{D}'(S)$. These processes are imbedded in $\mathcal{G}_{L^0}(\Omega, S)$ through the representatives $((H * \varphi_\varepsilon)|_S)_{\varepsilon\in(0,1]}$, where $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ is a family of mollifiers.

If in addition the map $\psi \to \langle H, \psi\rangle$ is linear and continuous from $\mathcal{D}(S)$ to $L^p(\Omega)$, then the imbedding defines an element of $\mathcal{G}_{L^p}(\Omega, S)$.

Possibly, logarithmic scales or additional cut-offs are needed to obtain processes as required in the existence results.

Examples: $H = \dot{W}(x)$, $H = \dot{W}(t)$ ... spatial or temporal white Gaussian white noise or Poisson noise, $H = \dot{\Pi}(x)$ or $H = \dot{\Pi}(t)$,

$H = \dot{W}(x,t)$ ... Gaussian space-time white noise,

$H = L(x)$ ... a Lévy process, e.g., Wiener or Poisson process $W(x)$, $\Pi(x)$,

$H(x) = \int_0^x \sigma \mathrm{e}^{-\alpha(x-y)}\,\mathrm{d}W(y)$ ... an Ornstein-Uhlenbeck process.

The 1D-wave equation with random field coefficient:

$$\partial_t^2 u - \lambda^2(x)\partial_x^2 u = 0.$$

Taking $\lambda(x)$ as a (cut-off) Ornstein-Uhlenbeck process is already beyond classical theory.

Note that $v = (\partial_t - \lambda\partial_x)u$, $w = (\partial_t + \lambda\partial_x)u$ produces the equivalent first order hyperbolic system

$$\begin{aligned}(\partial_t + \lambda\partial_x)v &= \tfrac{1}{2}(\partial_x\lambda)(v-w),\\ (\partial_t - \lambda\partial_x)w &= \tfrac{1}{2}(\partial_x\lambda)(v-w),\\ \partial_t u &= \tfrac{1}{2}(v+w).\end{aligned}$$

The 1D-wave equation with stochastic potential:

$$\partial_t^2 u - \partial_x^2 u = H(x,t)u.$$

Associated classical stochastic processes exist when $H(x,t) = \delta(x)\dot{W}(t)$ or $H(x,t) = \dot{\Pi}(x)$ (truncated).

Random transport equations:

$$\partial_t u + \lambda(x,t)\partial_x u = 0, \quad u(x,0) = u_0(x).$$

Interesting cases: $\lambda = \lambda(x)$ a Lévy process;

$\lambda = \lambda(x)$ limiting propagation speed of a randomly layered medium, such as the Goupillaud medium;

$\lambda = \lambda(x) = 1 + \dot{\Pi}(x)$ (truncated). In this case, the Colombeau random solution $u(x,t)$ has the deterministic solution $u_0(x-t)$ as associated distribution.

$\lambda = \dot{W}(t)$, temporal white noise. With $W_\varepsilon(t) = W * \varphi_\varepsilon(t)$, a representative is explicitly given by $u_\varepsilon(x,t) = u_0(x - W_\varepsilon(t))$. Here the Colombeau random solution $u(x,t)$ has the classical random field $u_0(x - W(t))$ as associated distribution.

Systems of linear stochastic differential equations:

$$X'(t) = a(t)X(t) + \big(b(t) + c(t)X(t)\big)H(t), \quad X(0) = X_0$$

with a, b, c smooth.

Case from the literature:

$$X'(t) = a(t)X(t) + b(t)W^+(t),$$

where $W^+(t)$ is positive noise, $W_\varepsilon^+(t) = \exp\big(\dot{W}_\varepsilon(t) - \frac{1}{2}\|\varphi_\varepsilon\|_{L^2(\mathbb{R})}\big)$.

The classical case $H(t) = \dot{W}(t)$:

The Colombeau random solution $X(t)$ has the classical process $Y(t)$ as associated distribution, the Stratonovich solution to the system of stochastic differential equations

$$\mathrm{d}Y(t) = a(t)Y(t)\,\mathrm{d}t + \big(b(t) + c(t)Y(t)\big)\,\mathrm{d}W(t).$$