Treating strong singularities in differential equations: very weak solution concept

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Very weak solutions

concept for treating strong singularities, such as Delta distribution, in equations:

PDE: hyperbolic systems, Heat, Schrödinger, Wave type equations: Acoustic wave, Landau Hamiltonian, Water wave equations that describing tsunamis

… with irregular coefficients in time and space

… and on groups and manifolds

The very weak solution concept

- It is simplified version of the Colombeau generalized function solution concept appropriate for the application.
- The fundamental idea:
  - model irregular objects in the (system of) equations by approximating nets of smooth functions with moderate asymptotics
  - treat regularised net of problems in a usual way and obtain net of solutions– "sequential solution"
  - if sequential solution is **moderate**, will be called *very weak solution*
  - for the uniqueness of *very weak solution* – use negligible nets

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  - for the uniqueness of *very weak solution* – use negligible nets
- Notions of moderate and negligible nets could be defined based on a locally convex topological vector space: for a locally convex topological vector space E with topology given by the family of seminorms $\{p_j\}_{j\in J}$, E-moderate nets are

$$\mathcal{M}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} : \forall j \in J, \exists N \in \mathbb{N}, p_j(u_\varepsilon) = O(\varepsilon^{-N})\}$$

and E-negligible

$$\mathcal{N}_E := \{(u_\varepsilon)_\varepsilon \in E^{(0,1]} : \forall j \in J, \forall q \in \mathbb{N}, p_j(u_\varepsilon) = O(\varepsilon^{q})\}$$

First treatment:

Garetto C., Ruzhansky M.,
Hyperbolic second order equations with non-regular time dependent coefficients.
Arch. Ration. Mech. Anal., 217 (2015), 113-154.

Works that show usefulness of concept:

Ruzhansky M., Tokmagambetov N.. **Very weak solutions of wave equation for Landau Hamiltonian with irregular electromagnetic field.** Lett. Math. Phys., 107:591-618, 2017.

Ruzhansky M. and Tokmagambetov N. **Wave equation for operators with discrete spectrum and irregular propagation speed.** Arch. Ration. Mech. Anal., 226: 1161–1207, 2017.

Ruzhansky M., Tokmagambetov N. **On a very weak solution of the wave equation for a Hamiltonian in a singular electromagnetic field.** Math. Notes, 103, 856-858, 2018.

Munoz C., Ruzhansky M., and Tokmagambetov N. **Wave propagation with irregular dissipation and applications to acoustic problems and shallow waters**, J. Math. Pures Appl., 9(123), 127–147, 2019.

More recent works:

C. Garetto.
On the wave equation with multiplicities and space-dependent irregular coefficients.
Preprint, arXiv:2004.09657 (2020).

Mathematical treatment: well- posedness in very weak sense. Singularities in coefficients depending on space variable for wave equation.

Nets are in C^\infty classes in time and space.

Unique vws equivalent to sol.in Colombeau sense.

M.E. Sebih, J.Wirth. **On a wave equation with singular dissipation.** Preprint, Arxiv:2002.00825 (2020).

A Altybay, M Ruzhansky, N Tokmagambetov. **A parallel hybrid implementation of the 2D acoustic wave equation,** International Journal of Nonlinear Sciences and Numerical Simulation, Int. J. Nonlinear Sci. Numer. Simul. 2020. to appear

Ruzhansky, Michael, Yessirkegenov, Nurgissa **Very weak solutions to hypoelliptic wave equations**. J. Differential Equations 268 (2020), no. 5, 2063–2088

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Definitions of moderate families, Theorems on existence and uniqueness,
consistency with the classical settings.

+ Numerical examples and analysis.
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- Altybay A., Ruzhansky M., Sebih M., Tokmagambetov N., **Tsunami propagation for singular topographies**. Arxiv : 2005.11931 (2020).
- Altybay A., Ruzhansky M., Sebih M., Tokmagambetov N., **The heat equation with singular potentials.** Arxiv : 2004.11255 (2020).
- Altybay A., Ruzhansky M., Sebih M., Tokmagambetov N., **Fractional Schrödinger equations with potentials of higher-order singularities.** Arxiv : 2004.10182 (2020).
- Altybay A., Ruzhansky M., Sebih M., Tokmagambetov N., **Fractional Klein-Gordon equation with strongly singular mass term.** Arxiv : 2004.10145 (2020).

Definition (Moderate nets)

Let $(X, \|\cdot\|_X)$ be a Banach space. A net of elements $(K_\varepsilon)_{\varepsilon\in(0,1]} \subset X$ is $X-$moderate if there exist $N \in \mathbb{N}_0$ and $C > 0$ such that for every $\varepsilon \in (0,1]$

$$\|K_\varepsilon\|_X \leq C\varepsilon^{-N}.$$

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Example (Heat equation with singular potential)

$$\left(\frac{\partial}{\partial t} - \triangle\right) u(t,x) + q(x)u(t,x) = f(t,x)$$
$$u(0,x) = u^0(x) \tag{HE}$$

- $q \in \mathrm{L}^\infty \implies u \in Y := C^1([0,T], L^2(\mathbb{R}^d)) \cap C([0,T], H^1(\mathbb{R}^d))$
- $q \in \mathcal{D}'(\mathbb{R}^d)$: net of functions $(u_\varepsilon)_\varepsilon \subset Y$ is **a very weak solution** to (HE) if there exist an $\mathrm{L}^\infty-$moderate regularisation $(q_\varepsilon)_\varepsilon$ of q such that for every $\varepsilon \in (0,1]$, u_ε solves (HE)$_\varepsilon$ ((HE) with q_ε replacing q) and $(u_\varepsilon)_\varepsilon$ is Y- moderate.

The 3D fractional Zener wave equation

- models wave propagation in viscoelastic media that occupies bounded open Lipschitz domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$. It is of the form:

$$\tau \varrho(x) L_t^{\alpha} u = Q_x u + G. \tag{FZWE}$$

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- for L^∞ density ϱ a unique weak solution is proved to exist

Lj. Oparnica and E. Süli, Well-posedness of the fractional Zener model for heterogenous viscoelastic materials, *Fractional Calculus and Applied Analysis*, 23(1), 126-166, 2020.

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- want to consider

$$\varrho(x) = \varrho_1(x) + \delta(x).$$

3D FZWE is derived from a system:

- the equation of motion

$$\varrho \ddot{u} = \operatorname{Div} \sigma + F,$$

where $\varrho = \varrho(x)$ denotes density of the media under consideration, $u = u(t,x)$ is the displacement, $\sigma = \sigma(t,x)$ is the stress tensor, and $F = F(t,x)$ is a specified load vector, $x \in \Omega$, $t \in (0,T]$

- the constitutive equation, fractional Zener model

$$(1 + \tau D_t^\alpha)\sigma = (1 + D_t^\alpha)[2\mu\varepsilon(u) + \lambda \operatorname{tr}(\varepsilon(u))I], \quad \tau \in (0,1], \alpha \in (0,1),$$

giving relation between the stress tensor σ and the strain tensor

$$\varepsilon(u) := \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}}),$$

where $\mu = \mu(x)$ and $\lambda = \lambda(x)$ are Lamé coefficients, and D_t^α is the fractional derivative of order $\alpha \in (0,1)$ in the sense of Caputo.

After Laplace transforming equations, elimination of the Laplace transform of σ and then inverting the Laplace transform one obtains so-called fractional Zener wave equation

$$\tau\varrho\ddot{u} + (1-\tau)\,\frac{\partial}{\partial t}(-\dot{e}_{\alpha,1} *_t \varrho\dot{u}) = \mathrm{Div}(2\mu\varepsilon(u) + \lambda\,\mathrm{tr}(\varepsilon(u))I) \,+\, G,$$

- $G := (\tau-1)\,\dot{e}_{\alpha,1}\,\varrho v_0 \,+\, e_{\alpha,1}\ \mathrm{Div}(\tau\sigma_0 - 2\mu\varepsilon(u_0) - \lambda\,\mathrm{tr}(\varepsilon(u_0))I) + \tau F + (\tau-1)\dot{e}_{\alpha,1} *_t F$
- $e_{\alpha,1}$, with $\alpha\in(0,1)$, is one parameter Mittag-Leffler function which is a completely monotonous function that satisfies

  $e_{\alpha,1}\geq 0$, $-\dot{e}_{\alpha,1}\geq 0$ and $\ddot{e}_{\alpha,1}\geq 0$ on $(0,T]$, with $\dot{e}_{\alpha,1}\in \mathrm{L}^1((0,T))$ and $\ddot{e}_{\alpha,1}\in \mathrm{L}^1_{\mathrm{loc}}((0,T))$ for all $T>0$.

Setting

$$L_t^\alpha u := \ddot{u} + (1-\tau)\,\frac{\partial}{\partial t}(-\dot{e}_{\alpha,1} * \dot{u})(t),$$

and

$$Q_x u := \mathrm{Div}(2\mu\varepsilon(u) + \lambda\,\mathrm{tr}(\varepsilon(u))I),$$

we get (FZWE).

Theorem: weak solution to 3D FZWE

Let $\tau \in (0,1]$, $\alpha \in (0,1)$, $u_0 \in [\mathrm{H}_0^1(\Omega)]^3$, $v_0 \in [\mathrm{L}^2(\Omega)]^3$, $\sigma_0 \in [\mathrm{L}^2(\Omega)]^{3\times 3}$, $F \in \mathrm{L}^2(0,T;[\mathrm{L}^2(\Omega)]^3)$, and coefficients ϱ, μ and λ be elements of $\mathrm{L}^\infty(\Omega)$ and ϱ, μ being bounded below away from zero.

Then, there exists $u \in \mathrm{C}_w([0,T];[\mathrm{H}_0^1(\Omega)]^3)$ satisfying

$$\tau \int_0^T (\varrho u(s,\cdot), \ddot{v}(s,\cdot))\,\mathrm{d}s - (1-\tau)\int_0^T ((-\dot{e}_{\alpha,1} *_s \varrho \dot{u})(s,\cdot), \dot{v}(s,\cdot))\,\mathrm{d}s$$

$$+ \int_0^T \big(2\mu\varepsilon(u(s,\cdot)) + \lambda\,\mathrm{tr}(\varepsilon(u(s,\cdot)))I\,, \varepsilon(v(s,\cdot))\big)\,\mathrm{d}s$$

$$= -\tau(\varrho u_0, \dot{v}(0,\cdot)) + \tau(\varrho v_0, v(0,\cdot)) + \int_0^T \langle G(s,\cdot), v(s,\cdot)\rangle\,\mathrm{d}s,$$

for all $v \in \mathrm{W}^{2,1}(0,T;[\mathrm{L}^2(\Omega)]^3) \cap \mathrm{L}^1(0,T;[\mathrm{H}_0^1(\Omega)]^3)$ with $v(T,\cdot) = 0$ and $\dot{v}(T,\cdot) = 0$.

Furthermore, u satisfies the energy inequality

$$\frac{\tau}{2}\|\dot u(t')\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{1}{2}\|\varepsilon(u(t'))\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\varepsilon(u(t')))\|^2_{\mathrm{L}^2_\lambda(\Omega)} + \frac{1-\tau}{2}\int_0^{t'} -\dot e_{\alpha,1}(s)\|\dot u(s)\|^2_{\mathrm{L}^2_\varrho(\Omega)}\,\mathrm{d}s \le 3A(t)\exp(t+1-e_{\alpha,1}(t)),$$

for all $t \in (0,T]$ and a.e. $t' \in (0,t]$, where $A(t)$ is defined for $t \in [0,T]$ by

$$A(t) := \frac{\tau^2+(1-\tau)^2}{2\tau}\|v_0\|^2_{\mathrm{L}^2_\varrho(\Omega)} + \frac{3}{2}\|\varepsilon(u_0)\|^2_{\mathrm{L}^2_\mu(\Omega)} + \frac{1}{2}\|\operatorname{tr}(\varepsilon(u_0))\|^2_{\mathrm{L}^2_\lambda(\Omega)} + \frac{3}{2}\|\kappa_0\|^2_{\mathrm{L}^2_{1/\mu}(\Omega)} + \frac{\tau^2+(1-\tau)^2}{\tau}\int_0^t \|F(s)\|^2_{\mathrm{L}^2_{1/\varrho}(\Omega)}\,\mathrm{d}s,$$

with $\kappa_0 = \tau\sigma_0 - 2\mu\varepsilon(u_0) - \lambda\operatorname{tr}(\varepsilon(u_0))I$. This implies

$$\|u(t)\|^2_{\mathrm{L}^2_\varrho(\Omega)} \le c\,A\left(t, \|u_0\|^2_{\mathrm{L}^2_\varrho(\Omega)}, \|v_0\|^2_{\mathrm{L}^2_\varrho(\Omega)}, \|F\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2_{1/\varrho}(\Omega))}\right)\exp(t)$$

The initial boundary-value problem (P)

$$\tau \varrho(x) L_t^\alpha u(t,x) = Q_x u(t,x) + G(t,x), \quad x \in \Omega,\ t \in (0,T], \tag{1}$$

L_t^α is convolution, integro-differential operator in t,
Q_x is an elliptic partial differential operator in x
G is a function depending on given initial data, and density ϱ is of the form

$$\varrho(x) = \varrho_1(x) + \delta(x), \tag{RHO}$$

ϱ_1 being function bounded below away from zero in $\mathrm{L}^\infty(\Omega)$. Equation (1) is subject to the initial conditions

$$u(0,x) = u_0(x), \qquad \partial_t u(0,x) = v_0(x), \quad x \in \Omega, \tag{2}$$

and a boundary condition

$$u(t,x) = 0 \qquad \text{for all } (t,x) \in (0,T] \times \partial\Omega. \tag{3}$$

Definition (Moderate nets)

A net of functions $(u_\epsilon)_{\epsilon\in(0,1]} \subset \mathrm{C}_w\left([0,T];[\mathrm{H}_0^1(\Omega)]^3\right)$ is **moderate** if there exist $N \in \mathbb{N}_0$, and $c > 0$ such that for all $\varepsilon \in (0,1]$ it holds

$$\|u_\varepsilon(t,\cdot)\|_{\mathrm{L}^2(\Omega)} \leq c\varepsilon^{-N}, \quad t \in [0,T].$$

Definition

The net $(u_\varepsilon)_{\varepsilon\in(0,1]} \subset \mathrm{C}_w\left([0,T];[\mathrm{H}_0^1(\Omega)]^3\right)$ is **a very weak solution** if

- there exists regularization of ϱ: net $(\varrho_\varepsilon)_{\varepsilon\in(0,1]}$ which is C^∞- moderate
- for all $\varepsilon \in (0,1]$, u_ε is a weak solution to $(P)_\varepsilon$

$$\begin{aligned}
&\tau\varrho_\varepsilon(x)L_t^\alpha u(t,x) = Q_x u(t,x) + G(t,x), \quad x \in \Omega,\, t \in (0,T],\\
&u(0,x) = u_0(x), \qquad \partial_t u(0,x) = v_0(x), \quad x \in \Omega\\
&u(t,x) = 0 \qquad \text{for all } (t,x) \in (0,T] \times \partial\Omega.
\end{aligned}$$

- $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is moderate.

Very weak solution for (P): Existence and consistence

Theorem (Existence)

Let ϱ *be given by* $\varrho(x) = \varrho_1(x) + \delta(x)$, *let* ρ_1, μ, λ *are elements of* $\mathrm{L}^\infty(\Omega)$ *and* ρ_1, μ *be bounded below away from zero, and let initial data and load vector satisfy* $u_0 \in [\mathrm{H}_0^1(\Omega)]^3$, $v_0 \in [\mathrm{L}^2(\Omega)]^3$, *and* $F \in \mathrm{L}^2\left(0, T; [\mathrm{L}^2(\Omega)]^3\right)$.

Then, initial-boundary value problem (P) has a very weak solution.

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Let $(u_\varepsilon)_{\varepsilon \in (0,1]}$ *be very weak solution and let* u *be weak solution. Then, as* $\varepsilon \to 0$ *net* $(u_\varepsilon)_{\varepsilon \in (0,1]}$ *converges to* u *in the space* $L^2\left([0, T]; \left[\mathrm{L}^2(\Omega)\right]^3\right)$.

Very weak solution for (P): Uniqueness

Let $\varrho(x) = \varrho_1(x) + \delta(x)$. Let ϱ_ε and $\tilde{\varrho}_\varepsilon$ are two regularisations of ϱ such that for all $q \in \mathbb{N}$ there exists c so that

$$\|\varrho_\varepsilon - \tilde{\varrho}_\varepsilon\|_{\mathrm{L}^\infty} \leq c\varepsilon^q.$$

Then for the two corresponding very weak solutions to problem (P) $(u_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ it holds that for all $N \in \mathbb{N}$ there exists C

$$\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^2} \leq C\varepsilon^N.$$

We say that (P) has a unique very weak solution.

For further work:

“The approach of very weak solutions opens up a whole new research area where one can deal with problems with singularities in a way that is consistent with stronger notions of solutions should they exist.“

More examples.

More numerical analysis but also questions of convergence…

Do we always have consistency?

Analysis of nets: regularity theory and microlocal analysis.

Microlocal and harmonic analysis allowing different types of singularities

Pseudo-differential operators with irregular coefficients.

Spectral problems for singular operators or in singular domains

“this is a very promising far-reaching research with further mathematical developments and many expected applications in other sciences “

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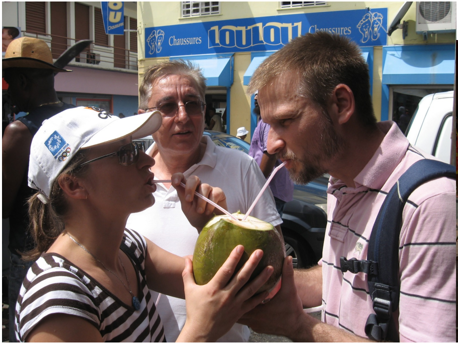
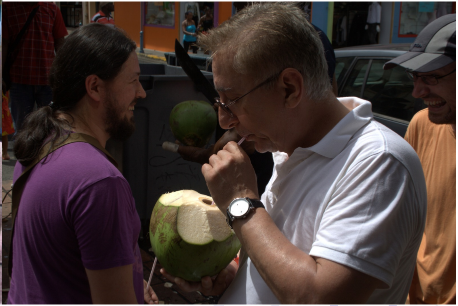
- Few photos...





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