Global Well-posedness of a Class of Strictly Hyperbolic Cauchy Problems with Coefficients Non-Absolutely Continuous in Time

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Dedicated to

My Spiritual Master Bhagavan Sri Sathya Sai Baba

Overview

Presentation Outline

Presentation Outline

Strict Hyperbolicity

Consider a Cauchy problem

$$\begin{cases} Lu(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ D_t^{k-1} u(0,x) = f_k(x), & k = 1, \dots, m \end{cases}$$

where the operator $L(t, x, \partial_t, D_x)$ is given by

$$L = D_t^m - \sum_{j=0}^{m-1} \sum_{|\alpha|+j \le m} a_{j,\alpha}(t,x) D_x^\alpha D_t^j.$$

Here $D_x^\alpha = (-i)^\alpha \partial_x^\alpha$. L is hyperbolic if

$$\tau^m - \sum_{j=0}^{m-1} \sum_{|\alpha|+j = m} a_{j,\alpha}(t,x) \xi^\alpha \tau^j = 0$$

has real roots $\tau_k(t,x,\xi)$. In addition, if they are distinct, then we have strict hyperbolicity.

Presentation Outline

Well-posedness Results

Authors	Regularity in $t \in [0, T]$	Regularity in $x \in \mathbb{R}^n$	Well-posed in	Loss of Derivatives
Colombini et al.[8]	C_{LL} C^{χ}	- -	H^s $H^{s,\varepsilon,\sigma}$	finite loss ∞ loss
Nishitani [13]	C^{χ}	G^{σ}	$H^{s,\varepsilon,\sigma}$	∞ loss
Cicognani & Lorenz [6]	C_L to C^{χ}	B^{∞}	$H^s_{\eta,\delta}(\mathbb{R}^n)$	arb. small ∞ loss

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Colombini et al.[7]	$C^1((0,T])$ $\frac{1}{t^q}$ $q = 1$ $q > 1$	- -	H^s $H^{s,\varepsilon,\sigma}$, $1 < \sigma < \frac{q}{q-1}$	finite loss ∞ loss
Cicognani[5]	$C^1((0,T])$ $\frac{1}{t^q}$ $q = 1$ $q > 1$	B^{∞} G^{σ}	H^s $H^{s,\varepsilon,\sigma}$, $1 < \sigma < \frac{q}{q-1}$	finite loss ∞ loss

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Kubo & Reissig[10]	Oscillating $\left(\frac{1}{t}\left(\log\frac{1}{t}\right)^{\gamma}\right)^{k}$	B^{∞}	H^s	arb. small loss finite loss

Global Well-posedness Results

Authors	Low-regularity in t	Well-posed in	Loss of Derivatives	Loss of Decay
Ascanelli & Cappiello[1]	C_{LL}	H^{s_1,s_2}	t-dependent finite loss	t-dependent poly. growth
Ascanelli & Cappiello[2]	C^χ	$\mathcal{S}_\sigma$, $\mathcal{S}'_\sigma$	Infinite loss	exponential growth

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Cappiello, Gourdin & Gramchev [15]	C_L to C_{LL}	H^{s_1,s_2}	arb. small finite loss	arb. small finite loss
Nori, Coriasco & Battisti [15]	Oscillating $\left(\frac{1}{t}\left(\log\frac{1}{t}\right)^{\gamma}\right)^{k}$	H^{s_1,s_2}	finite loss	finite loss

Presentation Outline

Our Model of Strictly Hyperbolic Operator

We deal with the Cauchy problem

$$\begin{cases} P(t,x,\partial_t,D_x)u(t,x) = f(t,x), & (t,x) \in [0,T] \times \mathbb{R}^n, \\ D_t^{k-1}u(0,x) = f_k(x), & k = 1,\dots,m \end{cases} \tag{1}$$

where the strictly hyperbolic operator $P(t,x,\partial_t,D_x)$ is given by

$$P = D_t^m - \sum_{j=0}^{m-1} \Big(A_{m-j}(t,x,D_x) + B_{m-j}(t,x,D_x) \Big) D_t^j \quad \text{with}$$

$$A_{m-j}(t,x,D_x) = \sum_{|\alpha|+j=m} a_{j,\alpha}(t,x) D_x^\alpha \text{ and}$$

$$B_{m-j}(t,x,D_x) = \sum_{|\alpha|+j<m} b_{j,\alpha}(t,x) D_x^\alpha.$$

These characteristic roots $\tau(t,x,\xi)$ are such that

$$\tau_1(t,x,\xi) < \tau_2(t,x,\xi) < \cdots < \tau_m(t,x,\xi), \quad \text{and}$$
$$C\langle x\rangle\langle\xi\rangle \leq |\tau_k(t,x,\xi)|.$$

Regularity of Coefficients

1. $a_{j,\alpha} \in C([0,T]; C^\infty(\mathbb{R}^n)) \cap C^1((0,T]; C^\infty(\mathbb{R}^n))$ satisfying

$$|D_x^\beta a_{j,\alpha}(t,x)| \le C^{|\beta|}\beta!^\sigma \langle x\rangle^{m-j-|\beta|}, \quad (t,x) \in [0,T]\times\mathbb{R}^n,$$
$$|D_x^\beta \partial_t a_{j,\alpha}(t,x)| \le C^{|\beta|}\beta!^\sigma \langle x\rangle^{m-j-|\beta|}\frac{1}{t^q}, \quad (t,x) \in (0,T]\times\mathbb{R}^n,$$

for $2 < \sigma < q/(q-1)$. Note that $2 < \sigma < q/(q-1) \implies q \in [1,2)$.

[1] Nicola, F., Rodino, L. (2011) Global Pseudo-differential Calculus on Euclidean Spaces, Birkhäuser Basel.

[2] Ascanelli, A., Cappiello, M. (2008) Hölder continuity in time for SG hyperbolic systems, J. Differential Equations 244 2091-2121.

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$$|D_x^\beta \partial_t a_{j,\alpha}(t,x)| \leq C^{|\beta|}\beta!^\sigma \langle x\rangle^{m-j-|\beta|}\frac{1}{t^q}, \quad (t,x) \in (0, T] \times \mathbb{R}^n,$$

for $2 < \sigma < q/(q-1)$. Note that $2 < \sigma < q/(q-1) \implies q \in [1,2)$.

2. $b_{j,\alpha} \in C^1([0, T]; C^\infty(\mathbb{R}^n))$ satisfy

$$|D_x^\beta b_{j,\alpha}(t,x)| \leq C^{|\beta|}\beta!^\sigma \langle x\rangle^{m-j-1-|\beta|}, \quad (t,x) \in [0, T]\times\mathbb{R}^n, \quad C_\beta > 0.$$

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Key Steps in the Result

Key Steps

1. Factorization of the Operator

[3] Cicognani, M., Lorenz, D. (2017) Strictly hyperbolic equations with coefficients low-regular in time and smooth in space, J. Pseudo-Differ. Oper. Appl.

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1. Factorization of the Operator
2. First order system[3]

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1. Factorization of the Operator
2. First order system[3]
3. Conjugation by an infinite order ψDO

[3] Cicognani, M., Lorenz, D. (2017) Strictly hyperbolic equations with coefficients low-regular in time and smooth in space, J. Pseudo-Differ. Oper. Appl.

Presentation Outline

Governing Metrics

The SG-metric

$$g_{x,\xi} = \frac{|dx|^2}{\langle x \rangle^2} + \frac{|d\xi|^2}{\langle \xi \rangle^2}.$$

governs the growth of $A_{m-j}(t,x,\xi)$ and $B_{m-j}(t,x,\xi)$, i.e.,

$$|\partial_\xi^\alpha D_x^\beta A_{m-j}(t,x,\xi)| \leq C^{|\beta|+|\alpha|}\alpha!\beta!^\sigma \langle x \rangle^{m-j-|\beta|} \langle \xi \rangle^{m-j-|\alpha|}.$$

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Conjugation

Since we are working in the Gevrey setting, to microlocally compensate the infinite loss of regularity (both derivatives and decay) we need to conjugate by an infinite order ψDO. After such a conjugation the metric governing the lower order terms is given by

$$\tilde{g}_{x,\xi} = \left(\frac{\langle \xi \rangle^{\frac{1}{\sigma}}}{\langle x \rangle^\gamma} \right)^2 |dx|^2 + \left(\frac{\langle x \rangle^{\frac{1}{\sigma}}}{\langle \xi \rangle^\gamma} \right)^2 |d\xi|^2, \quad \gamma = 1 - \frac{1}{\sigma}.$$

Presentation Outline

Metric on the Phase Space

Denote by $\omega(X, Y)$ the standard symplectic form on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$: if $X = (x, \xi)$ and $Y = (y, \eta)$, then

$$\omega(X, Y) = \xi \cdot y - \eta \cdot x.$$

We can identify σ with the isomorphism of $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ such that $\omega^* = -\omega$, with the formula $\omega(X, Y) = \langle \omega X, Y \rangle$. To a Riemannian metric g_X on $\mathbb{R}^{2n}$ (which is a measurable function of X), we associate the dual metric g_X^ω by

$$\forall T \in \mathbb{R}^{2n}, \quad g_X^\omega(T) = \sup_{0 \neq T' \in \mathbb{R}^{2n}} \frac{\langle \omega T, T' \rangle^2}{g_X(T')}.$$

Considering g_X as a matrix associated to positive definite quadratic form on $\mathbb{R}^{2n}$, $g_X^\omega = \omega^* g_X^{-1} \omega$.

[0] Lerner, N. 2010 Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators, Birkhäuser Basel.

The Planck function [11] is defined to be

$$h_g(x,\xi) := \inf_{0\neq T\in\mathbb{R}^{2n}} \left(\frac{g_X(T)}{g_X^\omega(T)}\right)^{1/2}.$$

The uncertainty principle is quantified as the upper bound $h_g(x,\xi) \leq 1$. We often make use of the strong uncertainty principle, that is, for some $\kappa > 0$, we have

$$h_g(x,\xi) \leq (1+|x|+|\xi|)^{-\kappa}, \quad (x,\xi) \in \mathbb{R}^{2n}.$$

1. For SG metric g, $h(x,\xi) = h_g(x,\xi) = (\langle x\rangle\langle\xi\rangle)^{-1}$.
2. For the metric $\tilde{g}$, $\tilde{h}(x,\xi) = h_{\tilde{g}}(x,\xi) = (\langle x\rangle\langle\xi\rangle)^{\frac{1}{\sigma}-\gamma}$. The metric $\tilde{g}$ satisfies the strong uncertainty principle when $\frac{1}{\sigma} - \gamma < 0$ or $2 < \sigma$.

Presentation Outline

We use an infinite order ψDO of the form

$$e^{\varepsilon h(x,D)^{-1/\sigma}},\ \varepsilon > 0,$$

where $h(x, D)^{-1/\sigma} = \langle x\rangle^{1/\sigma}\langle D_x\rangle^{1/\sigma}$ is an ψDO with symbol $h(x, \xi)^{-1/\sigma} = (\langle x\rangle\langle \xi\rangle)^{1/\sigma}$.

Infinite Order ψDO

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Sobolev Spaces Associated with Infinite Order ψDO

The Sobolev space $H^{s,\varepsilon,\sigma}(\mathbb{R}^n)$ for $\sigma > 2$, $\varepsilon > 0$ and $s = (s_1, s_2) \in \mathbb{R}^2$ is defined as

$$H^{s,\varepsilon,\sigma}(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : \langle x\rangle^{s_2}\langle D\rangle^{s_1} \exp\{\varepsilon\langle x\rangle^{1/\sigma}\langle D\rangle^{1/\sigma}\}v \in L^2(\mathbb{R}^n)\},$$

equipped with the norm $\|v\|_{s,\varepsilon,\sigma} = \|\langle\cdot\rangle^{s_2}\langle D\rangle^{s_1} \exp\left(\varepsilon\langle x\rangle^{1/\sigma}\langle D\rangle^{1/\sigma}\right)v\|_{L^2}$.

Presentation Outline

Subdivision of the Phase Space

Define $t_{x,\xi}$, for a fixed (x, ξ), as the solution to the equation

$$t^q = N\, h(x, \xi),$$

where N is the positive constant and q is the given order of singularity. Since $2 < \sigma < q/(q-1)$, we consider $\delta \in (0, 1)$ such that

$$\frac{1}{\sigma} = \frac{q-1+\delta}{q} = 1 - \frac{1-\delta}{q}.$$

Denote $\gamma = 1 - \frac{1}{\sigma}$. Using $t_{x,\xi}$ and the notation $J = [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ we define the interior region

$$\begin{aligned} Z_{int}(N) &= \{(t, x, \xi) \in J : 0 \le t \le t_{x,\xi}\} \\ &= \{(t, x, \xi) \in J : t^{1-\delta} \le N^\gamma \;\; h(x, \xi)^\gamma\}, \end{aligned}$$

and the exterior region

$$\begin{aligned} Z_{ext}(N) &= \{(t, x, \xi) \in J : t_{x,\xi} \le t \le T\} \\ &= \{(t, x, \xi) \in J : t^{1-\delta} \ge N^\gamma \;\; h(x, \xi)^\gamma\}. \end{aligned}$$

Presentation Outline

Set $\tau_j(t,x,\xi) = \tau_j(T,x,\xi)$ when $t > T$. We define the regularized root $\lambda_j(t,x,\xi)$ as

$$\lambda_j(t,x,\xi) = \int \tau_j\left(t - h(x,\xi)s, x, \xi\right)\rho(s)ds$$

where ρ is compactly supported smooth function satisfying $\int\limits_{\mathbb{R}} \rho(s)ds = 1$ and $0 \leq \rho(s) \leq 1$ with supp $\rho(s) \subset \mathbb{R}_{<0}$. Then

$$\begin{aligned}(\lambda_j - \tau_j)(t,x,\xi) &= \int (\tau_j(t - h(x,\xi)s, x, \xi) - \tau_j(t,x,\xi))\rho(s)ds \\ &= \frac{1}{h(x,\xi)} \int (\tau_j(s,x,\xi) - \tau_j(t,x,\xi))\rho((t-s)h(x,\xi)^{-1})ds.\end{aligned}$$

Decomposition of the Operator

We have

$$|\partial_\xi^\alpha D_x^\beta(\lambda_j - \tau_j)(t,x,\xi)| \le C^{|\alpha|+|\beta|}\alpha!(\beta!)^\sigma \langle x\rangle^{1-|\beta|}\langle \xi\rangle^{1-|\alpha|}$$

and note that in $Z_{ext}(N)$ we have

$$|\partial_\xi^\alpha D_x^\beta(\lambda_j - \tau_j)(t,x,\xi)| \le C^{|\alpha|+|\beta|}\alpha!(\beta!)^\sigma \langle x\rangle^{-|\beta|}\langle \xi\rangle^{-|\alpha|}\frac{1}{t^q}$$

In $Z_{int}(N)$, we have $(\langle x\rangle\langle \xi\rangle)^\gamma \le \frac{N^\gamma}{t^{1-\delta}}$ where $\gamma = 1 - \frac{1}{\sigma}$. So we can write

$$\begin{aligned}|\partial_\xi^\alpha D_x^\beta(\lambda_j - \tau_j)(t,x,\xi)| &\le C^{|\alpha|+|\beta|}\beta!^\sigma\alpha!\langle x\rangle^{1-|\beta|}\langle \xi\rangle^{1-|\alpha|}(\langle x\rangle\langle \xi\rangle)^\gamma \\ &\le C_1^{|\alpha|+|\beta|}\beta!^\sigma\alpha!\frac{N^\gamma}{t^{1-\delta}}\langle x\rangle^{\frac{1}{\sigma}-|\beta|}\langle \xi\rangle^{\frac{1}{\sigma}-|\alpha|}\end{aligned}$$

Decomposition of the Operator

Similarly, in $Z_{ext}(N)$, we have $t^{q/\sigma} \geq (N\, h(x,\xi))^{\frac{1}{\sigma}}$ and

$$\frac{1}{t^q} = \frac{1}{t^{1-\delta}} \frac{1}{t^{q/\sigma}} \leq \frac{1}{t^{1-\delta}} \left(\frac{\langle x \rangle \langle \xi \rangle}{N} \right)^{1/\sigma}.$$

$$|\partial_\xi^\alpha D_x^\beta (\lambda_j - \tau_j)(t,x,\xi)| \leq C_1^{|\alpha|+|\beta|} \beta!^\sigma \alpha! \frac{N^{-1/\sigma}}{t^{1-\delta}} \langle x \rangle^{\frac{1}{\sigma}-|\beta|} \langle \xi \rangle^{\frac{1}{\sigma}-|\alpha|}$$

Hence, in the whole of extended phase space we have

$$|\partial_\xi^\alpha D_x^\beta (\lambda_j - \tau_j)(t,x,\xi)| \leq N^\gamma C^{|\alpha|+|\beta|} \beta!^\sigma \alpha! \frac{1}{t^{1-\delta}} \langle x \rangle^{\frac{1}{\sigma}-|\beta|} \langle \xi \rangle^{\frac{1}{\sigma}-|\alpha|}.$$

Similarly, in $Z_{ext}(N)$, we have $t^{q/\sigma} \geq (N\ h(x,\xi))^{\frac{1}{\sigma}}$ and

$$\frac{1}{t^q} = \frac{1}{t^{1-\delta}} \frac{1}{t^{q/\sigma}} \leq \frac{1}{t^{1-\delta}} \left(\frac{\langle x \rangle \langle \xi \rangle}{N} \right)^{1/\sigma}.$$

$$|\partial_\xi^\alpha D_x^\beta (\lambda_j - \tau_j)(t,x,\xi)| \leq C_1^{|\alpha|+|\beta|} \beta!^\sigma \alpha! \frac{N^{-1/\sigma}}{t^{1-\delta}} \langle x \rangle^{\frac{1}{\sigma}-|\beta|} \langle \xi \rangle^{\frac{1}{\sigma}-|\alpha|}$$

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Conjugation

We conjugate the first order system equivalent to operator P in (1) by $e^{\Lambda(t)\langle x \rangle^{1/\sigma} \langle D \rangle^{1/\sigma}}$, where $\Lambda(t) = \frac{\lambda}{\delta}(T^\delta - t^\delta)$.

Presentation Outline

Theorem

Consider the strictly hyperbolic Cauchy problem ([1](#)) f_k belongs to $H^{s+(m-k)e,\Lambda_1,\sigma}, \Lambda_1 > 0$ for $k = 1, \cdots, m$ and $f \in C([0, T]; H^{s,\Lambda_2,\sigma})$, $\Lambda_2 > 0$. Then, there exists $\Lambda_0 > 0$, such that there is a unique solution

$$u \in \bigcap_{j=0}^{m-1} C^{m-1-j}\left([0, T]; H^{s+je,\Lambda^*,\sigma}\right)$$

where $\Lambda^* < \min\{\Lambda_0, \Lambda_1, \Lambda_2\}$.

Theorem

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where $\Lambda^* < \min\{\Lambda_0, \Lambda_1, \Lambda_2\}$. More specifically, for a sufficiently large λ and $\delta \in (0,1)$, we have the a-priori estimate

$$\sum_{j=0}^{m-1} \|\partial_t^j u(t,\cdot)\|_{s+(m-1-j)e,\Lambda(t),\sigma} \leq C\Bigg(\sum_{j=1}^{m} \|f_j\|_{s+(m-j)e,\Lambda(0),\sigma} + \int_0^t \|f(\tau,\cdot)\|_{s,\Lambda(\tau),\sigma}\, d\tau\Bigg)$$

for $0 \leq t \leq T \leq (\delta\Lambda^*/\lambda)^{1/\delta}$, $C = C_s > 0$ and $\Lambda(t) = \frac{\lambda}{\delta}(T^\delta - t^\delta)$.

A Generalization

In our work [14], we have assumed

$$|\partial_\xi^\alpha D_x^\beta A_{m-j}(t,x,\xi)| \leq C^{|\beta|+|\alpha|}\alpha!\beta!^\sigma \Phi(x)^{m-j-|\beta|}\langle\xi\rangle^{m-j-|\alpha|},$$

where $1 \leq \Phi(x) \lesssim \langle x\rangle$ and used the metric

$$g_{x,\xi} = \frac{|dx|^2}{\Phi(x)^2} + \frac{|d\xi|^2}{\langle\xi\rangle^2}.$$

Example: $\Phi(x) = \langle x\rangle^\kappa$ for some $\kappa \in [0,1]$.

Presentation Outline

In the literature [4] [5], the infinite order ψDO that is used is of the form

$$e^{\varepsilon_1 \langle x \rangle^{1/\sigma} + \varepsilon_2 \langle D_x \rangle^{1/\sigma}}, \quad \sigma > 1,\ \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2.$$

$$\mathcal{S}_{\sigma,\varepsilon}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : e^{\varepsilon_1 \langle x \rangle^{1/\sigma} + \varepsilon_2 \langle D_x \rangle^{1/\sigma}} u \in L^2(\mathbb{R}^n)\}.$$

[4] Cappiello, M. (2003) Pseudodifferential parametrices of infinite order for SG-hyperbolic problems, Rend. Sem. Mat. Univ. Politec. Torino 61 (4) 411–441.

[5] Ascanelli, A., Cappiello, M. (2008) Hölder continuity in time for SG hyperbolic systems, J. Differential Equations 244 2091-2121

Gelfand-Shilov Spaces and Infinite Order ψDO

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$\mathcal{S}_\sigma(\mathbb{R}^n)$

Denoting $\mathcal{S}_\sigma^\sigma(\mathbb{R}^n) = \mathcal{S}_\sigma(\mathbb{R}^n)$ and its dual by $\mathcal{S}'_\sigma(\mathbb{R}^n)$, we have that

$$\mathcal{S}_\sigma(\mathbb{R}^n) = \varinjlim_{\varepsilon \to (0,0)} \mathcal{S}_{\sigma,\varepsilon}(\mathbb{R}^n) \text{ and } \mathcal{S}'_\sigma(\mathbb{R}^n) = \varprojlim_{\varepsilon \to (0,0)} \mathcal{S}'_{\sigma,\varepsilon}(\mathbb{R}^n).$$

[4] Cappiello, M. (2003) Pseudodifferential parametrices of infinite order for SG-hyperbolic problems, Rend. Sem. Mat. Univ. Politec. Torino 61 (4) 411–441.

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$\mathcal{M}_\sigma(\mathbb{R}^n)$

In our analysis [14], we have used infinite order ψDO

$$e^{\varepsilon\langle x\rangle^{1/\sigma}\langle D\rangle^{1/\sigma}}, \quad \sigma > 2 \text{ and } \varepsilon > 0.$$

Let

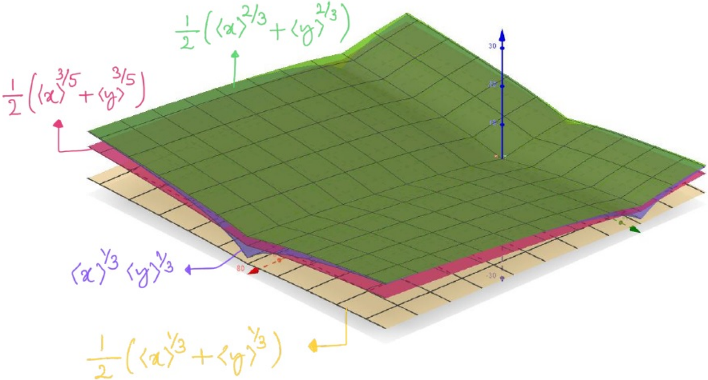
$$\mathcal{M}_{\sigma,\varepsilon}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : e^{\varepsilon\langle x\rangle^{1/\sigma}\langle D\rangle^{1/\sigma}} u \in L^2(\mathbb{R}^n)\},$$

where $\mathcal{M}'_{\sigma,\varepsilon}(\mathbb{R}^n)$, the dual of $\mathcal{M}_{\sigma,\varepsilon}(\mathbb{R}^n)$ is $\mathcal{M}_{\sigma,-\varepsilon}(\mathbb{R}^n)$. We define

$$\mathcal{M}_\sigma(\mathbb{R}^n) = \varinjlim_{\varepsilon\to 0} \mathcal{M}_{\sigma,\varepsilon}(\mathbb{R}^n) \text{ and } \mathcal{M}'_\sigma(\mathbb{R}^n) = \varprojlim_{\varepsilon\to 0} \mathcal{M}'_{\sigma,\varepsilon}(\mathbb{R}^n).$$

Note that $\frac{\varepsilon}{2}(\langle x\rangle^{1/\sigma}+\langle\xi\rangle^{1/\sigma}) \le \varepsilon(\langle x\rangle\langle\xi\rangle)^{1/\sigma} \le \frac{\varepsilon}{2}(\langle x\rangle^{2/\sigma}+\langle\xi\rangle^{2/\sigma})$. Thus,

we have $\mathcal{S}_{\frac{\sigma}{2},\frac{\varepsilon}{2}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{\sigma,\varepsilon}(\mathbb{R}^n) \hookrightarrow \mathcal{S}_{\sigma,\frac{\varepsilon}{2}}(\mathbb{R}^n)$.



Relation between $\mathcal{S}_\sigma(\mathbb{R}^n)$ and $\mathcal{M}_\sigma(\mathbb{R}^n)$

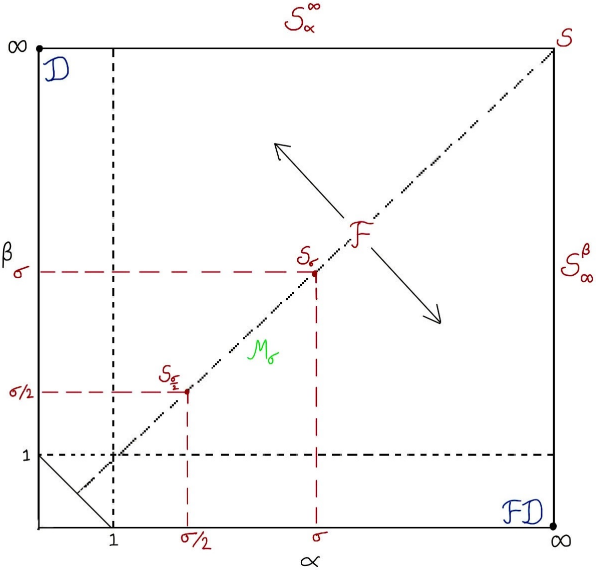
Taking inductive limit as $\varepsilon \to 0$, we have

$$\mathcal{S}_{\frac{\sigma}{2}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_\sigma(\mathbb{R}^n) \hookrightarrow \mathcal{S}_\sigma(\mathbb{R}^n).$$

As for the duals, taking projective limit as $-\varepsilon \to 0$, we have

$$\mathcal{S}'_\sigma(\mathbb{R}^n) \hookrightarrow \mathcal{M}'_\sigma(\mathbb{R}^n) \hookrightarrow \mathcal{S}'_{\frac{\sigma}{2}}(\mathbb{R}^n).$$

$\mathcal{S}_\sigma(\mathbb{R}^n)$ and $\mathcal{M}_\sigma(\mathbb{R}^n)$



Characterization by Perturbed Laplacian Operator

$\mathcal{S}_\sigma(\mathbb{R}^n)$

$\mathcal{S}_\sigma(\mathbb{R}^n)$ for $\sigma \geq \frac{1}{2}$ can be characterized[6] by Harmonic oscillator, $H = |x|^2 - \Delta$ (additive pertubration of Δ) as

$$\|H^N u\|_{L^\infty} \lesssim C^N (N!)^{2\sigma},$$

for some $C > 0$.

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[6] Toft, J. (2017) Images of function and distribution spaces under the Bargmann transform. J. Pseudo Differ. Oper. Appl. 8, 83–139

[7] Cappiello, M., Rodino, L. (2006) SG-Pseudodifferential Operators and Gelfand-Shilov Spaces, Rocky Mountain J. Math. 36(4) 1117-1148.

Characterization by Perturbed Laplacian Operator

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We are looking at characterizing the space $\mathcal{M}_\sigma(\mathbb{R}^n)$ using the operator $\mathcal{L} = \langle x \rangle^2 (1 - \Delta)$ (multiplicative perturbation of Δ)[7].

[6] Toft, J. (2017) Images of function and distribution spaces under the Bargmann transform. J. Pseudo Differ. Oper. Appl. 8, 83–139

[7] Cappiello, M., Rodino, L. (2006) SG-Pseudodifferential Operators and Gelfand-Shilov Spaces, Rocky Mountain J. Math. 36(4) 1117-1148.

Thank You

Presentation Outline

References I

[1] Ascanelli, A., Cappiello, M. (2006) Log-Lipschitz regularity for SG hyperbolic systems, J. Differential Equations 230 556–578.

[2] Ascanelli, A., Cappiello, M. (2008) Hölder continuity in time for SG hyperbolic systems, J. Differential Equations 244 2091-2121.

[3] Cappiello, M., Gourdin, D.,Gramchev, T. (2011) Cauchy Problems for Hyperbolic Systems in $\mathbb{R}^n$ with irregular principal symbol in time and for $|x| \to \infty$, J. Differential Equations 250 2624-2642.

[4] Cappiello, M., Rodino, L. (2006) SG-Pseudodifferential Operators and Gelfand-Shilov Spaces, Rocky Mountain J. Math. 36(4) 1117-1148.

[5] Cicognani, M., (2003) The Cauchy problem for strictly hyperbolic operators with non-absolutely continuous coefficients, Tsukuba J. Mathematics 27 1-12.

[6] Cicognani, M., Lorenz, D. (2017) Strictly hyperbolic equations with coefficients low-regular in time and smooth in space, J. Pseudo-Differ. Oper. Appl.

[7] Colombini, F., Del Santo, D., Kinoshita, T. (2002) Well-posedness of the Cauchy problem for a hyperbolic equation with non-Lipschitz coefficients, Ann. Sc. Norm. Super. Pisa Cl. Sci. 1 (2002) 327–358.

References II

[8] Colombini, F.,De Giorgi, E., Spagnolo, S. (1979) Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temp, Ann. Sc. Norm. Super. Pisa Cl. Sci. 4 (6) 511–559.

[9] Kajitani, K. (1983) Cauchy problem for nonstrictly hyperbolic systems in Gevrey classes, J. Math. Kyoto Univ. 23 (3) 599–616.

[10] Kubo, A., Reissig, M. (2003) Construction of Parametrix to Strictly Hyperbolic Cauchy Problems with Fast Oscillations in NonLipschitz Coefficients, 28:7-8, 1471-1502

[11] Lerner, N. 2010 Metrics on the Phase Space and Non-Selfadjoint Pseudo-Differential Operators,Birkhäuser Basel.

[12] Nicola, F., Rodino, L. (2011) Global Pseudo-differential Calculus on Euclidean Spaces, Birkhäuser Basel.

[13] Nishitani, T. (1983) Sur les équations hyperboliques à coefficients qui sont höldériens en t et de la classe de Gevrey en x, Bull. Sci. Math. 107 113–138

References III

[14] Rahul R. P., Uday Kiran N., (2020) Global Well-Posedness of a Class of Strictly Hyperbolic Cauchy Problems with Coefficients Non-Absolutely Continuous in Time (Preprint) arXiv:2007.07153.

[15] Uday Kiran, N., Coriasco, S., Battisti, U.. (2016) Hyperbolic operators with non-Lipschitz coefficients. In: Recent Advances in Theoretical and Computational Partial Differential Equations with Applications. Panjab University, Chandigarh.