Ellipticity and the Fredholm property in the Weyl-Hörmander calculus

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joined work with Stevan Pilipović

Outline of the problem

For $a \in \mathcal{S}(\mathbb{R}^{2n})$, the Weyl quantisation of a is:

$$a^w \varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a((x+y)/2,\xi)\varphi(y)dyd\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n);$$

$a^w : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous; in fact, it extends to a continuous mapping $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$

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  The SG-calculus (scattering calculus), when $\varphi(x,\xi) = \langle x\rangle^\rho$ and $\Phi(x,\xi) = \langle\xi\rangle^\rho$, $M(x,\xi) = \langle x\rangle^s\langle\xi\rangle^t$.

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- If the calculus satisfies the strong uncertainty principle, i.e. $\varphi(x,\xi)\Phi(x,\xi) \geq c\langle(x,\xi)\rangle^{\varepsilon}$, $\varepsilon > 0$, (the Shubin calculus, the *SG*-calculus), then elliptic operators have parametrices; i.e. there exists b such that $b^w a^w = \mathrm{Id} + R$, where $R : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ (regularising operator).
- The Sobolev space $H(M) = \{u \in \mathcal{S}'(\mathbb{R}^n) | \, a^w u \in L^2\}$, where a^w is elliptic operator of order M; furthermore $H(1) = L^2(\mathbb{R}^n)$.
  For the Shubin calculus when $M = \langle(x,\xi)\rangle^m$, $m \in \mathbb{Z}_+$,
  $$H(M) = \{u \in \mathcal{S}'(\mathbb{R}^n) | \, x^\beta D^\alpha u \in L^2(\mathbb{R}^n), \text{ for all } |\alpha| + |\beta| \leq m\}.$$
- If a is of order M then $a^w : H(M_1) \to H(M_1/M)$.
- A consequence of the existence of parametrices is that every elliptic operator a^w of order M restricts to a Fredholm mapping $H(M_1) \to H(M_1/M)$, for any M_1 and its index is independent of M_1
  (a continuous operator $A : H_1 \to H_2$ is called Fredholm if $\ker A$ and $\operatorname{coker} A = H_2/A(H_1)$ are finite dimensional; $\operatorname{ind} A = \dim\ker A - \dim\operatorname{coker} A$).
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- Let a be a 0-order symbol, i.e. bounded by a constant times $M(x,\xi)^0 = 1$. If a^w is bijective operator on $L^2(\mathbb{R}^n)$, is the inverse again a ΨDO? Yes! A result of Bony and Chemin verifies this for the Weyl-Hörmander calculus (under certain technical assumptions).
- This property of the calculus is called spectral invariance.
- If $\lambda \mapsto a_\lambda$ is C^k-mapping $(0 \leq k \leq \infty)$ of 0-order symbols such that each a_λ^w is invertible on $L^2(\mathbb{R}^n)$, is the same true for the mapping of the inverses $\lambda \mapsto b_\lambda$? $(b_\lambda^w a_\lambda^w = \mathrm{Id} = a_\lambda^w b_\lambda^w)$

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Hörmander metric

V-an n dimensional real vector space with V' its dual;
$W = V \times V'$ is symplectic with the symplectic form $[(x,\xi),(y,\eta)] = \langle \xi, y\rangle - \langle \eta, x\rangle$ (the phase space).
We denote the points in W with capital letters $X, Y, Z, \ldots$.
Let $X \mapsto g_X$ be a Borel measurable symmetric covariant 2-tensor field on W that is positive definite at every point; we employ the notation $g_X(T) = g_X(T,T)$, $T \in T_X W$.
$g^\sigma_X(T) = \sup_{S\in W\setminus\{0\}} [T,S]^2/g_X(S)$ is called the symplectic dual of g.

$X \mapsto g_X$ is a Hörmander metric if:

(*i*) (slow variation) there exist $C \geq 1$ and $r > 0$ such that for all $X, Y, T \in W$

$$g_X(X-Y) \leq r^2 \Rightarrow C^{-1} g_Y(T) \leq g_X(T) \leq C g_Y(T);$$

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Denote $\lambda_g(X) = \inf_{T\in W\setminus\{0\}} (g^\sigma_X(T)/g_X(T))^{1/2}$; it is Borel measurable and $\lambda_g(X) \geq 1$, $\forall X \in W$.

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Admissible weights. Symbol classes

A positive Borel measurable function M on W is said to be g-admissible if there are $C \geq 1$, $r > 0$ and $N \in \mathbb{N}$ such that for all $X, Y \in W$

$$g_X(X - Y) \leq r^2 \Rightarrow C^{-1}M(Y) \leq M(X) \leq CM(Y);$$
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$S(M, g)$ is the space of all $a \in \mathcal{C}^\infty(W)$ for which

$$\|a\|^{(k)}_{S(M,g)} = \sup_{l \leq k} \sup_{\substack{X \in W \\ T_1,\ldots,T_l \in W \setminus \{0\}}} \frac{|a^{(l)}(X; T_1, \ldots, T_l)|}{M(X) \prod_{j=1}^{l} g_X(T_j)^{1/2}} < \infty, \quad \forall k \in \mathbb{N}.$$

$S(M, g)$ is an (F)-space.

When $g_{x,\xi} = \varphi^{-2}|dx|^2 + \Phi^{-2}|d\xi|^2$, $S(M, g)$ reduces to the Beals-Fefferman classes; in this case $g^\sigma_{x,\xi} = \Phi^2|dx|^2 + \varphi^2|d\xi|^2$ and $\lambda_g(X) = \varphi(X)\Phi(X)$.

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ΨDOs with symbols in $S(M, g)$

When $a \in S(M, g)$, a^w is continuous operator on $\mathcal{S}(V)$ and it extends to a continuous operator on $\mathcal{S}'(V)$.

The composition $a^w b^w$ is the ΨDO $(a\#b)^w$ where

$$a\#b(X) = \frac{1}{\pi^{2n}} \int_W \int_W e^{-2i[X-Y_1, X-Y_2]} a(Y_1) b(Y_2) dY_1 dY_2.$$

The mapping $\# : S(M_1, g) \times S(M_2, g) \to S(M_1 M_2, g)$ is continuous.

When E is a Hausdorff locally compact topological space $\mathcal{C}(E; S(1, g))$ becomes a unital algebra (with unity $\mathbf{f}(\lambda) = 1$).

When E is a smooth manifold, $\mathcal{C}^k(E; S(1, g))$, $0 \leq k \leq \infty$, becomes a unital algebra. Furthermore, the smooth vector fields on E are derivations of the unital algebra $\mathcal{C}^\infty(E; S(1, g))$, i.e.

$$X(\mathbf{f}_1 \# \mathbf{f}_2) = X\mathbf{f}_1 \# \mathbf{f}_2 + \mathbf{f}_1 \# X\mathbf{f}_2$$

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The Sobolev space $H(M, g)$

Let M be an admissible weight. There exist $a \in S(M, g)$ and $b \in S(1/M, g)$ such that $a \# b = 1 = b \# a$.

The Sobolev space $H(M, g)$ is defined as

$$H(M, g) = \{u \in \mathcal{S}'(V) |\, a^w u \in L^2(V)\}.$$

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Additional hypothesis for spectral invariance

The Hörmander metric g is said to be geodesically temperate if there exist $C \geq 1$ and $N \in \mathbb{N}$ such that

$$g_X(T) \leq C g_Y(T)(1 + d(X, Y))^N, \quad \forall X, Y, T \in W,$$

where $d(\cdot, \cdot)$ stands for the geodesic distance on W induced by the symplectic intermediate $g^\#$.

The metrics of all of the frequently used calculi are geodesically temperate.

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Inverse smoothness in $S(1, g)$

Theorem

Assume that g is a geodesically temperate Hörmander metric. Let E be a Hausdorff topological space and $\mathbf{f} : E \to S(1, g)$ a continuous mapping. If for each $\lambda \in E$, $\mathbf{f}(\lambda)^w$ is invertible operator on $L^2(V)$, then there exists a unique continuous mapping $\tilde{\mathbf{f}} : E \to S(1, g)$ such that

$$\tilde{\mathbf{f}}(\lambda)\#\mathbf{f}(\lambda) = \mathbf{f}(\lambda)\#\tilde{\mathbf{f}}(\lambda) = 1, \;\; \forall \lambda \in E. \tag{1}$$

If E is a smooth manifold without boundary and $\mathbf{f} : E \to S(1, g)$ is of class $\mathcal{C}^N$, $0 \leq N \leq \infty$, then $\tilde{\mathbf{f}} : E \to S(1, g)$ is also of class $\mathcal{C}^N$.

Equivalence of ellipticity and the Fredholm property

Lemma

Let g be a Hörmander metric satisfying $\lambda_g \to \infty$ and M a g-admissible weight. If $a \in S(M, g)$ is elliptic than for any g-admissible weight M_1, a^w restricts to a Fredholm operator from $H(M_1, g)$ into $H(M_1/M, g)$ and its index is independent of M_1.

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Let g be a geodesically temperate Hörmander metric satisfying $\lambda_g \to \infty$ and M and M_1 two g-admissible weights. If $a \in S(M, g)$ is such that a^w restricts to a Fredholm operator from $H(M_1, g)$ into $H(M_1/M, g)$ then a is elliptic.

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Existence of parametrices

Theorem

Let g be a geodesically temperate Hörmander metric satisfying $\lambda_g \to \infty$ and M a g-admissible weight. If $a \in S(M, g)$ is elliptic then there are $r_1, r_2 \in \mathcal{S}(W)$ and elliptic $\tilde{a}_1, \tilde{a}_2 \in S(1/M, g)$ such that

$$\begin{array}{ccc} \tilde{a}_1 \# a = 1 + r_1 & \text{and} & a \# \tilde{a}_2 = 1 + r_2 \\ (i.e.\ \tilde{a}_1^w a^w = \mathrm{Id} + r_1^w & \text{and} & a^w \tilde{a}_2^w = \mathrm{Id} + r_2^w) \end{array}$$

and consequently a^w is globally regular. Furthermore, $r_1^w(\mathcal{S}'(V))$ and $r_2^w(\mathcal{S}'(V))$ are finite dimensional subspaces of $\mathcal{S}(V)$.
In particular, $\ker a^w$ is a finite dimensional subspace of $\mathcal{S}(V)$ and for any g-admissible weight M_1, $\ker(a^w{}_{|H(M_1, g; \tilde{V})}) = \ker a^w$.

- Consequently, the dimensions of the cokernels of the Fredholm operators $a^w{}_{|H(M_1,g)} : H(M_1, g) \to H(M_1/M, g)$ are also the same for any g-admissible weight M_1.
- All of the above results hold equally well for matrix valued symbols, i.e. for symbols in $S(M, g; \mathcal{L}(\mathbb{C}^\nu))$, $\nu \in \mathbb{Z}_+$.

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Fedosov-Hörmander integral formula for the index

If g satisfies the strong uncertainty principle:

$$\text{there are } C, \delta > 0 \text{ such that } \lambda_g(X) \geq C(1 + g_0(X))^{\delta},\ \forall X \in W,$$

and $a \in S(1, g; \mathcal{L}(\mathbb{C}^{\nu}))$ is elliptic, then $\operatorname{ind} a^w$ can be given by the Fedosov-Hörmander integral formula.

Proposition

Assume that the Hörmander metric g satisfies the strong uncertainty principle and let a be an elliptic symbol in $S(M, g; \mathcal{L}(\mathbb{C}^{\nu}))$ for some g-admissible weight M. Let D be any compact properly embedded codimension-0 submanifold with boundary in W which contains in its interior the set where a is not invertible. Then

$$\operatorname{ind} a^w = -\frac{(n-1)!}{(2n-1)!(2\pi i)^n} \int_{\partial D} \operatorname{tr}(a^{-1} da)^{2n-1}.$$

The orientation of D is the one induced by W, where the latter has the orientation induced by the symplectic form.

Remark

If we fix a basis for V and take the dual basis for V', the orientation on W is given by the nonvanishing $2n$-form $d\xi_1 \wedge dx^1 \wedge \ldots \wedge d\xi_n \wedge dx^n$.

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An illustrative example

Consider the operator

$$a^w = -\Delta + \langle x\rangle^{-2s}, \;\; 0 < s < 1.$$

with Weyl symbol $a(x,\xi) = |\xi|^2 + \langle x\rangle^{-2s}$.

- a^w is not elliptic in any of the "classical" symbolic calculi, but ... ;
- a^w is elliptic in the Weyl-Hörmander calculus for an appropriate choice of the metric, namely a is elliptic in $S(M,g)$ with $g_{x,\xi} = \langle x\rangle^{-2}|dx|^2 + \langle x\rangle^{2s}\langle\xi\rangle^{-2}|d\xi|^2$ and $M = a$ (one can prove that g is a Hörmander metric and M is g-admissible);
- the above results imply $a^w : H(M_1, g) \to H(M_1/M, g)$ is Fredholm, for every g-admissible weight M_1 and its index is independent of M_1. In fact, the Fedosov-Hörmander formula gives $\operatorname{ind} a^w = 0$;
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