Ultradifferentiable extension theorems

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Whitney jets

Let $E \subseteq \mathbb{R}^n$ be closed. Let $m \in \mathbb{N} \cup \{\infty\}$.

- $\mathcal{J}^m(E) := \{F = (F^\alpha)_{|\alpha| \le m} : F^\alpha \in C^0(E, \mathbb{R})\}$ – m-jets on E
- $j_E^m : C^m(U) \mapsto \mathcal{J}^m(E)$, $f \mapsto (\partial^\alpha f|_E)_\alpha$ – jet mapping
- $T_a^m F(x) := \sum_{|\alpha| \le m} \frac{(x-a)^\alpha}{\alpha!} F^\alpha(a)$ – Taylor polynomial of order $m < \infty$
- $R_a^m F := F - j_E^m T_a^m F$ – remainder term

A jet $F \in \mathcal{J}^m(E)$ is a Whitney jet of class C^m, $m < \infty$, on E, in symbols $F \in \mathcal{E}^m(E)$, if for each $|\alpha| \le m$,

$$(R_a^m F)^\alpha(b) = o(|a-b|^{m-|\alpha|}) \quad \text{as } |a-b| \to 0,\ a, b \in E.$$

Whitney jet of class C^∞: $\mathcal{E}^\infty(E) := \bigcap_{m \in \mathbb{N}} \pi_m^{-1}(\mathcal{E}^m(E))$, where $\pi_m : \mathcal{J}^\infty(E) \to \mathcal{J}^m(E)$ is the obvious truncation operator.

Whitney's classical extension theorem

Extension of Whitney jets [Whitney '34]

We have

$$\mathcal{E}^m(E) = j^m_E C^m(\mathbb{R}^n).$$

If $m < \infty$ there is a continuous linear section $\mathcal{E}^m(E) \to C^m(\mathbb{R}^n)$.

Continuous linear extension operators for $m = \infty$

- Continuous linear sections do not always exist, e.g. $E = \{0\}$.
- Continuous linear sections exist if:
  - E is a closed halfspace **[Mityagin '61]**, **[Seeley '64]**;
  - E is the closure of a Lipschitz domain **[Stein '70]**;
  - E is closed subanalytic and $\overline{\operatorname{int} E} = E$ **[Bierstone '78]**.
- E compact (for simplicity) admits an extension operator if and only if $\mathcal{E}^\infty(E)$ satisfies (DN) **[Tidten '79]**.
- Pawłucki, Pleśniak, Bos, Milman, Frerick, Jordá, Wengenroth, ...

The ultradifferentiable setting

How are growth constraints on the jets preserved by the extension?

Problem (one possible formulation)

Let us henceforth assume that E is compact. Let $M = (M_k)$ be a positive sequence. A jet $F = (F^\alpha)_\alpha \in \mathcal{J}^\infty(E)$ is a Whitney ultrajet of class $\mathcal{B}^{\{M\}}$, in symbols $F \in \mathcal{B}^{\{M\}}(E)$, if there exist $C, \rho > 0$ such that

$$|F^\alpha(a)| \le C\rho^{|\alpha|} M_{|\alpha|}, \quad \alpha \in \mathbb{N}^n,\ a \in E,$$

$$|(R_a^p F)^\alpha(b)| \le C\rho^{p+1} M_{p+1} \frac{|b-a|^{p+1-|\alpha|}}{(p+1-|\alpha|)!}, \quad p \in \mathbb{N},\ |\alpha| \le p,\ a, b \in E.$$

- Characterize the sequences M such that $j_E^\infty \mathcal{B}^{\{M\}}(\mathbb{R}^n) = \mathcal{B}^{\{M\}}(E)$.
- If a loss of regularity is unavoidable: Characterize the sequences N such that $j_E^\infty \mathcal{B}^{\{N\}}(\mathbb{R}^n) \supseteq \mathcal{B}^{\{M\}}(E)$.

$\mathcal{B}^{\{M\}}(\mathbb{R}^n) = \big\{ f \in C^\infty(\mathbb{R}^n) :\ \exists \rho > 0 : \sup_{\alpha \in \mathbb{N}^n} \frac{\|f^{(\alpha)}\|_{L^\infty(\mathbb{R}^n)}}{\rho^{|\alpha|} M_{|\alpha|}} < \infty \big\}.$

(We will mostly be concerned with Roumieu type classes.)

Short history of the problem

In the special case $E = \{0\}$ the problems were essentially solved by **[Petzsche '88]** and **[Schmets, Valdivia '04]**.

For general E:

- **[Bruna '80]** proved: Under the assumptions

  1. $M_k/k!$ is logarithmically convex (log-convex),
  2. M has moderate growth, i.e., $\exists C$: $M_{j+k} \le C^{j+k} M_j M_k$ for all j, k,

we have $j_E^\infty \mathcal{B}^{\{M\}}(\mathbb{R}^n) = \mathcal{B}^{\{M\}}(E)$ if and only if M satisfies the strong non-quasianalyticity condition

$$\sum_{\ell \ge k} \frac{M_{\ell-1}}{M_\ell} \lesssim \frac{k M_{k-1}}{M_k}.$$

Short history of the problem (continued)

- **[Chaumat, Chollet '94]** (and **[Langenbruch '94]** for convex E with non-empty interior): Under the assumptions

1. $M_k/k!$ is log-convex,
2. M has moderate growth,
3. $N_k/k!$ is log-convex,
4. N is non-quasianalytic, i.e., $\sum_k N_{k-1}/N_k < \infty$,

we have $j^\infty_E \mathcal{B}^{\{N\}}(\mathbb{R}^n) \supseteq \mathcal{B}^{\{M\}}(E)$ if and only if

$$\sum_{\ell \ge k} \frac{N_{\ell-1}}{N_\ell} \lesssim \frac{kM_{k-1}}{M_k}.$$

- At about the same time the analogous problems were studied for Braun–Meise–Taylor classes by several people (Bonet, Braun, Langenbruch, Meise, Taylor, ...)

Weight functions

• Ultradifferentiable classes introduced by **[Beurling '61]** and **[Björck '66]** (by decay conditions on the Fourier transform) and equivalently described by **[Braun, Meise, and Taylor '90]**. The growth condition is described by a weight function.

Weight functions

A continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ with $\omega|_{[0,1]} = 0$ is called weight function if it satisfies

- $\omega(2t) = O(\omega(t))$ as $t \to \infty$,
- $\omega(t) = O(t)$ as $t \to \infty$,
- $\log t = o(\omega(t))$ as $t \to \infty$,
- $\varphi(t) := \omega(e^t)$ is convex.

It is called non-quasianalytic if $\displaystyle\int_1^\infty \frac{\omega(t)}{t^2}\,dt < \infty.$

Ultradifferentiable classes

We consider the Young conjugate $\varphi^*(x) := \sup_{y\geq 0}(xy - \varphi(y))$, $x \geq 0$.

Braun–Meise–Taylor classes (BMT)

Let ω be a weight function and $\rho > 0$. Consider the Banach space

$$\mathcal{B}^\omega_\rho(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|^\omega_\rho := \sup_{x\in\mathbb{R}^n,\, \alpha\in\mathbb{N}^n} \frac{|\partial^\alpha f(x)|}{\exp(\frac{1}{\rho}\varphi^*(\rho|\alpha|))} < \infty \right\}$$

and the inductive limit $\mathcal{B}^{\{\omega\}}(\mathbb{R}^n) := \operatorname{ind}_{\rho\in\mathbb{N}} \mathcal{B}^\omega_\rho(\mathbb{R}^n)$.

Denjoy–Carleman classes (DC)

Let M be a weight sequence and $\rho > 0$. Consider the Banach space

$$\mathcal{B}^M_\rho(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|^M_\rho := \sup_{x\in\mathbb{R}^n,\, \alpha\in\mathbb{N}^n} \frac{|\partial^\alpha f(x)|}{\rho^{|\alpha|} M_{|\alpha|}} < \infty \right\},$$

and the inductive limit $\mathcal{B}^{\{M\}}(\mathbb{R}^n) := \operatorname{ind}_{\rho\in\mathbb{N}} \mathcal{B}^M_\rho(\mathbb{R}^n)$.

The extension problem for BMT classes

- **[Bonet, Braun, Meise, Taylor '91]**, **[Abanin '01]**: Let ω be a non-quasianalytic weight function. Then TFAE:

  1. $j_E^\infty \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) = \mathcal{B}^{\{\omega\}}(E)$ for every compact $E \subseteq \mathbb{R}^n$.
  2. $j_E^\infty \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) = \mathcal{B}^{\{\omega\}}(E)$ for some compact $E \subseteq \mathbb{R}^n$.
  3. ω is strong, i.e., $\exists C > 0 \ \forall t > 0 : \int_1^\infty \frac{\omega(tu)}{u^2}\, du \leq C\omega(t) + C$.

Question: Let ω be a non-quasianalytic weight function. Let σ be another weight function. Under which conditions do we have $j_E^\infty \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \supseteq \mathcal{B}^{\{\sigma\}}(E)$ for all compact $E \subseteq \mathbb{R}^n$?

- **[Bonet, Meise, Taylor '92]**: For $E = \{0\}$:

$$j_E^\infty \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \supseteq \mathcal{B}^{\{\sigma\}}(E) \Leftrightarrow \exists C > 0 \ \forall t > 0 : \int_1^\infty \frac{\omega(tu)}{u^2}\, du \leq C\sigma(t) + C$$

- **[Langenbruch '94]**: The equivalence holds for all convex compact E with non-empty interior.

Solution of the extension problem

Theorem [R., Schindl '20]

Let ω be a non-quasianalytic concave weight function. Let σ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$. Then TFAE:

1. $j^\infty_E \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \supseteq \mathcal{B}^{\{\sigma\}}(E)$ for every compact $E \subseteq \mathbb{R}^n$.
2. $\exists C > 0 \ \forall t > 0 : \displaystyle\int_1^\infty \frac{\omega(tu)}{u^2}\, du \le C\sigma(t) + C$.

The additional assumptions on ω and σ are natural:

- Any strong weight function is non-quasianalytic.
- Any strong weight function is equivalent to a concave weight function.
- Any strong weight function ω satisfies $\omega(t) \to o(t)$ as $t \to \infty$.

Thus the theorem is the "right" generalization of the case $\omega = \sigma$, where no loss of regularity occurs.

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Main ingredients

- Optimal cut-off functions of Bonet, Braun, Meise and Taylor
- Extension scheme of Dynkin adapted by Chaumat and Chollet
- Weight matrix framework for ultradifferentiability

Uniform approach to ultradifferentiability

Associated weight matrix

Let ω be a weight function. We associate the weight matrix $\mathfrak{W} = \{W^x\}_{x>0}$ by setting $W^x_k := \exp(\frac{1}{x}\varphi^*(xk))$ for $k \in \mathbb{N}$.

Equivalent description [R., Schindl '14]

We have as locally convex spaces

$$\mathcal{B}^{\{\omega\}}(\mathbb{R}^n) = \operatorname{ind}_{x>0} \mathcal{B}^{\{W^x\}}(\mathbb{R}^n) =: \mathcal{B}^{\{\mathfrak{W}\}}(\mathbb{R}^n).$$
$$\mathcal{B}^{(\omega)}(\mathbb{R}^n) = \operatorname{proj}_{x>0} \mathcal{B}^{(W^x)}(\mathbb{R}^n) =: \mathcal{B}^{(\mathfrak{W})}(\mathbb{R}^n).$$

TFAE (see also **[Bonet, Meise, Melikhov '07]**):

- $\mathcal{B}^{[\omega]}(\mathbb{R}^n) = \mathcal{B}^{[W^x]}(\mathbb{R}^n)$ for all $x > 0$.
- $\exists H \geq 1 \; \forall t \geq 0 : 2\omega(t) \leq \omega(Ht) + H$.
- W^x has moderate growth for some $x > 0$.
- W^x has moderate growth for all $x > 0$.

Challenges

Assumptions in the theorem of Chaumat and Chollet

1. $M_k/k!$ is log-convex, we say that M is strongly log-convex,
2. M has moderate growth,
3. $N_k/k!$ is log-convex,
4. N is non-quasianalytic.

Properties of the associated weight matrix $\mathfrak{W} = \{W^x\}_{x>0}$

1. W^x is log-convex (not strongly!) and $(W_k^x)^{1/k} \to \infty$.
2. $\vartheta_k^x := W_k^x / W_{k-1}^x$ satisfies $\vartheta^x \le \vartheta^y$ if $x \le y$, which entails $W^x \le W^y$.
3. For all $x > 0$ and all $k \in \mathbb{N}_{\ge 2}$, $\vartheta_{2k}^x \le \vartheta_k^{4x}$ (moderate growth in a very weak sense).

The log-convexity and the moderate growth assumption in the theorem of Chaumat and Chollet are too restrictive!

Associated functions

Let $m = (m_k)$ be a positive sequence with $m_0 = 1$ and $m_k^{1/k} \to \infty$. (We will apply this to $m_k := M_k/k!$.)

- We associate the function

$$h_m(t) := \begin{cases} \inf_{k\in\mathbb{N}} m_k t^k & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

It is increasing, continuous, positive for $t > 0$ and $= 1$ for large t.

- Let $m = (m_k)$ be log-convex. We associate the function

$$\begin{aligned} \Gamma_m(t) &:= \min\{k : h_m(t) = m_k t^k\}, \quad t > 0, \\ &= \min\left\{k : \frac{m_{k+1}}{m_k} \geq \frac{1}{t}\right\} \quad \text{since } m \text{ is log-convex.} \end{aligned}$$

It is decreasing, $\lim_{t\to 0}\Gamma_m(t) = \infty$, and $k \mapsto m_k t^k$ is decreasing for $k \leq \Gamma_m(t)$.

More associated weight matrices

Let σ be a weight function with $\sigma(t) = o(t)$ as $t \to \infty$ which is equivalent to a concave weight function.

- For the associated weight matrix $\mathfrak{S} = \{S^\xi\}_{\xi>0}$ we have $\mathcal{B}^{\{\sigma\}} = \mathcal{B}^{\{\mathfrak{S}\}}$.
- We also have $\mathcal{B}^{\{\sigma\}} = \mathcal{B}^{\{\underline{\mathfrak{S}}\}}$ where $\underline{\mathfrak{S}} = \{\underline{S}^\xi\}_{\xi>0}$ with $\underline{S}^\xi_k := k!\,\underline{s}^\xi_k$ and $(\underline{s}^\xi_k)$ is the log-convex minorant of $(s^\xi_k) = (S^\xi_k/k!)$. So each $\underline{S}^\xi$ is strongly log-convex! Moreover, $\underline{s}^\xi_{j+k} \le H^{j+k}\underline{s}^{2\xi}_j\underline{s}^{2\xi}_k$, for all $\xi > 0$ and all $j, k \in \mathbb{N}$ and thus $h_{\underline{s}^\xi}(t) \le h_{\underline{s}^{2\xi}}(Ht)^2$, for all $\xi > 0$ and all $t > 0$.
- For each $\xi > 0$ we define $V^\xi_k := k!\,v^\xi_k$ where $v^\xi_k := \min_{0\le j\le k} \underline{s}^{2\xi}_j\underline{s}^{2\xi}_{k-j}$.

Then, for each $\xi > 0$ and all $k \ge 1$, $\frac{v^\xi_{2k-1}}{v^\xi_{2k-2}} = \frac{v^\xi_{2k}}{v^\xi_{2k-1}} = \frac{\underline{s}^{2\xi}_k}{\underline{s}^{2\xi}_{k-1}}$, and thus:

  - V^ξ is strongly log-convex,
  - $2\Gamma_{\underline{s}^{2\xi}}(t) = \Gamma_{v^\xi}(t)$ for all $t > 0$, (not available with $\underline{\mathfrak{S}}$ only!)
  - $\mathcal{B}^{\{\sigma\}} = \mathcal{B}^{\{\mathfrak{V}\}}$ where $\mathfrak{V} = \{V^\xi\}_{\xi>0}$.

Sufficiency of the condition

(Necessity follows from the well understood case $E = \{0\}$.)

$$\left(\exists C \; \forall t : \int_1^\infty \frac{\omega(tu)}{u^2}\, du \le C\sigma(t) + C\right) \Rightarrow \left(j_E^\infty \mathcal{B}^{\{\omega\}}(\mathbb{R}^n) \supseteq \mathcal{B}^{\{\sigma\}}(E)\right)$$

Preparations:

• $\kappa(t) := \int_1^\infty \frac{\omega(tu)}{u^2}\, du$ is a concave weight function satisfying $\kappa(t) = o(t)$ and $\kappa(t) = O(\sigma(t))$ as $t \to \infty$, i.e., $\mathcal{B}^{\{\sigma\}} \subseteq \mathcal{B}^{\{\kappa\}}$.

• So we may assume without loss of generality that $\sigma = \kappa \ge \omega$ is concave and satisfies $\kappa(t) = o(t)$ as $t \to \infty$. Thus the facts from the previous slide are available and moreover

$$\underline{S}^\xi \le S^\xi \le W^\xi, \qquad \text{for all } \xi > 0.$$

Lemma

Let $E \subseteq \mathbb{R}^n$ be a non-empty compact set. There exists a collection of closed cubes $\{Q_i\}_{i\in\mathbb{N}}$ with sides parallel to the axes satisfying:

1. $\mathbb{R}^n \setminus E = \bigcup_{i\in\mathbb{N}} Q_i$.
2. The interiors of the Q_i are pairwise disjoint.
3. $\operatorname{diam} Q_i \leq d(Q_i, E) \leq 4 \operatorname{diam} Q_i$ for all $i \in \mathbb{N}$.
4. Let Q_i^* be the closed cube which has the same center as Q_i expanded by the factor $9/8$. For each $i \in \mathbb{N}$ the number of cubes Q_j^* which intersect Q_i^* is bounded by 12^{2n}.
5. There exist $b_1, B_1 > 0$ (independent of E) such that for all $i, j \in \mathbb{N}$ with $Q_i^* \cap Q_j^* \neq \emptyset$ we have $b_1 \operatorname{diam} Q_i \leq \operatorname{diam} Q_j \leq B_1 \operatorname{diam} Q_i$.

A special partition of unity

For $\sigma = \omega$ due to **[Bonet, Braun, Meise, Taylor '91]** and based on Hörmander's L^2-method and a Paley–Wiener theorem:

Proposition

Let $E \subseteq \mathbb{R}^n$ be compact and $\{Q_i\}_{i\in\mathbb{N}}$ be a family of Whitney cubes for E. For all $p \in \mathbb{N}_{>0}$ there exist $W \in \mathfrak{W}$, $M > 0$, $0 < r_0 < 1/2$, and a family of smooth functions $\{\varphi_{i,p}\}_{i\in\mathbb{N}}$ satisfying

1. $0 \le \varphi_{i,p} \le 1$ for all $i \in \mathbb{N}$,
2. $\operatorname{supp} \varphi_{i,p} \subseteq Q_i^*$ for all $i \in \mathbb{N}$,
3. $\sum_{i\in\mathbb{N}} \varphi_{i,p}(x) = 1$ for all $x \in \mathbb{R}^n \setminus E$,
4. if $d(Q_i, E) \le r_0/B_1$, then for all $\beta \in \mathbb{N}^n$ and $x \in \mathbb{R}^n \setminus E$,

$$|\varphi_{i,p}^{(\beta)}(x)| \le M W_{|\beta|} \exp\Big(\frac{A_1(n)}{p}\sigma^\star\Big(\frac{b_1 p}{A_2(n)}\operatorname{diam} Q_i\Big)\Big),$$

for constants $A_1(n) \le A_2(n)$ only depending on n.

The conjugate of a weight function

- Let σ be a weight function satisfying $\sigma(t) = o(t)$ as $t \to \infty$.

$$\sigma^\star(t) := \sup_{s \geq 0} \big(\sigma(t) - st\big), \quad t > 0.$$

$\sigma^\star$ is decreasing, continuous, convex with $\sigma^\star(t) \to \infty$ as $t \to 0$.

- There is a connection between $\sigma^\star$ and $h_{\underline{s}^\xi}$:

$$\forall \xi > 0 \; \exists C \geq 1 \; \forall t > 0 : \exp(\sigma^\star(t)) \leq \Big(\frac{e}{h_{\underline{s}^\xi}(t/C)}\Big)^C$$

The partition of unity is optimal: to substantiate this we look at Bruna's observation.

Bruna's observation

- Let $r, \lambda > 0$. Suppose $\varphi \in \mathcal{B}^{\{M\}}(\mathbb{R})$ is $= 1$ for $|x| \le r$ and $= 0$ for $|x| \ge (1+\lambda)r$. By Taylor's formula, for $x \in (r, r+\lambda r)$,

$$
\begin{aligned}
1 \xleftarrow{x \to r} |\varphi(x)| &\le \frac{|\varphi^{(k)}(\xi)|}{k!}(\lambda r)^k \quad \text{for some } \xi \in (x, r+\lambda r) \\
\implies \sup_x |\varphi^{(k)}(x)| &\ge \frac{k!}{(\lambda r)^k} \quad \forall k \\
\implies \sup_{x,k} \frac{|\varphi^{(k)}(x)|}{\rho^k M_k} &\ge \sup_k \frac{k!}{(\rho\lambda r)^k M_k} = \frac{1}{\inf_k (\rho\lambda r)^k m_k} = \frac{1}{h_m(\rho\lambda r)}
\end{aligned}
$$

- Bruna showed: for strongly log-convex non-quasianalytic M with moderate growth TFAE

1. $\sum_{\ell \ge k} \frac{M_{\ell-1}}{M_\ell} \lesssim \frac{k M_{k-1}}{M_k}$
2. $\forall r, \lambda, \rho > 0 \;\; \exists \varphi$ as above s.t. $|\varphi^{(k)}| \le \frac{\rho^k M_k}{h_m(B\rho\lambda r)}$ where B is independent of r, λ, ρ, i.e., there exist optimal bump functions.

The extension of a Whitney ultrajet

- Every Whitney ultrajet $F = (F^\alpha)$ of class $\mathcal{B}^{\{\sigma\}}$ on the compact set $E \subseteq \mathbb{R}^n$ is an element of $\mathcal{B}^{\{V^\xi\}}(E)$ for some $\xi > 0$.
- Let $p \in \mathbb{N}$ (to be specified later) and $\{\varphi_{i,p}\}_{i\in\mathbb{N}}$ the corresponding partition of unity relative to $\{Q_i\}_{i\in\mathbb{N}}$. Let $x_i := \operatorname{center}(Q_i)$.
- Then an extension of class $\mathcal{B}^{\{\omega\}}$ of F to a suitable neighborhood of E in $\mathbb{R}^n$ is provided by

$$f(x) := \begin{cases} \sum_{i\in\mathbb{N}} \varphi_{i,p}(x)\, T^{p(x_i)}_{\hat{x}_i} F(x), & \text{if } x \in \mathbb{R}^n \setminus E, \\ F^0(x), & \text{if } x \in E, \end{cases}$$

where $\hat{x}$ is any point in E with $d(x) := d(x,E) = |x - \hat{x}|$ and

$$p(x) := \max\{2\Gamma_{\underline{s}^{2\xi}}(Ld(x)) - 1, 0\}.$$

Here L is a positive constant (to be specified later).

Remarks

• The use of the Taylor polynomial $T_{\hat{x}_i}^{p(x_i)} F(x)$ with variable degree goes back to Dynkin and was used by Chaumat and Chollet.

• Its combination with the partition of unity $\{\varphi_{i,p}\}_{i \in \mathbb{N}}$ which is tailor-made for the BMT-case gives the optimal result.

• To check that f is the desired extension requires a series of intricate estimates in which all the special properties of the associated functions and weight matrices are used.

• The proof shows that for each $\rho > 0$, $\xi > 0$ there exist $M(\rho) > 0$, $W \in \mathfrak{W}$ and a continuous linear extension operator

$$\mathcal{B}_{\rho}^{V^{\xi}}(E) \to \mathcal{B}_{M(\rho)}^{W}(\mathbb{R}^n).$$

This extension operator depends on ρ and ξ and in general there is no continuous extension operator $\mathcal{B}^{\{\sigma\}}(E) \to \mathcal{B}^{\{\omega\}}(\mathbb{R}^n)$.

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The Beurling case

- For DC-classes the same condition $\sum_{\ell \geq k} \frac{N_{\ell-1}}{N_\ell} \lesssim \frac{kM_{k-1}}{M_k}$ characterizes the inclusion $j_E^\infty \mathcal{B}^{(N)}(\mathbb{R}^n) \supseteq \mathcal{B}^{(M)}(E)$ **[Chaumat, Chollet '94]**.
- **[Bonet, Braun, Meise, Taylor '91]**, **[Abanin '01]**: Let ω be a weight function. Then TFAE:

  1. $j_E^\infty \mathcal{B}^{(\omega)}(\mathbb{R}^n) = \mathcal{B}^{(\omega)}(E)$ for every compact $E \subseteq \mathbb{R}^n$.
  2. $j_E^\infty \mathcal{B}^{(\omega)}(\mathbb{R}^n) = \mathcal{B}^{(\omega)}(E)$ for some compact $E \subseteq \mathbb{R}^n$.
  3. ω is strong.

- **[Bonet, Meise, Taylor '92]**: For $E = \{0\}$:

$$j_E^\infty \mathcal{B}^{(\omega)}(\mathbb{R}^n) \supseteq \mathcal{B}^{(\sigma)}(E) \Leftrightarrow \exists C > 0 \ \forall t > 0 : \int_1^\infty \frac{\omega(tu)}{u^2}\, du \leq C\sigma(t) + C$$

This problem seems to be open for general E.

Extension operators

- **Roumieu case**: Usually extension operators do not exist. There is no continuous linear section for $j_E^\infty : \mathcal{B}^{\{M\}}(\mathbb{R}^n) \to \mathcal{B}^{\{M\}}(E)$ for compact convex E and any Gevrey class **[Langenbruch '88]**.
- **Beurling case**: Usually extension operators exist. Let ω be strong.
  - $E = [0,1]$ (or any bounded domain with real analytic boundary) has an extension operator **[Meise, Taylor '89]**.
  - $E = \{0\}$ has an extension operator if and only if

    $$\forall C > 0\ \exists \delta > 0\ \exists R_0 \geq 1\ \forall R \geq R_0 : \omega^{-1}(CR)\omega^{-1}(\delta R) \leq \omega^{-1}(R)^2,$$

    for instance Gevrey–Beurling classes **[Meise, Taylor '89]**.
  - Any closed E has an extension operator if $\{0\}$ has one **[Franken '93]**.
  - If $\{0\}$ has no extension operator then $\{(x,y) \in [0,1]^2 : y \leq |f(x)|\}$ has no extension operator for any $f \in \mathcal{E}^{(\omega)}(\mathbb{R})$ with $j^\infty_{\{0\}} f = 0$ **[Franken '94]**.

Extension operators (continued)

- A positive result in the mixed case. Let M be strongly log-convex of moderate growth and N strongly log-convex and non-quasianalytic such that

$$\Big(\frac{M_k}{N_k}\Big)^{1/k} \to 0.$$

Then TFAE:

1. $\frac{M_k}{kM_{k-1}} \sum_{\ell \ge k} \frac{N_{\ell-1}}{N_\ell} \to 0.$
2. $j^\infty_{\{0\}} \mathcal{B}^{(N)}(\mathbb{R}) \supseteq \mathcal{B}^{\{M\}}(\{0\}).$
3. For all compact $E \subseteq \mathbb{R}^n$ there is an continuous linear extension operator $\mathcal{B}^{\{M\}}(E) \to \mathcal{B}^{(N)}(\mathbb{R}^n)$.

[Chaumat, Chollet '94].

Concave weight functions

Theorem [R., Schindl '20], [Fürdös, Nenning, R., Schindl '20]

Let ω be a weight function satisfying $\omega(t) = o(t)$ as $t \to \infty$. TFAE:

1. ω is equivalent to a concave weight function.
2. $\exists C > 0 \; \exists t_0 > 0 \; \forall \lambda \geq 1 \; \forall t \geq t_0 : \omega(\lambda t) \leq C\lambda\, \omega(t)$.
3. There is a weight matrix $\mathfrak{S}$ consisting of strongly log-convex weight sequences such that $\mathcal{B}^{[\omega]} = \mathcal{B}^{[\mathfrak{S}]}$.
4. $\mathcal{B}^{[\omega]}$ is stable under composition.
5. $\mathcal{B}^{[\omega]}$ is stable under inverse/implicit functions.
6. $\mathcal{B}^{[\omega]}$ is stable under solving ODEs.
7. $\mathcal{B}^{[\omega]}$ can be described by almost analytic extensions.

Here $\mathcal{B}^{[\omega]}$ stands for the Roumieu class $\mathcal{B}^{\{\omega\}}$ and the Beurling class $\mathcal{B}^{(\omega)}$.

Almost analytic extensions

An almost analytic extension of a real function f is an extension F to the complex domain such that $\overline{\partial}F(z)$ has a certain growth rate as z approaches the real domain. This growth rate encodes regularity properties of f.

• Let $f : \mathbb{R} \to \mathbb{C}$. Then f is C^∞ if and only if it has an extension F to $\mathbb{C}$ such that $\overline{\partial}F(z)$ vanishes to infinite order on $\mathbb{R}$. (Used in Nirenberg's proof of the Malgrange preparation theorem.)

• **[Dynkin 70ies]**: DC-Roumieu classes admit a description by almost analytic extensions. (The weight sequence is strongly log-convex.)

$$f \in \mathcal{B}^{\{M\}}(U) \Leftrightarrow \begin{array}{l} \exists F \in C_c^\infty(\mathbb{C}^n) : F|_U = f, \\ \exists C, D > 0 : \overline{\partial}F(z) \leq C h_m(Dd(z, \overline{U})) \end{array}$$

• **[Petzsche, Vogt '84]**: Non-quasianalytic BMT-classes (of compact support) admit a description by almost analytic extensions.

Almost analytic extensions (continued)

We prove general ultradifferentiable almost analytic extension theorems in **[Fürdös, Nenning, R., Schindl '20]**; e.g.

Theorem

Let ω be a concave weight function satisfying $\omega(t) = o(t)$ as $t \to \infty$. Let $U \subseteq \mathbb{R}^n$ be a bounded quasiconvex domain. Then:

1. $f \in \mathcal{B}^{\{\omega\}}(U)$ if and only if there exist $F \in C_c^1(\mathbb{C}^n)$ and $\rho > 0$ such that $F|_U = f$ and
$$\sup_{z \in \mathbb{C}^n \setminus \overline{U}} |\overline{\partial} F(z)| \exp(\rho \omega^\star(d(z, \overline{U})/\rho)) < \infty. \tag{$*$}$$
2. $f \in \mathcal{B}^{(\omega)}(U)$ if and only if for all $\rho > 0$ there exists $F \in C_c^1(\mathbb{C}^n)$ such that $F|_U = f$ and $(*)$.

If ω is a strong weight function, then the extension F in 2. may be taken independent of $\rho > 0$.

Idea of Dynkin's extension scheme

Sufficiency: Bochner–Martinelli formula

Necessity: • Let $E := \overline{U}$, where U is a bounded quasiconvex domain, and $f \in \mathcal{B}^M_\rho(E)$.

• Define

$$G(z) := T^{p(z)}_{\hat{z}} f(z), \quad z \in \mathbb{C}^n \setminus E,$$

$$p(z) := \Gamma_m(C(n)\rho d(z,E)).$$

• The desired extension is of the form

$$F(z) := \frac{(2c_2)^{2n}}{\delta(z)^{2n}} \int \Psi\Big(\frac{2c_2(\zeta - z)}{\delta(z)}\Big) G(\zeta)\, d\mathcal{L}^{2n}(\zeta), \quad z \in \mathbb{C}^n \setminus E;$$

Ψ is a suitable cut-off function and $\delta \in C^\infty(E^c)$ the regularized distance:

1. $c_1 d(z,E) \le \delta(z) \le c_2 d(z,E)$ for all $z \notin E$,
2. for all α and $z \notin E$,

$$\big|\partial^\alpha \delta(z)\big| \le B_\alpha d(z,E)^{1-|\alpha|},$$

where the constants B_α, c_1, c_2 are independent of E.

Applications to microlocal analysis

[Fürdös, Nenning, R., Schindl '20]

• We treat the ultradifferentiable wave front set for $u \in \mathcal{D}'$ in the uniform weight matrix setting; this generalizes the wave front sets of Hörmander and of Albanese, Jornet, Oliaro.

• We obtain a characterization of the ultradifferentiable wave front set by almost analytic extensions. As a consequence we show: The ultradifferentiable wave front set is compatible with pullbacks by mappings of the corresponding ultradifferentiable class and hence the definition of the wave front set can be extended to ultradifferentiable manifolds.

• We get a general ultradifferentiable version of Bony's theorem, that is a characterization of the ultradifferentiable wave front set not only by almost analytic extensions but also in terms of the FBI transform.

Applications to microlocal analysis (continued)

- We obtain generalizations of the elliptic regularity theorem.

Remark: In the Beurling case one must in general assume that the coefficients of the linear operator are *strictly more regular* than the wave front set in question. There are however circumstances when the operator can be as regular as the wave front set. E.g.

Theorem

Let ω be a concave weight function and let $P(x,D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be a linear partial differential operator with $\mathcal{E}^{(\omega)}$-coefficients. Then

$$\mathrm{WF}_{(\omega)}\, u \subseteq \mathrm{WF}_{(\omega)}\, Pu \cup \mathrm{Char}\, P, \quad u \in \mathcal{D}'.$$

If P is elliptic, then $\mathrm{WF}_{(\omega)}\, u = \mathrm{WF}_{(\omega)}\, Pu$.

Applications to microlocal analysis (continued)

- As a corollary we get a general version of the quasianalytic Holmgren uniqueness theorem. In particular

Theorem

Let ω be a concave quasianalytic weight function. Let P be a linear partial differential operator with coefficients in $\mathcal{E}^{\{\omega\}}(\Omega)$. If X is a C^1-hypersurface in Ω that is non-characteristic at x_0 and $u \in \mathcal{D}'(\Omega)$ a solution of $Pu = 0$ that vanishes on one side of X near x_0, then $u \equiv 0$ in a full neighborhood of x_0.

Thank you for your attention!