ON A NONLINEAR STOCHASTIC FRACTIONAL HEAT EQUATION

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Introduction

We consider the fractional stochastic heat equation:

$$
\begin{aligned}
\partial_t U(t,x) &= K(t,x)\,{}^R\mathcal{D}_x^\alpha U(t,x) + f(x,t,U(x,t)) \\
&+ \sigma(t,x,U(x,t))P(x,t),\ \alpha \in (1,2), t>0,\ x \in \mathbf{R}, \\
\partial_t U(0,x) &= Q(x)
\end{aligned}
$$

where ${}^R\mathcal{D}_x^\alpha$ denotes the αth Riesz fractional derivative with respect to x and P and Q are certain Colombeau generalized stochastic processes.

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The generalized stochastic processes appearing in the equation are of Colombeau type. For instance, authors often choose P to be a (generalized) white noise process. Here, basically, P and Q can both be almost arbitrary Colombeau generalized stochastic processes of certain growth.

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We establish and prove the result concerning the existence and uniqueness of solution within certain Colombeau space.

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We consider the case when the Riesz fractional derivative is involved.

We establish and prove the result concerning the existence and uniqueness of solution within certain Colombeau space.

The solutions are obtained by using the theory of generalized uniformly continuous semigroups of operators.

The heat equation given above can be written in operator form:

$$\begin{aligned} \partial_t U(x,t) &= A^\alpha U(x,t) + f(x,t,U(x,t)) + \sigma(x,t,U(x,t))P(x,t), \\ U(x,0) &= U_0, \ \ t>0, \ x \in \mathbf{R}, \end{aligned}$$

where $A^\alpha U(x,t) = K(x,t)^R\mathcal{D}_x^\alpha U(x,t)$, $1<\alpha<2$.

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Instead of the original problem we solve an approximate problem

$$\begin{aligned} \partial_t U(x,t) &= \widetilde{A}^\alpha U(x,t) + f(x,t,U(x,t)) + \sigma(x,t,U(x,t))P(x,t), \\ U(x,0) &= U_0, \;\; t>0, \; x \in \mathbf{R}, \end{aligned}$$

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where $\widetilde{A}^\alpha$ is obtained from A^α by regularizing Riesz derivative.

We prove that A^α and $\widetilde{A}^\alpha$ are L^2-associated and that, if we suppose that solution of the original problem exists, it is L^2-associated to the solution of the approximate problem.

Fractional derivatives

Suppose that $u \in C_0^\infty(\mathbf{R})$ and $m-1 < \alpha < m$, where $m \in \mathbf{N}$.

The left Liouville fractional derivative of order α on the whole axis $\mathbf{R}$ is given by

$$(D_+^\alpha u)(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int\limits_{-\infty}^{x} \frac{u(\xi)}{(x-\xi)^{\alpha-m+1}} d\xi.$$

It is well known thta this definition can be extended to a continuous linear map from $H^\alpha(\mathbf{R})$ into $L^2(\mathbf{R})$. This extension of the left Liouville fractional derivative we denote by $\mathcal{D}_+^\alpha$.

Fractional derivatives

Similarly, the right Liouville fractional derivative of order α $(m-1<\alpha<m, m\in\mathbf{N})$, on the whole axis $\mathbf{R}$ is given by

$$(D_{-}^{\alpha}u)(x)=\frac{1}{\Gamma(m-\alpha)}\left(-\frac{d}{dx}\right)^{m}\int_{x}^{\infty}\frac{u(\xi)}{(\xi-x)^{\alpha-m+1}}d\xi.$$

Similarly, one can extend the definition to a continuous linear mapping $\mathcal{D}_{-}^{\alpha}$ from $H^{\alpha}(\mathbf{R})$ to $L^{2}(\mathbf{R})$.

Fractional derivatives

The αth Riesz fractional derivative, $m-1<\alpha<m$, $m\in\mathbf{N}$, denoted by ${}^{R}\mathcal{D}^{\alpha}$, is defined by using the left and right αth Liouville fractional derivative as

$$ {}^{R}\mathcal{D}^{\alpha}u(x) = -\frac{1}{2\cos\frac{\alpha\pi}{2}}(\mathcal{D}_{+}^{\alpha}u(x)+\mathcal{D}_{-}^{\alpha}u(x)). $$

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The Fourier transform of the αth Riesz fractional derivative is

$$\begin{aligned}\widehat{{}^{R}\mathcal{D}^{\alpha}}u(\xi) &= -\frac{1}{2\cos\frac{\alpha\pi}{2}}\left[(i\xi)^{\alpha}\hat{u}(\xi)+(-i\xi)^{\alpha}\hat{u}(\xi)\right]\\ &= -|\xi|^{\alpha}\hat{u}(\xi),\ u\in H^{\alpha}(\mathbf{R}).\end{aligned}$$

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If $u(x)\in\mathcal{S}(\mathbf{R})$ then
$-(-\Delta)^{\frac{\alpha}{2}}u(x) = {}^{R}\mathcal{D}^{\alpha}u(x),\ \alpha\in(0,1)\cup(1,2).$

Colombeau spaces

We define the following spaces:

$\mathcal{E}_M([0,\infty) : H^{2,\infty}(\mathbf{R}))$ is the space of all

$G_\varepsilon : (0,\infty)\times\mathbf{R} \mapsto \mathbf{C},\ G_\varepsilon(t,\cdot) \in H^{2,\infty}(\mathbf{R}),$ for every $t \in [0,\infty)$,

with the property that for every $T > 0$ there exist $C > 0$, $N \in \mathbf{N}$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T)} \|\partial^\alpha G_\varepsilon(t,\cdot)\|_{L^\infty(\mathbf{R})} \leq C\varepsilon^{-N}, \alpha \in \{0,1,2\},\ \ \varepsilon < \varepsilon_0.$$

We say that $\|\partial^\alpha G_\varepsilon\|_{L^\infty}$ is moderate or that it has a moderate bound.

Colombeau spaces

$\mathcal{N}([0,\infty) : H^{2,\infty}(\mathbf{R}))$ is the space of all $G_\varepsilon \in \mathcal{E}_M([0,\infty) : H^{2,\infty}(\mathbf{R}))$ with the property that for every $T > 0$ and $a \in \mathbf{R}$ there exist $C > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T)} \|\partial^\alpha G_\varepsilon(t,\cdot)\|_{L^\infty(\mathbf{R})} \leq C\varepsilon^a, \alpha \in \{0,1,2\}, \ \ \varepsilon < \varepsilon_0.$$

We say that $\|\partial^\alpha G_\varepsilon)\|_{L^\infty}$ is negligible or that it has $\mathcal{N}$- bound.

Spaces $\mathcal{E}_M([0,\infty):H^{2,\infty}(\mathbf{R}))$ and $\mathcal{N}([0,\infty):H^{2,\infty}(\mathbf{R}))$ are algebras and $\mathcal{N}([0,\infty):H^{2,\infty}(\mathbf{R}))$ is an ideal of $\mathcal{E}_M([0,\infty):H^{2,\infty}(\mathbf{R}))$.

The factor algebra

$$\mathcal{G}([0,\infty):H^{2,\infty}(\mathbf{R})) = \frac{\mathcal{E}_M([0,\infty):H^{2,\infty}(\mathbf{R}))}{\mathcal{N}([0,\infty):H^{2,\infty}(\mathbf{R}))}$$

is called the algebra of $H^{2,\infty}$-Colombeau generalized functions.

Colombeau spaces

$\mathcal{E}_M([0,\infty) : H^2(\mathbf{R}))$ is the space of all

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Colombeau spaces

Again, spaces $\mathcal{E}_M([0,\infty) : H^2(\mathbf{R}))$ and $\mathcal{N}([0,\infty) : H^2(\mathbf{R}))$ are algebras and $\mathcal{N}([0,\infty) : H^2(\mathbf{R}))$ is an ideal of $\mathcal{E}_M([0,\infty) : H^2(\mathbf{R}))$, so we can define the factor algebra

$$\mathcal{G}([0,\infty) : H^2(\mathbf{R})) = \frac{\mathcal{E}_M([0,\infty) : H^2(\mathbf{R}))}{\mathcal{N}([0,\infty) : H^2(\mathbf{R}))}$$

which is called the algebra of H^2-Colombeau generalized functions.

By omitting the variable t, one can similarly define the spaces $\mathcal{E}_M(H^2(\mathbf{R}))$, $\mathcal{N}(H^2(\mathbf{R}))$ and $\mathcal{G}(H^2(\mathbf{R}))$.

Colombeau generalized stochastic processes

A $\mathcal{G}_{H^{2,\infty}}$-Colombeau generalized stochastic process on a probability space (Ω, Σ, μ) is a mapping $U : \Omega \mapsto \mathcal{G}([0,\infty) : H^{2,\infty}(\mathbf{R}))$ such that there exists a function $\tilde{U} : (0,1) \times [0,\infty) \times \mathbf{R} \times \Omega \mapsto \mathbf{R}$ with the following properties:

1) For fixed $\varepsilon \in (0,1)$, $(t,x,\omega) \mapsto \tilde{U}(\varepsilon,t,x,\omega)$ is jointly measurable in $[0,\infty) \times \mathbf{R} \times \Omega$.

2) The mapping $\varepsilon \mapsto \tilde{U}(\varepsilon,t,x,\omega)$ is en element of $\mathcal{E}_M([0,\infty) : H^{2,\infty}(\mathbf{R}))$ almost surely in $\omega \in \Omega$, and it is a representative of $U(\omega)$.

The algebra of $\mathcal{G}_{H^{2,\infty}}$-Colombeau generalized stochastic processes on Ω will be denoted by $\mathcal{G}^{\Omega}([0,\infty) : H^{2,\infty}(\mathbf{R}))$.

Colombeau generalized stochastic processes

A $\mathcal{G}_{H^2}$-Colombeau generalized stochastic processes on a probability space (Ω, Σ, μ) is a mapping $U : \Omega \mapsto \mathcal{G}([0, \infty) : H^2(\mathbf{R}))$ such that there exists a function $\tilde{U} : (0, 1) \times [0, \infty) \times \mathbf{R} \times \Omega \mapsto \mathbf{R}$ with the following properties:

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The algebra of $\mathcal{G}_{H^2}$-Colombeau generalized stochastic processes on Ω will be denoted by $\mathcal{G}^{\Omega}([0, \infty) : H^2(\mathbf{R}))$.

Colombeau generalized stochastic processes

The smoothed white noise process is usually defined by $\dot{W}_\varepsilon = \dot{W} * h_\varepsilon$, where h_ε is a mollifier net. If we make a slight modification of the definition above, and define the smoothed white noise process as

$$\dot{W}_\varepsilon = \left(\dot{W} * h_\varepsilon\right)\zeta_\varepsilon,$$

where ζ_ε is a non-negative net of smooth, compactly supported cutoff functions converging to identity, then this white noise process is a representative of a $\mathcal{G}_{H^2}$-Colombeau generalized stochastic process. The cutoff procedure is necessary in order to obtain H^2-moderate properties of the $\dot{W}_\varepsilon$.

Colombeau generalized semigroups of operators

$\mathcal{S}E_M([0,\infty) : \mathcal{L}(E))$ is the space of nets

$$S_\varepsilon : [0,\infty) \to \mathcal{L}(E), \quad \varepsilon \in (0,1),$$

differentiable with respect to $t \in [0,\infty)$, with the property that for every $T > 0$ there exist $N \in \mathbf{N}$, $M > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T)} \left\| \frac{d^\gamma}{dt^\gamma} S_\varepsilon(t) \right\|_{\mathcal{L}(E)} \leq M\varepsilon^{-N}, \quad \varepsilon < \varepsilon_0, \quad \gamma \in \{0,1\}. \tag{1}$$

It is an algebra with respect to composition of operators.

Colombeau generalized semigroups of operators

$\mathcal{S}N([0,\infty) : \mathcal{L}(E))$ is the space of nets

$$N_\varepsilon : [0,\infty) \to \mathcal{L}(E), \quad \varepsilon \in (0,1),$$

differentiable with respect to $t \in [0,\infty)$, with the property that for every $T > 0$ and $a \in \mathbf{R}$ there exist $M > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T)} \left\| \frac{d^\gamma}{dt^\gamma} N_\varepsilon(t) \right\|_{\mathcal{L}(E)} \le M\varepsilon^a, \quad \varepsilon < \varepsilon_0, \quad \gamma \in \{0,1\}. \tag{2}$$

It is an ideal of $\mathcal{S}E_M$.

Colombeau generalized semigroups of operators

We define a Colombeau-type space by

$$\mathcal{S}G([0,\infty):\mathcal{L}(E)) = \frac{\mathcal{S}E_M([0,\infty):\mathcal{L}(E))}{\mathcal{S}N([0,\infty):\mathcal{L}(E))}. \qquad (3)$$

Elements of $\mathcal{S}G([0,\infty):\mathcal{L}(E))$ will be denoted by $S=[S_\varepsilon]$, where S_ε is a representative of the class.

Colombeau generalized semigroups of operators

$\mathcal{S}E_M(E)$ is the space of nets of linear continuous mappings

$$A_\varepsilon : E \to E, \quad \varepsilon \in (0,1),$$

with the property that there exists constants $N \in \mathbf{N}$, $M > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)} \leq M\varepsilon^{-N}, \quad \varepsilon < \varepsilon_0.$$

$\mathcal{S}N(E)$ is the space of nets of linear continuous mappings $A_\varepsilon : E \to E, \quad \varepsilon \in (0,1)$, with the property that for every

$a \in \mathbf{R}$, there exist $M > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)} \leq M\varepsilon^a, \quad \varepsilon < \varepsilon_0.$$

The Colombeau space of generalized linear operators on E is defined by

$$\mathcal{S}G(E) = \frac{\mathcal{S}E_M(E)}{\mathcal{S}N(E)}.$$

Elements of $\mathcal{S}G(E)$ will be denoted by $A = [A_\varepsilon]$, where A_ε is a representative of the class.

Colombeau generalized semigroups of operators

$S \in \mathcal{S}\mathrm{G}([0,\infty) : \mathcal{L}(E))$ is called a uniformly continuous Colombeau semigroup if it has a representative S_ε which is a uniformly continuous semigroup for every ε small enough, i.e.

1. $S_\varepsilon(0) = I$,
2. $S_\varepsilon(t_1 + t_2) = S_\varepsilon(t_1)S_\varepsilon(t_2)$, for every $t_1 \geq 0,\ t_2 \geq 0$,
3. $\lim_{t\to 0} \|S_\varepsilon(t) - I\| = 0$.

Let S_ε and $\widetilde{S}_\varepsilon$ be representatives of a uniformly continuous Colombeau semigroup S, with infinitesimal generators A_ε and $\widetilde{A}_\varepsilon$, respectively, for ε small enough. Then $A_\varepsilon - \widetilde{A}_\varepsilon \in \mathcal{S}N(E)$.

Colombeau generalized semigroups of operators

$A \in \mathcal{S}G(E)$ is called the infinitesimal generator of a uniformly continuous Colombeau semigroup $S \in \mathcal{S}\mathrm{G}([0,\infty) : \mathcal{L}(E))$ if A_ε is the infinitesimal generator of the representative S_ε, for every ε small enough.

Let A be the infinitesimal generator of a uniformly continuous Colombeau semigroup S, and B be the infinitesimal generator of a uniformly continuous Colombeau semigroup T. If $A = B$, then $S = T$.

Colombeau generalized semigroups of operators

Let h_ε be a positive net satisfying $h_\varepsilon \le \varepsilon^{-1}$. It is said that $A \in \mathcal{S}G(E)$ is of $h_\varepsilon-$type if it has a representative A_ε such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)} = \mathcal{O}(h_\varepsilon),\ \varepsilon \to 0.$$

Every $A \in \mathcal{S}G(E)$ of $h_\varepsilon-$type, where $h_\varepsilon \le C \log \frac{1}{\varepsilon}$, is the infinitesimal generator of some $T \in \mathcal{S}\mathrm{G}([0,\infty) : \mathcal{L}(E))$.

Note that a uniformly continuous Colombeau semigroup always has an infinitesimal generator and it is unique. That follows from the fact that its representative is a classical uniformly continuous semigroup for which there exists a unique infinitesimal generator.

Colombeau generalized semigroups of operators

In the existence and uniqueness of solution proof it will be necessary that corresponding generalized semigroup S is of $\log\frac{1}{\varepsilon}$-type, too. Therefore, the operator A must satisfy the stronger condition

Every uniformly continuous generalized semigroup $S \in \mathcal{S}\mathrm{G}([0,\infty) : \mathcal{L}(E))$ generated by $A \in \mathcal{S}G(E)$ of $h_\varepsilon-$type, where $h_\varepsilon = o(\log\log\frac{1}{\varepsilon})$, is of $\log\frac{1}{\varepsilon}$-type.

Regularized derivative

Let $m-1<\alpha<m$, $m\in\mathbf{N}$, and $G=[G_\varepsilon]$ be a Colombeau generalized function of h_ε-type, where h_ε is a positive net such that $h_\varepsilon\le\varepsilon^{-1}$.

Regularized αth Riesz derivative with respect to x of G, in notation ${}^R\widetilde{\mathcal{D}}_x^\alpha G$, is defined by the representative

$$ {}^R\widetilde{\mathcal{D}}_{h_\varepsilon}^\alpha G_\varepsilon = {}^R\mathcal{D}_x^\alpha G_\varepsilon * \phi_{h_\varepsilon} = G_\varepsilon \ * \ {}^R\mathcal{D}_x^\alpha \phi_{h_\varepsilon}, $$

where $\phi_{h_\varepsilon}(x)=h_\varepsilon\phi(xh_\varepsilon)$, $\phi\in C_0^\infty$, $\phi(\xi)\ge 0$, ϕ is symmetric function with $\int\phi(\xi)d\xi=1$.

Solution of the approximative problem

Function $f(x,t,u)$ is of a bounded type if satisfies:

(i) $f(x,t,u)$ is a global Lipschitz function with respect to x and u,

(ii) $f(x,t,u)$ has a bounded second order derivative with respect to u

(iii) $f(x,t,0)=0,$

(iv) $\partial_x f(x,t,u)$ is a global Lipschitz function with respect to u.

We solve a stochastic fractional heat equation within the Colombeau space $\mathcal{G}^{\Omega}([0,\infty):H^2(\mathbf{R}))$.

Solution of the approximative problem

Let functions $f(x,t,u)$, $\sigma(x,t,u)$ and their partial derivatives with respect to u and x be the functions of bounded type in the sense of previous definition. Let Colombeau generalized stochastic processes Q and P be such that $Q \in \mathcal{G}^{\Omega}(H^2(\mathbf{R}))$, $P \in \mathcal{G}^{\Omega}([0,\infty) : H^{2,\infty}(\mathbf{R}))$ and P is of $\log\frac{1}{\varepsilon}$-type. Let h_ε be a net satisfying $h_\varepsilon = o((\log\log\frac{1}{\varepsilon})^{1/5})$, as $\varepsilon \to 0$.

Solution of the approximative problem

Suppose that operator $\widetilde{A}^{\alpha} \in \mathcal{S}\mathrm{G}(H^2(\mathbf{R}))$ is represented by the nets of operators

$$\widetilde{A}^{\alpha}_{\varepsilon} : H^2(\mathbf{R}) \to \mathbf{H^2}(\mathbf{R}),$$
$$\widetilde{A}^{\alpha}_{\varepsilon} u = K^R_{\varepsilon} \widetilde{\mathcal{D}}^{\alpha}_{h_\varepsilon} u = K_{\varepsilon}(u \ * \ {}^R D^{\alpha}_x \phi_{h_\varepsilon}), \ \ 1 < \alpha < 2,$$

where $K_\varepsilon \in H^2(\mathbf{R})$, $\|K_\varepsilon\|_{H^2(\mathbf{R})} = \mathcal{O}((\log\log \frac{1}{\varepsilon})^{1/2})$, $\phi_{h_\varepsilon}(x) = h_\varepsilon \phi(x h_\varepsilon)$, $\phi \in C_0^\infty$, $\phi(\xi) \geq 0$, ϕ is symmetric function with $\int \phi(\xi) d\xi = 1$.

Solution of the approximative problem

Then for every $1 < \alpha < 2$ there exists a unique generalized solution $U \in \mathcal{G}^{\Omega}([0,\infty) : H^2(\mathbf{R}))$, to the Cauchy problem

$$\partial_t U(t) = \widetilde{A}^{\alpha} U(t) + f(\cdot, t, U) + \sigma(\cdot, t, U) P(\cdot, t), \qquad U(0) = Q,$$

and it is represented by

$$U_\varepsilon(t) = S_\varepsilon(t) Q_\varepsilon + \int_0^t S_\varepsilon(t-s) f(\cdot, s, U_\varepsilon) ds$$
$$+ \int_0^t S_\varepsilon(t-s) \sigma(\cdot, s, U_\varepsilon) P_\varepsilon(\cdot, s) ds,$$

where $S \in \mathcal{S}\mathrm{G}([0,\infty) : \mathcal{L}((H^2(\mathbf{R})))$ is an uniformly continuous Colombeau semigroup generated by $\widetilde{A}^{\alpha}$.

Association of non-regularized and regularized operators

Let $u \in H^2(\mathbf{R})$ and A be an operator given by
$A^\alpha u = K(x,t)^R\mathcal{D}_x^\alpha u$, where $K \in L^\infty(\mathbf{R})$. Further, let $\widetilde{A}_\varepsilon^\alpha$ be the operator given by
$\widetilde{A}_\varepsilon^\alpha u = K(x,t)^R\widetilde{\mathcal{D}}_{h_\varepsilon}^\alpha u = K_\varepsilon(x,t)({}^R\mathcal{D}_x^\alpha u * \phi_{h_\varepsilon})$, where
$K_\varepsilon(x,t) = K(x,t) * \phi_{h_\varepsilon}$ such that

$$\|\partial_x K(x,t)\|_{L^\infty} \leq g_\varepsilon^{-M}, \tag{4}$$

for some $M > 0$ and $g_\varepsilon = \log h_\varepsilon$, $h_\varepsilon < C \log \frac{1}{\varepsilon}$.

Then the operators A^α and $\widetilde{A}_\varepsilon^\alpha$ are L^2-associated when $\varepsilon \to 0$, i.e., for every $u \in H^2(\mathbf{R})$, the following holds

$$\|(A^\alpha - \widetilde{A}_\varepsilon^\alpha)u\|_{L^2} \to 0, \quad \varepsilon \to 0.$$

Association of non-regularized and regularized problems

Assume that there exists the solution, U_ε, of the non-regularized problem:

$$\partial_t U_\varepsilon(x,t) = K(x,t)^R\mathcal{D}_x^\alpha U_\varepsilon(x,t)$$
$$+ f(x,t,U_\varepsilon) + \sigma(x,t,U_\varepsilon)P_\varepsilon(x,t), \quad U_\varepsilon(0) = Q_\varepsilon,$$

where $K \in H^3(\mathbf{R})$ and $U_\varepsilon \in H^4(\mathbf{R})$, and let V_ε be a solution of the corresponding regularized equation with the same initial data:

$$\partial_t V_\varepsilon(x,t) = K(x,t)^R\mathcal{D}_x^\alpha V_\varepsilon(x,t) * \phi_{h_\varepsilon}(x)) + f(x,t,V_\varepsilon) +$$
$$\sigma(x,t,V_\varepsilon)P_\varepsilon(x,t), \quad V_\varepsilon(0) = Q_\varepsilon,$$

where h_ε, f and σ are as above.

Then, solutions U_ε and V_ε are L^2-associated almost surely, i.e., for every $T > 0$

$$\sup_{t\in[0,T)} \|U_\varepsilon(t) - V_\varepsilon(t)\|_{L^2} \to 0, \text{ as } \varepsilon \to 0, \text{ for almost all } \omega \in \Omega.$$

THANKS!