Wavelet expansions in Gelfand Shilov spaces

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joint work with S. Pilipović, N. Teofanov and J. Vindas

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Preliminaries

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- Gelfand-Shilov spaces are very convenient frameworks for investigation
- Continuity of the wavelet transform and convergence of wavelet and MRA expansions in Gelfand-Shilov spaces

S. Pilipović, D. Rakić, N. Teofanov, J. Vindas, *The Wavelet Transforms in Gelfand-Shilov spaces*, Collectanea Mathematica, 67 (3) (2016), 443–4605.

S. Pilipović, D. Rakić, N. Teofanov, J. Vindas, *Multiresolution expansions and wavelets in Gelfand-Shilov spaces*, RACSAM, 114:66 (2020).

Gelfand-Shilov spaces

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- A function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ belongs to the Gelfand-Shilov space $\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ if there exists a constant $h > 0$ such that

$$|x^\alpha \varphi^{(\beta)}(x)| \lesssim h^{-|\alpha+\beta|}\, \alpha!^{\rho_2} \beta!^{\rho_1}, \qquad x \in \mathbb{R}^d,\ \alpha, \beta \in \mathbb{N}^d.$$

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- The family of norms

$$p_h^{\rho_1,\rho_2}(\varphi) = \sup_{x\in\mathbb{R}^d,\alpha,\beta\in\mathbb{N}^d} \frac{h^{|\alpha+\beta|}}{\alpha!^{\rho_2}\beta!^{\rho_1}} |x^\alpha \partial^\beta \varphi(x)|, h > 0,$$

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  defines the canonical inductive limit topology of $\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$.
- The space $\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ is nontrivial if and only if $\rho_1 + \rho_2 > 1$ or $\rho_1 + \rho_2 = 1$ and $\rho_1, \rho_2 > 0$.

I. M. Gelfand, G. E. Shilov, *Generalized Functions II, III*, Academic Press, 1967.

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- Also, $\varphi \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ if and only if there exists a constant $h > 0$ and $c > 0$ such that

$$|\varphi^{(\beta)}(x)| \lesssim h^{-|\beta|}\beta!^{\rho_1} e^{-c|x|^{\frac{1}{\rho_2}}}, x \in \mathbb{R}^d, \beta \in \mathbb{N}^d.$$

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- By $\mathcal{D}^{\rho_1}(\mathbb{R}^d)$ is denoted the subspace of $\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ consisting of compactly supported Gevrey ultradifferentiable functions, it is non-trivial if and only if $\rho_1 > 1$ and $\mathcal{S}_0^{\rho_1}(\mathbb{R}^d) = \mathcal{D}^{\rho_1}(\mathbb{R}^d)$ and $\mathcal{S}_{\rho_2}^0(\mathbb{R}^d) = \mathcal{F}(\mathcal{D}^{\rho_2}(\mathbb{R}^d))$.

F. Nicola, L. Rodino, *Global Pseudo-Differential Calculus on Euclidean Spaces*, Pseudo-Differential Operators. Theory and Applications **4**, Birkhäuser Verlag, Basel, 2010.

Gelfand-Shilov spaces with moment condition

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- If an orthonormal wavelet belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, then all of its moments must vanish.
- By $(\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R}^d)$ is denoted the closed subspace of $\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ given by

$$(\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R}^d) = \left\{ \varphi \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x^\alpha \varphi(x) \mathrm{d}x = 0, \ \forall \alpha \in \mathbb{N}^d \right\}.$$

Space $(\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R}^d)$ is non-trivial if and only if $\rho_2 > 1$.

M. Holschneider, *Wavelets. An analysis tool.* The Clarendon Press, Oxford University Press, New York (1995)

E. Hernández, G. Weiss, *A first course on wavelets.* CRC Press, Boca Raton (1996)

The wavelet transform

The wavelet transform

- If $\psi \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ then the wavelet transform of an ultradistribution $f \in (\mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d))'$ with respect to the wavelet ψ is defined by

$$\mathcal{W}_\psi f(b,a) = \left\langle f(x), \frac{1}{a^d}\bar{\psi}\left(\frac{x-b}{a}\right)\right\rangle = \frac{1}{a^d}\int_{\mathbb{R}^d} f(x)\bar{\psi}\left(\frac{x-b}{a}\right)\,\mathrm{d}x,$$

where $(b,a) \in \mathbb{H}^{d+1} = \mathbb{R}^d \times \mathbb{R}_+$.

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where $(b,a) \in \mathbb{H}^{d+1} = \mathbb{R}^d \times \mathbb{R}_+$.

- Let $s > \rho_1 \geq 0$ and $t > \rho_2 > 0$. If $\psi \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$ and B is a bounded set in $(\mathcal{S}_t^s(\mathbb{R}^d))'$, then for each $k > 0$,

$$|\mathcal{W}_\psi f(b,a)| \lesssim e^{k\left(a^{\frac{1}{t-\rho_2}} + (\frac{1}{a})^{\frac{1}{s-\rho_1}} + |b|^{\frac{1}{t}}\right)}, \qquad (b,a) \in \mathbb{H}^{d+1},$$

uniformly for $f \in B$.

Continuity of the wavelet transform in Gelfand-Shilov spaces

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- A smooth function Φ belongs to $\mathcal{S}^s_{t,\tau_1,\tau_2}(\mathbb{H}^{d+1})$ if for every $\alpha \in \mathbb{N}$ there exists a constant $h > 0$ such that

$$\left|\partial_a^\alpha \partial_b^\beta \Phi(b,a)\right| \lesssim h^{|\beta|}\, \beta!^s\, e^{h\left(a^{1/\tau_1}+a^{-1/\tau_2}+|b|^{1/t}\right)},\ (b,a) \in \mathbb{H}^{d+1},\ \beta \in \mathbb{N}^d.$$

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- Let $\rho_1 \geq 0$, $\rho_2 > 1$, $\sigma = \rho_1 + \rho_2 - 1 > 0$ and let $s > \sigma$, $t > \sigma + 1$. Then the wavelet mapping

$$\mathcal{W} : (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R}^d) \times (\mathcal{S}^{s-\sigma}_{t-\sigma})_0(\mathbb{R}^d) \to \mathcal{S}^s_{t,t-\rho_2,s-\rho_1}(\mathbb{H}^{d+1}),$$

given by $\mathcal{W} : (\psi, \varphi) \mapsto \mathcal{W}_\psi \varphi$ is continuous.

Orthonormal wavelet

Orthonormal wavelet

- A function $\psi \in L^2(\mathbb{R})$ is called an orthonormal wavelet if $\{\psi_{m,n} : m \in \mathbb{Z}, n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where

$$\psi_{m,n}(x) = 2^{\frac{m}{2}}\psi(2^m x - n), \qquad m, n \in \mathbb{Z},\ x \in \mathbb{R}.$$

If additionally $\psi \in L^1(\mathbb{R})$, then $\int_{\mathbb{R}} \psi(x)dx = 0$.

Multiresolution analysis

Multiresolution analysis

- A multiresolution analysis (MRA) is an increasing sequence $\{V_m\}_{m\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $\bigcap_{m\in\mathbb{Z}} V_m = \{0\}$ and $\bigcup_{m\in\mathbb{Z}} V_m$ is dense in $L^2(\mathbb{R}^d)$;

(ii) $f(x) \in V_m \Leftrightarrow f(2x) \in V_{m+1},\ m \in \mathbb{Z}$;

(iii) $f(x) \in V_0 \Leftrightarrow f(x-n) \in V_0,\ \ n \in \mathbb{Z}^d$;

(iv) there exists a scaling function $\phi \in L^2(\mathbb{R}^d)$ such that $\{\phi(x-n)\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis of V_0.

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(iv) there exists a scaling function $\phi \in L^2(\mathbb{R}^d)$ such that $\{\phi(x-n)\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis of V_0.

- MRA is a classical way to construct orthonormal wavelets. Moreover, any orthonormal wavelet from $\mathcal{S}(\mathbb{R})$ is an MRA wavelet.

E. Hernández, G. Weiss, *A first course on wavelets.* CRC Press, Boca Raton (1996)

(ρ_1, ρ_2)-regular orthonormal wavelet

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- Let $\rho_1 \geq 0$ and $\rho_2 > 1$. An MRA is called (ρ_1, ρ_2)-regular if it possesses a scaling function $\phi \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R}^d)$.

(ρ_1, ρ_2)-regular orthonormal wavelet

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- An orthonormal wavelet ψ is called (ρ_1, ρ_2)-regular if $\psi \in (\mathcal{S}_{\rho_2}^{\rho_1})(\mathbb{R})$ and if it arises from a (ρ_1, ρ_2)-regular MRA.

Y. Meyer, *Wavelets and operators.* Cambridge University Press, Cambridge (1992)

S. Pilipović, N. Teofanov, *Multiresolution expansion, approximation order and quasiasymptotic of tempered distributions.* J. Math. Anal. Appl. 331, 455–471 (2007)

S. Kostadinova, J. Vindas, *Multiresolution expansions of distributions: pointwise convergence and quasiasymptotic behavior.* Acta Appl. Math. 138, 115–134 (2015)

K. Saneva, J. Vindas, *Wavelet expansions and asymptotic behavior of distributions.* J. Math. Anal. Appl. 370, 543–554 (2010)

Example

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- Let $\rho_2 > 1$ and $a < \pi/3$. Take $\varphi \in \mathcal{D}^{\rho_2}(\mathbb{R})$ such that $\operatorname{supp} \varphi \subseteq [-a, a]$, $\int_{-\infty}^{\infty} \varphi(\xi)\, d\xi = \pi/2$ and $\varphi_2(\xi) = (1/2)\varphi(\xi/2)$. The bell type function is defined by

$$b(\xi) = \sin\Big(\int_{-\infty}^{\xi-\pi} \varphi(t)dt\Big) \cos\Big(\int_{-\infty}^{\xi-2\pi} \varphi_2(t)dt\Big),$$

when $\xi > 0$ and extended evenly to $(-\infty, 0]$.

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  when $\xi > 0$ and extended evenly to $(-\infty, 0]$.
- Then, $b \in \mathcal{D}^{\rho_2}(\mathbb{R})$ and

$$\operatorname{supp} b \subseteq [-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3].$$

P.G. Lemarié, Y. Meyer, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamericana 2 (1986), 1–18.

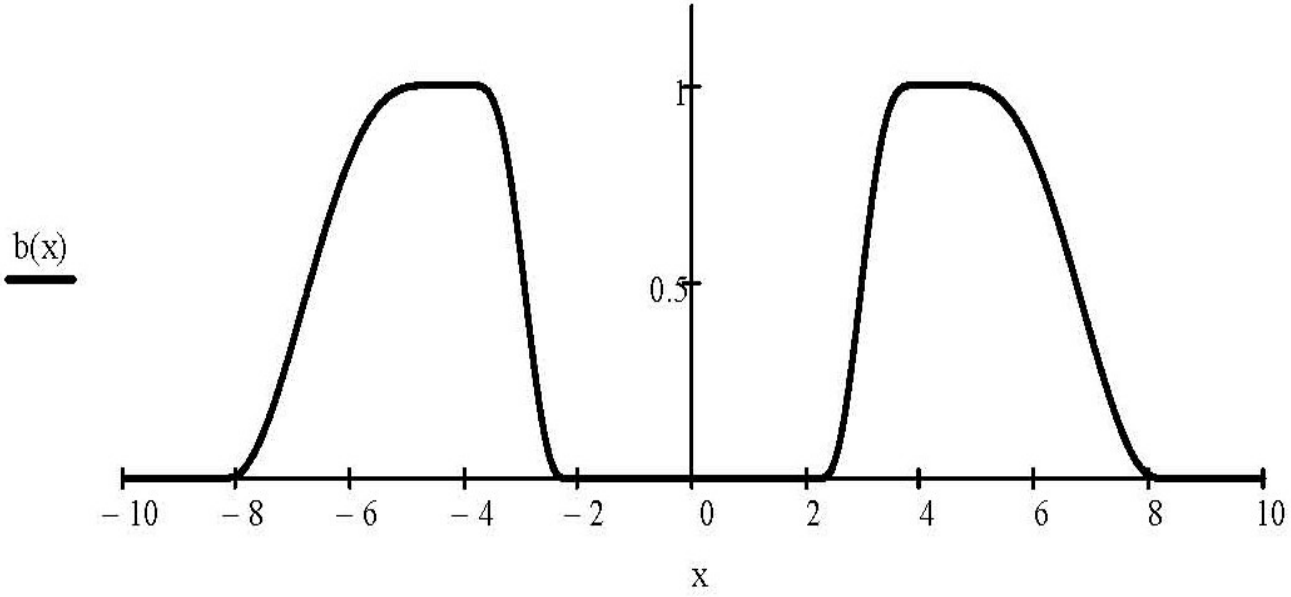
J. Dziubański, E. Hernández, *Band-limited wavelets with subexponential decay*, Canad. Math. Bull. 41 (1998), 398–403.

Example

The associated orthonormal wavelet is

$$\hat{\psi}(\xi) = e^{i\xi/2} b(\xi), \quad \xi \in \mathbb{R},$$

and $\psi \in \mathcal{F}(\mathcal{D}^{\rho_2}(\mathbb{R})) \subseteq \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R})$, for all $\rho_1 \geq 0$.



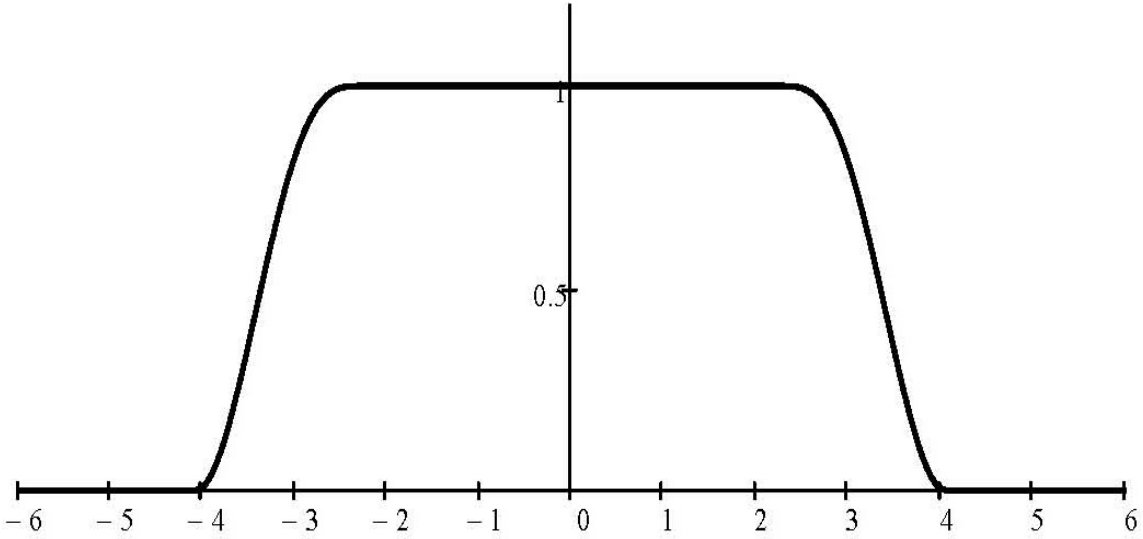
Example

Example

- An associated scaling function could be defined as

$$|\hat{\phi}(\xi)|^2 = \begin{cases} 1 & \text{if} \quad |\xi| \leq 2\pi/3, \\ b^2(2\xi) & \text{if} \quad 2\pi/3 \leq |\xi| \leq 4\pi/3, \\ 0 & \text{if} \quad |\xi| \geq 4\pi/3, \end{cases}$$

and $\arg \hat{\phi}(\xi) = \xi$. Then $\phi \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R})$, for any $\rho_1 \geq 0$.



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- Let $\psi \in (\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R})$ be a (ρ_1, ρ_2)-regular orthonormal wavelet, with scaling function $\phi \in \mathcal{S}_{\rho_2}^{\rho_1}(\mathbb{R})$. Take

$$\psi_\lambda(x) = \psi_{\epsilon,m,n}(x) = 2^{md/2}\psi_\epsilon(2^m x - n),\ x \in \mathbb{R}^d,\ \lambda \in \Lambda,$$

where $\Lambda = Q \times \mathbb{Z} \times \mathbb{Z}^d$, $Q = \{0,1\}^d \setminus (0,\dots,0)$.

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where $\Lambda = Q \times \mathbb{Z} \times \mathbb{Z}^d$, $Q = \{0,1\}^d \setminus (0,\ldots,0)$.

- Then, $\psi_\lambda \in (\mathcal{S}_{\rho_2}^{\rho_1})_0(\mathbb{R}^d)$ and $\{\psi_\lambda \mid \lambda \in \Lambda\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. The wavelet coefficients could be expressed in terms of the wavelet transform

$$c_\lambda^\psi(\varphi) = \langle \varphi, \bar{\psi}_\lambda \rangle = 2^{-\frac{md}{2}}\, \mathcal{W}_\psi \varphi(n2^{-m}, 2^{-m}),\ \lambda \in \Lambda.$$

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- $\{c_\lambda\}_{\lambda\in\Lambda}$ is in space of rapidly decreasing multi-sequences $\mathcal{W}^s_{t,\rho_1,\rho_2}(\Lambda)$ if and only if there exists $k \in \mathbb{N}$ such that the norm

$$\|\{c_\lambda\}\|_k^{\mathcal{W}^s_{t,\rho_1,\rho_2}} := \sup_{\lambda\in\Lambda} |c_\lambda| e^{k\left(\left(\frac{1}{2^m}\right)^{\frac{1}{t-\rho_2}} + (2^m)^{\frac{1}{s-\rho_1}} + \left|\frac{n}{2^m}\right|^{\frac{1}{t}}\right)}.$$

is finite.

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  is finite.

- Let $\psi \in (\mathcal{S}^{\rho_1}_{\rho_2})_0(\mathbb{R})$ be a (ρ_1, ρ_2)-regular orthonormal wavelet, where $\rho_1 \geq 0$ and $\rho_2 > 1$. Set $\sigma = \rho_1 + \rho_2 - 1$ and let $s > \sigma$ and $t > \sigma + 1$. Then, for every $h > 0$ there is $k > 0$ such that, for every $\varphi \in (\mathcal{S}^{s-\sigma}_{t-\sigma})_0(\mathbb{R}^d)$ with $p_h^{s-\sigma,t-\sigma}(\varphi) < \infty$,

$$\|\{c_\lambda^\psi(\varphi)\}\|_k^{\mathcal{W}^s_{t,\rho_1,\rho_2}} \lesssim p_h^{s-\sigma,t-\sigma}(\varphi).$$

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- If $\varphi \in (\mathcal{S}_{t-\sigma}^{s-\sigma})_0(\mathbb{R}^d)$, then

$$\varphi = \sum_{\lambda \in \Lambda} c_\lambda^\psi(\varphi)\, \psi_\lambda \text{ converges in } (\mathcal{S}_t^s)_0(\mathbb{R}^d).$$

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- If $f \in ((\mathcal{S}_t^s)_0(\mathbb{R}^d))'$, then its wavelet series expansion

$$f = \sum_{\lambda \in \Lambda} c_\lambda^\psi(f) \psi_\lambda$$

converges in (the strong dual topology of) $((\mathcal{S}_{t-\sigma}^{s-\sigma})_0(\mathbb{R}^d))'$.

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converges in (the strong dual topology of) $((\mathcal{S}_{t-\sigma}^{s-\sigma})_0(\mathbb{R}^d))'$.
- We have the Parseval identity

$$\langle f, \varphi \rangle = \sum_{\lambda \in \Lambda} c_\lambda^\psi(f)\, c_\lambda^{\bar{\psi}}(\varphi), \quad f \in ((\mathcal{S}_t^s)_0(\mathbb{R}^d))',\ \varphi \in (\mathcal{S}_{t-\sigma}^{s-\sigma})_0(\mathbb{R}^d).$$

References

J. Dziubański, E. Hernández, *Band-limited wavelets with subexponential decay.* Canad. Math. Bull. 41, 398–403 (1998)

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Y. Meyer, *Wavelets and operators.* Cambridge University Press, Cambridge (1992)

P.G. Lemarié, Y. Meyer, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamericana 2 (1986), 1–18.

F. Nicola, L. Rodino, *Global Pseudo-Differential Calculus on Euclidean Spaces*, Pseudo-Differential Operators. Theory and Applications **4**, Birkhäuser Verlag, Basel, 2010.

S. Pilipović, D. Rakić, N. Teofanov, J. Vindas, *The Wavelet Transforms in Gelfand-Shilov spaces*, Collectanea Mathematica, 67 (3) (2016), 443–4605.

S. Pilipović, D. Rakić, N. Teofanov, J. Vindas, *Multiresolution expansions and wavelets in Gelfand-Shilov spaces*, RACSAM, 114:66 (2020)

S. Pilipović, N. Teofanov, *Multiresolution expansion, approximation order and quasiasymptotic of tempered distributions.* J. Math. Anal. Appl. 331, 455–471 (2007)

K. Saneva, J. Vindas, *Wavelet expansions and asymptotic behavior of distributions.* J. Math. Anal. Appl. 370, 543–554 (2010)

Thank you for your attention !