Certain aspects of bilinear operators

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joint works with A. Abdeljawad and S. Coriasco/
F. Bastianoni / P. Balazs

The lecture is dedicated to Academician Stevan Pilipović
on the occasion of his 70th birthday.

- 1989 prof. Pilipović gave lectures on Analysis II.
  That was the most exciting course of my graduate studies.
- 1990's during my postgraduate studies I had a privilege to witness his amazing passion for scientific research.[1]
- 2000's the first steps of my scientific career under the influence of prof./Academician Pilipović, in parallel to my first serious international cooperation(s).
- since 2010's the new and equally exciting phase of our cooperation.

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part I
some results from

A. Abdeljawad, S. Coriasco, N. Teofanov,
Bilinear Pseudo-differential Operators with GevreyHörmander Symbols,
Mediterr. J. Math. 17 (August 2020), 120.

- The pseudo-differential operator $\mathrm{Op}_t(a)$, $t \in [0,1]$, is the linear and continuous operator on $\mathcal{S}(\mathbf{R}^d)$, defined by the formula

$$\mathrm{Op}_t(a)f(x) = \iint a(x - t(x-y), \xi) f(y) e^{2\pi i \langle x-y, \xi \rangle}\, dy d\xi, \quad x \in \mathbf{R}^d,$$

  and the definition extends uniquely to $a \in \mathcal{S}'(\mathbf{R}^{2d})$.
  Then $\mathrm{Op}_t(a)$ is continuous from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$.
- The above formula establishes the connection between the *symbol* a and the operator Op_t.[2]
- By choosing $t = 0$ and $t = 1/2$ we get the Kohn-Nirenberg and the Weyl correspondence respectively.

What about the symbol?

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- Hörmander classes are the common choice in the context of PDEs.
- In signal analysis (as explained in e.g. T. Strohmer's paper (ACHA, 2006)) symbols with "no reference to derivatives" appear more naturally.[3]
- We consider some symbol-global type symbols, closely related to Gelfand-Shilov type spaces.
- More precisely, we consider $a \in C^\infty(\mathbf{R}^{d_0+\cdots+d_k})$ which obey conditions of the form

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \dots, \partial_{\xi_k}^{\beta_k} a(x, \xi_1, \dots, \xi_k)| \lesssim h^{|\alpha+\beta_1+\cdots+\beta_k|} \alpha!^\sigma \prod_{j=1}^k \beta_j!^{s_j} \cdot \omega(x, \xi_1, \dots, \xi_k),$$

$\omega \in \mathcal{P}_E(\mathbf{R}^{d_0+d_1+\cdots+d_k})$, $\alpha \in \mathbf{N}^{d_0}$, $\beta_j \in \mathbf{R}^{d_j}$, $s_j, \sigma, h > 0, j = 1, \dots, k$.

- $a \in \Gamma_{(\omega)}^{\sigma, s_1, \dots, s_k}(\mathbf{R}^{d_0+\cdots+d_k})$ if the above condition holds for *some* $h > 0$.

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- A (continuous) function ω is called a *weight* on $\mathbf{R}^d$, if $\omega, 1/\omega \in L^\infty_{loc}(\mathbf{R}^d)$ are positive everywhere.
  If ω and v are weights on $\mathbf{R}^d$. Then ω is called *v-moderate* or *moderate*, if

$$\omega(x_1 + x_2) \lesssim \omega(x_1)v(x_2), \quad x_1, x_2 \in \mathbf{R}^d.$$

- By $\mathcal{P}_E(\mathbf{R}^{d_0+\cdots+d_k})$ we denote the set of all moderate weights on $\mathbf{R}^{d_0+\cdots+d_k}$, and $\mathcal{P}^0_{s_0,\dots,s_k}(\mathbf{R}^{d_0+\cdots+d_k})$ ($\mathcal{P}_{s_0,\dots,s_k}(\mathbf{R}^{d_0+\cdots+d_k})$) is the set of all weights $\omega \in \mathcal{P}_E(\mathbf{R}^{d_0+\cdots+d_k})$ such that

$$\omega(x_0 + y_0, \dots, x_k + y_k) \lesssim \omega(x_0, \dots, x_k) e^{r(|y_0|^{\frac{1}{s_0}} + \cdots + |y_k|^{\frac{1}{s_k}})}, \; x_j, y_j \in \mathbf{R}^{d_j}$$

  holds for every (for some) $r > 0$.
- If $\omega \in \mathcal{P}_E(\mathbf{R}^{d_0+d_1+\cdots+d_k})$, then there exists an "equivalent" weight $\omega_0 \in \mathcal{P}_E(\mathbf{R}^{d_0+d_1+\cdots+d_k}) \cap C^\infty(\mathbf{R}^{d_0+d_1+\cdots+d_k})$.

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- Symbols considered in this lecture are related to pseudodifferential operators of infinite order considered in the context of Gevrey type regularity, so we call them Gervey–Hörmander symbols.
- We refer to the work of Liess–Rodino, Matsuzawa, Zanghirati in 80's, and Boutet de Monvel, Krée and Volevic even before.
  In the 21st century similar "symbol global type" operators are considered by Coriasco, Cappiello, Pilipović, Teofanov, Prangoski, Toft, and others.
- In particular, our results can be considered as a bilinear extension of

M. Cappiello, J. Toft, *Pseudo-differential operators in a Gelfand-Shilov setting*, Math. Nachr. 290 (2017), 738–755.

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  However, due to the global nature of Gervey–Hörmander symbol classes, we apply a different technique to prove the continuity properties of considered operators when acting on modulation spaces.

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- Let $r, t \in [0,1]$, $r + t \leq 1$, let the bilinear symbol be given by

$$a_{r,t}(x,y,z,\xi,\eta) = a(x + r(y-x) + t(z-x), \xi, \eta),$$

and let the phase function ψ be defined by

$$\psi(x,y,z,\xi,\eta) = \langle y-x, \xi \rangle + \langle z-x, \eta \rangle, \quad x,y,z,\xi,\eta \in \mathbf{R}^d.$$

- Then the bilinear pseudo-differential operator $\mathrm{Op}_{r,t}(a)$ is defined by

$$\left(\mathrm{Op}_{r,t}(a)(f,g)\right)(x) = \iiiint e^{-2\pi i \psi(x,y,z,\xi,\eta)} a_{r,t}(x,y,z,\xi,\eta) f(y) g(z)\, dy dz d\xi d\eta, \; x \in \mathbf{R}^d. \quad (1)$$

- $\mathrm{Op}_{r,t}(a)$ is bilinear and continuous from $\mathcal{S}(\mathbf{R}^d) \otimes \mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$.

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$$\left(\mathrm{Op}_{r,t}(a)(f,g)\right)(x) =$$
$$\iiiint e^{-2\pi i \psi(x,y,z,\xi,\eta)} a_{r,t}(x,y,z,\xi,\eta) f(y) g(z)\, dy dz d\xi d\eta,\ x \in \mathbf{R}^d. \quad (1)$$

- $\mathrm{Op}_{r,t}(a)$ is bilinear and continuous from $\mathcal{S}(\mathbf{R}^d) \otimes \mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}'(\mathbf{R}^d)$.

- Let $r, t \in [0,1]$, $r+t \leq 1$, let the bilinear symbol be given by

$$a_{r,t}(x,y,z,\xi,\eta) = a(x + r(y-x) + t(z-x), \xi, \eta),$$

and let the phase function ψ be defined by

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- If $r = t = 0$, then the definition of $\mathrm{Op}_{0,0}(a)$ coincides with the bilinear pseudo-differential operator

$$T_a(f,g)(x) = \iint e^{2\pi i\langle x,\xi+\eta\rangle} a(x,\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)\, d\xi d\eta, \quad x \in \mathbf{R}^d,$$

considered by Bényi et al. (2005). The corresponding multilinear extension is studied by Molahajloo et al. (2016).

- Recall, if $k \in \mathbf{N}$, $\sigma = (\sigma_0, \dots, \sigma_k) > 0$, $\mathbf{s} = (s_0, \dots, s_k) > 0$, and $\mathbf{d} = d_0 + \cdots + d_k$. Then $F \in \mathcal{S}_{\mathbf{s}}^{\sigma}(\mathbf{R}^{\mathbf{d}})$ if

$$|\partial^\alpha F(x_0, \dots, x_k)| \lesssim h^{|\alpha|} \prod_{j=0}^{k} \alpha_j^{\sigma_j} e^{-r\left(|x_0|^{\frac{1}{s_0}} + \cdots + |x_k|^{\frac{1}{s_k}}\right)},$$

for every $\alpha = (\alpha_0, \dots, \alpha_k) \in \mathbf{N}^{\mathbf{d}}$, and for some $h, r > 0$.

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Theorem

Let there be given $s, \sigma > 0$ such that $s + \sigma \geq 1$, $\omega \in \mathcal{P}^0_{s,\sigma,\sigma}(\mathbf{R}^{3d})$, $r, t \in [0, 1]$, such that $r + t \leq 1$, and $a \in \Gamma^{\sigma,s,s}_{(\omega)}(\mathbf{R}^{3d})$.
Then $\mathrm{Op}_{r,t}(a)$ is continuous from $\mathcal{S}^\sigma_s(\mathbf{R}^d) \times \mathcal{S}^\sigma_s(\mathbf{R}^d)$ to $\mathcal{S}^\sigma_s(\mathbf{R}^d)$, and from $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d) \times (\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$ to $(\mathcal{S}^\sigma_s)'(\mathbf{R}^d)$.

Recall, $\omega \in \mathcal{P}^0_{s,\sigma,\sigma}(\mathbf{R}^{3d})$ means that

$$\omega(x_1 + x_2, \xi_1 + \xi_2, \eta_1 + \eta_2) \lesssim \omega(x_1, \xi_1, \eta_1) e^{r(|x_2|^{\frac{1}{s}} + |\xi_2|^{\frac{1}{\sigma}} + |\eta_2|^{\frac{1}{\sigma}})},$$

for every $r > 0$ and $a \in \Gamma^{\sigma,s,s}_{(\omega)}(\mathbf{R}^{3d})$ means that

$$|\partial^\alpha_x \partial^\beta_\xi \partial^\gamma_\eta a(x, \xi, \eta)| \lesssim h^{|\alpha+\beta+\gamma|} \alpha!^\sigma \beta!^s \gamma!^s \cdot \omega(x, \xi, \eta).$$

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Ingredients of the proof:

- Invariance property which implies that it is enough to consider the case $r = t = 0$, i.e. $\mathrm{Op}_{0,0}(a)(f, g)$.
- Use the idea of Bény et. al. and consider the linear pseudo-differential operator

$$\mathrm{Op}_{0,0}(a)(f, g)(x) = \int e^{2\pi i\langle x,\xi\rangle} a_g(x, \xi)\widehat{f}(\xi)\, d\xi,$$

where

$$a_g(x, \xi) = \int e^{2\pi i\langle x,\eta\rangle} a(x, \xi, \eta)\widehat{g}(\eta)\, d\eta.$$

- Show that $a_g \in \Gamma^{\sigma,s}_{(\tilde{\omega})}(\mathbf{R}^{2d})$ with $\tilde{\omega}(x, \xi) = \omega(x, \xi, 0) \in \mathcal{P}^0_{s,\sigma}(\mathbf{R}^{2d})$.
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The invariance property:

- Let $s_j, \sigma_j, j = 1, 2, 3$, be such that

$$s_j + \sigma_j \ge 1, \quad 0 < s_2,\, s_3 \le s_1, \quad \text{and} \quad 0 < \sigma_1 \le \sigma_2,\, \sigma_3 \tag{2}$$

  and let $r, t \in [0, 1]$ be such that $r + t \le 1$. Then $e^{-i\langle rD_\xi + tD_\eta, D_x\rangle}$ on $\mathcal{S}(\mathbf{R}^{3d})$ restricts to a homeomorphism on $\mathcal{S}_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3}(\mathbf{R}^{3d})$, and extends uniquely to a homeomorphism on $(\mathcal{S}_{s_1,\sigma_2,\sigma_3}^{\sigma_1,s_2,s_3})'(\mathbf{R}^{3d})$.
  If, in addition, $\omega \in \mathcal{P}^0_{s_1,\sigma_2,\sigma_3}(\mathbf{R}^{3d})$, then $a \in \Gamma_{(\omega)}^{\sigma_1,s_2,s_3}(\mathbf{R}^{3d})$ if and only if

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Remark:
By the definition of $\Gamma_{(\omega)}^{s_0,s_1,\dots,s_k}(\mathbf{R}^{d_0+\cdots+d_k})$ we conclude that if $\omega(x_0,x_1,\dots,x_k)$ is chosen to be

$$e^{-r(|x_0|^{\frac{1}{\sigma_0}}+\cdots+|x_k|^{\frac{1}{\sigma_k}})},$$

then

$$\Gamma_{(\omega)}^{s_0,s_1,\dots,s_k}(\mathbf{R}^{d_0+\cdots+d_k})=\mathcal{S}_{\mathbf{s}}^{\sigma}(\mathbf{R}^{\mathbf{d}}).$$

If $\omega_r(x_0,x_1,\dots,x_k)=(1+|x_0|^2+\dots|x_k|^2)^{-r/2}$ instead, then

$$\cap_{r>0}\Gamma_{(\omega_r)}^{0}(\mathbf{R}^{d_0+\cdots+d_k})=\mathcal{S}(\mathbf{R}^{\mathbf{d}}).$$

part II
some remarks

- Let $r, t \in [0,1]$, $r+t \le 1$, and let (by a slight abuse of notation) $(r,t) \in \mathbf{R}^{2d}$ denote the vector with the first d coordinates equal to r, and the others equal to t. The we consider the bilinear symbol given by

$$
\begin{aligned}
a_{r,t}(x,y,w) &= a(x+(r,t)(y-x), w) \\
&= a((x_1 + r(y_1 - x_1), x_2 + t(y_2 - x_2), w_1, w_2)),
\end{aligned}
$$

$x = (x_1, x_2), y = (y_1, y_2), w = (w_1, w_2) \in \mathbf{R}^{2d}$.

- Then the bilinear pseudo-differential operator $\mathrm{Op}_{r,t}(a)$ is defined by

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\left(\mathrm{Op}_{r,t}(a)(\overrightarrow{f})\right)(x) = \iint e^{-2\pi i \langle y-x, w\rangle} a_{r,t}(x,y,w) f_1(y_1) f_2(y_2)\, dy dw, \ x \in \mathbf{R}^{2d}, \quad (4)
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where $\overrightarrow{f}(y) = f_1(y_1) f_2(y_2)$.

- Assume that $x_1 = x_2 = x \in \mathbf{R}^d$, $\mathcal{R} : (x,x) \mapsto x$ (the trace mapping), and let $\Phi_{r,t} : (x,y) \mapsto \mathcal{R}(x,x) + \langle (r,t), (y-x) \rangle$.
Then (by considering the diagonal $x_1 = x_2$) $\mathrm{Op}_{r,t}(a \circ \Phi_{r,t})$ is the bilinear operator considered in part I.

- Let $r, t \in [0,1]$, $r+t \leq 1$, and let (by a slight abuse of notation) $(r,t) \in \mathbf{R}^{2d}$ denote the vector with the first d coordinates equal to r, and the others equal to t. The we consider the bilinear symbol given by

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$$= \iint e^{-2\pi i \langle y-x, w\rangle} a_{r,t}(x,y,w) f_1(y_1) f_2(y_2)\, dy dw, \;\; x \in \mathbf{R}^{2d}, \quad (4)$$

where $\overrightarrow{f}(y) = f_1(y_1) f_2(y_2)$.

- Assume that $x_1 = x_2 = x \in \mathbf{R}^d$, $\mathcal{R} : (x,x) \mapsto x$ (the trace mapping), and let $\Phi_{r,t} : (x,y) \mapsto \mathcal{R}(x,x) + \langle (r,t), (y-x) \rangle$.
Then (by considering the diagonal $x_1 = x_2$) $\mathrm{Op}_{r,t}(a \circ \Phi_{r,t})$ is the bilinear operator considered in part I.

- Let $r, t \in [0,1]$, $r+t \leq 1$, and let $f_1, f_2, g_1, g_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$, $\vec{f} = (f_1, f_2)$ and $\vec{g} = (g_1, g_2)$. Then the (r,t)*–bilinear Wigner transform* $W_{r,t}(\vec{f}, \vec{g})$ is given by

$$\begin{aligned} W_{r,t}(\vec{f}, \vec{g})(x, \omega) &= W_r(f_1, g_1)(x_1, \omega_1) \otimes W_t(f_2, g_2)(x_2, \omega_2) \\ &= \int_{\mathbb{R}^{2d}} e^{-2\pi i \langle \omega, s \rangle} \prod_{j=1,2} f_j(x_j + rs_j) \overline{g_j(x_j - (1-r)s_j)}\, ds, \end{aligned} \quad (5)$$

$s = (s_1, s_2) \in \mathbb{R}^d \times \mathbb{R}^d$.

- Then following formula holds (for $a_{r,t} \in (\mathcal{S}^{(1)})(\mathbb{R}^{4d})$):

$$\langle \mathrm{Op}_{r,t}(a)(\overrightarrow{f}), \vec{g} \rangle = \langle a_{r,t}, W_{r,t}(\vec{g}, \vec{f}) \rangle,$$

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$$A_{a,r,t}^{\vec{\varphi},\vec{\phi}} := \mathrm{Op}_{r,t}(a * W_{r,t}(\vec{\phi}, \vec{\varphi})).$$

- Let there be given $\varphi_1, \varphi_2, f_1, f_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$. Then the tensorized short-time Fourier transform is given by

$$V_{\varphi_1 \otimes \varphi_2}(f_1 \otimes f_2)(x, \omega) = \int_{\mathbb{R}^{2d}} (f_1 \otimes f_2)(t) \overline{(M_{\omega_1} T_{x_1} \varphi_1 \otimes M_{\omega_2} T_{x_2} \varphi_2)(t)} dt,$$

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- Let $\mathcal{H}$ be the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ of complex Hilbert spaces, and $(X, \mu) = (X_1 \times X_2, \mu_1 \otimes \mu_2)$ be the product of measure spaces with $\sigma-$finite positive measures μ_1, μ_2. The mapping $F : X \to \mathcal{H}$ is called a *continuous bilinear frame* of $\mathcal{H}$ with respect to (X, μ), if

  1) F is weakly-measurable, i.e., for all $\vec{f} \in \mathcal{H}$,

  $$x = (x_1, x_2) \to \langle \vec{f}, F(x) \rangle$$

  is a measurable function on X;

  2) there exist constants $A, B > 0$ such that

  $$A\|\vec{f}\|^2 \leq \int_X |\langle \vec{f}, F(x) \rangle|^2 \, d\mu(x) \leq B\|\vec{f}\|^2, \quad \forall \vec{f} \in \mathcal{H}. \tag{6}$$

  The constants A and B are called *continuous frame bounds*. If $A = B$, then F is called a *tight* continuous frame, if $A = B = 1$ a *Parseval* frame. The mapping F is called the *Bessel mapping* if only the second inequality in (6) holds.

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- If $F = F_1 \otimes F_2$ and $G = G_1 \otimes G_2$ are Bessel mappings for $\mathcal{H}$ with respect to (X, μ) and $m : X \to \mathbb{C}$ is a measurable function, then the operator $\mathbf{M}_{m,F,G} : \mathcal{H} \to \mathcal{H}$ weakly defined by

$$\begin{aligned}\langle \mathbf{M}_{m,F,G}\vec{f}, \vec{g}\rangle &= \langle \mathbf{M}_{m,F_1\otimes F_2,G_1\otimes G_2}\vec{f}, \vec{g}\rangle \\ &= \int_{X_1}\int_{X_2} m(x_1, x_2)\langle \vec{f}, (F_1 \otimes F_2)(x)\rangle\langle (G_1 \otimes G_2)(x), \vec{g}\rangle d\mu(x) \quad (7)\end{aligned}$$

for all $\vec{f}, \vec{g} \in \mathcal{H}$, is called *continuous bilinear Bessel multiplier* of F and G with respect to the mapping m, called the *symbol*.

The following notation is to be understood in weak sense:

$$\mathbf{M}_{m,F,G}\vec{f} := \int_X m(x)\langle \vec{f}, F(x)\rangle G(x)d\mu(x).$$

- We observe that the localization operators are an example of (linear) multipliers in the above sense, cf. P. Balazs, D. Bayer, and A. Rahimi (J. Phys A, 2012).

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